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EQUI-RIEMANN AND EQUI-RIEMANN TYPE INTEGRABLE FUNCTIONS WITH VALUES IN A BANACH SPACE

Abstract

In this paper we study equi-Riemann and equi-Riemann-type integrability of a collection of functions defined on a closed interval of $\mathbb{R}$ with values in a Banach space. We obtain some properties of such collections and interrelations among them. Moreover we establish equi-integrability of different types of collections of functions. Finally, we obtain relations among equi-Riemann integrability with other properties of a collection of functions.

1 Introduction

Riemann integration of Banach space valued functions defined on a closed bounded interval of $\mathbb{R}$ was first studied by L.M. Graves [10]. R.A. Gordon, in his survey article [8], compiled many results of Graves and others, as for example, Alexiewicz and Orlicz [3]. In our paper [19], there is an extensive study of Riemann and Riemann-type integrable functions with values in a Banach space.

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The main object of the present paper is to study equi-Riemann and equi-Riemann-type integrability, as for example, equi-Darboux, equi-Riemann-Dunford, equi-Riemann-Pettis integrability of a collection of functions defined on a closed bounded interval of $\mathbb{R}$ with values in a Banach space. Equi-integrability for different types of integrable functions has been studied by many authors, e.g., [1], [9], [11], [16], [18]. These works, in fact, motivate us to think in this line.

Following Alewine and Schechter [1], we define, at the outset, some terminologies relating to a collection of functions, in terms of which we define equi-Riemann and equi-Riemann-type integrable collections of functions and study their properties and interrelations. We obtain some Cauchy type criteria for equi-Riemann-type integrable collections of functions. We show that equi-Riemann integrability of a collection of functions with values in a Banach space becomes equivalent to that of a collection of real-valued functions. These results help us to prove many ones related to equi-Riemann and equi-Riemann-type integrable collections of functions. We show that if a collection of functions is equi-integrable in any of the above senses and pointwise bounded at some point in the interval of definition, then it is uniformly bounded therein. We establish the equi-integrability of collections of functions possessing some special properties.

We conclude the paper with some relations between equi-Riemann integrability, Birkhoff property and Bourgain property of a collection of functions.

Motivation of this work lies in its vast applicability in the study of convergence of nets and sequences of Riemann and Riemann-type integrable functions.

2 Notations, definitions, and preliminaries

Throughout the paper, X stands for a real Banach space with dual X^* (any other Banach space appeared in this paper will also be assumed to be a real Banach space). The closed unit ball of X and X^* will be denoted by B_X and B_{X^*} respectively, i.e., $B_X = \{x \in X : \|x\| \leq 1\}$ and $B_{X^*} = \{x^* \in X^* : \|x^*\| \leq 1\}$. A subset Γ of B_{X^*} is said to be a norming set for X if $\|x\| = \sup\limits_{x^* \in \Gamma} |x^*(x)|$ for all $x \in X$. Also $[a, b]$ stands for a closed bounded interval of $\mathbb{R}$, Σ for the σ-algebra of the Lebesgue measurable subsets of $[a, b]$ and λ for the Lebesgue measure on Σ so that $([a, b], \Sigma, \lambda)$ becomes a complete finite measure space. The set of all functions defined on $[a, b]$ into X is denoted by $X^{[a,b]}$.

We adopt the usual definitions of partitions and related notations of [8]. For any $\delta > 0$, we say that a (tagged) partition $\mathcal{P}$ of $[a, b]$ is δ-fine if $|\mathcal{P}| < \delta$.

Let $f \in X^{[a,b]}$ and let $E \subset [a,b]$. Then the oscillation of the function f on E, denoted as $\omega(f,E)$, is defined as $\omega(f,E) = \sup\{\|f(u) - f(v)\| : u, v \in E\}$. Now for any tagged partition $\mathcal{P} = \{(s_i, [t_{i-1}, t_i]) : 1 \leq i \leq n\}$ of $[a,b], f(\mathcal{P})$ will denote the Riemann sum $\sum_{i=1}^{n} f(s_i)(t_i - t_{i-1})$, and for any partition $\mathcal{P} = \{t_i : 0 \leq i \leq n\}$ of $[a,b]$, $\omega(f,\mathcal{P})$ will denote the oscillatory sum $\sum_{i=1}^{n} \omega(f, [t_{i-1}, t_i])(t_i - t_{i-1})$.

A function $f \in X^{[a,b]}$ is said to be R_Δ (resp. R_δ) integrable on $[a,b]$ if there exists a vector z in X with the following property: for each $\epsilon > 0$ there exists a partition $\mathcal{P}_\epsilon$ of $[a,b]$ (resp. a $\delta > 0$) such that $\|f(\mathcal{P}) - z\| < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that refines $\mathcal{P}_\epsilon$ (resp. $\mathcal{P}$ is a δ-fine tagged partition of $[a,b]$). It is shown that a function f is R_Δ integrable if and only if it is R_δ integrable [8, p. 924, Theorem 3]. A function f is said to be Riemann integrable if it is either R_δ or R_Δ integrable and the vector z, in the above definition, is called the Riemann integral of f over $[a,b]$ and is denoted by

$$R\text{-}\int_a^b f(t)dt \text{ or simply by } R\text{-}\int_a^b f dt.$$

A function $f \in X^{[a,b]}$ is said to be D_Δ (resp. D_δ) integrable on $[a,b]$ if for each $\epsilon > 0$ there exists a partition $\mathcal{P}_\epsilon$ of $[a,b]$ (resp. a $\delta > 0$) such that $\omega(f,\mathcal{P}) < \epsilon$ whenever $\mathcal{P}$ is a partition of $[a,b]$ that refines $\mathcal{P}_\epsilon$ (resp. $\mathcal{P}$ is a δ-fine partition of $[a,b]$). As in the case of R_δ and R_Δ integrable functions it can be shown that a function in $X^{[a,b]}$ is D_Δ integrable if and only if it is D_δ integrable. A function f is said to be Darboux integrable if it is either D_δ or D_Δ integrable. It can be shown that a Darboux integrable function is Riemann integrable and in this case the Riemann integral of f over $[a,b]$ is defined to be the Darboux integral of f over $[a,b]$.

A function $f \in X^{[a,b]}$ is said to be scalarly Riemann integrable on $[a,b]$ if x^*f is Riemann integrable on $[a,b]$ for each $x^* \in X^*$; it is said to be Riemann-Pettis integrable (in short, RP-integrable) on $[a,b]$ if it is scalarly Riemann integrable and Pettis integrable on $[a,b]$ with respect to Lebesgue measure. It is known that a scalarly Riemann integrable function is bounded and Dunford integrable and as such a scalarly Riemann integrable function is known as Riemann-Dunford integrable. The Dunford (resp. Pettis) integral of a Riemann-Dunford (resp. Riemann-Pettis) integrable function over $[a,b]$ is defined to be its Riemann-Dunford (resp. Riemann-Pettis) integral over $[a,b]$.

The collections of all Riemann, Darboux, Riemann-Dunford, Riemann-Pettis and Pettis integrable (with respect to Lebesgue measure) functions in

$X^{[a,b]}$ will be denoted by $R([a,b],X), D([a,b],X), RD([a,b],X), RP([a,b],X)$ and $P([a,b],X)$ respectively. In particular, if $X=\mathbb{R}$, we write $R[a,b]$ for $R([a,b],X)$ and similarly for other such collections.

It is known that $D([a,b],X) \subset R([a,b],X) \subset RP([a,b],X) \subset RD([a,b],X) \subset l^{\infty}([a,b],X)$ and $RP([a,b],X) \subset P([a,b],X)$. Further, for a finite-dimensional space X, we have

$$D([a,b],X) = R([a,b],X) = RP([a,b],X) = RD([a,b],X) \subset l^{\infty}([a,b],X).$$

For some standard results on Riemann, Darboux, Riemann-Dunford and Riemann-Pettis integrable functions, we refer to [8] and [19]. For definitions and certain properties of Bochner, Dunford and Pettis integrable functions, we refer to [5].

