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MEASURE, CATEGORY AND CONVERGENT SERIES

Abstract

The analogy between measure and Baire category is displayed first by a theorem of Steinhaus and its "dual," a theorem of Piccard. These two theorems are then applied to provide a double criterion for the unconditional convergence of a series in terms of the "measure size" and the "category size" of the set of its convergent subseries. As a further application, after a substantial preparatory section concerning essential separability of measurable and BP-measurable functions, the results about exhaustivity of BP_r-measurable and universally measurable additive maps on the Cantor group are established. In the last sections of the paper, two classical theorems about countable additivity of the universal measurable and BP_r-measurable additive maps are examined. The analogy in question is illustrated not only by the results themselves, but also by the proofs provided.

1 Introduction

There is a long line of research motivated by the analogy between measure and (Baire) category, see e.g. Oxtoby's survey [28]. In particular, the analogy may concern "dual" statements involving BP-measurable sets and measurable sets. In those statements one would like *1st category sets* to correspond to measure zero sets, and *2nd category sets* (those that are not 1st category) to sets of positive measure. We recall that a subset of a topological space is 1st category if it is a countable union of nowhere dense sets. An example of the duality in question is provided by the following two classical results.

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First is the Steinhaus Theorem (going back to [32]; in the group setting, to [35]).

Theorem 1.1. *Let G be a locally compact group with (left invariant, say) Haar measure χ and $A \subset G$. If A is of positive measure, then AA^{-1} is a neighborhood of unit in G.*

Here is an elementary proof due to Kestelman [22] (who still worked in $\mathbb{R}^n$), later rediscovered by Stromberg in [33].

Proof. By the inner regularity of χ, one can assume that A is compact. By the outer regularity of χ, one finds an open set U containing A such that $\chi(U \setminus A) < \frac{1}{2}\chi(A)$, and then a neighborhood V of unit such that $VA \subset U$. Given $z \in V$, $zA \subset VA \subset U$ and so

$$\chi(U \setminus zA) = \chi(U) - \chi(zA) = \chi(U) - \chi(A) < \frac{1}{2}\chi(A).$$

Hence $\chi(U \setminus (A \cap zA)) \leq \chi(U \setminus A) + \chi(U \setminus zA) < \chi(A) < \chi(U)$. Consequently, $\chi(A \cap zA) \neq 0$ and $A \cap zA \neq \emptyset$. Therefore there exists $y \in A$ such that $y = zx$ with $x \in A$, whence $yx^{-1} = z$ and $z \in AA^{-1}$. As z was an arbitrary member of V, this means that $V \subset AA^{-1}$. □

The dual statement is a theorem of Piccard [31], mostly known as the *Pettis Lemma* (because of [30]). Kelley has it as an exercise ([21], Chapter 6, Exercise P(b)) and calls the Banach–Kuratowski–Pettis Theorem.

Recall that a subset A in a topological space Y is said (to have the Baire Property or) to be *BP–measurable* if it can be written as a symmetric difference $A = O \vartriangle I$, where O is open and I is 1st category (in Y).

Theorem 1.2. *Let G be a topological group and A a BP-measurable subset of G. If A is of second category, then AA^{-1} is a neighborhood of unit in G.*

The following proof, due essentially to Bourbaki (compare the proof of [5, p. 69, Lemme 9], shows the duality we are after.

Proof. As A is 2nd category in G, so is G. By the Banach category theorem [28, Th. 16.1], every nonempty open set is 2nd category in G. In particular, if $A = O \vartriangle I$, then O is 2nd category and therefore not empty. It suffices to show that $OO^{-1} \subset AA^{-1}$. Let $x \in OO^{-1}$. Then the open set $O \cap xO$ is not empty and therefore 2nd category. Define

$$Z = (O \cap xO) \setminus (A \cap xA).$$

Since $Z \subset (O \setminus A) \cup x(O \setminus A)$, Z is 1st category. Consequently, $A \cap xA \neq \emptyset$. This means that $x \in AA^{-1}$ and $OO^{-1} \subset AA^{-1}$. □

These two theorems will provide a unifying tool in what follows. It is our aim to present some results, new and old, that are "dual" in a similar way as the theorems of Steinhaus and Piccard are. Perhaps even more importantly, we provide proofs that display and stress that sort of duality.

2 Series

In what follows, $\mathcal{P}(A)$ stands for all subsets of a set A, $\mathcal{F}(A)$ for all its finite subsets; $\mathbb{N} = \{1, 2, \dots\}$, $\mathcal{P} = \mathcal{P}(\mathbb{N}), \mathcal{F} = \mathcal{F}(\mathbb{N})$ and $\mathcal{F}_m = \mathcal{F}\{m, m+1, \dots\}$. By identifying subsets of $\mathbb{N}$ with their characteristic functions, subfamilies of $\mathcal{P}$ become subsets of the *Cantor group* $K = \{0,1\}^{\mathbb{N}}$, i.e., we regard the set of all sequences of $0's$ and $1's$ as the countable product of the group $\{0,1\}$ with addition mod. 2. K is a compact metric Abelian group whose addition in terms of $\mathcal{P}$ is the symmetric difference $\triangle$ of subsets in $\mathcal{P}$. Putting masses $1/2$ at the points 0 and 1 of $\{0,1\}$, the corresponding product measure χ on K is its Haar probability measure.

Using the identification, the topological properties of subfamilies in $\mathcal{P}$ always refer to the topological properties of the corresponding subsets of the Cantor group.

If X is a topological Abelian group, its topology can be defined by a family of (group) semi–norms. Then, in order to establish convergence of a (filter or) sequence in X, it is sufficient to do it with respect to each semi–norm separately. Equivalently, if X is Hausdorff, one can embed X into a product of complete normed groups and one can argue with respect to each coordinate group separately. For a possibility of such an embedding, see e.g. [6, Ch. 2,§1, no 3, Prop. 3], where a full proof is given for the case of a locally convex space, but the argument is general and works in a group setting as well.

