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Biharmonic Maps on Principal G-Bundles over Complete Riemannian Manifolds of Nonpositive Ricci Curvature

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Abstract. We show that, for every principal G-bundle over a complete Riemannian manifold of nonpositive Ricci curvature, if the projection of the G-bundle is biharmonic, then it is harmonic.

1. Introduction

Variational problems play a central role in geometry; a harmonic map is one of important variational problems, which is a critical point of the energy functional $E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v_g$ for smooth maps φ of (M, g) into (N, h). The Euler–Lagrange equations are given by the vanishing of the tension filed $\tau(\varphi)$ (see [6; 15; 18; 25; 38]). In 1983, Eells and Lemaire [9] extended the notion of harmonic maps to biharmonic maps, which are, by definition, critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g. \tag{1.1}$$

Jiang [17] studied the first and second variation formulas of E_2. In this area, extensive studies have been done (e.g., see [2; 3; 5; 10; 12; 13; 14; 15; 16; 20; 21; 24; 23; 31; 29; 30; 32; 35; 36; 37; 39; 40], etc.). Notice that harmonic maps are always biharmonic by definition. Chen [7] raised the so-called Chen conjecture, and later Caddeo, Montaldo, Piu, and C. Oniciuc [5] raised the generalized Chen conjecture.

Chen's conjecture. *Every biharmonic submanifold of the Euclidean space $\mathbb{R}^n$ is harmonic* (*minimal*).

The generalized Chen conjecture. *Every biharmonic submanifold of a Riemannian manifold of nonpositive curvature is harmonic* (*minimal*).

For the generalized Chen conjecture, Ou and Tang [33; 32] gave a counterexample in a Riemannian manifold of negative curvature. For the Chen conjecture, affirmative answers were known for the case of surfaces in the three-dimensional Euclidean space [7] and for the case of hypersurfaces of the four-dimensional Euclidean space [11; 8]. Akutagawa and Maeta [1] showed a supporting evidence to

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the Chen conjecture: *Any complete regular biharmonic submanifold of the Euclidean space* $\mathbb{R}^n$ *is harmonic (minimal).* The affirmative answers to the generalized Chen conjecture were shown [26; 27; 28] under the L^2-condition and completeness of (M, g).

In this paper, we treat with a principal G-bundle over a Riemannian manifold and show the following two theorems.

THEOREM 2.3. *Let* $\pi : (P, g) \to (M, h)$ *be a principal* G*-bundle over a Riemannian manifold* (M, h) *with negative definite Ricci tensor field. Assume that* P *is compact, so that* M *is also compact. If the projection* π *is biharmonic, then it is harmonic.*

THEOREM 2.4. *Let* $\pi : (P, g) \to (M, h)$ *be a principal* G*-bundle over a Riemannian manifold with nonpositive Ricci curvature. Assume that* (P, g) *is a noncompact complete Riemannian manifold and that the projection* π *has both finite energy* $E(\pi) < \infty$ *and finite bienergy* $E_2(\pi) < \infty$. *If* π *is biharmonic, then it is harmonic.*

We give two comments on these theorems: For the generalized Chen conjecture, the nonpositivity of the sectional curvature of the ambient space of biharmonic submanifolds is necessary. However, it should be emphasized that for the principal G-bundles, we need not the assumption of nonpositivity of the sectional curvature. We only assume the *nonpositivity of the Ricci curvature* of the domain manifolds in the proofs of Theorems 2.3 and 2.4. Second, in Theorem 2.4, the finiteness of the energy and bienergy is necessary. Otherwise, we can see the following counterexamples of Loubeau and Ou [22].

EXAMPLE 1 (cf. [34], [22, p. 62]). The inversion in the unit sphere $\phi : \mathbb{R}^n \backslash \{o\} \ni \mathbf{x} \mapsto \mathbf{x}/(|\mathbf{x}|^2) \in \mathbb{R}^n$ is biharmonic if $n = 4$. It is not harmonic since $\tau(\phi) = -4\mathbf{x}/(|\mathbf{x}|^4)$.

