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LIPSCHITZ RESTRICTIONS OF CONTINUOUS FUNCTIONS AND A SIMPLE CONSTRUCTION OF ULAM-ZAHORSKI C^1 INTERPOLATION

Abstract

We present a simple argument that for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ its restriction to some perfect set is Lipschitz. We will use this result to provide an elementary proof of the C^1 free interpolation theorem, that for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ there exists a continuously differentiable function $g\colon \mathbb{R} \to \mathbb{R}$ which agrees with f on an uncountable set. The key novelty of our presentation is that no part of it, including the cited results, requires from the reader any prior familiarity with the Lebesgue measure theory.

1 Introduction and background

The main result we like to discuss here is the following 1985 theorem of Agronsky, Bruckner, Laczkovich, and Preiss [1]. It implies that every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ must have some traces of differentiability, even though there exist continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that are nowhere differentiable (see e.g. [10, 22, 23]) or, even stronger, nowhere approximately and $\mathcal{I}$-approximately differentiable. In fact, the first coordinate of the classical Peano curve (i.e., $f_1\colon [0,1] \to [0,1]$, where $f = (f_1, f_2)\colon [0,1] \to [0,1]^2$ is a continuous surjection constructed by Peano) has these properties, see [6] or

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[7, Example 4.3.8]. Such a function cannot agree with a C^1 function on a set which is either of second category or of positive Lebesgue measure.

Theorem 1. *For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is a continuously differentiable function $g\colon \mathbb{R} \to \mathbb{R}$ such that the set $[f = g] = \{x \in \mathbb{R}\colon f(x) = g(x)\}$ is uncountable. In particular, $[f = g]$ contains a perfect set P and the restriction $f \restriction P$ is continuously differentiable.*

In the statement of Theorem 1 the differentiability of $h = f \restriction P$ is understood as the existence of its derivative, that is, of the function $h'\colon P \to \mathbb{R}$ defined, for every $p \in P$, as $h'(p) = \lim_{x \to p,\ x \in P} \frac{h(x)-h(p)}{x-p}$.

The story behind Theorem 1 spreads over a big part of the 20th century and is described in detail in [2] and [16]. Briefly, around 1940 S. Ulam asked, in Scottish Book, Problem 17.1, see [21], whether every continuous $f\colon \mathbb{R} \to \mathbb{R}$ agrees with some real analytic function on an uncountable set. Z. Zahorski showed, in his 1948 paper [25], that the answer is no: there exists a C^∞ (i.e., infinitely many times differentiable) function which can agree with every real analytic function on at most finite set of points. At the same paper Zahorski stated a problem, refereed to as Ulam-Zahorski problem: does every continuous $f\colon \mathbb{R} \to \mathbb{R}$ agrees with some C^∞ (or possibly C^n or D^n) function on some uncountable set? Clearly, Theorem 1 shows that Ulam-Zahorski problem has an affirmative answer for the C^1 class of functions. This is the best possible result in this direction, since A. Olevskiĭ constructed, in his 1994 paper [16], a continuous function which can agree with every C^2 function on at most countable set of points.

The format of our proof of Theorem 1 is relatively straightforward. First we provide a simple argument that for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ its restriction to some perfect set $P \subset \mathbb{R}$ is Lipschitz.[1] Here the key case, presented in Sec. 2, is when f is monotone. Then we will follow an argument of Morayne [15] to show that there is a perfect $Q \subset P$ for which $f \restriction Q$ satisfies the assumptions of Whitney's C^1 extension theorem [24]. At this point, to make the argument more accessible, we point the reader to a version of Whitney's C^1 extension theorem from [4], whose proof is elementary and simple.

[1] Of course this result follows immediately from Theorem 1, as g from Theorem 1 is Lipschitz on any bounded interval. However, we are after a simpler proof of Theorem 1, so using it to argue for our step to prove it is pointless.

2 Lipschitz restrictions of monotone continuous maps

In what follows f will always be a continuous function from $\mathbb{R}$ into $\mathbb{R}$, Δ will stand for the set $\{\langle x,x\rangle \colon x\in\mathbb{R}\}$, and $q\colon \mathbb{R}^2\setminus\Delta\to\mathbb{R}$ be the quotient function for f, that is, defined as $q(x,y)=\frac{f(x)-f(y)}{x-y}$. For $Q\subset\mathbb{R}$ we will use the symbol $q\restriction Q^2$ to denote the restriction of q to the set $Q^2\setminus\Delta$.

