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MULTIPLICATION OPERATORS ON THE SPACES OF FUNCTIONS OF BOUNDED p-VARIATION IN WIENER'S SENSE

Abstract

In this article, we make a comprehensive study about the properties of multiplication operators acting on the spaces of functions of bounded p-variation in Wiener's sense $WBV_p[0,1]$. We characterize all functions $u \in WBV_p[0,1]$ that define invertible, compact and Fredholm multiplication operators M_u on $WBV_p[0,1]$. Also we characterize when M_u has finite range and has closed range on $WBV_p[0,1]$.

1 Introduction

The classical space of functions of bounded variation on an interval $[a,b]$, denoted by $BV[a,b]$ was defined in 1881 by Camile Jordan in his celebrated article: *Sur la série de Fourier* in the volume 92 of Comptes rendus hebdomadaires des séances de l'Académie des sciences (see [14]). $BV[a,b]$ consists

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of all real functions f defined on $[a,b]$ such that its total variation $V_a^b(f) < \infty$, where

$$V_a^b(f) := \sup_P \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\} < \infty,$$

and the supremum is taken over all the partitions

$$P : a = x_0 < x_1 < \cdots < x_n = b$$

of the interval $[a,b]$. The space of functions of bounded variation is non-separable Banach space with the norm

$$\|f\|_{BV[a,b]} := \|f\|_\infty + \mathrm{V}_a^b(f),$$

where $\|f\|_\infty = \sup\{|f(t)| : t \in [a,b]\}$. This space of functions finds applications to solve value initial problems and it can be applied in other areas of the knowledge such as Physical and the Engineering (see for instance [7]).

There are many extensions or generalizations of the concept of functions of bounded variation. There are remarkable contributions made by Wiener [23], Riesz [19, 20], De La Vallée Poussin [8], Hardy [10], Korenblum [17], and Waterman [22]. We refer to the excellent monograph of Appell, Banas, and Merentes [4] where they collect the properties and relations of the different generalizations of $BV[a,b]$. In this note, we consider the extension given by Wiener in [23] which is defined as follows: Given a parameter $p \geq 1$, the space of functions of bounded p-variation in Wiener's sense, denoted by $WBV_p[a,b]$, consists of all real functions f defined on $[a,b]$ such that its total variation in the Wiener's sense $\mathrm{Var}_p^W(f;[a,b])$ defined by

$$\mathrm{Var}_p^W(f;[a,b]) := \sup_P \left(\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|^p \right)^{1/p} < \infty, \tag{1}$$

where, as before, the supremum is taken over all the partitions

$$P : a = x_0 < x_1 < \cdots < x_n = b$$

of $[a,b]$. Clearly, when $p = 1$ we have $WBV_1[a,b] = BV[a,b]$. The space $WBV_p[a,b]$ becomes a Banach space with the norm

$$\|f\|_{WBV_p([a,b])} := \|f\|_\infty + \mathrm{Var}_p^W(f;[a,b]). \tag{2}$$

This kind of space provided the first example of a non-reflexive space which is isometrically isomorphic to its double dual (see James [12, 13]). Furthermore, for all $f, g \in WBV_p[a,b]$ the following very useful identity holds:

$$\|fg\|_{WBV_p([a,b])} \leq \|f\|_{WBV_p([a,b])} \|g\|_{WBV_p([a,b])}. \tag{3}$$

This last relation tell us that $WBV_p[a,b]$ is a Banach algebra. Also, we have that $WBV_p[a,b]$ is embedded into $B[a,b]$, the space of all bounded functions defined on $[a,b]$, in fact, we have

$$|f(x)-f(y)| \leq \mathrm{Var}_p^W\,(f;[x,y])$$

for all $a \leq x < y \leq b$. In particular, this last relation implies that

$$\|f\|_\infty \leq |f(a)| + \mathrm{Var}_p^W\,(f;[a,b])$$

and that $\mathrm{Var}_p^W\,(f;[a,b]) = 0$ if and only if f is a constant function on $[a,b]$.

The fact that $WBV_p[a,b]$ is a Banach algebra allow us to define, for $u \in WBV_p[a,b]$ fixed, a linear operator M_u on $WBV_p[a,b]$ by the relation $M_u(f) = u \cdot f$ with $f \in WBV_p[a,b]$; which is known as *multiplication operator* with symbol u. Clearly, since the constant function $\mathbf{1}_{[a,b]}$ defined by $\mathbf{1}_{[a,b]}(t) = 1$ for all $t \in [a,b]$ belongs to $WBV_p[a,b]$ we have that M_u applies $WBV_p[a.b]$ into itself if and only if the symbol $u \in WBV_p[a,b]$. Hence, from now, we can suppose that the symbol u belongs to $WBV_p[a,b]$.

