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RANDOM CUTOUTS OF THE UNIT CUBE WITH I.U.D CENTERS

Abstract

Consider the random open balls $B_n(\omega) := B(\omega_n, r_n)$ with their centers ω_n being i.u.d. on the d -dimensional unit cube $[0, 1]^d$ and with their radii $r_n \sim cn^{-\frac{1}{d}}$ for some constant $0 < c < (\beta(d))^{-\frac{1}{d}}$, where $\beta(d)$ is the volume of the d dimensional unit ball. We call $[0, 1]^d - \bigcup_{n=1}^{\infty} B_n(\omega)$ a random cutout set. In this paper, we present an exposition of Zähle cutout model in [4] by a detailed study of such a random cutout set for the purpose of teaching and learning. We show that with probability one Hausdorff dimension of such random cut-out set is at most $d(1 - \beta(d)c^d)$ and frequently equals $d(1 - \beta(d)c^d)$.

1 Introduction

The term fractal was first introduced by Mandelbrot in 1975 and usually refers to sets which, in some sense, have a self-similar structure. The Cantor ternary set is one of the best known and most easily constructed fractals. Although Mandelbrot and others have modeled a great number of real objects using such fractals [1, 2], simple fractals can have limitations when modeling complex real phenomena. As pointed out in [2] in such cases random fractals can prove considerably more satisfactory. Some form of self-similarity is common for

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random fractal models as well, in particular this is true for those arising from stochastic processes.

Random fractals arise as sets derived from non-differentiable stochastic processes and fields, such as the zeros of Brownian motion or the zeros of other recurrent processes with independent stable increments. One important theme in modern stochastic geometry is to generate the “pure geometric” constructs of random fractals, but without the aid of random fields. The earliest investigation of this nature can be found in [3] where Mandelbrot constructed a random set on the real line by removing or “cutting out” a sequence of random intervals with decreasing length. This became known as the *random cutout set*. Mandelbrot’s dimension calculations for this random cutout set were based on what is referred to as a birth and death process. Subsequently, Zähle in [4] studied generalizations of Mandelbrot’s cutout model in higher dimensions. Here, the intervals removed in Mandelbrot’s cutout model are replaced by considerably more general random open sets, and the approach used in the dimension calculations is purely measure-geometric. In fact, Zähle’s entire approach is quite different from that found in [3]. Other constructive, purely geometric examples can be found in [5-13] and the references therein. In [4], the author presents various general conditions under which the essential dimension of a random cutout fractal can be found. However, these results and related conditions are sufficiently abstract (and general) that they can provide a steep initial learning curve to someone just learning the material. The purpose of the present paper is to provide a basic understanding of the Zähle’s machinery and results by making a detailed study of a specific case of Zähle’s cutout model. In Falconer’s book [10], there is an exposition of exactly such a case in the one-dimensional case, while the two dimensional case is left as an exercise (see Exercise 8.4). In this paper, we expand and extend Falconer’s one dimensional result to high-dimensional space and fill in several salient details. In all of this, it is important to highlight that although the form of the generalization from one dimensional space to dimensions $d \geq 2$ is rather straightforward, some of the proofs verifying that generalization remain significantly difficult. We begin by defining our basic framework.

Given a probability space (Ω, \mathcal{F}, P) and let $\{r_n\}_{n \geq 1}$ be a sequence of positive real numbers which is decreasing to zero. Let $\{\omega_n\}_{n \geq 1}$ be a sequence of independent and identically distributed random variables which are defined on (Ω, \mathcal{F}, P) and take values in the unit cube $[0, 1]^d$. We shall call

$$B_n(\omega) := B(\omega_n, r_n) := \{x \in [0, 1]^d : |\omega_n - x| < r_n\}$$

a random open ball throughout this paper, where $|\cdot|$ denote the Euclidean distance in \mathbb{R}^d . For $\omega \in \Omega$, we define $K_0 = K_0(\omega) = [0, 1]^d$, and recurrently,

$K_n(\omega) = K_{n-1}(\omega) - B_n(\omega)$ for $n \geq 1$. Then $\{K_n(\omega)\}_{n \geq 1}$ is a sequence of random compact sets and $K_{n+1}(\omega) \subset K_n(\omega)$. We call

$$K(\omega) = \bigcap_{n=0}^{\infty} K_n(\omega) = [0, 1]^d - \bigcup_{n=1}^{\infty} B_n(\omega)$$

a random cutout set. Note that this construction differs from cutout set of [14] in that open balls removed may overlap.