From now on, unless stated otherwise, X stands for a *Hausdorff topological Abelian group* and, if it is normed, then its (group) norm is denoted by $\|\cdot\|$, i.e., $X = (X, \|\cdot\|)$.

In order to set the terminology, recall a few known facts.

Let X be sequentially complete. The following conditions are equivalent for a sequence (x_n) in X.

(a) The series $\sum_n x_n$ is unconditionally convergent, i.e., converges for each ordering of its terms.

(b) The series $\sum_n x_n$ is subseries convergent, i.e., all its partial series are (subseries) convergent.

(c) With the convention $0 \cdot x = 0$ and $1 \cdot x = x$, the series $\sum \varepsilon_n x_n$ converges for all sequences (ε_n) of $0's$ and $1's$.

(d) The series $\sum_n x_n$ satisfies the Cauchy condition:

$$\forall\, U \;\exists\, m \;\forall\, F \in \mathcal{F}_m \; \sum_{n\in F} x_n \in U,$$

where U is a neighborhood of zero in X.

(e) If X is normed, then the Cauchy condition can be written as

$$\forall\, \varepsilon > 0 \;\exists\, m \;\forall\, F \in \mathcal{F}_m \; \Big\| \sum_{n\in F} x_n \Big\| \leqslant \varepsilon.$$

- For a sequence $\boldsymbol{x} = (x_n)$ in X, its *set of unconditional convergence* is defined as

$$\mathcal{C}(\boldsymbol{x}) = \mathcal{C}(x_n) = \{A \in \mathcal{P} : \sum_{n\in A} x_n \text{ is unconditionally convergent}\}.$$

It is identified with

$$C(\boldsymbol{x}) = \{(\varepsilon_n) \in K : \sum_n \varepsilon_n x_n \text{ is unconditionally convergent}\}.$$

Because of the identification, we use the symbol $\mathcal{C}(\boldsymbol{x})$ only.

Proposition 2.1. *Let $X = (X, \|\cdot\|)$ be complete. For every sequence*

$$\boldsymbol{x} \;=\; (x_n) \subset \; X,$$

the set $\mathcal{C}(\boldsymbol{x})$ is an $F_{\sigma\delta}$ subset of K.

Proof. By using the Cauchy condition (e) above, one has

$$(\varepsilon_j) \in \mathcal{C}(\boldsymbol{x}) \iff \forall\, k\, \exists\, n\, \forall\, F \in \mathcal{F}_n : \Big\| \sum_{j\in F} \varepsilon_j x_j \Big\| \leqslant \frac{1}{k},$$

hence

$$\mathcal{C}(\boldsymbol{x}) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{F\in\mathcal{F}_n} \Big\{ (\varepsilon_j) \in K : \Big\| \sum_{j\in F} \varepsilon_j x_j \Big\| \leqslant \frac{1}{k} \Big\}.$$

As, for each finite set F, the mapping $(\varepsilon_j) \to \|\sum_{j\in F} \varepsilon_j x_j\|$ from K into $\mathbb{R}$ is continuous, the sets on the right-hand side of the above formula are closed. Consequently, $\mathcal{C}(\boldsymbol{x})$ is an $F_{\sigma\delta}$ set. $\square$

Theorem 2.2. *Let X be sequentially complete and (x_n) a sequence in X. If $\mathcal{C} = \mathcal{C}(x_n)$ is either 2nd category or of positive Haar measure, then $\sum_n x_n$ is unconditionally convergent in X.*

PROOF. Let $\mathcal{C}$ be 2nd category in $\mathcal{P}$. Embed X into a product of complete normed groups. Assume, for a moment, that X is a complete normed group. Then, by Proposition 2.1, $\mathcal{C}$ is an $F_{\sigma\delta}$ set in $\mathcal{P}$ and so, in particular it is BP-measurable. By Theorem 1.2 the family $\mathcal{V} = \mathcal{C} \vartriangle \mathcal{C}$ is a neighbourhood of zero. Find a finite set $F \in \mathcal{V}$, and then n so large that $F \cup H \in \mathcal{V}$ with $H = \{n, n+1, n+2, \dots\}$ disjoint with F. Then $F \cup H = C_1 \vartriangle C_2$ with C_1, C_2 belonging to $\mathcal{C}$. Now $C_1 \setminus (C_1 \cap C_2)$ and $C_2 \setminus (C_1 \cap C_2)$ are clearly in $\mathcal{C}$ and so their disjoint union $F \cup H$ is in $\mathcal{C}$ as well. In particular, $\sum_{k=n}^{\infty} x_k$ is unconditionally convergent. As this is true for every coordinate group in the product, the original series satisfies the Cauchy condition for summability in X and, since X is sequentially complete, is unconditionally convergent there. If $\mathcal{C}$ is of positive Haar measure, the proof is the same applying Theorem 1.1. □

Remark 2.3. A family $\mathcal{I} \subset \mathcal{P}(A)$ is an *ideal in A*, if $B \subset C \in \mathcal{I} \Longrightarrow B \in \mathcal{I}$ and $B, C \in \mathcal{I} \Longrightarrow B \cup C \in \mathcal{I}$. Suppose $\mathcal{I}$ is an ideal in $\mathbb{N}$ containing finite sets. The proof above shows that *if $\mathcal{I}$ is either 2nd category BP–measurable or is of positive Haar measure in $\mathcal{P}$, then $\mathcal{I} = \mathcal{P}$*, compare [11].

Remark 2.4. By noticing that $\mathcal{I}$ is also a subgroup of $\mathcal{P}$, one can apply instead of the Piccard theorem an earlier result of Banach [2, Ch. I, Th. 1].