If $f \in X^{[a,b]}$ is Riemann or Darboux or Riemann-Dunford or Riemann-Pettis integrable on $[a,b]$, then it is so on every closed subinterval of $[a,b]$. If $f \in R([a,b],X)$, then the function $F \in X^{[a,b]}$, defined by $F(t) = R\text{-}\int_a^t f dt$, $t \in [a,b]$, is called the indefinite Riemann integral of f. Indefinite integrals of other types of integrable functions are defined similarly. If each member of a collection of functions $\mathcal{F} \subset X^{[a,b]}$ is integrable in any of the above senses, then we denote, by $\mathcal{F}_1$, the collection of all indefinite integrals of members of $\mathcal{F}$.

Let $\mathcal{F} \subset X^{[a,b]}$ and let $\Gamma \subset X^*$. Then we write $Z_{\mathcal{F},\Gamma} = \{x^*f : x^* \in \Gamma, f \in \mathcal{F}\}$. Clearly $Z_{\mathcal{F},\Gamma} \subset \mathbb{R}^{[a,b]}$. If, in particular, $\Gamma = B_{X^*}$, then we denote $Z_{\mathcal{F},\Gamma}$ by $Z_{\mathcal{F}}$. If $\mathcal{F} = \{f\}$−a singleton set, then we write $Z_{f,\Gamma}$ and Z_f in places of $Z_{\{f\},\Gamma}$ and $Z_{\{f\}}$ respectively. For $x^* \in X^*$, we write $Z_{\mathcal{F},x^*}$ in place of $Z_{\mathcal{F},\{x^*\}}$.

3 Main results

We begin with the following definitions in the line of [1]:

Definition 3.1. *Let $f \in X^{[a,b]}$. Let $\mathcal{P}$ be any partition of $[a,b]$ and let $\delta > 0$. Then*

(1) $\theta_{\mathcal{P}}(f) = \sup\{\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| : \mathcal{P}_1, \mathcal{P}_2$ are tagged partitions of $[a,b]$ that refine $\mathcal{P}\}$.

(2) $\theta'_{\mathcal{P}}(f) = \sup\{\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| : \mathcal{P}_1, \mathcal{P}_2$ are tagged partitions of $[a,b]$ that have the same points as $\mathcal{P}\}$.

(3) $\theta_{\delta}(f) = \sup\{\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| : \mathcal{P}_1, \mathcal{P}_2$ are δ-fine tagged partitions of $[a,b]\}$.

(4) $\omega_{\mathcal{P}}(f) = \sup\{\omega(f, \mathcal{P}') : \mathcal{P}'$ is a partition of $[a,b]$ that refines $\mathcal{P}\}$.

(5) $\omega_\delta(f) = \sup\{\omega(f, \mathcal{P}) : \mathcal{P}$ *is a* δ*-fine partition of* $[a, b]\}$.

If $\mathcal{F} \subset X^{[a,b]}$, then we define $\theta_{\mathcal{P}}(\mathcal{F}) = \sup\limits_{f\in\mathcal{F}} \theta_{\mathcal{P}}(f)$. Similarly $\theta'_{\mathcal{P}}(\mathcal{F})$, $\theta_\delta(\mathcal{F})$, $\omega_{\mathcal{P}}(\mathcal{F})$, $\omega_\delta(\mathcal{F})$, $\omega(\mathcal{F}, \mathcal{P})$ are defined.

Lemma 3.2. *Let* $\mathcal{P}$ *and* $\mathcal{P}'$ *be any two tagged partitions of* $[a, b]$ *and let* $\mathcal{P}'$ *refine* $\mathcal{P}$. *Then* $\|f(\mathcal{P}) - f(\mathcal{P}')\| \leq \omega(f, \mathcal{P})$ *for each* $f \in X^{[a,b]}$.

Proof. Let $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i]) : 1 \leq i \leq n\}$ be a tagged partition of $[a, b]$. Let $u_0^k, u_1^k, \cdots, u_{n_k-1}^k, u_{n_k}^k$ be the points of $\mathcal{P}'$ in $[t_{k-1}, t_k]$, $k = 1, 2, \cdots, n$, i.e., $t_{k-1} = u_0^k < u_1^k < \cdots < u_{n_k-1}^k < u_{n_k}^k = t_k$, $k = 1, 2, \cdots, n$ and let v_i^k be the tag of $\mathcal{P}'$ in $[u_{i-1}^k, u_i^k]$, $i = 1, 2, \cdots, n_k$. Then for any $f \in X^{[a,b]}$, we have

$$\begin{aligned}
\|f(\mathcal{P}) - f(\mathcal{P}')\| &= \left\| \sum_{k=1}^{n} \left\{ f(\xi_k)(t_k - t_{k-1}) - \sum_{i=1}^{n_k} f(v_i^k)(u_i^k - u_{i-1}^k) \right\} \right\| \\
&\leq \sum_{k=1}^{n} \sum_{i=1}^{n_k} \|f(\xi_k) - f(v_i^k)\|(u_i^k - u_{i-1}^k) \\
&\leq \sum_{k=1}^{n} \sum_{i=1}^{n_k} \omega(f, [t_{k-1}, t_k])(u_i^k - u_{i-1}^k) \\
&= \sum_{k=1}^{n} \omega(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = \omega(f, \mathcal{P}).
\end{aligned}$$

□

Theorem 3.3. *Let* $\mathcal{F} \subset X^{[a,b]}$. *Then*

(a) for each partition $\mathcal{P}$ *of* $[a, b]$, $\theta'_{\mathcal{P}}(\mathcal{F}) \leq \theta_{\mathcal{P}}(\mathcal{F}) \leq 2\theta'_{\mathcal{P}}(\mathcal{F})$ *and* $\theta'_{\mathcal{P}}(\mathcal{F}) \leq \omega(\mathcal{F}, \mathcal{P}) = \omega_{\mathcal{P}}(\mathcal{F})$,

(b) for each $\delta > 0$, $\theta_{\mathcal{P}}(\mathcal{F}) \leq \theta_\delta(\mathcal{F}) \leq 2\omega_\delta(\mathcal{F})$ *and* $\omega(\mathcal{F}, \mathcal{P}) \leq \omega_\delta(\mathcal{F})$ *for any* δ*-fine partition* $\mathcal{P}$ *of* $[a, b]$.

Proof. (*a*) $\theta'_{\mathcal{P}}(\mathcal{F}) \leq \theta_{\mathcal{P}}(\mathcal{F})$: Obvious.

$\theta_{\mathcal{P}}(\mathcal{F}) \leq 2\theta'_{\mathcal{P}}(\mathcal{F})$: We follow the method of proof of [8, p. 925-926, Theorem 5]. Let $f \in \mathcal{F}$ and $\mathcal{P} = \{t_i : 0 \leq i \leq n\}$ be a partition of $[a, b]$. Let $\mathcal{P}_0$ be the tagged partition $\{(t_i, [t_{i-1}, t_i]) : 1 \leq i \leq n\}$ of $[a, b]$. For each i, let W_i be the set $\{(t_i - t_{i-1})f(t) : t \in [t_{i-1}, t_i]\}$ and let $W = \sum\limits_{i=1}^{n} W_i$.

Let $x \in W - W$. Then there exist two tagged partitions $\mathcal{P}_1, \mathcal{P}_2$ of $[a, b]$ that have the same points as $\mathcal{P}$ such that $x = f(\mathcal{P}_1) - f(\mathcal{P}_2)$ implying that

$\|x\| \leq \theta'_{\mathcal{P}}(\mathcal{F})$. Hence it follows that $\|x\| \leq \theta'_{\mathcal{P}}(\mathcal{F})$ for all $x \in co(W-W)$ where $co(W-W)$ is the convex hull of $W-W$.

Now let $\mathcal{P}_1, \mathcal{P}_2$ be any two tagged partitions of $[a,b]$ that refine $\mathcal{P}$. Then proceeding as in the proof of [8, p. 925-926, Theorem 5] we can show that

$$f(\mathcal{P}_0) - f(\mathcal{P}_1) \in co(W-W),\ f(\mathcal{P}_0) - f(\mathcal{P}_2) \in co(W-W).$$

Hence it follows from above that

$$\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| \leq \|f(\mathcal{P}_1) - f(\mathcal{P}_0)\| + \|f(\mathcal{P}_0) - f(\mathcal{P}_2)\| \leq 2\theta'_{\mathcal{P}}(\mathcal{F}).$$

Since $f \in \mathcal{F}$ is arbitrary and $\mathcal{P}_1, \mathcal{P}_2$ are arbitrary tagged partitions of $[a,b]$ that refine $\mathcal{P}$, the result follows.

