Michigan Math. J. 67 (2018), 743–756

Extactic Divisors for Webs and Lines on Projective Surfaces

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Abstract. Given a web (multifoliation) and a linear system on a projective surface, we construct divisors cutting out the locus where some element of the linear system has abnormal contact with the leaf of the web. We apply these ideas to reobtain a classical result by Salmon on the number of lines on a projective surface. In a different vein, we investigate the numbers of lines and disjoint lines contained in a projective surface and tangent to a contact distribution.

1. Introduction

1.1. Extactic Divisors

A smooth point x of a (germ of) plane curve $C \subset \mathbb{P}^2$ is an extactic point of order n if there exists a plane curve of degree n that intersects C with multiplicity $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))$ at x. For example, inflection points are extactic points of order one, and the extactic points of order two are sextactic points. In the literature, the extactic points of order n are also called n-flexes.

In [15], we can find a construction of divisors on $\mathbb{P}^2$ attached to a foliation $\mathcal{F}$ that intersect the leaves of $\mathcal{F}$ precisely at the extactic points. These are called the extactic divisors of $\mathcal{F}$. The nth extactic divisor of $\mathcal{F}$ is defined whenever the general leaf of $\mathcal{F}$ is not contained in an algebraic curve of degree at most n. Of course, if a curve C is contained in an algebraic curve of degree d, then every point of C is an extactic point of order $n \geq d$. In particular, the nth extactic divisor of a foliation $\mathcal{F}$ contains all $\mathcal{F}$-invariant algebraic curves of degree at most n. The extactic divisors proved to be useful in the study of the Liouvillian integrability of polynomial differential equations; see, for instance, [2] and [3].

In Section 3, we revisit the construction of extactic divisors and reformulate it using the language of invariant jet differentials. Moreover, we show how to extend this construction to webs on surfaces. It is worth noting that the inflection divisor for webs on $\mathbb{P}^2$ was treated before by the first author in his Phd thesis.

Theorem A. *Let $\mathcal{W}$ be a d-web of degree r on $\mathbb{P}^2$. If the number of algebraic curves of degree at most n invariant by $\mathcal{W}$ is finite, then the nth extactic divisor of*

Received December 14, 2016. Revision received August 23, 2017.

$\mathcal{W}$ *has degree*

$$\frac{n}{8}\cdot[(n+1)(n+2)(4d+(n+3)(r-d))$$
$$+(n+3)(n^2+3n-2)(d-1)(d+2r)].$$

As in [16], we carry the construction of the extactic divisor on arbitrary surfaces and for arbitrary linear systems. As an application, we give in Section 4 a proof of a classical result of Salmon, which provides a bound for the number of lines on projective surfaces contained in $\mathbb{P}^3$.

1.2. Involutive Lines on Projective Surfaces

Questions in experimental physics lead one of us to the study of involutive lines (lines tangent to a contact structure) contained in surfaces in $\mathbb{P}^3$; see [12]. In Section 5, we study the numbers of involutive lines in a surface $S\subset\mathbb{P}^3$.

If we set $\ell_i(d)$ as the number of involutive lines a degree d smooth surface in $\mathbb{P}^3$ can have, then our bound takes the following form.

THEOREM B. *If $S\subset\mathbb{P}^3$ is a smooth surface of degree $d\geq 3$ in $\mathbb{P}^3$, then the number of involutive lines in S is at most $3d^2-4d$, that is, $\ell_i(d)\leq 3d^2-4d$. Moreover,*

$$\overline{\ell_i}=\limsup_{d\to\infty}\frac{\ell_i(d)}{d^2}\in[1,3].$$

The study of collections of pairwise disjoint lines (skew lines) on projective surfaces is much more recent. Set $s\ell(d)$ as the number of skew-lines a smooth surface of degree d can have. Miyaoka [14] proved that $s\ell(d)\leq 2d(d-2)$ when $d\geq 4$. There are quartics containing 16 skew lines (Kummer surfaces), and thus $s\ell(4)=16$. Rams [17] exhibited examples of smooth surfaces that imply $s\ell(d)\geq d(d-2)+2$ for $d\geq 6$; and Boissière and Sarti [1] improved Rams' lower bound to $s\ell(d)\geq d(d-2)+4$ when d is odd and greater than or equal to 7.