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive real numbers which are decreasing to zero. We say that they are equivalent, and denote it by $a_n \sim b_n$, if $\frac{a_n}{b_n} \rightarrow 1$ as $n \rightarrow \infty$. In this paper we are interested in the case that each ω_n is uniformly distributed on $[0, 1]^d$ and

$$r_n \sim \frac{c}{\sqrt[d]{n}} \quad (1)$$

for some constant c with $0 < c < \frac{1}{\sqrt[d]{\beta(d)}}$, where $\beta(d) = \Gamma(\frac{1}{2})^d / \Gamma(\frac{d}{2} + 1)$ is the volume of the d dimensional unit ball. As we shall show in Property 3.1 of Section 3, the Lebesgue measure of set $K(\omega)$ is almost surely (a.s. for short) zero in this case. Obviously, this is a specific case of Zähle cutout model in [4]. As mentioned just now, we shall present a detailed study for such case for the purpose of teaching and learning. Our result is the following Theorem 1.1 which extends the result of [10], which takes place in \mathbb{R} .

Theorem 1. *Suppose $K(\omega)$ is a random cut-out set defined as above. Then*

$$P\{\dim_H K(\omega) \leq \overline{\dim}_B K(\omega) \leq d(1 - \beta(d)c^d)\} = 1,$$

and $P\{\dim_H K(\omega) = d(1 - \beta(d)c^d)\} > 0$, where \dim_H denotes the Hausdorff dimension.

Another interesting motivation of studying random cut-out sets is that it is helpful to the study of Dvoretzky's problem and related topics, see [3, 15]. Dvoretzky's problem [16] was posed in 1956. Subsequently, it attracted the attention of Levy, Kahane, Erdős, Billard, Mandelbrot, et al. In 1972, L. Shepp [17, 18] gave a complete solution to this problem. For further information on Dvoretzky's problem, please refer to [19]. We refer the reader to [20-23] and the references therein for more recent developments. Furthermore, the model studied in this paper is quite similar to fractal percolation and related continuous Poissonian cutout fractal models. Since there is a vast literature on the geometric and dimension theoretic properties of such random fractals (see [26, 27] and the references therein), our study is also valuable to the study of such topics.

The rest of this paper is organized as follows. Section 2 is the preliminaries. In Section 2.1, we recall the potential theoretic method. In Section 2.2, we recall the definition of Martingale and the related convergence theorems. In Section 2.3, we introduce a sequence of random measures and show that it is weakly converges to a random measure with probability 1 by using the martingale convergence theorem. The proof of Theorem 1.1 is given in Section 3.

2 Preliminary

2.1 Potential theoretic method

We recall a technique for calculating Hausdorff dimensions that is widely used both in theory and in practice. Let $s \geq 0$, we call

$$I_s(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-y|^s} d\mu(x) d\mu(y)$$

s -energy of a measure μ on \mathbb{R}^d . Denoting the s -dimension Hausdorff measure by \mathcal{H}^s . The following Lemma is often seen in literature, here we refer reader to see [10, 11] for more details.

Lemma 2. *Suppose $E \subseteq \mathbb{R}^d$. If there exists a finite measure μ on E with $I_s(\mu) < \infty$ and $\mu(E) > 0$, then $\mathcal{H}^s(E) = \infty$ and $\dim_H E \geq s$.*

2.2 Martingales and the convergence theorems

We denote by \mathbb{E} the expectation throughout this paper. The word ‘martingale’ comes from the name of a classical betting system (involving doubling one’s stake after every lost game), and it is natural to think intuitively of martingales in the context of gambling. For example, a gambler plays a sequence of games against a casino. If X_1, X_2, \dots is a sequence of random variables, we may think of X_k as the gambler’s capital after k trials in a succession of games. Having survived the first k trials, the expected value of the gambler’s capital X_{k+1} after trial $k+1$ is $\mathbb{E}(X_{k+1}|X_1, \dots, X_k)$. If this equals X_k , the game is “fair” since the expected gain on trial $n+1$ is $\mathbb{E}(X_{k+1} - X_k|X_1, \dots, X_k) = X_k - X_k = 0$. If $\mathbb{E}(X_{k+1}|X_1, \dots, X_k) \geq X_k$, the game is “favorable,” and if $\mathbb{E}(X_{k+1}|X_1, \dots, X_k) \leq X_k$, the game is “unfavorable.”

