INROADS

Real Analysis Exchange
Vol. 43(1), 2017, pp. 221–222
DOI: 10.14321/realanalexch.41.1.0221

Alan Chang, Department of Mathematics, University of Chicago, Chicago, IL 60637, U.S.A. email: ac@math.uchicago.edu

ON THE MINKOWSKI SUM OF TWO CURVES

Abstract

We answer a question posed by Miklós Laczkovich in [1] on the Minkowski sum of two curves.

Let $A \subset \mathbb{R}^2$ be a continuous curve from $(0,0)$ to $(1,0)$. Let $B \subset \mathbb{R}^2$ be a continuous curve from $(0,0)$ to $(0,1)$. Let $S = A + B = \{a + b : a \in A, b \in B\}$. What is the minimum possible area (i.e., two dimensional Lebesgue measure) of S? The answer is the following.

Proposition 1. *The minimum possible area of S is* 1.

Proof. We claim that

$$\bigcup_{(m,n)\in\mathbb{Z}^2} (S + (m,n)) = \mathbb{R}^2. \tag{1}$$

To prove (1), define $A' = \bigcup_{m\in\mathbb{Z}}(A + (m,0))$ and $B' = \bigcup_{n\in\mathbb{Z}}(B + (0,n))$. We need to show $A' + B' = \mathbb{R}^2$. Since A is a continuous curve from $(0,0)$ to $(1,0)$, A' is a continuous curve extending infinitely far in both horizontal directions. Similarly, B' is a continuous curve extending infinitely far in both vertical directions. By the Jordan curve theorem (e.g., [2, Theorem 63.4]), $A' + B' = \mathbb{R}^2$. And as Laczkovich remarked to me at the time, it is enough to use the Jordan curve theorem for polygons. This completes the proof of (1).

By (1), it follows that the image of S under the quotient map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is surjective. Thus, S has area has at least 1. $\square$

Mathematical Reviews subject classification: Primary: 26A03, 26A04; Secondary: 26A05
Key words: first keyword, second keyword, and so on
Received by the editors June 29, 2017
Communicated by: Paul D. Humke

References

[1] M. Laczkovich, *Problem Session*, Summer Symposium in Real Analysis XXXIX, Northfield, MN, June 9, 2015.

[2] J. Munkres, *Topology, Second Edition*. Prentice Hall, Upper Saddle River, NJ, 2000.