$\theta'_{\mathcal{P}}(\mathcal{F}) \leq \omega(\mathcal{F}, \mathcal{P})$: For any partition $\mathcal{P}$ of $[a,b]$ and for any two tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a,b]$ that have the same points as $\mathcal{P}$, it is easy to note that $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| \leq \omega(f, \mathcal{P}) \leq \omega(\mathcal{F}, \mathcal{P})$ for all $f \in \mathcal{F}$. Taking supremum over all the tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a,b]$ that have the same points as $\mathcal{P}$ and over $\mathcal{F}$, the result follows.

$\omega(\mathcal{F}, \mathcal{P}) = \omega_{\mathcal{P}}(\mathcal{F})$: It is easy to note that for any partition $\mathcal{P}$ of $[a,b]$, $\omega(f, \mathcal{P}) = \omega_{\mathcal{P}}(f)$ for all $f \in \mathcal{F}$ whence the result follows by taking supremum over $\mathcal{F}$.

(b) $\theta_{\mathcal{P}}(\mathcal{F}) \leq \theta_\delta(\mathcal{F})$: Obvious.

$\theta_\delta(\mathcal{F}) \leq 2\omega_\delta(\mathcal{F})$: Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any two δ-fine tagged partitions of $[a,b]$. Let $\mathcal{P}$ be any tagged partition of $[a,b]$ that refines both $\mathcal{P}_1$ and $\mathcal{P}_2$. Then for any $f \in \mathcal{F}$, it follows from the previous lemma that

$$\begin{aligned}\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| &\leq \|f(\mathcal{P}_1) - f(\mathcal{P})\| + \|f(\mathcal{P}) - f(\mathcal{P}_2)\| \\ &\leq \omega(f, \mathcal{P}_1) + \omega(f, \mathcal{P}_2) \\ &\leq \omega_\delta(f) + \omega_\delta(f) \\ &= 2\omega_\delta(f).\end{aligned}$$

Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are arbitrary δ-fine partitions of $[a,b]$ and $f \in \mathcal{F}$ is arbitrary, the result follows.

$\omega(\mathcal{F}, \mathcal{P}) \leq \omega_\delta(\mathcal{F})$: Obvious. □

Lemma 3.4. *Let $\mathcal{F} \subset \mathbb{R}^{[a,b]}$. If each $f \in \mathcal{F}$ is bounded on $[a,b]$, then for each partition $\mathcal{P}$ of $[a,b]$, $\omega(\mathcal{F}, \mathcal{P}) = \theta'_{\mathcal{P}}(\mathcal{F})$.*

Proof. Let $\mathcal{P} = \{t_i : 0 \leq i \leq n\}$ be any partition of $[a,b]$ and let $\epsilon > 0$ be arbitrary.

Let $f \in \mathcal{F}$. Let $M_i,\ m_i$ be its supremum and infimum respectively on $[t_{i-1}, t_i],\ i = 1, 2, \cdots, n$. Then there exist points $\alpha_i, \beta_i \in [t_{i-1}, t_i]$ such that

$f(\alpha_i) > M_i - \frac{\epsilon}{2(b-a)}$ and $f(\beta_i) < m_i + \frac{\epsilon}{2(b-a)}$ for $i = 1, 2, \cdots, n$. Now $\mathcal{P}_1 = \{(\alpha_i, [t_{i-1}, t_i]) : 1 \leq i \leq n\}$ and $\mathcal{P}_2 = \{(\beta_i, [t_{i-1}, t_i]) : 1 \leq i \leq n\}$ are two tagged partitions of $[a, b]$ that have the same points as $\mathcal{P}$.

Now

$$\begin{aligned}
\omega(f, \mathcal{P}) &= \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) \\
&< \sum_{i=1}^{n} \left[f(\alpha_i) + \frac{\epsilon}{2(b-a)} - f(\beta_i) + \frac{\epsilon}{2(b-a)} \right] (t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} f(\alpha_i)(t_i - t_{i-1}) - \sum_{i=1}^{n} f(\beta_i)(t_i - t_{i-1}) + \epsilon \\
&\leq |f(\mathcal{P}_1) - f(\mathcal{P}_2)| + \epsilon \\
&\leq \theta'_{\mathcal{P}}(\mathcal{F}) + \epsilon.
\end{aligned}$$

Since $f \in \mathcal{F}$ is arbitrary, we have $\omega(\mathcal{F}, \mathcal{P}) \leq \theta'_{\mathcal{P}}(\mathcal{F}) + \epsilon$. Also by part (a) of the previous theorem, $\theta'_{\mathcal{P}}(\mathcal{F}) \leq \omega(\mathcal{F}, \mathcal{P})$. Since $\epsilon > 0$ is arbitrary, the result follows. □

Definition 3.5. *Let* $\mathcal{F} \subset X^{[a,b]}$.

(a) Let $E \subset [a, b]$*. Then the oscillation of* $\mathcal{F}$ *on* E*, denoted as* $\omega(\mathcal{F}, E)$*, is defined as* $\omega(\mathcal{F}, E) = \sup_{f \in \mathcal{F}} \omega(f, E)$.

(b) Let $s \in [a, b]$*. Then the oscillation of* $\mathcal{F}$ *at* s*, denoted as* $\omega(\mathcal{F}, s)$*, is defined as* $\omega(\mathcal{F}, s) = \sup_{f \in \mathcal{F}} \sup_{t \in [a,b]} \|f(t) - f(s)\|$.

It is evident that $\sup_{s \in [a,b]} \omega(\mathcal{F}, s) = \omega(\mathcal{F}, [a, b])$.

The following result is analogous to a result of Alewine and Schechter [1, p. 32, Corollary 6.2], the proof of which is straightforward and so omitted.

Lemma 3.6. *Let* $\mathcal{F} \subset X^{[a,b]}$*. Then* $\mathcal{F}$ *is uniformly bounded on* $[a, b]$ *in each of the following cases:*

(a) If the oscillation of $\mathcal{F}$ *at some point in* $[a, b]$ *is finite and if* $\mathcal{F}$ *is pointwise bounded at that point.*

(b) If $\mathcal{F}$ *is finite and its oscillation at some point in* $[a, b]$ *is finite.*

(c) If the oscillation of $\mathcal{F}$ *on* $[a, b]$ *is finite and if* $\mathcal{F}$ *is pointwise bounded at some point in* $[a, b]$.

Theorem 3.7. *Let $\mathcal{F} \subset X^{[a,b]}$. Let us consider the following statements:*

(a) The oscillation of $\mathcal{F}$ on $[a,b]$ is finite.

(b) $\omega(\mathcal{F},\mathcal{P})$ is finite for some/all partitions $\mathcal{P}$ of $[a,b]$.

(c) $\theta_{\mathcal{P}}(\mathcal{F})$ is finite for some/all partitions $\mathcal{P}$ of $[a,b]$.

(d) For each $\epsilon > 0$ and for each partition $\mathcal{P}$ of $[a,b]$, there exists a $\delta > 0$ such that $\omega_\delta(\mathcal{F}) < \omega(\mathcal{F},\mathcal{P}) + \epsilon$ and $\theta_\delta(\mathcal{F}) < 2\theta_{\mathcal{P}}(\mathcal{F}) + \epsilon$.

Then (a) $\Longleftrightarrow$(b) $\Longleftrightarrow$(c) $\Longrightarrow$(d).

PROOF. $(a) \Longrightarrow (b)$ & $(a) \Longrightarrow (c)$ For any partition $\mathcal{P}$ of $[a,b]$, it is easy to note that $\omega(\mathcal{F},\mathcal{P}) \leq (b-a)\omega(\mathcal{F},[a,b])$. Also by Theorem 3.3 (a), $\theta_{\mathcal{P}}(\mathcal{F}) \leq 2\omega(\mathcal{F},\mathcal{P})$. Hence the results follow.