EXAMPLE 2 (cf. [22, p. 70]). Let (M^2, h) be a Riemannian surface, and let $\beta : M^2 \times \mathbb{R} \to \mathbb{R}^*$ and $\lambda : \mathbb{R} \to \mathbb{R}^*$ be two positive C^∞ functions. Consider the projection $\pi : (M^2 \times \mathbb{R}^*, g = \lambda^{-2} h + \beta^2\, dt^2) \ni (p, t) \mapsto p \in (M^2, h)$. Here, we take $\beta = c_2 e^{\int f(x)\, dx}$, $f(x) = -c_1(1 + e^{c_1 x})/(1 - e^{c_1 x})$ with $c_1, c_2 \in \mathbb{R}^*$, and $(M^2, h) = (\mathbb{R}^2, dx^2 + dy^2)$. Then,

$$\pi : (\mathbb{R}^2 \times \mathbb{R}^*, dx^2 + dy^2 + \beta^2(x)\, dt^2) \ni (x, y, t) \mapsto (x, y) \in (\mathbb{R}^2, dx^2 + dy^2)$$

gives a family of *proper biharmonic* (i.e., biharmonic but not harmonic) Riemannian submersions.

2. Preliminaries

2.1. Harmonic Maps and Biharmonic Maps

We first prepare the materials for the first and second variational formulas for the bienergy functional and biharmonic maps. Let us recall the definition of a harmonic map $\varphi : (M, g) \to (N, h)$ of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h), which is an extremal of the *energy functional* defined by

$$E(\varphi) = \int_M e(\varphi) v_g,$$

where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of φ; that is, for any variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\left.\frac{d}{dt}\right|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0, \tag{2.1}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ given by $V(x) = \frac{d}{dt}|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$ $(x \in M)$, and the *tension field* is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, g), and $B(\varphi)$ is the second fundamental form of φ defined by

$$\begin{aligned} B(\varphi)(X, Y) &= (\widetilde{\nabla} d\varphi)(X, Y) \\ &= (\widetilde{\nabla}_X d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y) \end{aligned} \tag{2.2}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Here, ∇ and ∇^h are Levi-Civita connections on TM and TN of (M, g) on (N, h), respectively, and $\overline{\nabla}$ and $\widetilde{\nabla}$ are the induced ones on $\varphi^{-1}TN$ and $T^*M \otimes \varphi^{-1}TN$, respectively. By (2.1), φ is *harmonic* if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that φ is harmonic. Then,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\varphi_t) = \int_M h(J(V), V) v_g, \tag{2.3}$$

where J is an elliptic differential operator, called the *Jacobi operator* acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \overline{\Delta} V - \mathcal{R}(V), \tag{2.4}$$

where $\overline{\Delta} V = \overline{\nabla}^* \overline{\nabla} V = -\sum_{i=1}^m \{\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} V - \overline{\nabla}_{\nabla_{e_i} e_i} V\}$ is the *rough Laplacian,* $\mathcal{R}$ is the linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}(V) = \sum_{i=1}^m R^N(V, d\varphi(e_i))\, d\varphi(e_i)$, and R^N is the curvature tensor of (N, h) given by $R^h(U, V) = \nabla^h{}_U \nabla^h{}_V - \nabla^h{}_V \nabla^h{}_U - \nabla^h{}_{[U,V]}$ for $U, V \in \mathfrak{X}(N)$.

Eells and Lemaire [9] proposed polyharmonic (k-harmonic) maps, and Jiang [17] studied the first and second variation formulas of biharmonic maps. Let us consider the *bienergy functional* defined by

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g, \tag{2.5}$$

where $|V|^2 = h(V, V)$, $V \in \Gamma(\varphi^{-1}TN)$.

The first variation formula of the bienergy functional is given by

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V) v_g, \tag{2.6}$$

where

$$\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}(\tau(\varphi)) - \mathcal{R}(\tau(\varphi)), \tag{2.7}$$

which is called the *bitension field* of φ, and J is given in (2.4).