3 Essential separability

Let f be a function between topological spaces Y and X. It has the Baire Property or is *BP–measurable*, if for every open set V in X, the set $f^{-1}(V)$ is BP–measurable in Y. Suppose μ is a finite (positive) measure on Y. Then f is *μ–measurable*, if for every open set V in X, the set $f^{-1}(V)$ is μ–measurable in Y.

We will say that f is *essentially separably valued* in each of the following two (dual) cases. In the BP–measurable case: if there exists a 1st category set $A \subset Y$ such that that $f(Y \setminus A)$ is separable. Dually, in the measurable case: if there exists a set $A \subset Y$ of μ–measure zero such that $f(Y \setminus A)$ is separable. A subset in a topological space is called *residual* if it is the complement of a 1st category set.

Lemma 3.1. *Let X be a metric space. If a function $f : Y \to X$ has the property that each open cover of X admits a countable subfamily $\mathcal{B} = \{B_1, B_2, \dots\}$*

such that $\mu(\bigcup_{n=1}^{\infty} f^{-1}(B_n))$ is of full measure (resp. is residual), then f is essentially separably valued.

PROOF. For each $k \in \mathbb{N}$, cover X with $1/k$ balls and choose countable subfamilies $\mathcal{B}_k = \{B_{k1}, B_{k2}, \dots\}$ according to the assumption of the lemma. Set $A_{kn} = f^{-1}(B_{kn})$, $n = 1, 2, \dots$. Then $Y_1 = \bigcap_k \bigcup_n A_{kn}$ is of full measure (resp. residual) and its image $X_1 = f(Y_1)$ is separable. Indeed, one can find a point in $B \cap X_1$ for each ball $B \in \bigcup \mathcal{B}_k$ and these points form a countable dense subset of X_1. □

The following fact can already be deduced from [26], see [18]. Here, its somewhat more modern treatment is adapted from an unpublished manuscript of Prikry, cited as reference number 27 in [23]. We recall that a topological space is said to be *2nd countable* if it admits a countable base for open sets.

Lemma 3.2. *Let Y be a 2nd countable topological space equipped with a finite regular Borel measure μ. Let $\{A_\gamma : \gamma \in \Gamma\}$ be a disjoint family of subsets of Y such that for any $\gamma \in \Gamma$, $\mu(A_\gamma) = 0$ (resp. A_γ is 1st category). If $\bigcup_{\gamma \in \Gamma} A_\gamma$ has positive measure (resp. is 2nd category having the Baire property), then there exists $\Delta \subset \Gamma$ such that $\bigcup\{A_\gamma : \gamma \in \Delta\}$ is not μ–measurable (resp. is not BP–measurable) relative to $\bigcup_{\gamma \in \Gamma} A_\gamma$.*

PROOF. Well–order Γ and let β be the least element in Γ such that the outer measure μ^* of the union $\bigcup_{\gamma<\beta} A_\gamma$ is positive (resp. 2nd category). We may assume that $\bigcup_{\gamma<\beta} A_\gamma$ is μ–measurable (resp. BP–measurable). Indeed, if not, the lemma holds. By the definition of β, for any $\alpha < \beta$, $\mu(\bigcup_{\gamma<\alpha} A_\gamma) = 0$ (resp. $\bigcup_{\gamma<\alpha} A_\gamma$ is 1st category). Consequently, without loss of generality, we may assume

$$\bigcup_{\gamma<\Gamma} A_\gamma = Y \text{ and } \mu(\bigcup_{\gamma<\beta} A_\gamma) = 0 \text{ (resp. is 1st category) for all } \beta < \Gamma.$$

For any $\beta < \Gamma$, find a G_δ set P_β of measure zero containing $\bigcup_{\gamma<\beta} A_\gamma$ (resp. a residual G_δ set P_β disjoint with $\bigcup_{\gamma<\beta} A_\gamma$). Define $E \subset Y \times Y$ by

$$E = \bigcup_{\gamma \in \Gamma} (A_\gamma \times P_\gamma)$$

and take a horizontal cross–section E_y of E.

Let α be such that $y \in A_\alpha$. Then, for any $\beta > \alpha$, $y \in P_\beta$ and, consequently, $E_y \supset \bigcup_{\alpha<\beta<\Gamma} A_\beta$. Hence, E_y is of full measure, provided it is measurable. In the category case, if $\beta > \alpha$, $y \in \bigcup_{\gamma<\beta} A_\gamma \subset Y \setminus P_\beta$, that is, $E_y \cap \bigcup_{\alpha<\beta<\Gamma} A_\beta = \emptyset$, and E_y is of 1st category.

Clearly, by the very definition of E, every vertical cross–section E^y has measure zero (resp. is residual). We will get a contradiction with the Fubini Theorem (resp. its category analogue, the Kuratowski–Ulam Theorem [28, Chapter 15]) by showing that E is $\mu \times \mu$–measurable (resp. BP–measurable) in $Y \times Y$.

To this end, for every $\beta < \Gamma$, set $P_\beta = \bigcap_{n=1}^\infty P_{\beta_n}$, where P_{β_n} are open decreasing sets, and

$$E_n = \bigcup_{\beta < \Gamma} (A_\beta \times P_{\beta_n}),\ n \in \mathbb{N}.$$

Then $E = \bigcap E_n$ and it suffices to show that E_n are μ–measurable (resp. BP–measurable).