$(b) \Longrightarrow (a)$ & $(c) \Longrightarrow (a)$ Let $\mathcal{P} = \{t_k : 0 \leq k \leq n\}$ be a partition of $[a,b]$ with $n+1$ points such that $\theta_{\mathcal{P}}(\mathcal{F})$ is finite and let δ be the length of the smallest sub-interval of $\mathcal{P}$. Let $f \in \mathcal{F}$. Then clearly

$$\|f(s) - f(t)\| \leq \frac{\theta_{\mathcal{P}}(\mathcal{F})}{\delta}$$

for all $s,t \in [t_{i-1},t_i], i = 1,2,...,n$. If $s \in [t_{j-1},t_j]$ and $t \in [t_{k-1},t_k]$ where $j < k$, then

$$\begin{aligned}\|f(s) - f(t)\| \leq \|f(s) - f(t_j)\| &+ \|f(t_j) - f(t_{j+1})\| + \|f(t_{j+1}) - f(t_{j+2})\| \\ &+ ... + \|f(t_k) - f(t)\| \\ \leq \frac{n\theta_{\mathcal{P}}(\mathcal{F})}{\delta}&.\end{aligned}$$

Thus for all $s,t \in [a,b]$,

$$\|f(s) - f(t)\| \leq \frac{n\theta_{\mathcal{P}}(\mathcal{F})}{\delta} \leq \frac{2n\omega(\mathcal{F},\mathcal{P})}{\delta}$$

by Theorem 3.3 (a), whence the results follow.

$(a) \Longrightarrow (d)$ First part: We follow the method of proof of [8, p. 924-925, Theorem 3].

Let the oscillation of $\mathcal{F}$ on $[a,b]$ be finite. Let $\epsilon > 0$ and let $\mathcal{P} = \{t_k : 0 \leq k \leq n\}$ be any partition of $[a,b]$ with $n+1$ points. Let $\delta = \frac{\epsilon}{4n[\omega(\mathcal{F},[a,b])+1]}$. Then $\delta > 0$ as $\omega(\mathcal{F},[a,b])$ is finite.

Let $\mathcal{P}'$ be a δ-fine partition of $[a,b]$ and let $\mathcal{P}_1 = \mathcal{P} \bigcup \mathcal{P}'$. Then $\mathcal{P}_1$ is a refinement of $\mathcal{P}'$. Let $\{[c_k,d_k] : 1 \leq k \leq K\}$ be the intervals of $\mathcal{P}'$ that contain points of $\mathcal{P}$ in their interiors. We note that $K \leq n-1$. In the interval $[c_k, d_k]$, let $c_k = u_0^k < u_1^k < \cdots < u_{n_k-1}^k < u_{n_k}^k = d_k$ where $\{u_i^k : 1 \leq i \leq n_k - 1\}$ are the points of $\mathcal{P}$ in (c_k, d_k). Then for any $f \in \mathcal{F}$, we have

$$\begin{aligned}
|\omega(f,\mathcal{P}')-&\omega(f,\mathcal{P}_1)| \\
&= \left|\sum_{k=1}^{K}\left\{\omega(f,[c_k,d_k])(d_k-c_k) - \sum_{i=1}^{n_k}\omega(f,[u_{i-1}^k,u_i^k])(u_i^k-u_{i-1}^k)\right\}\right| \\
&\leq \sum_{k=1}^{K}\sum_{i=1}^{n_k}|\omega(f,[c_k,d_k]) - \omega(f,[u_{i-1}^k,u_i^k])|(u_i^k-u_{i-1}^k) \\
&\leq \sum_{k=1}^{K}\sum_{i=1}^{n_k}[\omega(f,[c_k,d_k]) + \omega(f,[u_{i-1}^k,u_i^k])](u_i^k-u_{i-1}^k) \\
&\leq 2\omega(\mathcal{F},[a,b])\sum_{k=1}^{K}(d_k-c_k) \\
&\leq 2\omega(\mathcal{F},[a,b])(n-1)\delta \\
&< \frac{\epsilon}{2}
\end{aligned}$$

and hence

$$\begin{aligned}
\omega(f,\mathcal{P}') &\leq |\omega(f,\mathcal{P}') - \omega(f,\mathcal{P}_1)| + \omega(f,\mathcal{P}_1) \\
&< \frac{\epsilon}{2} + \omega(\mathcal{F},\mathcal{P}),
\end{aligned}$$

since $\mathcal{P}_1$ is a refinement of $\mathcal{P}$. Therefore taking supremum over $\mathcal{F}$ and over all δ-fine partitions $\mathcal{P}'$ of $[a,b]$, we have $\omega_\delta(\mathcal{F}) \leq \frac{\epsilon}{2} + \omega(\mathcal{F},\mathcal{P}) < \omega(\mathcal{F},\mathcal{P}) + \epsilon$.

Second part: The oscillation of $Z_\mathcal{F}$ is clearly finite. Hence by first part there is a $\delta > 0$ such that $\omega_\delta(Z_\mathcal{F}) < \omega(Z_\mathcal{F},\mathcal{P}) + \frac{\epsilon}{2}$. Hence by Theorem 3.3 and Lemma 3.4, we have

$$\begin{aligned}
\theta_\delta(\mathcal{F}) &= \theta_\delta(Z_\mathcal{F}) \\
&\leq 2\omega_\delta(Z_\mathcal{F}) \\
&< 2\left(\omega(Z_\mathcal{F},\mathcal{P}) + \frac{\epsilon}{2}\right) \\
&= 2\theta'_\mathcal{P}(Z_\mathcal{F}) + \epsilon \\
&\leq 2\theta_\mathcal{P}(Z_\mathcal{F}) + \epsilon \\
&= 2\theta_\mathcal{P}(\mathcal{F}) + \epsilon.
\end{aligned}$$

□

Definition 3.8. *A collection of functions, $\mathcal{F}$, in $X^{[a,b]}$ is said to be equi-Riemann integrable on $[a,b]$ if for each $f \in \mathcal{F}$ there exists a vector $z_f \in X$ with the following property: for any $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[a,b]$ such that for any $f \in \mathcal{F}$*

$$\|f(\mathcal{P}') - z_f\| < \epsilon$$

for all tagged partitions $\mathcal{P}'$ of $[a,b]$ that refine $\mathcal{P}$.

From the above definition, it follows that if a collection of functions, $\mathcal{F}$, in $X^{[a,b]}$, is equi-Riemann integrable on $[a,b]$, then $\mathcal{F} \subset R([a,b],X)$ and

$$z_f = R\text{-}\int_a^b f dt.$$

From the very definition, it follows that any finite collection of functions in $R([a,b],X)$ is equi-Riemann integrable on $[a,b]$ and any subcollection of an equi-Riemann integrable collection of functions in $X^{[a,b]}$ is equi-Riemann integrable on $[a,b]$. Also it is obvious that finite union of equi-Riemann integrable collections of functions is equi-Riemann integrable.

The following example shows that an arbitrary, even a countable, union of equi-Riemann integrable collections of functions is not necessarily equi-Riemann integrable.

Example 3.9. *Let us consider the collection $\mathcal{F} = \{f_n\}$ of real-valued functions defined on $[0,1]$ where*

$$f_n(t) = \begin{cases} n & if\ t = \frac{1}{n} \\ 0 & elsewhere \end{cases}$$

for $n \in \mathbb{N}$. Then it is clear that $f_n \in R[0,1]$. Thus $\mathcal{F}$ is the countable union of equi-Riemann integrable collections of functions. But it is easy to verify that $\mathcal{F}$ is not equi-Riemann integrable on $[0,1]$.

Theorem 3.10. *Let $\mathcal{F} \subset X^{[a,b]}$. Then the following statements are equivalent:*

(a) $\mathcal{F}$ is equi-Riemann integrable on $[a,b]$.

(b) For each $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $\theta_{\mathcal{P}}(\mathcal{F}) < \epsilon$.

(c) For each $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $\theta'_{\mathcal{P}}(\mathcal{F}) < \epsilon$.

(d) For each $\epsilon > 0$, there exists a $\delta > 0$ such that $\theta_{\delta}(\mathcal{F}) < \epsilon$.

PROOF. $(a) \Longrightarrow (b)$ Let $\mathcal{F}$ be equi-Riemann integrable on $[a,b]$. Let $\epsilon > 0$. Then there is a partition $\mathcal{P}$ of $[a,b]$ such that for any $f \in \mathcal{F}$

$$\left\| f(\mathcal{P}') - \left(R\text{-}\int_a^b f dt \right) \right\| < \frac{\epsilon}{4}$$

for all tagged partitions $\mathcal{P}'$ of $[a,b]$ that refine $\mathcal{P}$.