A smooth map φ of (M, g) into (N, h) is said to be *biharmonic* if $\tau_2(\varphi) = 0$. By definition every harmonic map is biharmonic. We say that an immersion $\varphi : (M, g) \to (N, h)$ is *proper biharmonic* if it is biharmonic but not harmonic (minimal).

2.2. The Principal G-Bundle

Recall several notions on principal G-bundles. A manifold $P = P(M, G)$ is a principal fiber bundle over M with a compact Lie group G, where $p = \dim P$, $m = \dim M$, and $k = \dim G$. By definition a Lie group G acts on P by right-hand side denoted by $(G, P) \ni (a, u) \mapsto u \cdot a \in P$, and, for each point $u \in P$, the tangent space $T_u P$ admits a subspace $G_u := \{A^*{}_u \mid A \in \mathfrak{g}\}$, the vertical subspace at u, and each $A \in \mathfrak{g}$ defines the fundamental vector field $A^* \in \mathfrak{X}(P)$ by

$$A^*{}_u := \left.\frac{d}{dt}\right|_{t=0} u\exp(tA) \in T_u P.$$

A Riemannian metric g on P is called *adapted* if it is invariant under all the right action of G, that is, $R_a{}^*g = g$ for all $a \in G$. An adapted Riemannian metric on P always exists because for every Riemannian metric g' on P, we can define the new metric g on P by

$$g_u(X_u, Y_u) = \int_G g'(R_{a*}X_u, R_{a*}Y_u)\, d\mu(a),$$

where $d\mu(a)$ is a bi-invariant Haar measure on G. Then, $R_a{}^*g = g$ for all $a \in G$. Each tangent space $T_u P$ has the orthogonal direct decomposition of the tangent space $T_u P$,

(a) $T_u P = G_u \oplus H_u$,

where the subspace G_u of P_u satisfies

(b) $G_u = \{A^*{}_u \mid A \in \mathfrak{g}\}$,

and the subspace H_u of P_u satisfies

(c) $H_{u \cdot a} = R_{a*}H_u,\ a \in G,\ u \in P$;

the subspace H_u of P_u is called the *horizontal subspace* at $u \in P$ with respect to g.

In the following, we fix a locally defined orthonormal frame field $\{e_i\}_{i=1}^p$ corresponding (a), (b) in such a way that

- $\{e_i\}_{i=1}^m$ is a locally defined orthonormal basis of the horizontal subspace H_u $(u \in P)$, and
- $\{e_i = A^*{}_{m+i}\}_{i=1}^k$ is a locally defined orthonormal basis of the vertical subspace G_u $(u \in P)$ for an orthonormal basis $\{A_{m+i}\}_{i=1}^k$ of the Lie algebra $\mathfrak{g}$ of a Lie group G with respect to the $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle$.

For each decomposition (a), we can define the $\mathfrak{g}$-valued 1-form ω on P by

$$\omega(X_u) = A, \quad X_u = {X_u}^{\mathrm{V}} + {X_u}^{\mathrm{H}},$$

where

$${X_u}^{\mathrm{V}} \in G_u, \qquad {X_u}^{\mathrm{H}} \in H_u, \qquad {X_u}^{\mathrm{V}} = {A_u}^*$$

for $u \in P$ and a unique $A \in \mathfrak{g}$. This 1-form ω on P is called a *connection form* of P.

Then, there exist a unique Riemannian metric h on M and an $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that

$$g(X_u, Y_u) = h(\pi_* X_u, \pi_* Y_u) + \langle \omega(X_u), \omega(Y_u) \rangle, \quad X_u, Y_u \in T_u P, u \in P,$$

namely,

$$g = \pi^* h + \langle \omega(\cdot), \omega(\cdot) \rangle.$$

We call this Riemannian metric g on P an *adapted* Riemannian metric on P.

Let us recall the following definitions.