Let $\mathcal{G}$ be a countable base of open sets in Y and fix E_n, $n \in \mathbb{N}$. For $G \in \mathcal{G}$, let

$$\mathcal{B}_G = \{A_\beta : G \subset P_{\beta_n}\}.$$

Write $B_G = \bigcup \mathcal{B}_G$. As $E_n = \bigcup_{G \in \mathcal{G}} (B_G \times G)$, the set E_n is μ–measurable (resp. BP–measurable), if each B_G is so. However, if B_G is not, then the proposition holds. Indeed, $\mathcal{B}_G$ is then a subfamily of $\{A_\gamma : \gamma \in \Gamma\}$ whose union is not μ–measurable (resp. not BP–measurable). □

Remark 3.3. The assumption of regularity of μ is automatically satisfied if Y is a metric space.

Theorem 3.4. *Let Y be a 2nd countable topological space, μ a finite regular Borel measure on Y, X a metric space, and $f : Y \to X$ a μ–measurable (resp. BP–measurable) map. Then f is essentially separably valued.*

Proof. If μ is trivial, there is nothing to prove. Let $\mu(Y) > 0$ and let $\mathcal{B}$ be an open cover of X. By the Stone Theorem [13, 4.4.1], there exists an open refinement $\mathcal{D}$ of $\mathcal{B}$ which is σ–disjoint, i.e., such that $\mathcal{D} = \bigcup_{n=1}^\infty \mathcal{D}^n$, where $\mathcal{D}^n$ are disjoint families in X. Given $n \in \mathbb{N}$, there is only a *sequence* $(D_k^n), k = 1, 2, \ldots$ of all sets in $\mathcal{D}^n$ for which $\mu(f^{-1}(D_k^n)) > 0$. We note that the 2nd countability of Y is not needed here; the fact that any family of disjoint sets of positive measure is countable is used.

Now, $\mathcal{A}^n = \{A = f^{-1}(D) : D \in \mathcal{D}^n \setminus \{D_k^n : k \in \mathbb{N}\}\}$ is a family of disjoint μ–zero sets in Y. Since the union of any subfamily of open sets is open, the union of any subfamily of sets in $\mathcal{A}^n$ is μ–measurable. Hence $\bigcup\{A : A \in \mathcal{A}^n\}$ is a set of measure zero by Lemma 3.2. Denoting $\mathcal{A} = \bigcup_n \mathcal{A}^n$, one gets $\mu(\bigcup\{A : A \in \mathcal{A}\}) = 0$. As $\mathcal{D}$ is a cover of Y, $\bigcup\{f^{-1}(D_k^n) : n \in \mathbb{N}, k \in \mathbb{N}\}$ is a set of full measure in Y. But $\mathcal{D}$ is a refinement, so for every D_k^n one can find $B \in \mathcal{B}$ containing it. This defines a countable subfamily of $\mathcal{B}$, making an application of Lemma 3.1 possible which ends the proof.

Let now f be BP–measurable. We may assume that the space Y is 2nd category, because otherwise there is nothing to prove. By the Banach Category Theorem [28, 16.1], Y can be represented as a union of a 1st category set Y_0 and its complement Y_1, where Y_0 is the union of all 1st category open subsets of Y. Consider two (disjoint) members of $\mathcal{D}^n$, say D_1 and D_2, whose inverse images by f are both 2nd category. As f is BP-measurable, we can write $f^{-1}(D_1) = (O_1 \setminus I_1) \cup J_1$ and $f^{-1}(D_2) = (O_2 \setminus I_2) \cup J_2$, where O_1, O_2 are open sets and the other four sets are 1st category. Observe that O_1 and O_2 are 2nd category. Since $O_1 \cap O_2 \subset I_1 \cup I_2$, and the latter union is 1st category, the open set $O_1 \cap O_2$ is contained in Y_0. This means that $O_1 \cap Y_1$ and $O_2 \cap Y_1$ are disjoint nonempty open sets in Y_1. Consequently, given $n \in \mathbb{N}$, the family of all sets in $\mathcal{D}^n$ whose inverse images by f are 2nd category is countable and, therefore, can again be written $(D_k^n), k = 1, 2, \ldots$. This time, to conclude that there was only a countable number of such sets, we used the fact that Y_1 is 2nd countable.

Hence $\mathcal{A}^n = \{A = f^{-1}(D) : D \in \mathcal{D}^n \setminus \{D_k^n : k \in \mathbb{N}\}\}$ is a family of disjoint 1st category sets. By Lemma 3.2 again, the union of the sets in $\mathcal{A}^n$ is 1st category. Keeping the notation from the first part of the proof, the union of the sets in $\mathcal{A}$ is 1st category and so $\bigcup\{f^{-1}(D_k^n) : n \in \mathbb{N}, k \in \mathbb{N}\}$ is a residual set in Y. This permits it to finish the proof as in the first part. □

Remark 3.5. Assuming appropriate separation axioms on the topology of Y, there exist results stronger than Theorem 3.4, see e.g. [15] and [16]. The main reason for its inclusion in the form above is the desire to expose the duality of our proof of the "Baire counterpart" with the proof of the "measure part." The latter proof can be treated as more or less standard. It can be traced back at least as far as to the proof of Th. 2 in [3, Appendix 3].

4 Exhaustivity

Recall that X is a Hausdorff topological Abelian group and let $\boldsymbol{m} : \mathcal{P} \to X$ be a finitely additive measure or, for short, an *additive map*, that is, a map such that $\boldsymbol{m}(A \cup B) = \boldsymbol{m}(A) + \boldsymbol{m}(B)$ for disjoint sets A and B. It is called *exhaustive* whenever $\boldsymbol{m}(E_n) \to 0$ for each sequence of disjoints sets (E_n) in $\mathcal{P}$.

We first prove a result showing that both, BP–measurable and χ–measurable additive maps enjoy a sort of "weak exhaustivity" property.