Let $\mathcal{P}_1, \mathcal{P}_2$ be any two tagged partitions of $[a,b]$ that refine $\mathcal{P}$. Then for any $f \in \mathcal{F}$, we have

$$\begin{aligned} \|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| &\leq \left\| f(\mathcal{P}_1) - \left(R\text{-}\int_a^b f dt \right) \right\| + \left\| f(\mathcal{P}_2) - \left(R\text{-}\int_a^b f dt \right) \right\| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} \end{aligned}$$

which implies that $\theta_{\mathcal{P}}(f) \leq \frac{\epsilon}{2}$. Hence taking supremum over $\mathcal{F}$, we have $\theta_{\mathcal{P}}(\mathcal{F}) \leq \frac{\epsilon}{2} < \epsilon$.

$(b) \Longrightarrow (a)$ Let (b) hold. Then by [8, p. 925, Theorem 5], $\mathcal{F} \subset R([a,b],X)$.

Now let $\epsilon > 0$. Then by hypothesis, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $\theta_{\mathcal{P}}(\mathcal{F}) < \frac{\epsilon}{2}$.

Let $f \in \mathcal{F}$. Then $f \in R([a,b],X)$ and hence there exists a partition $\mathcal{P}'$ of $[a,b]$ such that

$$\left\| f(\mathcal{P}'') - \left(R\text{-}\int_a^b f dt \right) \right\| < \frac{\epsilon}{2}$$

for all tagged partitions $\mathcal{P}''$ of $[a,b]$ that refine $\mathcal{P}'$.

Let $\mathcal{P}_0 = \mathcal{P} \cup \mathcal{P}'$. Let $\mathcal{P}_1$ be a tagged partition of $[a,b]$ that refines $\mathcal{P}$ and let $\mathcal{P}_2$ be another tagged partition of $[a,b]$ that refines $\mathcal{P}_0$. Then $\mathcal{P}_2$ refines both $\mathcal{P}$ and $\mathcal{P}'$. So we have

$$\begin{aligned} \left\| f(\mathcal{P}_1) - \left(R\text{-}\int_a^b f dt \right) \right\| &\leq \|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| + \left\| f(\mathcal{P}_2) - \left(R\text{-}\int_a^b f dt \right) \right\| \\ &< \theta_{\mathcal{P}}(\mathcal{F}) + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This is true for all $f \in \mathcal{F}$. Hence $\mathcal{F}$ is equi-Riemann integrable on $[a,b]$.

$(b) \Longleftrightarrow (c)$ Follows from Theorem 3.3 (a).

$(b) \Longrightarrow (d)$ From hypothesis it follows that $\theta_{\mathcal{P}}(\mathcal{F})$ is finite for some partition $\mathcal{P}$ of $[a,b]$ and the result follows from Theorem 3.7 ($(c) \Longrightarrow (d)$).

$(d) \Longrightarrow (b)$ Follows from Theorem 3.3 (b). □

From Theorem 3.10 ((a) $\Longrightarrow$ (b)) and Theorem 3.7 ((c) $\Longrightarrow$ (a)), we have the following result:

Corollary 3.11. *Oscillation of an equi-Riemann integrable collection of functions on its interval of definition is finite.*

Definition 3.12. *A collection of functions, $\mathcal{F}$, in $X^{[a,b]}$, is said to be equi-Darboux integrable on $[a,b]$ if for any $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $\omega(\mathcal{F},\mathcal{P}) = \omega_{\mathcal{P}}(\mathcal{F}) < \epsilon$.*

Theorem 3.13. *Let $\mathcal{F} \subset X^{[a,b]}$. Then $\mathcal{F}$ is equi-Darboux integrable on $[a,b]$ if and only if for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\omega_\delta(\mathcal{F}) < \epsilon$.*

Proof. Let $\mathcal{F}$ be equi-Darboux integrable on $[a,b]$. Then it follows that $\omega(\mathcal{F},\mathcal{P})$ is finite for some partition $\mathcal{P}$ of $[a,b]$ and hence the result follows from Theorem 3.7 ((b) $\Longrightarrow$ (d)).

The converse part follows from Theorem 3.3 (b). □

From the very definition and also from the above theorem, it follows that an equi-Darboux integrable collection of functions is contained in $D([a,b],X)$.

A Darboux integrable function in $X^{[a,b]}$ is Riemann integrable, but the converse is not necessarily true. For real-valued functions the two notions coincide. For a collection of functions we have the following analogous result:

Theorem 3.14. *Let $\mathcal{F} \subset X^{[a,b]}$. If $\mathcal{F}$ is equi-Darboux integrable on $[a,b]$, then it is equi-Riemann integrable on $[a,b]$. If $X = \mathbb{R}$, then the converse is also true.*

Proof. First part follows from Theorem 3.3 (a) and Theorem 3.10 ((c) $\Longrightarrow$ (a)).

Second part follows from Theorem 3.10 ((a) $\Longrightarrow$ (c)) and Lemma 3.4. □

An equi-Riemann integrable collection of functions in $X^{[a,b]}$ is not necessarily equi-Darboux integrable as a Riemann integrable function in $X^{[a,b]}$ is not necessarily Darboux integrable.

Corollary 3.15. *Let $\mathcal{F} \subset X^{[a,b]}$. Then the following statements are equivalent:*

(a) $\mathcal{F}$ is equi-Riemann integrable on $[a,b]$.

(b) $Z_{\mathcal{F},\Gamma}$ is equi-Riemann integrable on $[a,b]$ for some/all norming subsets Γ of B_{X^*}.

(c) $Z_{\mathcal{F},\Gamma}$ is equi-Darboux integrable on $[a,b]$ for some/all norming subsets Γ of B_{X^*}.

PROOF. $(a) \iff (b)$ It should be noted that if Γ is a norming subset of B_{X^*}, then for any partition $\mathcal{P}$ of $[a,b]$, $\theta_{\mathcal{P}}(\mathcal{F}) = \theta_{\mathcal{P}}(Z_{\mathcal{F},\Gamma})$. Hence the result follows from Theorem 3.10 ($(a) \iff (b)$).

$(b) \iff (c)$ Follows from Theorem 3.14. □

Let $\mathcal{F} \subset X^{[a,b]}$. Then the variation of $\mathcal{F}$ on $[a,b]$ is defined as

$$V(\mathcal{F},[a,b]) = \sup_{f\in\mathcal{F}} V(f,[a,b]),$$

$V(f,[a,b])$ being the variation of f on $[a,b]$. If $V(\mathcal{F},[a,b])$ is finite, then $\mathcal{F}$ is said to be of equi-bounded variation on $[a,b]$.

The collection $\mathcal{F}$ is said to be of equi-weak bounded variation on $[a,b]$ if for each $x^* \in X^*$, $Z_{\mathcal{F},x^*}$ is of equi-bounded variation on $[a,b]$.

It can be easily verified that $\mathcal{F}$ is of equi-weak bounded variation if and only if for any norming subset Γ of B_{X^*}, $Z_{\mathcal{F},\Gamma}$ is of equi-bounded variation.

In the line of [2, p. 52, Theorem 2.2] we obtain the following result.

Lemma 3.16. *Let $\mathcal{F} \subset X^{[a,b]}$ and let $\delta > 0$. Then $\omega_\delta(\mathcal{F}) \le \delta V(\mathcal{F},[a,b])$.*

It is known that if a function in $X^{[a,b]}$ is of bounded variation (resp. weak bounded variation), then it is Darboux (resp. Riemann) integrable. For a collection of functions we have the following analogous results:

Theorem 3.17. *Let $\mathcal{F} \subset X^{[a,b]}$. If $\mathcal{F}$ is of equi-bounded (resp. equi-weak bounded) variation on $[a,b]$, then it is equi-Darboux (resp. equi-Riemann) integrable on $[a,b]$.*

PROOF. First part: Follows from the previous lemma and Theorem 3.13.

Second part: For any norming subset Γ of B_{X^*}, $Z_{\mathcal{F},\Gamma}$ is of equi-bounded variation and hence by part (a), equi-Darboux integrable on $[a,b]$. Hence the result follows from Corollary 3.15 ($(c) \implies (a)$). □

It follows from definition that a collection of functions of equi-bounded variation is of equi-weak bounded variation. The converse is true for a collection of real-valued functions. The following example shows that the converse is not true, in general, even if the collection is, in addition, uniformly bounded and if each of its members is of bounded variation.