DEFINITION 2.1. (1) The projection $\pi : (P, g) \to (M, h)$ is *harmonic* if the tension field vanishes, $\tau(\pi) = 0$, and

(2) the projection $\pi : (P, g) \to (M, h)$ is *biharmonic* if, the bitension field vanishes, $\tau_2(\pi) = J(\tau(\pi)) = 0$.

Here, J is the Jacobi operator for the projection π given by

$$J(V) := \overline{\Delta} V - \mathcal{R}(V), \quad V \in \Gamma(\pi^{-1}TM),$$

where

$$\begin{aligned}
\overline{\Delta} V &:= -\sum_{i=1}^{p} \{ \overline{\nabla}_{e_i} (\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V \} \\
&= -\sum_{i=1}^{m} \{ \overline{\nabla}_{e_i} (\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V \} \\
&\quad - \sum_{i=1}^{k} \{ \overline{\nabla}_{A^*_{m+i}} (\overline{\nabla}_{A^*_{m+i}} V) - \overline{\nabla}_{\nabla_{A^*_{m+i}} A^*_{m+i}} V \}
\end{aligned}$$

for $V \in \Gamma(\pi^{-1}TM)$, that is, $V(x) \in T_{\pi(x)}M$ $(x \in P)$. Here, $\{e_i\}_{i=1}^{p}$ is a local orthonormal frame field on (P, g) given as follows: $\{e_i\}_{i=1}^{m}$ is an orthonormal horizontal field on the principal G-bundle $\pi : (P, g) \to (M, h)$, and $\{e_{m+i,u} = A^*_{m+i,u}\}_{i=1}^{k}$ $(u \in P)$ is an orthonormal frame field on the vertical space $G_u = \{A^*_u \mid A \in \mathfrak{g}\}$ $(u \in P)$ corresponding to an orthonormal basis $\{A_{m+i}\}_{i=1}^{k}$ of $(\mathfrak{g}, \langle\ ,\ \rangle)$.

If $\pi : (P, g) \to (M, h)$ is harmonic, then it is clearly biharmonic. Our main interest is to ask under what conditions the reverse holds.

PROBLEM 2.2. *If the projection π of a principal G-bundle $\pi : (P, g) \to (M, h)$ is biharmonic, then is π harmonic or not?*

In this paper, we show that this answer is affirmative when the Ricci curvature of the base manifold (M, h) is negative definite. Indeed, we show the following:

THEOREM 2.3. *Let $\pi : (P, g) \to (M, h)$ be a principal G-bundle over a Riemannian manifold (M, h) with negative definite Ricci tensor field. Assume that P is compact, so that M is also compact. If the projection π is biharmonic, then it is harmonic.*

THEOREM 2.4. *Let $\pi : (P, g) \to (M, h)$ be a principal G-bundle over a Riemannian manifold with nonpositive Ricci curvature. Assume that (P, g) is a noncompact complete Riemannian manifold and that the projection π has both finite energy $E(\pi) < \infty$ and finite bienergy $E_2(\pi) < \infty$. If π is biharmonic, then it is harmonic.*

3. Proof of Theorem 2.3

In this section, we prove Theorem 2.3 in the case that Riemannian manifold (M, h) is compact and the Ricci tensor of (M, h) is negative definite. In Section 4, we will prove Theorem 2.4 in the case of a noncompact complete Riemannian manifold (M, h).

Let us first consider a principal G-bundle $\pi : (P, g) \to (M, h)$ whose total space P is compact. Assume that the projection $\pi : (P, g) \to (M, h)$ is biharmonic, that is, by definition, $J(\tau(\pi)) \equiv 0$, where $\tau(\pi)$ is the tension field of π defined by

$$\tau(\pi) := \sum_{i=1}^{p} \{\nabla^h_{e_i} \pi_* e_i - \pi_*(\nabla_{e_i} e_i)\}, \tag{3.1}$$

and the Jacobi operator J is defined by

$$JV := \overline{\Delta} V - \mathcal{R}(V) \quad (V \in \Gamma(\pi^{-1}TM)) \tag{3.2}$$

with the rough Laplacian

$$\overline{\Delta} V := -\sum_{i=1}^{p} \{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V\} \tag{3.3}$$

and

$$\mathcal{R}(V) := R^h(V, \pi_* e_i)\pi_* e_i, \tag{3.4}$$

where $\{e_i\}_{i=1}^p$ is a locally defined orthonormal frame field on (P, g).