Proposition 4.1. *Let $\boldsymbol{m} : \mathcal{P} \to X$ be a Haar measurable (resp. BP–measurable) additive map. Then $\boldsymbol{m}(\{n\}) \to 0$.*

PROOF. By embedding X into a product of normed groups, we may assume that X is metric. Let U be a neighborhood of 0 in X and find another neighborhood of 0 in X such that $V - V \subset U$. As $\boldsymbol{m}$ is essentially separable by Theorem 3.4, there is a sequence (x_n) of points in X such that the inverse images of the translates, $\boldsymbol{m}^{-1}(x_n + V)$, cover $\mathcal{P}$ apart, perhaps, of a measure zero (resp. 1st category) subset of $\mathcal{P}$. Hence one of them, say $\boldsymbol{m}^{-1}(x_k + V)$, is of positive χ–measure (resp. 2nd category). By the Steinhaus (resp. Piccard) Theorem, the family $\mathcal{A} = \boldsymbol{m}^{-1}(x_k + V) \vartriangle \boldsymbol{m}^{-1}(x_k + V)$ is a neighborhood of $\emptyset$ in $\mathcal{P}$.

$\mathcal{A}$ being a neighborhood of $\emptyset$, there exists $m \in \mathbb{N}$ so large that for $n > m$, $\{n\} \in \mathcal{A}$. One therefore has $\{n\} = E \vartriangle F$ with $\boldsymbol{m}(E) \in x_k + V$ and $\boldsymbol{m}(F) \in x_k + V$. Since $\{n\}$ is a singleton, $E \subset F$ or vice versa. Suppose the latter; then $\{n\} = E \setminus F$ and $\boldsymbol{m}(E \setminus F) = \boldsymbol{m}(E) - \boldsymbol{m}(F) \in x_k + V - (x_k + V) = V - V$. Hence, if m is large enough, $\boldsymbol{m}(\{n\}) \in U$ for $n > m$. As U was arbitrary, this means that $\boldsymbol{m}(\{n\}) \to 0$. □

A set E in a topological space Y is BP_r*–measurable* or has the Baire property in the restricted sense, if for every subspace A of Y, the set $A \cap E$ is BP–measurable relatively to A. A function f from Y into a topological space X is BP_r*–measurable* or has the Baire property in the restricted sense, if for every open subset O in X, the set $f^{-1}(O)$ is BP_r–measurable in Y.

Proposition 4.2. *If $\boldsymbol{m} : \mathcal{P} \to X$ is BP_r–measurable, then it is exhaustive.*

PROOF. Let (E_n) be a sequence of disjoint sets in $\mathcal{P}$. Define the map $\boldsymbol{j} : \mathcal{P}(\mathbb{N}) \to \mathcal{P}$ by putting for each $F \subset \mathbb{N}$

$$\boldsymbol{j}(F) = \bigcup_{n \in F} E_n.$$

To avoid confusion, denote the domain Cantor group of $\boldsymbol{j}$ by $\mathcal{R}$. The map $\boldsymbol{j}$ is a continuous injection from $\mathcal{R}$ into $\mathcal{P}$ and, in fact, a homeomorphism onto its image, $\mathcal{Z}$ say, because $\mathcal{R}$ is compact. By the assumption on $\boldsymbol{m}$, if $O \subset X$ is open, then $\boldsymbol{m}^{-1}(O) \cap \mathcal{Z}$ is BP–measurable relative to $\mathcal{Z}$. Hence its inverse image by $\boldsymbol{j}$ is still BP–measurable in $\mathcal{R}$. Thus, $\boldsymbol{mj} : \mathcal{R} \to X$ is BP–measurable. By "Baire part" of Proposition 4.1, $\boldsymbol{mj}(\{n\}) = \boldsymbol{m}(E_n) \to 0$. □

Recall that a (finite positive) Borel measure μ on a Hausdorff space Y is said to be *Radon measure* if it is inner regular with respect to compact sets [16]. A set E in Y is *universally measurable* if it is measurable for every Radon measure μ on Y. A function f from Y into a topological space X is *universally measurable* if for every open subset O in X, the set $f^{-1}(O)$ is universally measurable in Y.

For historical reasons, let us mention that the notion of universal measurability was defined *precisely* as a measure analogue of BP_r–measurability (by Szpilrajn-Marczewski in [34] under the name of absolute measurability).

Proposition 4.3. *If $\boldsymbol{m} : \mathcal{P} \to X$ is universally measurable, then it is exhaustive.*

PROOF. Keeping the notation from the proof of the preceding Proposition 4.2, let χ be the Haar measure on $\mathcal{R}$. Further, let $\mu = \chi \boldsymbol{j}^{-1}$ be its image measure on $\mathcal{P}$ defined the usual way. Since μ is a Radon measure ([16, 418 I]) and $\boldsymbol{m}$ is universally measurable, we conclude that $\boldsymbol{m}$ is $\chi\boldsymbol{j}^{-1}$-measurable. This means that $\boldsymbol{mj} : \mathcal{R} \to X$ is χ-measurable. It follows from the Proposition 4.1 that $\boldsymbol{mj}(\{n\}) = \boldsymbol{m}(E_n) \to 0$. $\square$

5 Countable additivity

Again, X is a Hausdorff topological Abelian group throughout. Let $\sum_n x_n$ be a subseries convergent series in X. Then the *canonical measure* $\boldsymbol{m} : \mathcal{P} \to X$ (or *canonical map* $\boldsymbol{m} : K \to X$) connected with it, is defined by

$$\boldsymbol{m}(E) = \sum_{n \in E} x_n \text{ for } E \subset \mathbb{N}.$$

An additive map $\boldsymbol{m} : \mathcal{P} \to X$ is a measure (i.e., countably additive) iff it is continuous iff it is the canonical map of the series $\sum_n x_n$, where $x_n = \boldsymbol{m}(\{n\})$.