We recall the notion of martingales[10, 24]. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$ an increasing sequence of sub σ -fields of \mathcal{F} . Assume that for each k , X_k is an integrable random variable on $(\Omega, \mathcal{F}_k, P)$. We say

that $\{X_k\}_{k \geq 0}$, or more precisely, $\{(\mathcal{F}_k, X_k)\}_{k \geq 0}$ is a martingale if for all $k = 0, 1, 2, \dots$,

$$\mathbb{E}(X_{k+1}|\mathcal{F}_k) = X_k, \text{ a.s.} \quad (2)$$

Condition (2) implies essentially that, whatever happens in the first k steps of the process, the expectation of X_{k+1} nevertheless equals X_k . In the gambling example described as above, X_0 represents the initial capital and \mathcal{F}_k the set of all possible outcomes of the first k trials. Then (2) means that regardless of what happens in the first k trials, the expected value of the gambler's capital X_{k+1} after the trial $k+1$ equals the capital X_k before that game; this reflects the fairness of the game.

Remark 1. *In measure theory, the technical definition of conditional expectation $\mathbb{E}(X|\mathcal{G})$ is quite complicated (for details see [24]), where X is a random variable on (Ω, \mathcal{F}, P) and \mathcal{G} a sub σ -field of \mathcal{F} . In this paper it is enough to think of $\mathbb{E}(X_{k+1}|\mathcal{F}_k)$ as the mean value of X_{k+1} calculated as though X_0, \dots, X_k are already known. As mentioned in [10], the properties of conditional expectation that we shall use are very natural in terms of this interpretation.*

Much of the theory remains true if (2) is weakened to inequality. We say that $\{(\mathcal{F}_k, X_k)\}_{k \geq 0}$ a submartingale if for $k = 0, 1, 2, \dots$, $\mathbb{E}(X_{k+1}|\mathcal{F}_k) \geq X_k$ a.s., a supermartingale if $\mathbb{E}(X_{k+1}|\mathcal{F}_k) \leq X_k$ a.s.. For non-negative supermartingals, that is with $X_k \geq 0$ for all k , we will concern the following convergence theorem.

Lemma 3. [10] *Suppose $\{X_k\}_{k \geq 0}$ is a non-negative supermartingale. Then there exists a non-negative random variable X on (Ω, \mathcal{F}, P) such that X_k converges to X a.s.. Moreover, $0 \leq \mathbb{E}(X) \leq \inf_k \mathbb{E}(X_k)$.*

A disadvantage of Lemma 3 is that it is possible to have a non-negative martingale X_k with $\mathbb{E}(X_k) = \mathbb{E}(X_0)$ for each $k \geq 1$, but with limit $X = 0$ a.s.. For many applications we need to be able to conclude that $X > 0$ with positive probability. The following convergence theorem shows that we can ensure that this is so under relatively mild conditions. We say that $\{(\mathcal{F}_k, X_k)\}_{k \geq 0}$ is an L^2 -bounded martingale if it is a martingale with

$$\sup_{0 \leq k < \infty} \mathbb{E}(X_k^2) < \infty.$$

Lemma 4. [10] *Suppose $\{X_k\}_{k \geq 0}$ is an L^2 -bounded martingale. Then there exists a random variable X on (Ω, \mathcal{F}, P) such that X_k converges to X a.s., with*

$$\mathbb{E}(|X - X_k|) \leq \mathbb{E}((X - X_k)^2)^{1/2} \rightarrow 0$$

as $k \rightarrow \infty$. In particular, $\mathbb{E}(X) = \mathbb{E}(X_k)$ for all $k \geq 0$.

2.3 Random measure

Let $\mathcal{B}(\mathbb{R}^d)$ denote the family of all Borel sets of \mathbb{R}^d . We say that μ is a random measure with respect to probability space (Ω, \mathcal{F}, P) if μ is \mathcal{F} -measurable, that is, μ is a function which associates with each $\omega \in \Omega$ a measure μ_ω on \mathbb{R}^d such that, for all $A \in \mathcal{B}(\mathbb{R}^d)$, the function

$$\omega \rightarrow \mu_\omega(A)$$

from Ω to $[0, +\infty)$ is $(\mathcal{F}, \mathcal{B}([0, +\infty))$ -measurable. Usually, we also say that μ is a random measure on \mathbb{R}^d for the purpose of highlighting \mathbb{R}^d .