The tangent space P_u $(u \in P)$ is canonically decomposed into the orthogonal direct sum of the vertical subspace $G_u = \{A_u{}^* \mid A \in \mathfrak{g}\}$ and the horizontal subspace H_u: $P_u = G_u \oplus H_u$. Then, we have

$$\begin{aligned} \tau_2(\pi) &= \overline{\Delta}\tau(\pi) - \sum_{i=1}^{p} R^h(\tau(\pi), \pi_* e_i)\pi_* e_i \\ &= \overline{\Delta}\tau(\pi) - \sum_{i=1}^{m} R^h(\tau(\pi), \pi_* e_i)\pi_* e_i \end{aligned}$$

$$-\sum_{i=1}^{k} R^h(\tau(\pi), \pi_* A^*_{m+i})\pi_* A^*_{m+i}$$
$$= \overline{\Delta}\tau(\pi) - \sum_{i=1}^{m} R^h(\tau(\pi), \pi_* e_i)\pi_* e_i,$$

where $p = \dim P$, $m = \dim M$, and $k = \dim G$, respectively. Then, we obtain

$$\begin{aligned} 0 &= \int_P \langle J(\tau(\pi)), \tau(\pi)\rangle v_g \\ &= \int_P \langle \overline{\nabla}^* \overline{\nabla}\tau(\pi), \tau(\pi)\rangle v_g - \int_P \sum_{i=1}^{m} \langle R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi)\rangle v_g \\ &= \int_P \langle \overline{\nabla}\tau(\pi), \overline{\nabla}\tau(\pi)\rangle v_g - \int_P \sum_{i=1}^{m} \langle R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi)\rangle v_g. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \int_P \langle \overline{\nabla}\tau(\pi), \overline{\nabla}\tau(\pi)\rangle v_g &= \int_P \sum_{i=1}^{m} \langle R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi)\rangle v_g \\ &= \int_P \sum_{i=1}^{m} \langle R^h(\tau(\pi), e'_i)e'_i, \tau(\pi)\rangle v_g \\ &= \int_P \mathrm{Ric}^h(\tau(\pi)) v_g, \end{aligned} \tag{3.5}$$

where $\{e'_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, h) satisfying $\pi_* e_i = e'_i$, and $\mathrm{Ric}(X)$ is the Ricci curvature of (M, h) along $X \in T_x M$. The left-hand side of (3.5) is nonnegative, and then both sides of (3.5) must vanish if the Ricci curvature of (M, h) is nonpositive. Therefore, we obtain

$$\begin{cases} \overline{\nabla}_X \tau(\pi) = 0 \quad (\forall X \in \mathfrak{X}(P)), \text{ i.e., } \tau(\pi) \text{ is parallel, and} \\ \mathrm{Ric}^h(\tau(\pi)) = 0. \end{cases} \tag{3.6}$$

Let us consider the 1-form α on M defined by

$$\alpha(Y)(\pi(x)) := \langle \tau(\pi)(x), Y_{\pi(x)}\rangle \quad (Y \in \mathfrak{X}(M), x \in P). \tag{3.7}$$

Then, for all $Y, Z \in \mathfrak{X}(M)$, we have

$$\begin{aligned} (\nabla^h_Z \alpha)(Y) &= Z(\alpha(Y)) - \alpha(\nabla^h_Z Y) \\ &= Z\langle \tau(\pi), Y\rangle - \langle \tau(\pi), \nabla^h_Z Y\rangle \\ &= \langle \overline{\nabla}_Z \tau(\pi), Y\rangle + \langle \tau(\pi), \nabla^h_Z Y\rangle - \langle \tau(\pi), \nabla^h_Z Y\rangle \\ &= 0, \end{aligned} \tag{3.8}$$

which implies that α is a parallel 1-form on (M, h). Since we assume that the Ricci tensor of (M, h) is negative definite, α must vanish (by Bochner's theorem; see [4], and [19, p. 55]). Thus, $\tau(\pi) \equiv 0$, that is, the projection of the principal

G-bundle $\pi : (P, g) \to (N, h)$ must be harmonic. We obtain Theorem 2.3 in the case that M is compact and the Ricci tensor is negative definite.