Theorem 2. *Assume that $f\colon\mathbb{R}\to\mathbb{R}$ is monotone and continuous on a non-trivial interval $[a,b]$. For every $L>|q(a,b)|$ there exists a closed uncountable set $P\subset[a,b]$ such that $f\restriction P$ is Lipschitz with constant L.*

The difficulty in proving Theorem 2 without measure theoretical tools comes from the fact that there exist strictly increasing continuous functions $f\colon\mathbb{R}\to\mathbb{R}$ which posses finite or infinite derivative at every point, but that the derivative of f is infinite on a dense G_δ-set. The first example of such function was given by Pompeiu in [18]. More recent description of such functions can be found in [20, sec. 9.7] and [5]. These examples show that a perfect set in Theorem 2 should be nowhere sense. Thus we will use a measure theoretical approach, in which the measure theoretical tools will be present only implicitly or, as in case of Fact 5, given together with a simple proof.

We extract the proof of next theorem from the proof, presented in [8], of a Lebesgue theorem that every monotone function $f\colon\mathbb{R}\to\mathbb{R}$ is differentiable almost everywhere.

Our proof of Theorem 2 is based on the following 1932 result of Riesz [19], known as the rising sun lemma. For reader's convenience we include its short proof.

Lemma 3. *If g is a continuous function from a non-trivial interval $[a,b]$ into $\mathbb{R}$, then the set $U=\{x\in[a,b)\colon g(x)<g(y)$ for some $y\in(x,b]\}$ is open in $[a,b)$ and $g(c)\le g(d)$ for every open connected component (c,d) of U.*

PROOF. It is clear that U is open in $[a,b)$. To see the other part, let (c,d) be a component of U. By continuity of g, it is enough to prove that $g(p)\le g(d)$ for every $p\in(c,d)$. Assume by way of contradiction that $g(d)<g(p)$ for some $p\in(c,d)$ and let $x\in[p,b]$ be a point at which $g\restriction[p,b]$ achieves the maximum. Then $g(d)<g(p)\le g(x)$ and so we must have $x\in[p,d)\subset U$, as otherwise d would belong to U. But $x\in U$ contradicts the fact that $g(x)\ge g(y)$ for every $y\in(x,b]$. □

Remark 4. In Lemma 3 we also have $g(c)\ge g(d)$, since $c\in[a,b)\setminus U$. But we do not actually need this fact.

For an interval I let $\ell(I)$ be its length. We need the following simple well-known observations.

Fact 5. *Let $a<b$ and $\mathcal{J}$ be a family of open intervals with $\bigcup\mathcal{J}\subset(a,b)$.*

(i) If $[\alpha,\beta]\subset\bigcup\mathcal{J}$, then $\sum_{I\in\mathcal{J}}\ell(I)>\beta-\alpha$.

(ii) If the intervals in $\mathcal{J}$ are pairwise disjoint, then $\sum_{I\in\mathcal{J}}\ell(I)\leq b-a$.

PROOF. (i) By compactness of $[\alpha,\beta]$ we can assume that $\mathcal{J}$ is finite, say of size n. Then (i) follows by an easy induction on n: if $(c,d)=J\in\mathcal{J}$ contains β, then either $c\leq\alpha$, in which case (i) is obvious, or $\alpha<c$ and, by induction, $\sum_{I\in\mathcal{J}}\ell(I)=\ell(J)+\sum_{I\in\mathcal{J}\setminus\{J\}}\ell(I)>\ell([c,\beta])+\ell([\alpha,c])=\beta-\alpha$.

(ii) Once again, it is enough to show (ii) for finite $\mathcal{J}$, say of size n, by induction. Then, there is $(c,d)=J\in\mathcal{J}$ to the right of any $I\in\mathcal{J}\setminus\{J\}$. Hence, by induction, $\sum_{I\in\mathcal{J}}\ell(I)=\ell(J)+\sum_{I\in\mathcal{J}\setminus\{J\}}\ell(I)\leq(b-c)+(c-a)=b-a$. $\square$

PROOF OF THEOREM 2. If there exists a nontrivial interval $[c,d]\subset[a,b]$ on which f is constant, then clearly $P=[c,d]$ is as needed. So, we can assume that f is strictly monotone on $[a,b]$. Also, replacing f with $-f$, if necessary, we can also assume that f is strictly increasing.