In the last decades there has been a growing interest in a deeper study of the properties of multiplication operators acting on different spaces of functions. It is remarkable the works of Halmos [9], Abrahamse [1], Takagi and Yokouchi [21], Komal and Gupta [15]. Recently, Castillo, Ramos-Fernández, and Rafeiro [5] characterized boundedness, invertibility, compactness, and closedness of the range of multiplication operators on variable Lebesgue spaces. Also, the essential norm of multiplication operator on Lorentz sequence spaces was recently estimated by Castillo, Ramos-Fernández, and Salas-Brown [6].

There is an important work due to Hudzik, Kumar, and Kumar about the properties of multiplication operator on a very general setting which generalize many of the results of the authors mentioned above (see [11]), namely on Köthe-type spaces, where a Banach space X is said a Köthe space if it is a space of measurable functions defined on a measurable space (Ω, Σ, μ) satisfying the following properties:

1. Every $f \in X$ is locally integrable.
2. If $|f(t)| \leq |g(t)|$ almost everywhere in Ω with f, g measurable and $g \in X$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$.
3. For each $A \in \Sigma$ with $\mu(A) < \infty$, the characteristic function of A, denoted by $\mathbf{1}_A$ belongs to X.

The L_p spaces, the Orlicz spaces, the Lorentz spaces, and the Orlicz-Lorentz spaces are important examples of Köthe-type spaces. However, the spaces $BV[a,b]$, $WBV_p[a,b]$ among other are not Köthe spaces.

Recently, Astudillo-Villalba and Ramos-Fernández in [3] made a comprehensive study about the properties of multiplication operators acting on $BV[0,1]$. The main goal of this article is to show that modifying the techniques used by Astudillo-Villalba and Ramos-Fernández in [3], we can characterize the properties of multiplication operators acting on the space of functions of bounded p-variation in Wiener's sense. More precisely, in Section 2, we characterize invertible multiplication operators on $WBV_p[a,b]$, we show that all surjective multiplication operators on $WBV_p[a,b]$ are also injective and that all bounded below multiplication operators on $WBV_p[a,b]$ are also bijective. Section 3 is about the compactness of multiplication operators M_u acting on $WBV_p[a,b]$, our characterization is given in terms of the zero set of the symbol u. We also show that M_u has range finite if and only if Z_u, the zero set of u, is a finite set. The Section 4 is dedicated to characterize the symbols u that define multiplication operators with closed range on $WBV_p[a,b]$. Finally, in Section 5 we characterize all symbols u that induce Fredholm multiplication operators on $WBV_p[a,b]$. Without loss of generality and for our convenience, through this note we will work with space $WBV_p[0,1]$ instead of the more general case $WBV_p[a,b]$.

2 Invertible and bounded below multiplication operators on $WBV_p[a,b]$

In this section we show that the class of invertible multiplication operators with symbols in $WBV_p[a,b]$ coincides with the class of bounded below multiplication operators with symbols in $WBV_p[a,b]$ and that all surjective multiplication operators acting on $WBV_p[a,b]$ is invertible. First of all, we characterize injective multiplication operators acting on $WBV_p[a,b]$, our characterization can be enunciate in terms of $\operatorname{supp}(u)$, the support of u which is defined as the set

$$\operatorname{supp}(u) = \{t \in [0,1] : \ u(t) \neq 0\}.$$

Then the set $Z_u = [0,1] \setminus \operatorname{supp}(u)$ are the zeros of the function u. Our first result can be enunciate as follows:

Proposition 1. *Suppose that $u \in WBV_p[0,1]$, then*

$$M_u : WBV_p([0,1]) \to WBV_p([0,1])$$

is injective if and only if $\operatorname{supp}(u) = [0,1]$.

Proof. Indeed, if $\operatorname{supp}(u) = [0,1]$ and $f \in \operatorname{Ker}(M_u)$ then $u(t) \cdot f(t) = 0$ for all $t \in [0,1]$ and hence $f \equiv 0$, that is, $\operatorname{Ker}(M_u) = \{0\}$ and M_u is injective on

$WBV_p[0,1]$. Conversely, if there exists a $t_0 \in [0,1]$ such that $u(t_0) = 0$, then we can define the function

$$f(t) = \begin{cases} 1 & \text{if } t = t_0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

which is non-null, belongs to $WBV_p[0,1]$ since $1 \leq \mathrm{Var}_p^W(f;[0,1]) \leq 2^{1/p}$ (note that $\mathrm{Var}_p^W(f;[0,1]) = 1$ if $t_0 = 1$ or $t_0 = 0$ and $\mathrm{Var}_p^W(f;[0,1]) = 2^{1/p}$ if $t_0 \in (0,1)$) and satisfies $u(t) \cdot f(t) = 0$ for all $t \in [0,1]$. This last implies that $f \in \mathrm{Ker}(M_u) \neq \{0\}$ and M_u is not injective on $WBV_p([0,1])$. □

Now we show the main result of this section. We recall that an operator $T : X \to X$, where X is a Banach space, is said to be *bounded below* if there exists a constant $L > 0$ such that $\|Tf\| \geq L\|f\|$ for all $f \in X$. It is known that $T : X \to X$ is bounded if and only if $T : X \to X$ is injective and $\mathrm{Ran}(T)$ is a closed subset of X.