4. Proof of Theorem 2.4

We prove Theorem 2.4 for a noncompact and complete Riemannian manifold (P, g) and a Riemannian manifold (M, h) with nonpositive Ricci curvature.

(*Step 1*) We first take a cutoff function η on (P, g) for a fixed point $p_0 \in P$ as follows:

$$\begin{cases} 0 \leq \eta \leq 1 & (\text{on } P), \\ \eta = 1 & (\text{on } B_r(p_0)), \\ \eta = 0 & (\text{outside } B_{2r}(p_0)), \\ |\nabla \eta| \leq \frac{2}{r} & (\text{on } P), \end{cases} \tag{4.1}$$

where $B_r(p_0)$ is the ball in (P, g) of radius r around p_0.

Now assume that the projection $\pi : (P, g) \to (M, h)$ is biharmonic. Namely, we have, by definition,

$$\begin{aligned} 0 = J_2(\pi) &= J_\pi(\tau(\pi)) \\ &= \overline{\Delta}\tau(\pi) - \sum_{i=1}^{p} R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \end{aligned} \tag{4.2}$$

where $\{e_i\}_{i=1}^p$ is a local orthonormal frame field on (P, g), and $\overline{\Delta}$ is the rough Laplacian defined by

$$\overline{\Delta} V := \overline{\nabla}^* \overline{\nabla} V = -\sum_{i=1}^{p} \{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V\} \tag{4.3}$$

for $V \in \Gamma(\pi^{-1}TM)$.

(*Step 2*) By (4.2) we have

$$\begin{aligned} \int_P \langle \overline{\nabla}^* \overline{\nabla} \tau(\pi), \eta^2 \tau(\pi) \rangle v_g &= \int_P \eta^2 \left\langle \sum_{i=1}^{p} R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi) \right\rangle v_g \\ &= \int_P \eta^2 \sum_{i=1}^{p} \langle R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi) \rangle v_g \\ &= \int_P \eta^2 \sum_{i=1}^{m} \langle R^h(\tau(\pi), e'_i)e'_i, \tau(\pi) \rangle v_g \\ &= \int_P \eta^2 \operatorname{Ric}^h(\tau(\pi)) v_g, \end{aligned} \tag{4.4}$$

where $\{e'_i\}_{i=1}^m$ is a local orthonormal frame field on (M, h), and $\operatorname{Ric}^h(u)$ $u \in T_y M$ $(y \in M)$ is the Ricci curvature of (M, h), which is nonpositive by our assumption.

(*Step 3*) Therefore, we obtain

$$
\begin{aligned}
0 &\geq \int_P \langle \overline{\nabla}^* \overline{\nabla} \tau(\pi), \eta^2 \tau(\pi) \rangle v_g \\
&= \int_P \langle \overline{\nabla} \tau(\pi), \overline{\nabla}(\eta^2 \tau(\pi)) \rangle v_g \\
&= \int_P \sum_{i=1}^{p} \langle \overline{\nabla}_{e_i} \tau(\pi), \overline{\nabla}_{e_i}(\eta^2 \tau(\pi)) \rangle v_g \\
&= \int_P \sum_{i=1}^{p} \{ \eta^2 \langle \overline{\nabla}_{e_i} \tau(\pi), \overline{\nabla}_{e_i} \tau(\pi) \rangle + e_i(\eta^2) \langle \overline{\nabla}_{e_i} \tau(\pi), \tau(\pi) \rangle \} v_g \\
&= \int_P \eta^2 \sum_{i=1}^{p} |\overline{\nabla}_{e_i} \tau(\pi)|^2 v_g \\
&\quad + 2 \int_P \sum_{i=1}^{p} \langle \eta \overline{\nabla}_{e_i} \tau(\pi), e_i(\eta) \tau(\pi) \rangle v_g .
\end{aligned}
\tag{4.5}
$$