Example 3.18. *Let $\{r_1, r_2, \cdots, r_n, \cdots\}$ be an enumeration of the rational numbers in $[0,1]$. For each $k \in \mathbb{N}$, let us define a function $f_k : [0,1] \to c_0$ by*

$$f_k(t) = \begin{cases} e_j & if\ t = r_j,\ j = 1,2,\cdots,k \\ \theta & elsewhere \end{cases}$$

where e_j is the jth unit vector of c_0 for $j = 1, 2, \cdots$.

Then it is easy to verify that $\{f_k\}$ is a uniformly bounded collection of functions of equi-weak bounded variation, each f_k being of bounded variation. But it is not of equi-bounded variation.

We know that if $f \in X^{[a,b]}$ is Darboux integrable or is of bounded variation on $[a, b]$, then $\|f\|$ is also so. For a collection of functions we have the following result whose proof is very easy and so omitted:

Theorem 3.19. *If $\mathcal{F} \subset X^{[a,b]}$ is equi-Darboux integrable (resp. of equi-bounded variation) on $[a, b]$, then $\{\|f\| : f \in \mathcal{F}\}$ is equi-Darboux integrable (resp. of equi-bounded variation) on $[a, b]$.*

The results in the above theorem are true neither for equi-Riemann integrable collections of functions nor for collections of functions of equi-weak bounded variation. In fact, these are not true for a single function [8, p. 931, Example 14].

Theorem 3.20. *Balanced convex hull of an equi-Riemann (resp. equi-Darboux) integrable collection of functions is equi-Riemann (resp. equi-Darboux) integrable.*

PROOF. Let

$$\mathcal{H} = \left\{ \sum_{i=1}^{n} \lambda_i f_i : \lambda_i \text{ is real with } \sum_{i=1}^{n} |\lambda_i| \le 1, f_i \in \mathcal{F}, i = 1, 2, \cdots, n;\ n \in \mathbb{N} \right\}.$$

Then it can be easily shown that $\mathcal{H}$ is equi-Riemann (resp. equi-Darboux) integrable and hence the results follow. □

Theorem 3.21. *Let X_i be a Banach space and let $\mathcal{F}_i \subset X_i^{[a,b]}$, $i = 1, 2, \cdots, n$. Let $\prod_{i=1}^{n} X_i$ be equipped with the summation or max norm so that it becomes a Banach space. Then $\prod_{i=1}^{n} \mathcal{F}_i$, as a collection of functions in $\left(\prod_{i=1}^{n} X_i\right)^{[a,b]}$, is equi-Riemann (resp. equi-Darboux) integrable on $[a, b]$ if and only if $\mathcal{F}_i$ is equi-Riemann (resp. equi-Darboux) integrable on $[a, b]$, for $i = 1, 2 \cdots, n$.*

PROOF. Let $X = \prod_{i=1}^{n} X_i$ and let $f = (f_1, f_2, \cdots, f_n) \in \prod_{i=1}^{n} \mathcal{F}_i$. Then for any two tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a, b]$ having the same points, it can be easily verified that for $k = 1, 2, \cdots, n$,

$$\|f_k(\mathcal{P}_1) - f_k(\mathcal{P}_2)\|_{X_k} \leq \|f(\mathcal{P}_1) - f(\mathcal{P}_2)\|_X \leq \sum_{i=1}^{n} \|f_i(\mathcal{P}_1) - f_i(\mathcal{P}_2)\|_{X_i}$$

(for both the norms of X) and hence for any partition $\mathcal{P}$ of $[a, b]$, we have

$$\theta'_{\mathcal{P}}(f_k) \leq \theta'_{\mathcal{P}}(f) \leq \sum_{i=1}^{n} \theta'_{\mathcal{P}}(f_i).$$

Also for any partition $\mathcal{P} = \{t_i : 0 \leq i \leq n\}$ of $[a, b]$, we have, for $k = 1, 2, \cdots, n$,

$$\omega(f_k, [t_{i-1}, t_i]) \leq \omega(f, [t_{i-1}, t_i]) \leq \sum_{i=1}^{n} \omega(f_i, [t_{i-1}, t_i])$$

(for both the norms of X) which implies that

$$\omega(f_k, \mathcal{P}) \leq \omega(f, \mathcal{P}) \leq \sum_{i=1}^{n} \omega(f_i, \mathcal{P}).$$

Hence the results follow from Theorem 3.10 ($(a) \Longleftrightarrow (c)$) and Definition 3.12 respectively. $\square$

Corollary 3.22. *Let $X_i,\ i = 1, 2, \cdots, n$ be n Banach spaces. Then*

(a) $\displaystyle\prod_{i=1}^{n} R([a, b], X_i) = R\left([a, b], \prod_{i=1}^{n} X_i\right)$.

(b) $\displaystyle\prod_{i=1}^{n} D([a, b], X_i) = D\left([a, b], \prod_{i=1}^{n} X_i\right)$.

From Theorem 3.21 and Theorem 3.14, we have the following result:

Theorem 3.23. *Let $X_i,\ i = 1, 2, \cdots, n$ be n Banach spaces. If the notions of equi-Riemann integrability and equi-Darboux integrability coincide for functions with values in $X_i,\ i = 1, 2, \cdots, n$, then they coincide for functions with values in $\prod_{i=1}^{n} X_i$, and conversely.*

Corollary 3.24. *Let X_i, $i = 1, 2, \cdots, n$ be n Banach spaces. If X_i has the property of Lebesgue for $i = 1, 2, \cdots, n$, then $\prod_{i=1}^{n} X_i$ has the property of Lebesgue, and conversely.*

The following result follows from Theorem 3.14 and Theorem 3.23:

Corollary 3.25. *Let n be any positive integer. Then the notions of equi-Riemann integrability and equi-Darboux integrability coincide for functions with values in $\mathbb{R}^n$. In particular, $\mathbb{R}^n$ has the property of Lebesgue.*

Definition 3.26. *A collection of functions, $\mathcal{F}$, in $X^{[a,b]}$, is said to be*

(a) equi-scalarly Riemann integrable on $[a, b]$ if for each $x^* \in X^*$, $Z_{\mathcal{F},x^*}$ is equi-Riemann integrable on $[a, b]$.

(b) equi-Riemann-Pettis integrable (in short, equi-RP integrable) on $[a, b]$ if $\mathcal{F}$ is equi-scalarly Riemann integrable on $[a, b]$ and $\mathcal{F} \subset P([a, b], X)$.

It is obvious that an equi-scalarly Riemann integrable collection of functions in $X^{[a,b]}$ is contained in $RD([a, b], X)$ and hence such a collection of functions is said to be equi-Riemann-Dunford integrable (in short, equi-RD integrable). Also an equi-Riemann-Pettis integrable collection of functions is contained in $RP([a, b], X)$.

It is clear that an equi-Riemann integrable collection of functions is equi-Riemann-Pettis integrable and an equi-Riemann-Pettis integrable collection of functions is equi-Riemann-Dunford integrable.

It should be noted that all these notions are equivalent for real-valued functions.

It follows from [8, p. 944, Theorem 31] that in a weakly sequentially complete Banach space the notions of equi-Riemann-Dunford integrability and equi-Riemann-Pettis integrability coincide.

Lemma 3.27. *Oscillation of an equi-Riemann-Dunford integrable collection of functions on its interval of definition is finite.*

Proof. Let $\mathcal{F}$ be an equi-Riemann-Dunford integrable collection of functions in $X^{[a,b]}$. Then it follows from Corollary 3.11 that for each $x^* \in X^*$, oscillation of $Z_{\mathcal{F},x^*}$ is finite on its interval of definition. Hence the result follows by an application of Uniform Boundedness Principle. □

We know that a Riemann integrable and hence a Darboux integrable function on $[a, b]$ is bounded. The following examples show that neither equi-Riemann nor equi-Darboux integrability of a collection of functions implies its uniform boundedness, and that uniform boundedness of a collection of Darboux integrable functions implies neither equi-Riemann nor equi-Darboux integrability.