Fix $L>|q(a,b)|=\frac{f(b)-f(a)}{b-a}$ and define $g\colon\mathbb{R}\to\mathbb{R}$ as $g(t)=f(t)-Lt$. Then $g(a)=f(a)-La>f(b)-Lb=g(b)$. Let $m=\sup\{g(x)\colon x\in[a,b]\}$ and $\bar{a}=\sup\{x\in[a,b]\colon g(x)=m\}$. Then $f(\bar{a})-L\bar{a}=g(\bar{a})\geq g(a)>g(b)=f(b)-Lb$, so $a\leq\bar{a}<b$ and we still have $L>|q(\bar{a},b)|=\frac{f(b)-f(\bar{a})}{b-\bar{a}}$. Moreover, $\bar{a}$ does not belong to the set

$$U=\{x\in[\bar{a},b)\colon g(y)>g(x)\text{ for some }y\in(x,b]\}$$

from Lemma 3 applied to g on $[\bar{a},b]$. In particular, U is open in $\mathbb{R}$ and the family $\mathcal{J}$ of all connected components of U contains only open intervals (c,d) for which, by Lemma 3, $g(c)\leq g(d)$.

The set $P=[\bar{a},b]\setminus U\subset[a,b]$ is closed and for any $x<y$ in P we have $f(y)-Ly=g(y)\leq g(x)=f(x)-Lx$, that is, $|f(y)-f(x)|=f(y)-f(x)\leq Ly-Lx=L|y-x|$. In particular, f is Lipschitz on P with constant L. It is enough to show that P is uncountable.

To see this notice that for every $J=(c,d)\in\mathcal{J}$ we have $f(d)-Ld=g(d)\geq g(c)=f(c)-Lc$, that is, $\ell(f[J])=f(d)-f(c)\geq L(d-c)=L\ell(J)$. Since the intervals in the family $\mathcal{J}^*=\{f[J]\colon\mathcal{J}\in\mathcal{J}\}$ are pairwise disjoint and contained in the interval $(f(\bar{a}),f(b))$, by Fact 5(ii) we have $\sum_{J^*\in\mathcal{J}^*}\ell(J^*)\leq f(b)-f(\bar{a})$. So, $\sum_{J\in\mathcal{J}}\ell(J)\leq\frac{1}{L}\sum_{J\in\mathcal{J}}\ell(f[J])=\frac{1}{L}\sum_{J^*\in\mathcal{J}^*}\ell(J^*)\leq\frac{f(b)-f(\bar{a})}{L}<b-\bar{a}$. Thus, by Fact 5(i), $P=[\bar{a},b]\setminus U=[\bar{a},b]\setminus\bigcup\mathcal{J}\neq\emptyset$. However, we need more,

that P cannot be contained in any countable set, say $\{x_n : n \in \mathbb{N}\}$. To see this, fix $\delta > 0$ such that $\frac{f(b)-f(\bar{a})}{L} + \delta < b - \bar{a}$, for every $n \in \mathbb{N}$ choose an interval $(c_n, d_n) \ni x_n$ of length $2^{-n}\delta$, and put $\hat{\mathcal{J}} = \mathcal{J} \cup \{(c_n, d_n) : n < \omega\}$. Then

$$\sum_{J\in\hat{\mathcal{J}}} \ell(J) = \sum_{J\in\mathcal{J}} \ell(J) + \sum_{n\in\mathbb{N}} \ell((c_n, d_n)) \leq \frac{f(b) - f(\bar{a})}{L} + \delta < \beta - \alpha$$

so, by Fact 5(i), $U \cup \bigcup_{n\in\mathbb{N}}(c_n, d_n) \supset U \cup \{x_n : n \in \mathbb{N}\}$ does not contain $[\bar{a}, b]$. In other words, $P = [\bar{a}, b] \setminus U$ is uncountable, as needed. $\square$

Remark 6. A presented proof of Theorem 2 actually gives a stronger result, that the set $[a, b] \setminus P$ can have arbitrary small Lebesgue measure.

3 Perfect set on which the difference quotient map is uniformly continuous

The next proposition is a version of a theorem of Morayne [15], which implies that the conclusion of Proposition 7 holds when f, defined on a perfect subset of $\mathbb{R}$, is Lipschitz (i.e., the quotient map for such f has bounded range). The key innovation in Proposition 7 is that we prove this result without assuming that f, or some restriction of it, is Lipschitz.

Proposition 7. *For every continuous $f : \mathbb{R} \to \mathbb{R}$ there exists a perfect set $Q \subset \mathbb{R}$ such that the quotient map $q \restriction Q^2$ is bounded and uniformly continuous.*

PROOF. If f is monotone on some non-trivial interval $[a, b]$, then, by Theorem 2, there exists a perfect set $P \subset \mathbb{R}$ such that $f \restriction P$ is Lipschitz. Thus, by Morayne's theorem applied to $f \restriction P$, there exists a perfect $Q \subset P$ for which the quotient map q is as needed. On the other hand, if f is monotone on no non-trivial interval, then, by a 1953 theorem of Padmavally [17] (compare also [14, 13, 9]) there exists a perfect set $Q \subset \mathbb{R}$ on which f is constant. Of course, the quotient map on such Q is as desired. $\square$

4 The main result

The following theorem is a restatement of Theorem 1 in a slightly different language.