(*Step 4*) Therefore, by (4.5) we have

$$
\begin{aligned}
\int_P \eta^2 \sum_{i=1}^{p} |\overline{\nabla}_{e_i} \tau(\pi)|^2 v_g &\leq -2 \int_P \sum_{i=1}^{p} \langle \eta \overline{\nabla}_{e_i} \tau(\pi), e_i(\eta) \tau(\pi) \rangle v_g \\
&= -2 \int_P \sum_{i=1}^{p} \langle V_i, W_i \rangle v_g ,
\end{aligned}
\tag{4.6}
$$

where $V_i := \eta \overline{\nabla}_{e_i} \tau(\pi)$ and $W_i := e_i(\eta) \tau(\pi)$ $(i = 1, \ldots, p)$. Then, we estimate the right-hand side of (4.6) by the Cauchy–Schwarz inequality:

$$
\pm 2 \langle V_i, W_i \rangle \leq \varepsilon |V_i|^2 + \frac{1}{\varepsilon} |W_i|^2 , \tag{4.7}
$$

which follows from

$$
0 \leq \left| \sqrt{\varepsilon} V_i \pm \frac{1}{\sqrt{\varepsilon}} W_i \right|^2 = \varepsilon |V_i|^2 \pm 2 \langle V_i, W_i \rangle + \frac{1}{\varepsilon} |W_i|^2 .
$$

Therefore, we estimate the right-hand side of (4.6) as follows:

$$
\begin{aligned}
\text{RHS of (4.6)} := -\int_P \sum_{i=1}^{p} \langle V_i, W_i \rangle v_g \\
\leq \varepsilon \int_P \sum_{i=1}^{p} |V_i|^2 v_g + \frac{1}{\varepsilon} \int_P \sum_{i=1}^{p} |W_i|^2 v_g .
\end{aligned}
\tag{4.8}
$$

(*Step 5*) By putting $\varepsilon = \frac{1}{2}$ we have

$$\int_P \eta^2 \sum_{i=1}^p |\overline{\nabla}_{e_i} \tau(\pi)|^2 v_g \leq \frac{1}{2} \int_P \sum_{i=1}^p \eta^2 |\overline{\nabla}_{e_i} \tau(\pi)|^2 v_g + 2 \int_P \sum_{i=1}^p e_i(\eta)^2 |\tau(\pi)|^2 v_g. \tag{4.9}$$

Therefore, we obtain

$$\frac{1}{2} \int_P \eta^2 \sum_{i=1}^p |\overline{\nabla}_{e_i} \tau(\pi)|^2 v_g \leq 2 \int_P |\nabla \eta|^2 |\tau(\pi)|^2 v_g. \tag{4.10}$$

Substituting (4.1) into (4.10), we obtain

$$\int_P \eta^2 \sum_{i=1}^p |\overline{\nabla}_{e_i} \tau(\pi)|^2 v_g \leq 4 \int_P |\nabla \eta|^2 |\tau(\pi)|^2 v_g \leq \frac{16}{r^2} \int_P |\tau(\pi)|^2 v_g. \tag{4.11}$$

(*Step 6*) Taking the limit as $r \to \infty$, by the completeness of (P, g) and $E_2(\pi) = \frac{1}{2} \int_P |\tau(\pi)|^2 v_g < \infty$ we obtain that

$$\int_P \sum_{i=1}^p |\overline{\nabla}_{e_i} \tau(\pi)|^2 v_g = 0, \tag{4.12}$$

which implies that

$$\overline{\nabla}_X \tau(\pi) = 0 \quad (\forall X \in \mathfrak{X}(P)). \tag{4.13}$$

(*Step 7*) Therefore, we obtain

$$|\tau(\pi)| \text{ is constant, say } c, \tag{4.14}$$

because

$$X|\tau(\pi)|^2 = 2\langle \overline{\nabla}_X \tau(\pi), \tau(\pi) \rangle = 0 \quad (\forall X \in \mathfrak{X}(P))$$

by (4.13).