A situation of interest arises in connection with theorems of Orlicz–Pettis type. On X two Hausdorff group topologies α and β are considered, with $\alpha \subset \beta$. One knows that a series $\sum_n x_n$ is subseries convergent in (X, α). One seeks criteria on β so that the series is also convergent in (X, β). The canonical map of the series, $\boldsymbol{m} : \mathcal{P} \to X$, is α–continuous, but is, a priori, only an additive map into (X, β).

The following theorem goes back to [25], [29] and [17].

Theorem 5.1. *Let α, β be Hausdorff group topologies on X with $\alpha \subset \beta$ and β sequentially complete. Suppose the identity map $\iota : (X, \alpha) \to (X, \beta)$ is universally measurable. If $\sum x_n$ is subseries convergent in (X, α), then it is so in (X, β).*

PROOF. It is sufficient to show that the canonical continuous map $\boldsymbol{m} : \mathcal{P} \to X$ associated with the series in (X, α) is exhaustive into (X, β). Indeed, then $\sum_n x_n$ satisfies the Cauchy condition for summability in (X, β) and therefore is subseries convergent there to the same limit as in (X, α). Consider a

Radon measure μ on $\mathcal{P}$ and its image measure $\nu = \mu\boldsymbol{m}^{-1}$ in (X, α). As ν is Radon ([16, 418 I]), the identity ι is ν–measurable. Hence, $\iota\boldsymbol{m}$ is μ–measurable or, which is the same, $\boldsymbol{m}$ into (X, β) is μ–measurable. We have shown that $\boldsymbol{m} : \mathcal{P} \to (X, \beta)$ is universally measurable and, therefore, exhaustive by Proposition 4.3. □

Problem 5.2. Is the "dual statement" to the above theorem, in which one assumes the identity ι to be BP_r–measurable, true?

Remark 5.3. For the origins of the Orlicz–Pettis Theorem see [20] and [14]. As for the Problem 5.2, there exist some results in this direction. For instance, in [10, Theorem 2.1] the measurability assumption about ι is even weaker, but a very strong completeness of β is needed.

If ι is Borel measurable, there is no problem. Though this case is already covered since Borel measurable maps are universally measurable, it leads us to a theorem of N.J.M. Andersen and J.P.R. Christensen [1]. But before we state the theorem, let us digress and present another double result whose "Baire part" will be needed.

Consider an equivalence relation $\sim$ in a product $Y = \prod_{i \in I} Y_i$ defined by:

$$a \sim b \text{ if the set } \{i \in I : a(i) \neq b(i)\} \text{ is finite.}$$

A subset A of Y is a *tail set* if $y \in A$ and $y \sim z$ implies $z \in A$. A function $f : Y \to X$ *is compatible with* $\sim$ if it is constant on equivalence classes $y^{\sim}$ for $y \in Y$.

Suppose now that Y is a product of probability spaces with the product probability measure μ (resp. Y is a product of Baire spaces each of which has a countable pseudo–base). For the record, a family $\mathcal{B}$ of non–empty open sets in a topological space is a *pseudo–base* if every non–empty open set contains at least one member of $\mathcal{B}$. The so–called 0-1 *law* of probability theory (resp. *topological* 0-1 *law*) says that, if $A \subset Y$ is a μ–measurable (resp. BP–measurable) tail set, then $\mu(A) = 0$ or 1 (resp. A is 1st category or residual subset of Y), see [19, Section 46(3)], [27, Theorem 4].

Proposition 5.4. *Let Y be a product of Baire spaces, each of which has a countable pseudo–base (resp. product of probability spaces with product measure μ), and X be a metric space. Assume $f : Y \to X$ is BP–measurable (resp. μ–measurable) and compatible with $\sim$. Then f is constant on a residual (resp. μ–measure one) subset of Y.*

Proof. Let $\mathcal{B}_k$ be an open cover of X by balls of radius $1/k$. By the already invoked Stone Theorem [13, 4.4.1], there exists an open refinement $\mathcal{D}$ of $\mathcal{B}_k$

with $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}^n$, where $\mathcal{D}^n$ are disjoint families in X. Given $n \in \mathbb{N}$, let $\mathcal{E}^n$ be the subfamily of all sets D in $\mathcal{D}^n$ for which $f^{-1}(D)$ is 2nd category (resp. of positive measure) and therefore *residual* (resp. *of μ–measure one*). $\mathcal{A}^n = \{A = f^{-1}(D) : D \in \mathcal{D}^n \setminus \mathcal{E}^n\}$ is a disjoint family of 1st category (resp. measure zero) sets in Y and therefore, by Lemma 3.2, the union $Y_0^n = \bigcup\{A : A \in \mathcal{A}^n\}$ is 1st category (resp. measure zero) set. Hence $Y_0 = \bigcup\{Y_0^n : n \in \mathbb{N}\}$ is 1st category (resp. measure zero) in Y. As $\mathcal{D}$ is a cover, $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}^n$ is a family covering $Y_1 = Y \setminus Y_0$, a residual (measure one) set in Y. By choosing for each $D \in \mathcal{E}$ a ball in $\mathcal{B}_k$ containing it, we conclude that:

For each open cover $\mathcal{B}_k$ of X by balls of radius $1/k$ there exists a 1st category (resp. measure one) set $Y_0(k)$ and a subfamily $\mathcal{B}'_k$ of $\mathcal{B}_k$ covering $Y_1(k) = Y \setminus Y_0(k)$ such that for each $B \in \mathcal{B}'_k$, $f^{-1}(B)$ is residual (resp. measure one) in Y.