Theorem 2. *Suppose that $u \in WBV_p[0,1]$. The following statements are equivalents:*

(1) $\mathrm{Ran}(M_u) = WBV_p([0,1])$*, that is, M_u is surjective on $WBV_p[0,1]$,*

(2) $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ *is bijective (with inverse continuous),*

(3) $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ *is bounded below,*

(4) $\inf\limits_{t\in[0,1]} (|u(t)|) > 0.$

PROOF. (1)$\Rightarrow$ (2): Suppose that M_u is surjective on $WBV_p[0,1]$. If $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is not injective, then by Proposition 1, there exists $t_0 \in [0,1]$ such that $u(t_0) = 0$. Thus the function f defined in (4) belongs to $WBV_p[0,1]$. Hence, since que $\mathrm{Ran}(M_u) = WBV_p([0,1])$, there exists a function $h \in WBV_p[0,1]$ such that $f = u \cdot h$. In particular, evaluating at $t = t_0$ we find that $1 = f(t_0) = u(t_0)\,h(t_0) = 0$. Which is a contradiction and $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ must be bijective.

(2)$\Rightarrow$ (3): Is true for any operator.

(3)$\Rightarrow$ (4): If $\inf\limits_{t\in[0,1]} (|u(t)|) = 0$, then there exists a sequence $\{t_n\} \subset [0,1]$ such that $0 \leq |u(t_n)| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence we can define the functions $\{f_n\}_{n\in\mathbb{N}}$, on $[0,1]$ by

$$f_n(t) = \begin{cases} 1 & \text{if } t = t_n \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Then $f_n \in WBV_p[0,1]$ and $2 \leq \|f_n\|_{WBV_p([0,1])} \leq 2^{1/p}+1$ for all $n \in \mathbb{N}$. Furthermore, $\mathrm{Var}_p^W(u \cdot f_n,[0,1]) = |u(t_n)|\mathrm{Var}_p^W(f_n,[0,1])$ for all $n \in \mathbb{N}$ and therefore

$$\|u \cdot f_n\|_{WBV_p([0,1])} = |u(t_n)|\,\|f_n\|_{WBV_p([0,1])} \leq \frac{1}{n}\|f_n\|_{WBV_p([0,1])},$$

for all $n \in \mathbb{N}$. Which tell us that $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is not bounded below on $WBV_p[0,1]$.

(4)$\Rightarrow$ (1): If $\inf\limits_{t\in[0,1]}(|u(t)|) > 0$, then since $u \in WBV_p([0,1])$ it follows that $\frac{1}{u} \in WBV_p([0,1])$. Hence, for each $f \in WBV_p([0,1])$, the function $h = \frac{1}{u} \cdot f \in WBV_p([0,1])$ and satisfies $M_u h = u \cdot h = f$. This shows that $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is surjective. $\square$

3 Compact multiplication operators on $WBV_p[0,1]$

In this section we characterize the symbols $u \in WBV_p[0,1]$ which define compact multiplication operators M_u acting on $WBV_p[0,1]$. We recall that an operator $T : X \to X$ is said to be *compact* if $T(x_n)$ has a convergent subsequence in X for all bounded sequence $\{x_n\} \subset X$. It is known that the limit of compact operators is also a compact operator and that the identity operator $I : X \to X$ defined by $If = f$ is compact if and only if $\dim(X) < \infty$. A continuous operator $S : X \to X$ is said to has *finite range* if $\dim(\mathrm{Ran}(S)) < \infty$. It is known that all operator having finite range is a compact operator. In the following result, we characterize the symbols u which define multiplication operators M_u having finite range on $WBV_p[0,1]$.

Theorem 3. *Suppose that* $u \in WBV_p[0,1]$. *Then*

$$M_u : WBV_p([0,1]) \to WBV_p([0,1])$$

has finite range if and only if $\mathrm{supp}(u)$ *is a finite set.*

Proof. If $\mathrm{supp}(u)$ is an infinite set, then we can choose a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathrm{supp}(u)$ such that $t_n \neq t_m$ for all $n \neq m$. Thus, the functions $\{h_n\}$ defined by

$$h_n(t) = \begin{cases} u(t) & \text{if } t = t_n \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

belong to $\mathrm{Ran}(M_u)$, since $h_n = u \cdot f_n$ with f_n the functions defined in (5) and they are clearly linearly independent. Therefore $\dim(\mathrm{Ran}(M_u)) = \infty$.