(*Step 8*) In the case that $\mathrm{Vol}(P, g) = \infty$ and $E_2(\pi) < \infty$, c must be zero since if $c \neq 0$, then

$$E_2(\pi) = \frac{1}{2} \int_P |\tau(\pi)|^2 v_g = \frac{c}{2} \mathrm{Vol}(P, g) = \infty,$$

a contradiction.

Thus, if $\mathrm{Vol}(P, g) = \infty$, then $c = 0$, that is, $\pi : (P, g) \to (M, h)$ is harmonic.

(*Step 9*) In the case $E(\pi) < \infty$ and $E_2(\pi) < \infty$, let us define the 1-form $\alpha \in A^1(P)$ on P by

$$\alpha(X) := \langle d\pi(X), \tau(\pi) \rangle \quad (X \in \mathfrak{X}(P)). \tag{4.15}$$

Then, we obtain

$$\int_P |\alpha| v_g = \int_P \left(\sum_{i=1}^p |\alpha(e_i)|^2 \right)^{1/2} v_g \leq \int_P |d\pi| |\tau(\pi)| v_g$$

$$\leq \left(\int_P |d\pi|^2 v_g\right)^{1/2} \left(\int_P |\tau(\pi)|^2 v_g\right)^{1/2}$$
$$= 2\sqrt{E(\pi)E_2(\pi)} < \infty. \tag{4.16}$$

For the function $\delta\alpha := -\sum_{i=1}^{p} (\nabla_{e_i}\alpha)(e_i) \in C^\infty(P)$, we have

$$\begin{aligned}
-\delta\alpha &= \sum_{i=1}^{p} (\nabla_{e_i}\alpha)(e_i) = \sum_{i=1}^{p} \{e_i(\alpha(e_i)) - \alpha(\nabla_{e_i} e_i)\} \\
&= \sum_{i=1}^{p} \{e_i \langle d\pi(e_i), \tau(\pi)\rangle - \langle d\pi(\nabla_{e_i} e_i), \tau(\pi)\rangle\} \\
&= \sum_{i=1}^{p} \{\langle \overline{\nabla}_{e_i}\, d\pi(e_i), \tau(\pi)\rangle + \langle d\pi(e_i), \overline{\nabla}_{e_i}\tau(\pi)\rangle - \langle d\pi(\nabla_{e_i} e_i, \tau(\pi)\rangle\} \\
&= \left\langle \sum_{i=1}^{p} \{\overline{\nabla}_{e_i}\, d\pi(e_i) - d\pi(\nabla_{e_i} e_i)\}, \tau(\pi)\right\rangle + \sum_{i=1}^{p} \langle d\pi(e_i), \overline{\nabla}_{e_i}\tau(\pi)\rangle \\
&= \langle \tau(\pi), \tau(\pi)\rangle + \langle d\pi, \overline{\nabla}\tau(\pi)\rangle \\
&= |\tau(\pi)|^2
\end{aligned} \tag{4.17}$$

since $\overline{\nabla}\tau(\pi) = 0$. By (4.17) we obtain

$$\int_P |\delta\alpha| v_g = \int_P |\tau(\pi)|^2 v_g = 2E_2(\pi) < \infty. \tag{4.18}$$

By (4.16), (4.18), and the completeness of (P, g), we can apply Gaffney's theorem, which implies that

$$0 = \int_P (-\delta\alpha) v_g = \int_P |\tau(\pi)|^2 v_g. \tag{4.19}$$

Thus, we obtain

$$\tau(\pi) = 0, \tag{4.20}$$

that is, $\pi : (P, g) \to (M, h)$ is harmonic. We obtain Theorem 2.4.

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