In the higher-dimensional case, Debarre [5] and Starr [18] (independently) proved that smooth hypersurfaces in $\mathbb{P}^{2m+1}$ contain a finite number of linearly embedded $\mathbb{P}^m$, m-planes for short. Starr (loc. cit.) observes that there is a naive upper bound that grows like $d^{(m+1)^2}$ and suggests that there should exist a polynomial bound with leading term $((3m+1)!/2-1)d^{m+1}$.

The problem of bounding the number of pairwise disjoint m-planes does not seem to be studied so far. Concerning pairwise disjoint involutive m-planes, we prove the following bound.

THEOREM C. *If $X\subset\mathbb{P}^{2m+1}$ is a smooth hypersurface of degree $d\geq 3$ in $\mathbb{P}^{2m+1}$, then the maximal number of pairwise disjoint involutive m-planes in X is at most $(d-1)^{m+1}+1$. Moreover, when X is a surface in $\mathbb{P}^3$, that is, $m=1$, the bound is sharp.*

When $m>2$, we do not know if the bound is sharp.

2. Jet Differentials

2.1. Jet Spaces

Let (X, V) be a directed complex manifold, that is, X is a manifold, and $V \subset TX$ is a subbundle of the tangent bundle of X. The space of k-jets of germs of curves tangent to V, denoted by $J_k V$, is by definition the set of equivalence classes $j_k(f)$ of germs of curves $f : (\mathbb{C}, 0) \to X$ that are everywhere tangent to V (i.e., $f'(t) \in V$ for every $t \in (\mathbb{C}, 0)$) modulo the equivalence relation $f \sim g$ if and only if all the derivatives of f and g at 0 coincide up to order k. The space $J_k V$ is a vector bundle over X of rank $k \operatorname{rank} V$.

Notice that $J_1 V$ is naturally isomorphic to (the total space of) V.

2.2. Jet Differentials

Let us recall the definition of jet differentials of order k and degree m introduced in [8]; for more detail, the reader can also see [7]. These are sections of vector bundles $\mathrm{E}^{\mathrm{GG}}_{k,m}(V^*)$ over X with fibers equal to the space of polynomials $Q(f', f'', \ldots, f^{(k)})$ over the fibers of $J^k V$ of weighted degree m with respect to the $\mathbb{C}^*$-action

$$\lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).$$

For i ranging from 0 to $[m/k]$, set S_i as the set of polynomials having degree with respect to $f^{(k)}$ at most i. Since the degree of Q with respect to $f^{(k)}$ is at most $[m/k]$, we have the following filtration for $\mathrm{E}^{\mathrm{GG}}_{k,m}(V^*)$:

$$\mathrm{E}^{\mathrm{GG}}_{k-1,m}(V^*) = S_0 \subset S_1 \subset \cdots \subset S_{[m/k]} = \mathrm{E}^{\mathrm{GG}}_{k,m}(V^*).$$

For $1 \le i \le [m/k]$, the quotient S_i/S_{i-1} is isomorphic to $\mathrm{E}^{\mathrm{GG}}_{k-1,m-ki}(V^*) \otimes \operatorname{Sym}^i V^*$. Since $\mathrm{E}^{\mathrm{GG}}_{1,m}(V^*)$ is nothing but $\operatorname{Sym}^m V^*$, we can proceed inductively to obtain a filtration of $\mathrm{E}^{\mathrm{GG}}_{k,m}(V^*)$ satisfying

$$\operatorname{Grad}^\bullet \mathrm{E}^{\mathrm{GG}}_{k,m}(V^*) = \bigoplus_{i_1+2i_2+\cdots+ki_k=m} \operatorname{Sym}^{i_1} V^* \otimes \cdots \otimes \operatorname{Sym}^{i_k} V^*.$$

2.3. Action of Jet Differentials on Vector Fields

Let $\mathcal{L}$ be a line bundle over X, and let $\sigma \in H^0(X, \mathrm{E}^{\mathrm{GG}}_{k,m}(V^*) \otimes \mathcal{L})$ be a jet differential of order k and degree m with coefficients in $\mathcal{L}$. Given a holomorphic vector field $v \in H^0(X, V) \subset H^0(X, TX)$ everywhere tangent to V, we can define the action of σ on v as follows. For any point $x \in X$, there exists a unique germ $f_x : (\mathbb{C}, 0) \to (X, x)$ such that $v(f_x(t)) = f_x'(t)$. We set $\sigma(v)$ as the section of $\mathcal{L}$ that at $x \in X$ is obtained by applying σ to $j_k(f_x(t))$.