Conversely, if $\mathrm{supp}(u) = \{t_1, \cdots, t_m\}$ is a finite set, then the functions $\{h_1, h_2, \cdots, h_m\}$ defined as in (6) are vectors, linearly independent in $\mathrm{Ran}(M_u)$.

Furthermore, if $f \in \mathrm{Ran}(M_u)$ then $f = M_u g = u \cdot g$ for some $g \in WBV_p([0,1])$. Thus, if we consider the scalars $\alpha_k = g(t_k)$, with $k = 1, 2, \cdots, m$, we have

$$f = \sum_{k=1}^{m} \alpha_k h_k$$

and $\dim(\mathrm{Ran}(M_u)) = m < \infty$. This shows our result. $\square$

Our characterization of the compactness of M_u acting on $WBV_p[0,1]$ is given in terms of certain subsets of $\mathrm{supp}(u)$ which we define now. Given $\epsilon > 0$ we set

$$E_\epsilon = \big\{ t \in [0,1] : |u(t)| \geq \epsilon \big\}. \tag{7}$$

Then we have the following property:

Proposition 4. *For each $\epsilon > 0$, the set*

$$X_{E_\epsilon} = \{ f \in WBV_p([0,1]) : f(t) = 0 \quad \forall t \in [0,1] \setminus E_\epsilon \}$$

is an M_u-invariant closed subspace of $WBV_p([0,1])$.

PROOF. Clearly, X_{E_ϵ} is a M_u-invariant subspace of $WBV_p([0,1])$. Furthermore, if $\{f_n\}$ is a sequence of functions in X_{E_ϵ} such that $\|f_n - f\|_{WBV_p([0,1])} \to 0$ as $n \to \infty$, then $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Hence, we can conclude that $f(t) = 0$ for all $t \in [0,1] \setminus E_\epsilon$ and $f \in X_{E_\epsilon}$. This shows that X_{E_ϵ} is closed. $\square$

We can now enunciate the main result of this section:

Theorem 5. *Suppose that $u \in WBV_p([0,1])$. The operator M_u is compact on $WBV_p([0,1])$ if and only if for each $\epsilon > 0$ the set E_ϵ is finite.*

PROOF. Let us suppose first that $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is a compact operator. Fix $\epsilon > 0$. By Proposition 4, the space X_{E_ϵ} is closed in $WBV_p([0,1])$, hence the inclusion operator $i_{E_\epsilon} : X_{E_\epsilon} \to WBV_p([0,1])$ defined by $i_{E_\epsilon} f = f$ is continuous and therefore, the composition

$$M_u \circ i_{E_\epsilon} : X_{E_\epsilon} \to WBV_p([0,1])$$

is a compact operator. Clearly, since X_{E_ϵ} is M_u-invariant, we have $\mathrm{Ran}\,(M_u \circ i_{E_\epsilon}) \subset X_{E_\epsilon}$. Also, if $f \in X_{E_\epsilon}$, then the function $h : [0,1] \to \mathbb{R}$ defined by

$$h(t) = \begin{cases} \frac{f(t)}{u(t)} & \text{if } t \in E_\epsilon \\ 0 & \text{otherwise} \end{cases}$$

belongs to $WBV_p[0,1]$ since

$$\begin{aligned}\mathrm{Var}_p^W(h;[0,1])^p \leq \left[\frac{1}{\epsilon^p} + \left(\frac{2\|u\|_\infty}{\epsilon^2}\right)^p\right]\mathrm{Var}_p^W(f;[0,1])^p \\ + \left[\frac{2\|f\|_\infty}{\epsilon^2}\right]^p \mathrm{Var}_p^W(u;[0,1])^p\end{aligned}$$

and clearly, $f(t) = u(t)\cdot h(t)$ for all $t \in [0,1]$. This means that $\mathrm{Ran}\,(M_u \circ i_{E_\epsilon}) = X_{E_\epsilon}$ and the operator $M_u \circ i_{E_\epsilon} : X_{E_\epsilon} \to X_{E_\epsilon}$ is a surjective and compact operator.

We affirm that $M_u \circ i_{E_\epsilon} : X_{E_\epsilon} \to X_{E_\epsilon}$ is injective. Indeed, if $f \in \mathrm{Ker}(M_u \circ i_{E_\epsilon})$, then $f \in X_{E_\epsilon}$ which implies that $f(t) = 0$ for all $t \in [0,1] \setminus E_\epsilon$; while from the fact that $u \cdot f = 0$, we obtain that $f(t) = 0$ for all $t \in E_\epsilon$. Therefore $\mathrm{Ker}(M_u \circ i_{E_\epsilon}) = \{0\}$ and $M_u \circ i_{E_\epsilon} : X_{E_\epsilon} \to X_{E_\epsilon}$ is a bijective and compact operator. Hence, we have arrived to the following conclusion: X_{E_ϵ} is a finite dimensional space.