In particular, as Y is a Baire space by [27, Theorem 3], we see that the families $\mathcal{B}'_k, k \in \mathbb{N}$, are nonempty. Choose a ball B_{2n_1} in $\mathcal{B}'_2$ and note that $f^{-1}(B_{2n_1})$ is a residual (resp. measure one) subset of Y. On the second step, choose a ball $B_{3n_2} \in \mathcal{B}'_3$. Again, $f^{-1}(B_{3n_2})$ is residual (resp. measure one) in Y and so is $f^{-1}(B_{2n_1}) \cap f^{-1}(B_{3n_2})$. Continuing, we will find a nested sequence of balls (B_{kn_k}) with radius $1/k \to 0$ and such that $Z = \bigcap\{f^{-1}(B_{kn_k}) : k \in \mathbb{N}\}$ is a residual (resp. measure one) set in Y. Pick $y \in Z$. Then $x = f(y) = \bigcap_{k=1}^{\infty} B_{kn_k}$ and $f^{-1}(x) = \bigcap_{k=1}^{\infty} f^{-1}(B_{kn_k}) = Z$. Hence f is constant on a residual (resp. measure one) set Z. $\square$

Here is the Andersen–Christensen Theorem in its generalized (and corrected, see Remark 5.6 below) form.

Theorem 5.5. *Let X be sequentially complete and $\boldsymbol{m} : \mathcal{P} \to X$ a BP_r–measurable additive map. Then $\boldsymbol{m}$ is the canonical measure of the subseries convergent series $\sum_n \boldsymbol{m}(\{n\})$.*

In the arguments that follow, the structure of the Cantor group K as product $\{0,1\}^{\mathbb{N}}$ and the continuity of $\boldsymbol{m}$ on K rather than the countable additivity of $\boldsymbol{m}$ on $\mathcal{P}$ will be exploited. For this reason, it is more convenient to think in terms of $\boldsymbol{m} : K \to X$ and change the notation accordingly. The group operation of addition *mod.* 2 in K is denoted by $+$ and $a, b, c, \ldots$ are the elements of K. Then F corresponds to $\mathcal{F}$, 0 to $\emptyset$, n to the singleton $\{n\}$, ab to $A \cap B$, $a - ab$ to $A \setminus B$, and e will be reserved for $\{1, 1, \ldots\}$.

PROOF. Given $\boldsymbol{m}$, by the Proposition 4.2 and the sequential completeness of X, the series $\sum_n \boldsymbol{m}(\{n\})$ is subseries convergent. Its canonical map $\boldsymbol{m}'$ from K to X is continuous. Consider $\boldsymbol{m}'' = \boldsymbol{m} - \boldsymbol{m}'$ on K. It is obvious that $\boldsymbol{m}''$ is an additive map vanishing on F. We claim that it is BP_r–measurable. As usual, we may embed X into a product of normed groups, and therefore it will be enough to assume that in the next step of the proof X is metric.

Let $Z \subset K$. We need to show that $(\boldsymbol{m}-\boldsymbol{m}')|Z$ is BP–measurable. Clearly, $\boldsymbol{m}|Z$ is BP_r–measurable and $\boldsymbol{m}'|Z$ is continuous. Using Theorem 3.4, find $H_1 \subset Z$, 1st category in Z, such that $\boldsymbol{m}|(Z \setminus H_1)$ is separably valued. As $\boldsymbol{m}|(Z \setminus H_1)$ is BP–measurable, by [24, Section 32, II] there exists a set H_2, first category in $Z \setminus H_1$ and therefore also in Z, such that $\boldsymbol{m}|(Z \setminus (H_1 \cup H_2))$ is continuous. Hence $\boldsymbol{m}-\boldsymbol{m}'$ is continuous apart of 1st category set which can be taken to be an F_σ subset of Z. Let O be an open set in X. Observe that $(f-g)^{-1}(O)$ can be written as a disjoint union of a 1st category set in Z and an open subset of a G_δ in Z and so a G_δ subset of Z. It follows by [24, Section 11.IV.2] that the inverse image under consideration is a BP–measurable subset of Z, as needed.

The proof will be complete if we can show that $\boldsymbol{m}''$ is identically zero. This will be done in the next Proposition (the proposition is implicit in the original proof of the theorem in [1]). $\square$

Remark 5.6. Andersen and Christensen claim that the assumption of sequential completeness of X is not needed. However, without any completeness assumption on X, the countably additive and therefore continuous function $\boldsymbol{m}'$ takes values in the completion, $\hat{X}$. Consequently, $\boldsymbol{m}-\boldsymbol{m}'$ also takes values in $\hat{X}$ and we do not know whether it is BP_r–measurable, a fact needed to conclude that it vanishes. Assuming Borel measurability does not improve the situation.

Proposition 5.7. *Let X be normed, and $\boldsymbol{m} : K \to X$ be an additive BP_r–measurable map. If $\boldsymbol{m}|F = 0$, then $\boldsymbol{m}$ is identically zero.*

Proof. By Proposition 5.4, $\boldsymbol{m}$ is constant on a dense G_δ subset A of K. Let x be that constant. Observing that $A \cap e + A$, as an intersection of dense G_δ sets, is nonempty, we deduce that $e = a + b$ with $a \in A$, $b \in A$ and $ab = 0$ (i.e. a and b are disjoint as elements of $\mathcal{P}$). It follows that $\boldsymbol{m}(e) = 2x$. At this point, we need the following lemma (Lemma after Theorem 2 in [7, p. 247]):

If A is a dense G_δ subset of K, then there exist a, b, c in A such that $c = a + b$ with a, b disjoint.

So, if the lemma holds, $x = 0$ and $\boldsymbol{m}(\mathbb{N}) = 0$. Let now a be an arbitrary element of K. If $a \in F$, then $\boldsymbol{m}(a) = 0$. If $a \notin F$, then denote by A the support of a in $\mathbb{N}$, i.e., $\{n \in \mathbb{N} : a(n) \neq 0\}$. The restriction of $\boldsymbol{m}$ to $K_a = \{0,1\}^A$ is again BP_r–measurable. Hence, by the proof above, $\boldsymbol{m}(a) = 0$.