For any complex number $\lambda \in \mathbb{C}$, we have that $\sigma(\lambda v) = \lambda^m \sigma(v)$. However, if $a \in H^0(X, \mathcal{O}_X)$ is a nonconstant function (of course, we assume that X is not compact here), then there is no obvious similar relation between $\sigma(av)$ and $a^m \sigma(v)$ when $k \ge 2$.

2.4. Multiplication and Differentiation of Jet Differentials

Given two jet differentials $\sigma_i \in H^0(X, \mathrm{E}^{\mathrm{GG}}_{k_i,m_i}(V^*))$, we can multiply them to obtain an element of $H^0(X, \mathrm{E}^{\mathrm{GG}}_{k,m}(V^*))$ with $k = \max\{k_1, k_2\}$ and $m = m_1 + m_2$ that sends the kth jet $j_k(f)$ to $\sigma_1(j_{k_1}(f)) \cdot \sigma_2(j_{k_2}(f))$. Thus we have an $\mathcal{O}_X$-linear (commutative) multiplication morphism

$$\mathrm{E}^{\mathrm{GG}}_{k_1,m_1}(V^*) \otimes_{\mathcal{O}_X} \mathrm{E}^{\mathrm{GG}}_{k_2,m_2}(V^*) \longrightarrow \mathrm{E}^{\mathrm{GG}}_{k,m}(V^*).$$

There is also a linear differentiation morphism of $\mathbb{C}$-sheaves (cf. [8])

$$D : \mathrm{E}^{\mathrm{GG}}_{k,m}(V^*) \longrightarrow \mathrm{E}^{\mathrm{GG}}_{k+1,m+1}(V^*),$$
$$\sigma \longmapsto \left(j_{k+1}(f) \mapsto \frac{d}{dt}\sigma(j_k(f)) \right).$$

In terms of the action of jet differentials on vector fields, we have that

$$(D\sigma)(v) = v(\sigma(v)).$$

Notice that D is not $\mathcal{O}_X$-linear but satisfies the Leibniz rule

$$D(\sigma_1 \cdot \sigma_2) = \sigma_1 \cdot D\sigma_2 + \sigma_2 \cdot D\sigma_1.$$

2.5. Invariant Jet Differentials

Demailly [6] defined a subbundle $\mathrm{E}_{k,m}(V^*)$ of the bundle of jet differentials $\mathrm{E}^{\mathrm{GG}}_{k,m}(V^*)$ whose sections consist of jet differentials of order k and degree m, which are invariant by reparameterizations tangent to the identity. More explicitly, a jet differential σ is an invariant jet differential if and only if at every point $x \in X$ it satisfies

$$\sigma(j_k(f)) = \sigma(j_k(f \circ \varphi))$$

for any germ $f : (\mathbb{C}, 0) \to (X, x)$ and any germ of diffeomorphism $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ with $\varphi'(0) = 1$. The sections of $\mathrm{E}_{k,m}(V^*)$ are called invariant jet differentials of order k and degree m.

2.6. Action on Foliations

Since invariant jet differentials are jet differentials by definition, they act on vector fields as explained in Section 2.3. Given an invariant jet differential with coefficients in a line bundle $\mathcal{L}$, say $\sigma \in H^0(X, \mathrm{E}_{k,m}(V^*) \otimes \mathcal{L})$, its invariance under reparameterizations implies that, for any $v \in H^0(X, V) \subset H^0(X, TX)$ and any holomorphic function $f \in H^0(X, \mathcal{O}_X)$ (constant or not), we have the identity