If E_ϵ is an infinite set, then it has a sequence $\{t_n\}$ such that $t_n \neq t_m$ for $n \neq m$. Then the sequence of functions $\{f_n\}$ as in (5) are linearly independent and belong to X_{E_ϵ}. This gives us a contradiction to the fact that X_{E_ϵ} is a finite dimensional space.

Suppose now that for each $\epsilon > 0$ the set E_ϵ is finite. Then, since

$$\mathrm{supp}(u) = \bigcup_{n=1}^{\infty} E_{\frac{1}{n}} = \bigcup_{n=1}^{\infty} \left\{t \in [0,1] : |u(t)| \geq \frac{1}{n}\right\},$$

we conclude that $\mathrm{supp}(u)$ is a finite set or a denumerable set. If $\mathrm{supp}(u)$ is a finite set, then Theorem 3 implies that $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ has finite range and therefore is a compact operator. If $\mathrm{supp}(u)$ is an infinite denumerable set, namely

$$\mathrm{supp}(u) = \{t_1, t_2, \cdots, t_n, \cdots\} \subset [0,1].$$

Then we have

$$\left(\sum_{k=1}^{\infty} |u(t_k)|^p\right)^{1/p} \leq \mathrm{Var}_p^W(u;[0,1]) \leq \left(2\sum_{k=1}^{\infty} |u(t_k)|^p\right)^{1/p},$$

and the series $\sum_{k=1}^{\infty} |u(t_k)|^p$ converges absolutely. Thus, any rearrangement of $\{t_n\}_{n\in\mathbb{N}}$ does not affect its value. For each $n \in \mathbb{N}$, we set $E_n = \{t_1, t_2, \cdots, t_n\}$, and we define the symbol u_n by

$$u_n(t) = u(t) \cdot \chi_{E_n}(t) = \begin{cases} u(t) & \text{if } t \in E_n \\ 0 & \text{otherwise.} \end{cases}$$

Then $u_n \in WBV_p([0,1])$ since $\mathrm{Var}_p^W\big(u_n; [0,1]\big) \leq (2n)^{1/p} \|u\|_\infty$, and $\mathrm{supp}(u_n)$ is a finite set. Theorem 3 implies that $M_{u_n} : WBV_p([0,1]) \to WBV_p([0,1])$ has finite range for each $n \in \mathbb{N}$. Furthermore, we have

$$u_n(t) - u(t) = \begin{cases} 0 & \text{if } t \in \{t_1, t_2, \cdots, t_n\} \\ -u(t) & \text{if } t \in \{t_{n+1}, t_{n+2}, \cdots\} \\ 0 & \text{otherwise,} \end{cases}$$

and hence

$$\mathrm{Var}_p^W\big(u_n - u; [0,1]\big) \leq \left(2 \sum_{k=n+1}^{\infty} |u(t_k)|^p\right)^{1/p} \to 0 \quad \text{as } n \to \infty,$$

since the series $\sum_{k=1}^{\infty} |u(t_k)|^p$ is convergent. We conclude that

$$\|M_{u_n} - M_u\| = \|M_{u_n - u}\| = \|u_n - u\|_{WBV_p([0,1])} \to 0 \quad \text{as } n \to \infty$$

and $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is a compact operator. □

4 Multiplication operators with closed range on $WBV_p[0,1]$

In this section we characterize all symbols $u \in WBV_p[0,1]$ which define multiplication operators $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ with closed range. The key of our result lie in to consider the following subspace: For $u \in WBV_p([0,1])$, we define the set $X_{\mathbf{Z}_u}$ by

$$X_{\mathbf{Z}_u} = \{f \in WBV_p([0,1]) : \ f(t) = 0 \ \forall t \in \mathbf{Z}_u\}. \tag{8}$$

Then, we have the following property:

Proposition 6. *If $\mathbf{Z}_u \neq \emptyset$ then $X_{\mathbf{Z}_u}$ is a M_u-invariant and closed subspace of $WBV_p([0,1])$. Furthermore,* $\mathrm{Ran}(M_u) \subset X_{\mathbf{Z}_u}$.