For simplicity, we write

$$p_n = \prod_{i=1}^n (1 - \beta(d)r_i^d), \quad n = 1, 2, \dots,$$

where $\beta(d)$ and r_i as described in introduction. Denoting the Lebesgue measure by \mathcal{L} on \mathbb{R}^d . We define a sequence of random measures μ_n with respect to probability space (Ω, \mathcal{F}, P) by setting

$$\mu_n(A) = \begin{cases} p_n^{-1} \mathcal{L}(A \cap K_n(\omega)), & n = 1, 2, \dots, \\ \mathcal{L}(A \cap [0, 1]), & n = 0, \end{cases} \quad (3)$$

for any $A \in \mathcal{B}(\mathbb{R}^d)$, where $\{K_n(\omega)\}_{n \geq 1}$ is a sequence of random compact sets as described in introduction. Recall that a right-semiclosed interval in \mathbb{R}^d is a set of the form $(a, b] = \{(x_1, \dots, x_d) \in \mathbb{R}^d : a_i < x_i \leq b_i \text{ for all } i = 1, 2, \dots, d\}$, where $-\infty \leq a_i \leq b_i < \infty$; by convention we also count (a, ∞) as right-semiclosed for $-\infty \leq a_i < \infty$.

Lemma 5. *Let $\{\mu_n\}_{n \geq 0}$ be a sequence of random measures with respect to probability space (Ω, \mathcal{F}, P) described as above, then there exists a random measure μ such that almost surely, μ_n is weakly convergent to μ .*

PROOF. Let \mathcal{F}_n denote the σ -field underlying the random positions of the centers of $B_1(\omega), B_2(\omega), \dots, B_n(\omega)$. (Formally \mathcal{F}_n is the σ -field generated by a k -fold product of Borel subsets of $[0, 1]^d$.) For each right-semiclosed interval A with rational endpoints, since all ω_n are *independent and uniformly distributed* (i.u.d. for short) on $[0, 1]^d$, we have

$$\begin{aligned} \mathbb{E}(\mu_{n+1}(A) | \mathcal{F}_n) &= \mathbb{E}(p_{n+1}^{-1} \mathcal{L}(A \cap K_n(\omega) \cap ([0, 1]^d - B_{n+1}(\omega)) | \mathcal{F}_n) \\ &= p_{n+1}^{-1} \mathcal{L}([0, 1]^d - B_{n+1}(\omega)) \mathcal{L}(A \cap K_n(\omega)) \\ &= p_{n+1}^{-1} (1 - \beta(d)r_{n+1}^d) p_n \mu_n(A) \\ &= \mu_n(A) \text{ a.s..} \end{aligned}$$

Thus $\{\mu_n(A)\}_{n \geq 0}$ is a non-negative martingale, so by Lemma 3 there exists a random variable $\mu(A)$ such that almost surely, $\mu_n(A)$ converges to $\mu(A)$. We denote by \mathcal{C} the set of all right-semiclosed intervals with rational endpoints in $\mathcal{B}(\mathbb{R}^d)$. Then $\mathcal{B}(\mathbb{R}^d)$ is the smallest σ -field containing the sets of \mathcal{C} , see [24] for details. Since \mathcal{C} is a countable set, so there exists a set $D \subset \Omega$ with $P(D) = 0$ such that

$$\lim_{n \rightarrow \infty} \mu_n(A) = \mu(A), \quad \text{all } \omega \notin D, A \in \mathcal{C}.$$

Now we extend μ on \mathcal{C} to a Borel measure supported by $K(\omega)$ in the usual way, where $K(\omega)$ is described as in introduction. Then for any $\omega \notin D$ there exists a measure μ such that μ_n weak converges to μ or, what is the same, for every $A \in \mathcal{B}(\mathbb{R}^d)$ such that $\mu(\partial(A)) = 0$ ($\mu(\partial(A))$ denotes the boundary of A),

$$\lim_{n \rightarrow \infty} \mu_n(A) = \mu(A).$$

For $\omega \in D$ define $\mu_\omega = 0$. We denote by \mathfrak{M} the usual σ -field in the family of all Randon measures on $\mathcal{B}(\mathbb{R}^d)$. Since the map

$$\nu \mapsto \nu(A), A \in \mathcal{B}(\mathbb{R}^d),$$

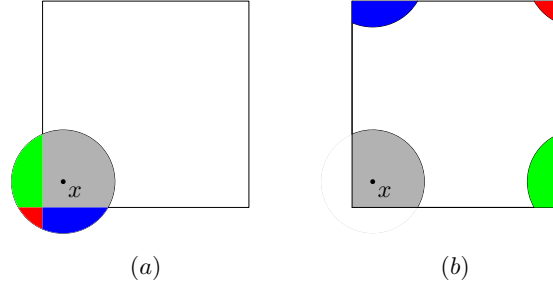
is $(\mathfrak{M}, \mathcal{B}(\mathbb{R}))$ -measurable, the map $\omega \mapsto \mu_\omega$ is a random measure. This completes the proof. \square