Example 3.28. *Let, for each positive integer* n*,* $f_n(t) = n$ *for all* $t \in [0,1]$*. Then the collection* $\{f_n : n \in \mathbb{N}\}$ *is equi-Darboux integrable in* $\mathbb{R}^{[0,1]}$*, but it is not uniformly bounded on* $[0,1]$*.*

Example 3.29. *Let* $\{r_1, r_2, \cdots, r_n, \cdots\}$ *be an enumeration of the rational numbers in* $[0,1]$*. Let us define for each* $n \in \mathbb{N}$*, the function* $f_n : [0,1] \to \mathbb{R}$ *by*

$$f_n(t) = \begin{cases} 1 & \text{for } t \in \{r_1, r_2, \cdots, r_n\} \\ 0 & \text{elsewhere.} \end{cases}$$

Then the collection $\{f_n : n \in \mathbb{N}\}$ *is uniformly bounded on* $[0,1]$*, and for each* $n \in \mathbb{N}$*,* $f_n \in D[0,1]$*, but the collection is not equi-Darboux integrable in* $\mathbb{R}^{[0,1]}$*.*

However we have the following result which follows from Lemma 3.27 and Lemma 3.6 (c):

Theorem 3.30. *Let* $\mathcal{F} \subset X^{[a,b]}$ *be equi-Riemann-Dunford integrable on* $[a,b]$ *and pointwise bounded at some point in* $[a,b]$*. Then it is uniformly bounded on* $[a,b]$*.*

We know that if a function is Darboux or Riemann or Riemann-Dunford or Riemann-Pettis integrable on $[a,b]$ (resp. on $[a,c]$ and $[c,b]$, for some $c \in [a,b]$), then it is so on every closed subinterval of $[a,b]$ (resp. on $[a,b]$). Proceeding similarly we have the following result:

Theorem 3.31. *Let* $\mathcal{F} \subset X^{[a,b]}$*.*

(a) If $\mathcal{F}$ *is equi-Darboux (resp. equi-Riemann, equi-Riemann-Dunford, equi-Riemann-Pettis) integrable on* $[a,b]$*, then it is so on every closed subinterval of* $[a,b]$*.*

(b) Let $c \in [a,b]$*. If* $\mathcal{F}$ *is equi-Darboux (resp. equi-Riemann, equi-Riemann-Dunford, equi-Riemann-Pettis) integrable on* $[a,c]$ *as well as on* $[c,b]$*, then it is so on* $[a,b]$*.*

It is known that a function is Bochner (resp. Pettis) integrable on a closed interval if and only if it is so on every subinterval of it. But if a function is Bochner (resp. Pettis) integrable on every closed subinterval of (a,b), then it is not necessarily Bochner (resp. Pettis) integrable on $[a,b]$. As for example, the function $f(t) = \frac{1}{t}$ for $t \in (0,1]$ and $f(0) = 0$ is Lebesgue integrable on every closed subinterval of $(0,1)$, but it is not Lebesgue integrable on $[0,1]$.

In this regard, we have the following result:

Theorem 3.32. *Let $f \in X^{[a,b]}$. Then*

(a) f is Pettis integrable on $[a,b]$ if and only if it is scalarly integrable on $[a,b]$, Pettis integrable on every closed subinterval of (a,b) and Z_f is relatively weakly compact in $L_1([a,b],\lambda)$,

(b) f is Riemann-Pettis integrable on $[a,b]$ if and only if it is Riemann-Dunford integrable on $[a,b]$ and Pettis integrable on every closed subinterval of (a,b).

Proof. (a) The necessary part is trivial.

For the sufficient part, let $E \in \Sigma$ be arbitrary with $\lambda(E) > 0$. Let $\delta = \frac{\lambda(E)}{4} > 0$ and let $A = [a+\delta, b-\delta] \cap E$. Then A is a measurable subset of $[a+\delta, b-\delta]$ with $\lambda(A) > 0$ and it can be easily verified that $cor_f(A) \subset cor_f(E)$. Now, by hypothesis, f is Pettis integrable on $[a+\delta, b-\delta]$ which implies that $cor_f(A) \neq \phi$ [13, p. 543, Theorem 4.10] and hence $cor_f(E) \neq \phi$. Hence the result follows from [13, p. 543, Theorem 4.10].

(b) Follows from part (a) as f is bounded and hence Z_f is relatively weakly compact in $L_1([a,b],\lambda)$ whenever f is Riemann-Dunford integrable on $[a,b]$. □

Let us recall that if a function is bounded on $[a,b]$ and Riemann integrable on every closed subinterval of (a,b), then it is Riemann integrable on $[a,b]$. For a collection of functions we have the following result:

Corollary 3.33. *Let $\mathcal{F} \subset X^{[a,b]}$ be uniformly bounded in some neighbourhood of a as well as in some neighbourhood of b.*

(a) If $\mathcal{F}$ is equi-Darboux (resp. equi-Riemann) integrable on every closed subinterval of (a,b), then it is equi-Darboux (resp. equi-Riemann) integrable on $[a,b]$.

(b) If $\mathcal{F}$ is equi-Riemann-Dunford (resp. equi-Riemann-Pettis) integrable on every closed subinterval of (a,b), then it is equi-Riemann-Dunford (resp. equi-Riemann-Pettis) integrable on $[a,b]$.

In each case, $\mathcal{F}$ is uniformly bounded on $[a,b]$.

Proof. (a) For the first part, let $\epsilon > 0$. According to hypothesis, there exists a $\delta > 0$ such that $\mathcal{F}$ is uniformly bounded on $[a, a+\delta)$ as well as on $(b-\delta, b]$. Hence there exists an $M > 0$ such that $\|f(t)\| \leq M$ for all $t \in [a, a+\delta) \cup (b-\delta, b]$ and for all $f \in \mathcal{F}$.

Let $c \in (a, a+\delta)$ and let $d \in (b-\delta, b)$ such that $c-a = b-d < \frac{\epsilon}{12M}$. Then by hypothesis $\mathcal{F}$ is equi-Darboux integrable on $[c,d]$ and hence there exists a partition $\mathcal{P}_1 = \{c = t_0, t_1, t_2, \cdots, t_n = d\}$ of $[c,d]$ such that

$$\omega(\mathcal{F}, \mathcal{P}_1) < \frac{\epsilon}{6}.$$

Now $\mathcal{P} = \{a, c = t_0, t_1, t_2, \cdots, t_n = d, b\}$ is a partition of $[a,b]$ and for all $f \in \mathcal{F}$,

$$\begin{aligned}
\omega(f, \mathcal{P}) &= \omega(f,[a,c])(c-a) + \sum_{i=1}^{n} \omega(f,[t_{i-1},t_i])(t_i - t_{i-1}) + \omega(f,[d,b])(b-d) \\
&< \frac{2M\epsilon}{12M} + \omega(f, \mathcal{P}_1) + \frac{2M\epsilon}{12M} \\
&\le \frac{\epsilon}{3} + \omega(\mathcal{F}, \mathcal{P}_1) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{6} = \frac{\epsilon}{2}.
\end{aligned}$$

Taking supremum over $\mathcal{F}$, we have $\omega(\mathcal{F}, \mathcal{P}) \le \frac{\epsilon}{2} < \epsilon$. Hence the result follows from Definition 3.12.

Second part follows from Corollary 3.15 ($(a) \iff (c)$) and the first part. (b) First part follows by an application of part (a) to $Z_{\mathcal{F},x^*}$ for each $x^* \in X^*$. Second part follows from the first part and Theorem 3.32 (b). Uniform boundedness of $\mathcal{F}$ in each case follows from Theorem 3.30. □

For definitions and some fundamental properties of regulated functions and equi-regulated collection of functions in $X^{[a,b]}$, we refer to [12] and [17]. A very basic reference for the properties of the equi-regulated sets of regulated functions is the paper [6].

It is well known that a regulated function on $[a,b]$ is Darboux integrable thereon. The following example shows that an equi-regulated collection of functions is not necessarily equi-Darboux integrable.