PROOF. Clearly X_{Z_u} is contained in $Ran(Mu)$ and it is a Mu-invariant subspace of $WBV_p([0;1])$. Also, if $\{f_n\}$ is a sequence in $X_{\mathbf{Z}_u}$ such that

$$\|f_n - f\|_{WBV_p([0,1])} \to 0$$

as $n \to \infty$, then $\|f_n - f\|_\infty \to 0$ as $n \to \infty$ and hence we can deduce that $f(t) = 0$ for all $t \in \mathbf{Z}_u$. That is, $X_{\mathbf{Z}_u}$ is closed in $WBV_p([0,1])$. □

Next, we are going to enunciate and to show the main result of this section.

Theorem 7. *Suppose that $u \in WBV_p[0,1]$. The operator*

$$M_u : WBV_p([0,1]) \to WBV_p([0,1])$$

has closed range if and only if there exists a $\delta > 0$ such that $|u(t)| \geq \delta$ for all $t \in \operatorname{supp}(u)$.

PROOF. Suppose first that $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ has closed range and that for each $n \in \mathbb{N}$ we can find $t_n \in \operatorname{supp}(u)$ such that

$$0 < |u(t_n)| < \frac{1}{n^2}.$$

In particular, the above inequality implies that the series $\sum_{k=1}^{\infty} |u(t_k)|$ is absolutely convergent and that $\{t_n\}_{n\in\mathbb{N}}$ is an infinite set. Furthermore, since $\{t_n\}_{n\in\mathbb{N}} \subset \operatorname{supp}(u) \subset [0,1]$, by passing to a subsequence we may assume that $\{t_n\}_{n\in\mathbb{N}}$ is an ordered and convergent set (recall that from a convergent sequence we can build a monotone subsequence). Next, for each $n \in \mathbb{N}$ we consider the function

$$h_n(t) = \begin{cases} u(t) & \text{if } t \in A_{2n+1} \\ 0 & \text{otherwise} \end{cases}$$

where $A_{2n+1} = \{t_1, t_3, t_5, \cdots, t_{2n+1}\}$. Clearly $h_n = u \cdot \chi_{A_{2n+1}} \in \operatorname{Ran}(M_u)$ since

$$\operatorname{Var}_p^W\left(\chi_{A_{2n+1}}; [0,1]\right) \leq (2(n-1))^{1/p}.$$

Furthermore, if $n, m \in \mathbb{N}$ and $n > m$, then

$$(h_n - h_m)(t) = \begin{cases} u(t) & \text{if } t \in \{t_{2m+3}, \cdots, t_{2n+1}\} \\ 0 & \text{otherwise} \end{cases}$$

and hence

$$\|h_n - h_m\|_{WBV_p([0,1])} \leq \left(2 \sum_{k=2m+3}^{2n+1} |u(t_k)|^p\right)^{1/p} \leq 2^{1/p} \sum_{k=2m+3}^{2n+1} |u(t_k)|$$
$$\leq 2^{1/p} \sum_{k=2m+3}^{\infty} \frac{1}{k^2} \quad \to 0 \text{ as } m \to \infty.$$

Thus, $\{h_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the closed set $\mathrm{Ran}(M_u)$ and there exists a function $f \in WBV_p([0,1])$ such that $\|h_n - u \cdot f\|_{WBV_p([0,1])} \to 0$ as $n \to \infty$. In particular, $\left\|u \cdot \chi_{A_{2n+1}} - u \cdot f\right\|_{\infty} \to 0$ as $n \to \infty$ and since $u(t) \neq 0$ for all $t \in A_{2n+1} \subset \mathrm{supp}(u)$, we conclude that

$$f(t) = \begin{cases} 1 & \text{if } t \in A_{2n+1} \\ 0 & \text{if } t \in \mathrm{supp}(u) \setminus A_{2n+1}. \end{cases}$$

But, if we consider the partition $P_n = \{0, t_1, t_2, t_3, \cdots, t_{n-1}, t_n, 1\}$, then we have

$$\mathrm{Var}_p^W(f;[0,1]) \geq \left(\sum_{k=2}^{n} |f(t_k) - f(t_{k-1})|^p\right)^{1/p} = \left(\sum_{k=2}^{n} 1\right)^{1/p} = (n-1)^{1/p}.$$

Which implies that f does not belong to $WBV_p[0,1]$ and we have a contradiction. Therefore there exists a $\delta > 0$ such that $|u(t)| \geq \delta$ for all $t \in \mathrm{supp}(u)$.