3 Proof of Theorem 1

Recall that $\{\omega_n\}_{n \geq 1}$ is a sequence of i.u.d. random variables which are defined on probability space (Ω, \mathcal{F}, P) and take values in $[0, 1]^d$, $\{r_n\}_{n \geq 1}$ is a sequence of positive real numbers which is decreasing to zero and

$$K(\omega) = [0, 1]^d - \bigcup_{n=1}^{\infty} B_n(\omega)$$

is a random cutout set, where $B_n(\omega) := B(\omega_n, r_n)$ is a random open ball with center ω_n and radius r_n for each n . In this paper, it is convenient to identify $[0, 1]^d$ with the d-dimensional torus. For example, we identify the corresponding edge of unit square $[0, 1]^2$ on plane, that is, if $\omega_n = x = (x_1, x_2)$ ($0 \leq x_1, x_2 < r_n$) is a center of open disc $B_n(\omega)$ showed as Figure 1(a), then $B_n(\omega)$ is taken to consist of four fields that are showed by grey($B_n(\omega) \cap [0, 1]^2$), red($(B_n(\omega) + (1, 1)) \cap [0, 1]^2$), blue($(B_n(\omega) + (0, 1)) \cap [0, 1]^2$) and green($(B_n(\omega) + (1, 0)) \cap [0, 1]^2$) in Figure 1(b) respectively.

Figure 1: $d = 2$

For $x = (x_1, x_2, \dots, x_d), y = (y_1, y_2, \dots, y_d) \in [0, 1]^d$, we write

$$d(x, y) = \sqrt{\sum_{i=1}^d (\min(|x_i - y_i|, 1 - |x_i - y_i|))^2},$$

that is the distance between x and y when identifying $[0, 1]^d$ with the d -dimensional torus. In particular, for $d = 1$, $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$, that is the distance between x and y with 0 and 1 identified. We first estimate the probability that a given point is in $K_n(\omega)$.

Proposition 6. *For any $x \in [0, 1]^d$, we have*

$$P(x \in K_n(\omega)) = p_n = \prod_{i=1}^n (1 - \beta(d)r_i^d), \quad n = 1, 2, \dots$$

PROOF. Since each ω_i is uniformly distributed on $[0, 1]^d$, it follows that for all $x \in [0, 1]^d$ and $i = 1, 2, \dots$,

$$P(x \in B_i(\omega)) = \beta(d)r_i^d.$$

Note that $x \in K_n(\omega)$ if and only if $x \notin B_i(\omega)$ for all $i = 1, 2, \dots, n$. But the events $(x \notin B_i(\omega))_{i=1}^n$ are independent, so

$$P(x \in K_n(\omega)) = \prod_{i=1}^n P(x \notin B_i(\omega)) = \prod_{i=1}^n (1 - \beta(d)r_i^d).$$

□

Proposition 7. (a) If $\sum_{n=1}^{\infty} r_n^d < \infty$, then $P(\mathcal{L}(K(\omega)) > 0) > 0$; (b) If $\sum_{n=1}^{\infty} r_n^d = \infty$, then $P(\mathcal{L}(K(\omega)) = 0) = 1$.

PROOF. Denote by χ_A the characteristic function of a set A . By Lebesgue's Dominated Convergence Theorem, Fubini Theorem and Proposition 6, we have

$$\begin{aligned}
\mathbb{E}(\mathcal{L}(K(\omega))) &= \int \mathcal{L}\left(\bigcap_{n=1}^{\infty} ([0, 1]^d - B_n(\omega_n, r_n))\right) dP(\omega) \\
&= \lim_{m \rightarrow \infty} \int \mathcal{L}\left(\bigcap_{n=1}^m ([0, 1]^d - B_n(\omega_n, r_n))\right) dP(\omega) \\
&= \lim_{m \rightarrow \infty} \int \int \prod_{n=1}^m \chi_{[0, 1]^d - B_n(\omega_n, r_n)}(y) dy dP(\omega) \\
&= \lim_{m \rightarrow \infty} \int \int \chi_{[0, 1]^d - \cup_{n=1}^m B_n(\omega_n, r_n)}(y) dy dP(\omega) \\
&= \lim_{m \rightarrow \infty} \int \int \chi_{K_n(\omega)}(y) dy dP(\omega) \\
&= \lim_{m \rightarrow \infty} \int \int \chi_{K_n(\omega)}(y) dP(\omega) dy \\
&= \lim_{m \rightarrow \infty} \prod_{n=1}^m (1 - \beta(d)r_n^d) \\
&\leq \lim_{m \rightarrow \infty} e^{-\sum_{n=1}^m \beta(d)r_n^d}.
\end{aligned}$$