$$\sigma(fv) = f^m \sigma(v) \in H^0(X, \mathcal{L}).$$

In fact, if $g : (\mathbb{C}, 0) \to (X, x)$ is such that $g'(t) = v(g(t))$, then the integral curve of fv through x is a reparameterization $g \circ \varphi$ with $\varphi'(t) = f(g(t))$. Therefore, σ acts not only on vector fields but also on vector fields with coefficients in line bundles. If v now belongs to $H^0(X, V \otimes \mathcal{M})$, where $\mathcal{M}$ is an arbitrary line bundle, then

$$\sigma(v) \in H^0(X, \mathcal{L} \otimes \mathcal{M}^{\otimes m}).$$

2.7. Action on Webs (When Rank V Is Equal to 2)

Suppose now that V has rank two. If $\sigma \in H^0(X, \mathrm{E}_{k,m}(V^*) \otimes \mathcal{L})$ is an invariant jet differential, then given a symmetric vector field $v \in H^0(X, \mathrm{Sym}^d V \otimes \mathcal{M})$ with coefficients in a line bundle $\mathcal{M}$, we can define the action of σ on v as follows. Over a generic point, the symmetric vector field v can be locally written as the product of d vector fields $v_1, \dots, v_d$: $v = v_1 \cdots v_d$. The set Δ where this decomposition is not possible is a closed analytic subset called the discriminant of v; it is described locally by the vanishing of the classical discriminant of homogeneous binary forms. The local decomposition is of course not unique since we may replace v_i by $a_i v_i$, where a_i are holomorphic functions satisfying $\prod a_i = 1$. Nevertheless, we can choose one such decomposition and set

$$\sigma(v) = \prod \sigma(v_i).$$

Since $\sigma(a_i v_i) = a_i^m \sigma(v_i)$, it follows that a different decomposition of v leads to the same result. However, this expression does not make sense a priori at the analytic subset Δ where the decomposition of v in a product of vector fields fails to exist. Nevertheless, it is clear that the result can be extended meromorphically through Δ. Therefore, if σ is an element of $H^0(X, \mathrm{E}_{k,m}(V^*) \otimes \mathcal{L})$ and $v \in H^0(X, \mathrm{Sym}^d V \otimes \mathcal{M})$, then $\sigma(v)$ is a meromorphic section of $\mathcal{L}^{\otimes d} \otimes \mathcal{M}^{\otimes m}$.

3. Extactic Divisors

3.1. Extactic Divisors for Foliations

Let X be a projective manifold, and a let $\mathcal{N}$ be a line bundle on X. Consider a linear system $|W| \subset \mathbb{P}H^0(X, \mathcal{N})$ of dimension $k \geq 1$ on X defined by sections of $\mathcal{N}$, that is, $W \subset H^0(X, \mathcal{N})$ is a vector space of dimension $k+1$. For any germ $f : (\mathbb{C}, 0) \to X$, define

$$\sigma_W(f) = \det \begin{pmatrix} f_0(t) & f_1(t) & \cdots & f_k(t) \\ f_0'(t) & f_1'(t) & \cdots & f_k'(t) \\ \vdots & & & \\ f_0^{(k)}(t) & f_1^{(k)}(t) & \cdots & f_k^{(k)}(t) \end{pmatrix},$$

where $f_i(t) = s_i(f(t))$ for functions $s_0, \dots, s_k$ expressing a basis of W in a trivialization of $\mathcal{N}$ at a neighborhood of $f(0)$. Changing the open set on X, we see that these local expressions patch together to give an invariant jet differential of order k and degree $m = k(k+1)/2$ with coefficients in $\mathcal{N}^{\otimes k+1}$, that is, σ_W can be interpreted as an element of $H^0(X, \mathrm{E}_{k,m}(T^*X) \otimes \mathcal{N}^{\otimes k+1})$.