Example 3.34. *Let us consider the sequence $\{f_n\}$ of functions defined on $[0,1]$ by $f_n(t) = 0$ for $t \in (0,1]$ and $f_n(0) = n$. Then the sequence $\{f_n\}$ is not uniformly Henstock integrable on $[0,1]$ [7, p. 724] and hence not equi-Darboux integrable on $[0,1]$. It is easy to verify that $\{f_n\}$ is equi-regulated on $[0,1]$.*

However we have the following result in this regard:

Theorem 3.35. *Let $\mathcal{F} \subset X^{[a,b]}$ be equi-regulated on $[a,b]$. Then the following statements hold:*

(a) *if the oscillation of $\mathcal{F}$ on $[a, b]$ is finite, then $\mathcal{F}$ is equi-Darboux integrable on $[a, b]$,*

(b) *if $\mathcal{F}$ is pointwise bounded on $[a, b]$, then it is uniformly bounded and equi-Darboux integrable on $[a, b]$.*

In each case, $\mathcal{F}_1$ is uniformly equi-differentiable from the right on $[a, b)$ and from the left on $(a, b]$.

PROOF. (a) Let the oscillation, $\omega(\mathcal{F}, [a, b])$, of $\mathcal{F}$ on $[a, b]$ be finite. Let $\epsilon > 0$. Then there exists a partition $\mathcal{P} = \{t_i : 0 \leq i \leq n\}$ of $[a, b]$ such that for all $f \in \mathcal{F}$, $\|f(t') - f(t'')\| < \frac{\epsilon}{2(b-a)}$ for all $t', t'' \in (t_{i-1}, t_i)$ [12, p. 11-12, Theorem 1.2] which implies that $\omega(f, (t_{i-1}, t_i)) \leq \frac{\epsilon}{2(b-a)}$ for $i = 1, 2, \cdots, n$.

Let $\delta > 0$ be such that $\delta < \frac{\epsilon}{4n[\omega(\mathcal{F},[a,b])+1]}$ and $t_{i-1} + \delta < t_i - \delta$, $i = 1, 2, \cdots, n$.

Let $\mathcal{P}' = \{a = s_0 < s_1 < s_2 < \cdots < s_{2n+1} = b\}$ be a partition of $[a, b]$ where $s_{2i+1} = t_i + \delta, i = 0, 1, \cdots, n-1, s_{2i} = t_i - \delta, i = 1, 2, \cdots, n$.

Then for any $f \in \mathcal{F}$, we have

$$\begin{aligned}
\omega(f, \mathcal{P}') &= \sum_{i=1}^{2n+1} \omega(f, [s_{i-1}, s_i])(s_i - s_{i-1}) \\
&= \sum_{k=0}^{n} \omega(f, [s_{2k}, s_{2k+1}])(s_{2k+1} - s_{2k}) \\
&\qquad + \sum_{j=1}^{n} \omega(f, [s_{2j-1}, s_{2j}])(s_{2j} - s_{2j-1}) \\
&\leq \sum_{k=0}^{n} \omega(f, [s_{2k}, s_{2k+1}])(s_{2k+1} - s_{2k}) \\
&\qquad + \sum_{j=1}^{n} \omega(f, (t_{j-1}, t_j))(s_{2j} - s_{2j-1}) \\
&\text{(since } [s_{2j-1}, s_{2j}] \subset (t_{j-1}, t_j) \text{ for } j = 1, 2, \cdots, n) \\
&\leq \omega(\mathcal{F}, [a, b]) \sum_{k=0}^{n} (s_{2k+1} - s_{2k}) + \frac{\epsilon}{2(b-a)} \sum_{j=1}^{n} (s_{2j} - s_{2j-1}) \\
&\text{(since } \omega(f, [s_{2k}, s_{2k+1}]) \leq \omega(\mathcal{F}, [a, b]) \text{ for } k = 0, 1, 2, \cdots, n) \\
&< \omega(\mathcal{F}, [a, b]) \frac{\epsilon}{2[\omega(\mathcal{F}, [a, b]) + 1]} + \frac{\epsilon}{2(b-a)}(b-a) \\
&< \epsilon
\end{aligned}$$

which implies that $\mathcal{F}$ is equi-Darboux integrable on $[a, b]$.

(b) Follows from [17, p. 4, Lemma 3.3] and part (a).

The last part follows in a straightforward way. □

It should be noted that a Darboux integrable function is not necessarily regulated. Hence an equi-Darboux integrable collection of functions is not necessarily equi-regulated.

It is known that a function of bounded variation on a closed interval is regulated and has at most a countable number of points of discontinuity. The following example shows that a collection of functions of equi-bounded variation may neither be equi-regulated nor be equicontinuous almost everywhere.

Example 3.36. *Let*

$$f_s(t) = \begin{cases} 1 & \text{for all } s = t \text{ in } [a, b] \\ 0 & \text{if } s \neq t \text{ in } [a, b]. \end{cases}$$

Clearly $\{f_s : s \in [a, b]\}$ *is of equi-bounded variation and hence equi-Darboux integrable in* $\mathbb{R}^{[a,b]}$*, but it is not equi-regulated. Also it is equicontinuous at no points of* $[a, b]$*.*

The above example also shows that an equi-Darboux integrable collection of functions may be equicontinuous at no points in contrast with the fact that a Darboux integrable function is continuous almost everywhere.

It is known that a differentiable, even a weakly differentiable, function on $[a, b]$ is Darboux integrable. The following example shows that a uniformly equi-differentiable collection of functions is not necessarily equi-Darboux integrable:

Example 3.37. *For each* $n \in \mathbb{N}$*, let us consider the real function* $f_n(t) = nt$*,* $t \in [a, b]$*. It is easy to verify that* $\{f_n : n \in \mathbb{N}\}$ *is uniformly equi-differentiable on* $[a, b]$ *but is not equi-Darboux integrable thereon.*

However we have the following result:

Theorem 3.38. *An equicontinuous (resp. equi-weakly continuous) collection of functions in* $X^{[a,b]}$ *is equi-Darboux (resp. equi-Riemann-Pettis) integrable on* $[a, b]$ *and the collection of indefinite integrals of its members is uniformly equi-differentiable (resp. uniformly equi-weakly differentiable) on* $[a, b]$*.*

Proof. First part: It should be noted that an equicontinuous collection of functions in $X^{[a,b]}$ is equi-regulated and has finite oscillation on $[a, b]$. Hence the result follows from Theorem 3.35 (a).

Second part: Follows from first part and [19, p. 419, Corollary 21]. □

We conclude the paper with the following results on relations among equi-Riemann integrability, Birkhoff property [4, p. 264, Definition 2] and Bourgain property [14, p. 518, Definition 10] of a collection of functions.

In the following discussion, we shall assume that $b - a = 1$ to make $([a,b], \Sigma, \lambda)$ a complete probability space.

The following result follows from the very definition of Birkhoff property, Theorem 3.10 ($(a) \Longrightarrow (b)$) and [4, p. 265, Lemma 2.3].

Theorem 3.39. *Let $\mathcal{F} \subset \mathbb{R}^{[a,b]}$. Let us consider the following statements:*

(a) $\mathcal{F}$ is equi-Riemann integrable on $[a,b]$.

(b) $\mathcal{F}$ has the Birkhoff property.

(c) $\mathcal{F}$ has the Bourgain property.

Then (a) $\Longrightarrow$(b) $\Longrightarrow$(c).

In view of [14, p. 520], we shall say that a collection of functions $\mathcal{F} \subset X^{[a,b]}$ has Bourgain property if the family $Z_{\mathcal{F}}$ has the Bourgain property.

From Corollary 3.15 ($(a) \Longrightarrow (b)$) and above theorem ($(a) \Longrightarrow (c)$) we have the following result:

Corollary 3.40. *If $\mathcal{F} \subset X^{[a,b]}$ is equi-Riemann integrable on $[a,b]$, then $\mathcal{F}$ has the Bourgain property.*

As a particular case of the above result we have the following one [15, p. 57, Proposition 6.0.1].

Corollary 3.41. *If $f \in R([a,b], X)$, then f has the Bourgain property.*

The converse of the above result is not necessarily true which follows from the fact that a Bochner integrable function has the Bourgain property [14, p. 520, Example 12] but is not necessarily Riemann integrable.

From [14, p. 521, Theorem 13], the following result follows:

Theorem 3.42. *Each Riemann-Dunford integrable function in $(X^*)^{[a,b]}$ having the Bourgain property is Riemann-Pettis integrable. Hence if an equi-Riemann-Dunford integrable collection of functions $\mathcal{F}$ in $(X^*)^{[a,b]}$ has the Bourgain property, then it is equi-Riemann-Pettis integrable.*

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