Since if $\sum_{k=1}^{\infty} r_n^d < \infty$ then $0 < \mathbb{E}(\mathcal{L}(K(\omega))) < \infty$, if $\sum_{k=1}^{\infty} r_n^d = \infty$ then $\mathbb{E}(\mathcal{L}(K(\omega))) = 0$, the desired result follows. This completes this proof. \square

Remark 2. Proposition 6 and Proposition 7 depend on condition 'i.u.d.', and not on the fact that r_n decreases to zero.

Next, we estimate the probability that a given pair of points is in $K_n(\omega)$.

Lemma 8. For any $\varepsilon > 0$, there exists a constant $L > 0$ such that

$$\frac{P(x \in K_n(\omega), y \in K_n(\omega))}{p_n^2} \leq Ld(x, y)^{-d\beta(d)c^d(1+\varepsilon)}. \quad (4)$$

for all $x, y \in [0, 1]^d$ and $n = 1, 2, \dots$.

PROOF. It follows from the O'Stolz Theorem [25], $\sum_{i=1}^{\infty} r_i^d = \infty$ and (1) that

$$\log p_n = \sum_{i=1}^n \log(1 - \beta(d)r_i^d) \sim -\sum_{i=1}^n \frac{\beta(d)c^d}{i} \sim -\beta(d)c^d \log n. \quad (5)$$

Thus by (1),

$$\log p_n \sim -\beta(d)c^d \log n \sim d\beta(d)c^d \log r_n.$$

For any $\varepsilon > 0$, we have from (5) that there exists a constant $L_1 > 0$ such that for all $n = 1, 2, \dots$,

$$\prod_{i=1}^n (1 - \beta(d)r_i^d) = p_n \geq L_1 r_n^{d\beta(d)c^d(1+\varepsilon)}. \quad (6)$$

For any $x, y \in [0, 1]^d$, we consider the positions of ω_i when $B_i(\omega)$ excludes both x and y . Since ω_i is uniformly distributed on $[0, 1]^d$, we have

$$P(x \notin B_i(\omega), y \notin B_i(\omega)) \leq \begin{cases} 1 - \beta(d)r_i^d & d(x, y) \leq r_i, \\ (1 - \beta(d)r_i^d)^2 & d(x, y) > r_i. \end{cases}$$

Thus

$$\frac{P(x \notin B_i(\omega), y \notin B_i(\omega))}{(1 - \beta(d)r_i^d)^2} \leq \begin{cases} (1 - \beta(d)r_i^d)^{-1} & d(x, y) \leq r_i, \\ 1 & d(x, y) > r_i. \end{cases}$$

By (6) and the independence of random open balls that are moved, we have

$$\begin{aligned} \frac{P(x \in K_n(\omega), y \in K_n(\omega))}{p_n^2} &= \prod_{i=1}^n \frac{P(x \notin B_i(\omega), y \notin B_i(\omega))}{(1 - \beta(d)r_i^d)^2} \\ &\leq \prod_{i: d(x, y) \leq \frac{\varepsilon}{i^{\frac{1}{d}}}} (1 - \beta(d)r_i^d)^{-1} \\ &\leq (p_{i(d(x, y))})^{-1} \\ &\leq L_1^{-1} (r_{i(d(x, y))})^{-d\beta(d)c^d(1+\varepsilon)}, \end{aligned}$$

where $i(d(x, y))$ is the largest positive integer i such that $d(x, y) \leq r_i$. From $r_{n+1}/r_n \rightarrow 1 (n \rightarrow \infty)$, we have $r_{i(d(x, y))} \sim d(x, y)$. So there exists a suitable constant $L > 0$ such that

$$\frac{P(x \in K_n(w), y \in K_n(w))}{p_n^2} \leq L d(x, y)^{-d\beta(d)c^d(1+\varepsilon)}$$

□

Lemma 9. *Let μ be a random measure defined in Lemma 5. Then $\mu(K(w)) > 0$ is of positive probability.*