Next, suppose that there exists a $\delta > 0$ such that $|u(t)| \geq \delta$ for all $t \in \mathrm{supp}(u)$. We affirm that $\mathrm{Ran}(M_u) = X_{\mathbf{Z}_u}$. Clearly $\mathrm{Ran}(M_u) \subseteq X_{\mathbf{Z}_u}$, hence it is enough to show that $X_{\mathbf{Z}_u} \subseteq \mathrm{Ran}(M_u)$. Indeed, for each $f \in X_{\mathbf{Z}_u}$, we have that $f \in WBV_p[0,1]$ and we can define the function

$$g(t) = \begin{cases} \frac{f(t)}{u(t)} & \text{if } t \in [0,1] \setminus \mathbf{Z}_u \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Thus, if $P : 0 = t_0 < t_1 < \cdots < t_n = 1$ is any partition of $[0,1]$, then by considering separately the cases $t_k, t_{k-1} \in \mathbf{Z}_u$, $t_k \in \mathbf{Z}_u$ and $t_{k-1} \notin \mathbf{Z}_u$, $t_k \in \mathbf{Z}_u$ and $t_{k-1} \notin \mathbf{Z}_u$ and $t_k \notin \mathbf{Z}_u$ and $t_{k-1} \in \mathbf{Z}_u$, we obtain

$$\sum_{k=1}^{n} |g(t_k) - g(t_{k-1})|^p \leq \left[\frac{1}{\delta^p} + \left(\frac{2\|u\|_\infty}{\delta^2}\right)^p\right] \mathrm{Var}_p^W(f;[0,1])^p$$
$$+ \left[\frac{2\|f\|_\infty}{\delta^2}\right]^p \mathrm{Var}_p^W(u;[0,1])^p,$$

where we have used the well known identity $(a+b)^p \le 2^p(a^p+b^p)$ valid for all $a, b \ge 0$. Therefore, $g \in WBV_p[0,1]$, $f = u \cdot g \in \mathrm{Ran}\,(M_u)$ and $\mathrm{Ran}(M_u) = X_{\mathbf{Z}_u}$ as we affirmed. The result follows since $X_{\mathbf{Z}_u}$ is a closed set of $WBV_p[0,1]$ by Proposition 6. □

5 Fredholm multiplication operators on $WBV_p[0,1]$

In this section we characterize all symbols $u \in WBV_p[0,1]$ which define Fredholm multiplication operators on $WBV_p[0,1]$. We recall that a bounded operator $T : X \to X$ is said to be Fredholm if $\dim(\mathrm{Ker}(M_u)) < \infty$ and $\mathrm{codim}(\mathrm{Ran}(M_u)) = \dim(X/\mathrm{Ran}(M_u)) < \infty$. The operator T is called upper semi-Fredholm if $\dim(\mathrm{Ker}(M_u)) < \infty$ and $\mathrm{Ran}(M_u)$ is a closed set of X, while T is lower semi-Fredholm if $\mathrm{codim}(\mathrm{Ran}(M_u)) < \infty$. It is known that the condition $\mathrm{codim}(\mathrm{Ran}(M_u)) < \infty$ implies that $\mathrm{Ran}(M_u)$ is a closed set of X. Clearly, $T : X \to X$ is Fredholm if and only if $T : X \to X$ is lower and upper Fredholm. The following is the main result of this section.

Theorem 8. *Suppose that $u \in WBV_p[0,1]$. The following statements are equivalents:*

(1) $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is upper semi-Fredholm,

(2) $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is lower semi-Fredholm,

(3) $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is Fredholm,

(4) $\mathbf{Z}_u$ is a finite set and there exists a $\delta > 0$ such that $|u(t)| \ge \delta$ for all $t \in \mathrm{supp}(u)$.

Proof. (1)⇒ (4): Let us suppose that $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is upper semi-Fredholm. If $\mathbf{Z}_u$ is an infinite set, then we can find a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbf{Z}_u$ such that $t_n \ne t_m$. Thus the functions f_n defined as in (5) are linearly independent in $WBV_p[0,1]$ and belong to $\mathrm{Ker}\,(M_u)$ which give us a contradiction to the fact that $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is upper semi-Fredholm. Finally, since $\mathrm{Ran}(M_u)$ is a closed set of $WBV_p[0,1]$ by Theorem 7 we conclude that there exists a $\delta > 0$ such that $|u(t)| \ge \delta$ for all $t \in \mathrm{supp}(u)$.

(4)⇒ (2): By Theorem 7 we have that the operator $\mathrm{Ran}\,(M_u)$ is a closed set of $WBV_p[0,1]$, then the quotient space

$$Q = WBV_p[0,1]/\mathrm{Ran}\,(M_u) = \{f + \mathrm{Ran}\,(M_u) : f \in WBV_p[0,1]\}$$

is a Banach space. We are going to show that $\dim(Q) < \infty$. Indeed, since $\mathbf{Z}_u$ is a finite set, then we can write $\mathbf{Z}_u = \{t_1, t_2, \cdots, t_m\}$ and the functions f_k

with $k = 1, 2, \cdots, m$ defined as in (5) are linearly independent in $WBV_p[0,1]$. Hence, the set

$$B = \{f_k + \text{Ran}\,(M_u) : k = 1, \cdots, m\}$$

is linearly independent in Q. Furthermore, if $\tilde{g} \in Q$ then $\tilde{g} = g + \text{Ran}(M_u)$ for some $g \in WBV_p([0,1])$ and we can consider the scalars $\alpha_k = g\,(t_k)$ with $k = 1, 2, \cdots, m$. We affirm that $g - \sum_{k=1}^{m} \alpha_k f_k \in \text{Ran}(M_u)$. Bearing this in mind, we define the function

$$h(t) = \begin{cases} \frac{g(t)}{u(t)} & \text{if } t \notin \mathbf{Z}_u \\ 0 & \text{otherwise.} \end{cases}$$