Theorem 8. *For every continuous function $f : \mathbb{R} \to \mathbb{R}$ there exists a perfect set $Q \subset \mathbb{R}$ such that $f \restriction Q$ can be extended to C^1 function $F : \mathbb{R} \to \mathbb{R}$.*

Let $Q \subset \mathbb{R}$ be as Proposition 7. It is well known, see e.g. [12], that uniform continuity of $q \restriction Q^2$ implies that the assumptions of the Whitney's C^1 extension theorem (see [24]) are satisfied, that is, $f \restriction Q$ has a desired C^1 extension $F\colon \mathbb{R} \to \mathbb{R}$. The problem with the citation [12], and many other papers containing needed extension result, is that the proofs presented there can hardly be considered simple. Thus, we like conclude the extendability of $f \restriction Q$, having uniformly continuous $q \restriction Q^2$, to C^1 extension $F\colon \mathbb{R} \to \mathbb{R}$ from the following recent result of Ciesielska and Ciesielski [4] which has simple elementary proof.

For a bounded open interval J let I_J be the closed middle third of J and for a perfect set $Q \subset \mathbb{R}$ let

$$\hat{Q} = Q \cup \bigcup \{I_J \colon J \text{ is a bounded connected component of } \mathbb{R} \setminus Q\}.$$

Proposition 9. [4] *Let $f\colon Q \to \mathbb{R}$, where Q is a perfect subset of $\mathbb{R}$, and put $\hat{f} = \bar{f} \restriction \hat{Q}$, where $\bar{f}\colon \mathbb{R} \to \mathbb{R}$ is a linear interpolation of $f \restriction Q$. If $f \restriction Q$ is differentiable, then there exists a differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ of $\hat{f}$. Moreover, F is C^1 if, and only if, $\hat{f}$ is continuously differentiable.*

PROOF OF THEOREM 8. If $Q \subset \mathbb{R}$ is from Proposition 7, then $q \restriction Q^2$, defined on $Q^2 \setminus \Delta$, can be extended to uniformly continuous $\bar{q}$ on Q^2 and $f\colon Q \to \mathbb{R}$ is continuously differentiable with $(f \restriction Q)'(x) = \bar{q}(x,x)$ for every $x \in Q$. By Proposition 9, $\hat{f}$ is differentiable (as a restriction of differentiable F). In particular, $\hat{f}'(x) = F'(x) = (f \restriction Q)'(x)$ for every $x \in Q$ and $\hat{f}'(x) = \bar{q}(c,d)$ whenever $x \in I_J$, where $J = (c,d)$ is a bounded connected component of $\mathbb{R} \setminus Q$.

By Proposition 9, we need to show that $\hat{f}'$ is continuous. Clearly $\hat{f}'$ is continuous on $\hat{Q} \setminus Q$, as it is locally constant on this set. So, let $x \in Q$ and let $\varepsilon > 0$. We need to find an open U containing x such that $|\hat{f}'(x) - \hat{f}'(y)| < \varepsilon$ whenever $y \in \hat{Q} \cap U$. Since $\bar{q}$ is continuous, there exists an open $V \in \mathbb{R}^2$ containing $\langle x, x\rangle$ such that $|\hat{f}'(x) - \bar{q}(y,z)| = |\bar{q}(x,x) - \bar{q}(y,z)| < \varepsilon$ whenever $\langle y, z\rangle \in Q^2 \cap V$. Let U_0 be open interval containing x such that $U_0^2 \subset V$ and let $U \subset U_0$ be an open set containing x such that: if $U \cap I_J \neq \emptyset$ for some bounded connected component $J = (c,d)$ of $\mathbb{R} \setminus Q$, then $c, d \in U_0$. We claim that U is as needed. Indeed, let $y \in \hat{Q} \cap U$. If $y \in Q$, then $\langle y, y\rangle \in U^2 \subset V$ and $|\hat{f}'(x) - \hat{f}'(y)| = |\bar{q}(x,x) - \bar{q}(y,y)| < \varepsilon$. Also, if $y \in I_J$ for some bounded connected component $J = (c,d)$ of $\mathbb{R} \setminus Q$, then $\langle c, d\rangle \in U_0^2 \subset V$ and, once again, $|\hat{f}'(x) - \hat{f}'(y)| = |\bar{q}(x,x) - \bar{q}(c,d)| < \varepsilon$. $\square$

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