PROOF. Note that the inequality (4) implies that

$$\begin{aligned} \frac{\mathbb{E}(\chi_{K_n(w) \times K_n(w)}(x, y))}{p_n^2} &= \frac{P(x \in K_n(w), y \in K_n(w))}{p_n^2} \\ &\leq Ld(x, y)^{-d\beta(d)c^d(1+\epsilon)}. \end{aligned} \tag{7}$$

Since $0 < c < \frac{1}{\sqrt[\beta(d)]{\beta(d)}}$, we can choose ϵ such that $d\beta(d)c^d(1 + \epsilon) < d$. It follows from (3) and (7) that

$$\begin{aligned} \mathbb{E}((\mu_n([0, 1]^d))^2) &= p_n^{-2} \mathbb{E}((\mathcal{L}(K_n(\omega)))^2) \\ &= p_n^{-2} \mathbb{E}(\iint \chi_{K_n(\omega)}(x) \times \chi_{K_n(\omega)}(y) dx dy) \\ &= p_n^{-2} \mathbb{E}(\iint \chi_{K_n(\omega) \times K_n(\omega)}(x, y) dx dy) \\ &= p_n^{-2} \mathbb{E}(\mathbb{E}(\chi_{K_n(\omega) \times K_n(\omega)}(x, y))) \\ &\leq L \int_{[0,1]^d} \int_{[0,1]^d} d(x, y)^{-d\beta(d)c^d(1+\epsilon)} dx dy < \infty. \end{aligned}$$

Thus $\{\mu_n([0, 1]^d)\}_{n \geq 0}$ is an L^2 -bounded martingale, so by Lemma 4,

$$\mathbb{E}(\mu(K(\omega))) = \mathbb{E}(\mu([0, 1]^d)) = \mathbb{E}(\mu_0([0, 1]^d)) = 1,$$

giving that $P(\mu(K(\omega)) > 0) > 0$. This completes the proof. □

The proof of Theorem 1. Given $\delta > 0$ and let $K(\omega)_\delta$ denote the δ -neighborhood of $K(\omega)$, that is,

$$K(\omega)_\delta = \{x \in \mathbb{R}^d : |x - y| \leq \delta \text{ for some } y \in K(\omega)\}.$$

Denoting the largest positive integer with $r_k > \delta$ by $k(\delta)$. Let $\widetilde{B}_j(\omega)$ be an open ball with the same center as $B_j(\omega)$ and radius $r_j - \delta$ for $j \leq k(\delta)$. Then if $x \in K(\omega)_\delta$ and $j \leq k(\delta)$, then $x \notin \widetilde{B}_j(\omega)$. By the independence of the random open balls that are removed, we have that for any $x \in [0, 1]^d$,

$$\begin{aligned} P(x \in K(\omega)_\delta) &\leq P(x \notin \bigcup_{j=1}^{k(\delta)} \widetilde{B}_j(\omega)) = P(x \in \bigcap_{j=1}^{k(\delta)} ([0, 1]^d - \widetilde{B}_j(\omega))) \\ &= \prod_{j=1}^{k(\delta)} P(x \notin \widetilde{B}_j(\omega)) = \prod_{j=1}^{k(\delta)} (1 - \beta(d)(r_j - \delta)^d). \end{aligned} \tag{8}$$

By the maximality of $k(\delta)$, $r_{k(\delta)} > \delta \geq r_{k(\delta)+1}$, which gives $r_{k(\delta)} \sim \delta$. So $\frac{c}{\sqrt[k(\delta)]{k(\delta)}} \sim \delta$ by 1. We thus get by using (5)

$$\begin{aligned}
\log \prod_{j=1}^{k(\delta)} (1 - \beta(d)(r_j - \delta)^d) &= \sum_{j=1}^{k(\delta)} \log(1 - \beta(d)(r_j - \delta)^d) \\
&\leq - \sum_{j=1}^{k(\delta)} \beta(d)(r_j - \delta)^d = -\beta(d) \sum_{j=1}^{k(\delta)} (r_j - \delta)^d \\
&= -\beta(d) \sum_{j=1}^{k(\delta)} \sum_{i=0}^d C_d^i r_j^i (-\delta)^{d-i} \\
&= -\beta(d) \sum_{j=1}^{k(\delta)} r_j^d - \beta(d) \sum_{j=1}^{k(\delta)} \sum_{i=1}^{d-1} C_d^i r_j^i (-\delta)^{d-i} + \\
&\quad + (-1)^{d+1} \beta(d) \delta^d k(\delta)
\end{aligned}$$