It remains to prove the lemma. Here is Christensen's proof of it. Consider the maps $g, h : K \times K \to K$ defined by

$$g(a,b) = a - ab \text{ and } h(a,b) = ab.$$

As g, h are surjective, open and continuous, $g^{-1}(A)$ and $h^{-1}(A)$ are dense G_δ sets in $K \times K$. Choose

$$(a, b) \in g^{-1}(A) \cap h^{-1}(A) \cap (A \times K)$$

and put $x = a$, $y = g(a, b)$ and $z = h(a, b)$. □

Problem 5.8. Is the "universal measurability" dual of Theorem 5.5 true?

Remark 5.9. J. P. R. Christensen (assuming the Continuum Hypothesis) claims having an example that solves Problem 5.8 in the negative, see [8, Theorem 6.1] and the example following it. However, if true, his claim would need a better proof.

6 Analogy breaks

We will need the following consequence of the main result of Section 3.

Corollary 6.1. *Let Y be a second countable Baire space, X a metric space, and f a function from Y to X. The following conditions are equivalent.*

(a) *f is BP–measurable.*

(b) *f is essentially separable.*

(c) *f is continuous apart a 1st category set.*

Proof. (a) implies (b) is Theorem 3.4. (a) implies (c). There exists a 1st category set $E \subset Y$ such that the restriction $f|(Y \setminus E)$ has a separable image. As Y is Baire, $Y \setminus E$ is dense in Y. Let O be an open set in X. By [24, §11,V, Theorem 2], if $f^{-1}(O)$ is BP–measurable in Y, then $f^{-1}(O) \cap (Y \setminus E)$ is so relative to $Y \setminus E$. This means that the restriction $f|(Y \setminus E)$ is BP–measurable relative to $Y \setminus E$. By the proof of the necessity part of [24, §32, II], we can find a 1st category set F in $Y \setminus E$ such that $f|(Y \setminus (E \cup F))$ is continuous. But F is then also 1st category in Y, i.e., (c) holds. Now, the sufficiency part of the just invoked proof of Kuratowski gives (c) implies (a). □

The Corollary above overlaps with the main theorem of [12] (our Y does not have to be Čech complete).

Theorem 6.2. *Suppose an additive map $\boldsymbol{m} : K \to X$ is BP–measurable. Then $\boldsymbol{m}$ is exhaustive.*

Proof. By embedding X into a product of normed groups, we may assume that X is metric. Suppose that $\boldsymbol{m} : K \to X$ is not exhaustive. Then there exists a disjoint sequence of $a_n \in K$ such that $\boldsymbol{m}(a_n) \not\to 0$. In view of Corollary 6.1, there exists a dense G_δ subfamily A of K on which $\boldsymbol{m}$ is continuous. As the set $F \cup \{a_n : n \in \mathbb{N}\}$ is countable, we may assume that A is translation invariant with respect to its elements. Let $b \in A$ and define $b'_n = b + a_n + ba_n$ and $b''_n = b + ba_n = b \setminus ba_n$. Both b'_n and b''_n are in A, $b'_n \to b$ and $b''_n \to b$ there. Hence $\boldsymbol{m}(b'_n) - \boldsymbol{m}(b''_n) \to 0$. On the other hand, b''_n and a_n are disjoint and $b'_n = b''_n + a_n$. Consequently, $\boldsymbol{m}(b'_n) - \boldsymbol{m}(b''_n) = \boldsymbol{m}(a_n) \not\to 0$. A contradiction. □

The result sharpens Proposition 4.2. But from our point of view, it is in a sense "too good"– it is at this point that the analogy between measure and category breaks down.

Example 6.3. Denote by A the set of points $a = (\varepsilon_n) \in K$ for which $\lim_n \frac{1}{n} \sum \varepsilon_n = \frac{1}{2}$, i.e., the set of normal numbers. By Borel's normal number theorem [4, Theorem 1.2], $\chi(A) = 1$. Now switch the interpretation from K to $\mathcal{P}$ (and so $\mathcal{A}$ will correspond to A). Identifying sets with their characteristic functions, consider $lin(\mathcal{A})$ and $lin(\mathcal{P})$. Let $\mathcal{B}$ be a set of linearly independent vectors contained in $\mathcal{A}$ which is maximal with respect to inclusion in $\mathcal{A}$. As easily checked, $\mathcal{B}$ is a Hamel basis of $lin(\mathcal{A})$. Let $\mathcal{H}$ be a Hamel basis consisting of vectors in $\mathcal{P}$, containing $\mathcal{B}$, and spanning $lin(\mathcal{P})$. Define a linear functional $\boldsymbol{m}$ on $lin(\mathcal{P})$ such that $\boldsymbol{m}$ is zero on $lin(\mathcal{A})$, and is unbounded on $\mathcal{H}$. This is possible because the co–dimension of the linear subspace $lin(\mathcal{A})$ in $lin(\mathcal{P})$ is infinite and the coefficients of basic vectors in $\mathcal{H} \setminus \mathcal{B}$ can be chosen arbitrarily. Consider the restriction of $\boldsymbol{m}$ to $\mathcal{P}$, keeping the notation $\boldsymbol{m}$. *The additive map* $\boldsymbol{m}$ *is measurable with respect to* χ (since it is χ–almost everywhere continuous), but it is unbounded, and so *it is not exhaustive on* $\mathcal{P}$.

Remark 6.4. The technique used to prove Theorem 6.2 is a modification of an argument due to Andersen and Christensen (see the beginning of the proof of [1, Theorem 1]). Example 6.3 is a somewhat polished result of Constantinescu [9, Proposition 16].

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