Then $h \in WBV_p[0,1]$ since there exists a $\delta > 0$ such that $|u(t)| \geq \delta$ for all $t \in \text{supp}(u)$. Furthermore, if $t \in \text{supp}(u)$ then $f_k(t) = 0$ for all $k = 1, 2, \cdots, m$ and we have

$$g(t) - \sum_{k=1}^{m} \alpha_k f_k(t) = g(t) = u(t) \cdot h(t),$$

while if $t \in \mathbf{Z}_u$ then $t = t_j$ for some $j \in \{1, 2, \cdots, m\}$ and we obtain

$$g(t) - \sum_{k=1}^{m} \alpha_k f_k(t) = g(t_j) - \alpha_j f_j(t_j) = \alpha_j - \alpha_j = 0 = u(t) \cdot h(t).$$

We conclude then that $g - \sum_{k=1}^{m} \alpha_k f_k = u \cdot h \in \text{Ran}(M_u)$ which means that

$$\widetilde{g} = \sum_{k=1}^{m} \alpha_k f_k + \text{Ran}\,(M_u)\,.$$

and B is a basis for Q as we affirmed. This shows that

$$\text{codim}\,(\text{Ran}(M_u)) = \dim\,(WBV_p([0,1])/\text{Ran}\,(M_u)) = m < \infty.$$

(2) $\Rightarrow$ (3): Since $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is lower semi-Fredholm, only we need to show that $\dim(\text{Ker}(M_u)) < \infty$. As in the proof of (4)$\Rightarrow$(2), the fact that $\text{codim}\,(\text{Ran}(M_u)) < \infty$ implies that $\mathbf{Z}_u$ is a finite set, namely $\mathbf{Z}_u = \{t_1, t_2, \cdots, t_m\}$. Then the functions f_k with $k = 1, 2, \cdots, m$ defined as in (5) belong to $\text{Ker}\,(M_u)$ and they are linearly independent in $WBV_p[0,1]$. Furthermore, for any $g \in \text{Ker}(M_u)$ we have

$$g = \sum_{k=1}^{m} \alpha_k f_k,$$

where $\alpha_k = g(t_k)$, which tell us that $\dim(\mathrm{Ker}(M_u)) = m < \infty$.
(3)⇒(1): Always is true by definition. □

Next we give some consequences. From Theorems 8 and 7 we have the following corollary:

Corollary 9. *Suppose that* $u \in WBV_p[0,1]$*. The operator*

$$M_u : WBV_p([0,1]) \to WBV_p([0,1])$$

is Fredholm if and only if $\mathbf{Z}_u$ *is a finite set and* $\mathrm{Ran}(M_u)$ *is a closed subset of* $WBV_p[0,1]$*.*

Also we can characterize the invertibility in terms of Fredholmness of $M_u : WBV_p([0,1]) \to WBV_p([0,1])$:

Corollary 10. *Suppose that* $u \in WBV_p[0,1]$*. The operator*

$$M_u : WBV_p([0,1]) \to WBV_p([0,1])$$

is invertible (with continuous inverse) if and only if $\mathrm{codim}(\mathrm{Ran}(M_u)) < \infty$ *and* $\mathrm{supp}(u) = [0,1]$*.*

Proof. Clearly, if $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is invertible, then $\mathrm{codim}(\mathrm{Ran}(M_u)) < \infty$ and $\mathrm{supp}(u) = [0,1]$ since it is Fredholm and injective. Conversely, if $\mathrm{codim}(\mathrm{Ran}(M_u)) < \infty$ then $M_u : WBV_p([0,1]) \to WBV_p([0,1])$ is lower semi-Fredholm and by Theorem 8 we have that $\mathbf{Z}_u$ is a finite set and there exists a $\delta > 0$ such that $|u(t)| \geq \delta$ for all $t \in \mathrm{supp}(u)$. But by hypothesis $\mathrm{supp}(u) = [0,1]$ which tell us that $\mathbf{Z}_u = \emptyset$ and $|u(t)| \geq \delta$ for all $t \in [0,1]$. Hence the proof follows now by Theorem 2. □

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