Since

$$\begin{aligned}
\frac{-\beta(d) \sum_{j=1}^{k(\delta)} \sum_{i=1}^{d-1} C_d^i r_j^i (-\delta)^{d-i}}{-\beta(d) c^d \log(k(\delta))} &\leq \frac{-\beta(d) \sum_{i=1}^{d-1} C_d^i (-\delta)^{d-i} \sum_{j=1}^{k(\delta)} r_j^d}{-\beta(d) c^d \log(k(\delta))} \\
&\sim \sum_{i=1}^{d-1} C_d^i (-\delta)^{d-i} \rightarrow 0, \quad \text{as } \delta \rightarrow 0,
\end{aligned}$$

we see that

$$-\beta(d) \sum_{j=1}^{k(\delta)} \sum_{i=0}^d C_d^i r_j^i (-\delta)^{d-i} \sim -\beta(d) c^d \log k(\delta) \sim d\beta(d) c^d \log \delta. \quad (9)$$

Given $\varepsilon > 0$, it follows from (8) and (9) that there exists a constant M such that for any $\delta \leq 1$,

$$\mathbb{E}(\mathcal{L}(K(\omega)_\delta)) = P(x \in K(\omega)_\delta) \leq M \delta^{d\beta(d) c^d - \varepsilon}.$$

Thus

$$\mathbb{E} \left(\sum_{\delta=2^{-k}: k=1,2,\dots} \mathcal{L}(K(\omega)_\delta) \delta^{-d\beta(d) c^d + 2\varepsilon} \right) \leq M \sum_{\delta=2^{-k}: k=1,2,\dots} \delta^\varepsilon < \infty.$$

This implies that

$$P \left(\sum_{\delta=2^{-k}: k=1,2,\dots} \mathcal{L}(K(w)_\delta) \delta^{-d\beta(d) c^d + 2\varepsilon} < \infty \right) = 1$$

and with probability one there exists a constant $M' > 0$ such that for any positive integer k , we have $\mathcal{L}(K(w)_{2^{-k}})(2^{-k})^{-d\beta(d)c^d+2\varepsilon} \leq M'$, and thus $\mathcal{L}(K(w)_\delta)\delta^{-d\beta(d)c^d+2\varepsilon}$ is bounded for any $0 < \delta < 1$. It follows from the (2.5) of [10] that

$$P(\overline{\dim}_B K(\omega) \leq d(1 - \beta(d)c^d) + 2\varepsilon) = 1.$$

Since ε is arbitrary, we conclude that

$$P(\overline{\dim}_B K(\omega) \leq d(1 - \beta(d)c^d)) = 1.$$

It remains to determine the lower bound. For $\varepsilon > 0$, let μ_n and μ be the random measures on $K_n(w)$ and $K(w)$ introduced in Lemma 5, respectively. By Lemma 5 and Fatou's Lemma [24], and using (7), we have

$$\begin{aligned} \mathbb{E}\left(\iint |x - y|^{-s} d\mu(x)\mu(y)\right) &\leq \mathbb{E}\left(\lim_{n \rightarrow \infty} \iint |x - y|^{-s} d\mu_n(x)\mu_n(y)\right) \\ &\leq \liminf_{n \rightarrow \infty} \mathbb{E}\left(\iint |x - y|^{-s} d\mu_n(x)\mu_n(y)\right) \\ &= \liminf_{n \rightarrow \infty} p_n^{-2} \mathbb{E}\left(\iint |x - y|^{-s} \chi_{K_n(w) \times K_n(w)}(x, y) dx dy\right) \\ &\leq L \int_{[0,1]^d} \int_{[0,1]^d} d(x, y)^{-s} d(x, y)^{-d\beta(d)c^d(1+\varepsilon)} dx dy \\ &= L \int_{[0,1]^d} \int_{[0,1]^d} d(x, y)^{-(s+d\beta(d)c^d(1+\varepsilon))} dx dy \\ &< \infty \end{aligned}$$

provided that $s < d(1 - \beta(d)c^d)(1 + \varepsilon)$. This implies that for any $s < d(1 - \beta(d)c^d)$,

$$P\left(\int \int |x - y|^{-s} d\mu(x)\mu(y) < \infty\right) = 1.$$

By Lemma 9 and Lemma 2, $P(\dim_H K(\omega) \geq d(1 - \beta(d)c^d)) > 0$. This completes the proof.

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