Alternatively, we can interpret σ_W as follows. On any projective space $\mathbb{P}^k$ we have a natural invariant jet differential

$$\sigma \in H^0(\mathbb{P}^k, \mathrm{E}_{k,k(k+1)/2}(\Omega^1_{\mathbb{P}^k}) \otimes K^*_{\mathbb{P}^k})$$

defined as follows. Let $\gamma : (\mathbb{C}, 0) \to \mathbb{P}^k$ be a germ and consider an arbitrary lifting $\hat{\gamma} : (\mathbb{C}, 0) \to \mathbb{C}^{k+1} - \{0\}$ under the natural projection $\pi : \mathbb{C}^{k+1} - \{0\} \to \mathbb{P}^k$. The jet differential σ maps γ to $\pi_* \hat{\gamma}'(t) \wedge \pi_* \hat{\gamma}''(t) \wedge \cdots \wedge \pi_* \hat{\gamma}^{(k)}(t) \in \gamma^* \bigwedge^k T\mathbb{P}^k \simeq$

$\gamma^* K^*_{\mathbb{P}^k}$. If we now consider the rational map $\varphi : X \dashrightarrow \mathbb{P}W^* \simeq \mathbb{P}^k$ associated with $|W|$, then σ_W is nothing but the pull-back of σ under φ.

PROPOSITION 3.1. *If $f : (\mathbb{C}, 0) \to X$ is a nonconstant germ such that $\sigma_W(f(t))$ vanishes identically, then the image of any representative of f is contained in an element of the linear system W.*

Proof. If $\sigma_W(f(t)) = 0$, then the local functions $f_0(t), \dots, f_k(t)$ have zero Wronskian and therefore are linearly dependent. □

Given a foliation by curves $\mathcal{F}$ defined by a vector field $v \in H^0(X, TX \otimes T^*\mathcal{F})$, the zero divisor of $\sigma_W(v) \in H^0(X, \mathcal{N}^{\otimes k+1} \otimes (T^*\mathcal{F})^{\otimes m})$ (when different from X) is called in [15] the extactic divisor of $\mathcal{F}$ with respect to the linear system W.

PROPOSITION 3.2. *With the previous notation, if $\sigma_W(v)$ vanishes identically, then every leaf of $\mathcal{F}$ is contained in an element of the linear system $|W|$, and there exists a nonconstant rational function $h \in \mathbb{C}(X)$ constant along the leaves of $\mathcal{F}$.*

Proof. See [15, Theorem 3]. □

3.2. Extactic Divisors for Webs on Surfaces

We suppose in this part that X is a surface; in particular, the bundle TX has rank two.

LEMMA 3.3. *Let $\mathcal{M}$ be a line bundle on X. If v is a holomorphic section of* $\mathrm{Sym}^d\, TX \otimes \mathcal{M}$, *then the discriminant of v is a section $\Delta(v)$ of $K_X^{-d(d-1)} \otimes \mathcal{M}^{\otimes 2(d-1)}$, that is,*

$$\Delta(v) \in H^0(X, \det(TX)^{d(d-1)} \otimes \mathcal{M}^{\otimes 2(d-1)}).$$

Proof. See [16, Section 1.3.4]. □

The following lemma is valid even if X is not a surface since rank V is equal to 2. Recall that in this case a symmetric vector field can be locally written as the product of vector fields $v = v_1 \cdots v_d$.

LEMMA 3.4. *Let X be a projective manifold, and let $|W| \subset \mathbb{P}H^0(X, \mathcal{N})$ be a linear system of dimension $k \geq 1$. If σ_W is the associated jet differential and $v \in H^0(X, \mathrm{Sym}^d\, V \otimes \mathcal{M})$ is a symmetric vector field with $\Delta(v) \neq 0$, then the polar divisor of $\prod \sigma_W(v_i)$ satisfies*

$$\left(\prod \sigma_W(v_i)\right)_\infty \leq \frac{k(k-1)}{2}(\Delta(v))_0.$$

Proof. Let H be an irreducible component of the discriminant of v. At a neighborhood $U \simeq \mathbb{D}^n$ of a sufficiently general point $p \in H$, we can choose local holomorphic coordinates $x_1, \dots, x_{n-1}, y$ such that $\Delta(v) = y^\delta \times u$ where $\delta \geq 1$ is an integer and u is an unity of $\mathcal{O}_{X,p}$. Hence $H = \{y = 0\}$. Since V has rank two, we can decompose v at a neighborhood of p as $w \times v_1 \cdots v_{d-r}$ where w is a local section of $\mathrm{Sym}^r V$ and $v_1, \dots, v_{d-r}$ are local sections of V. We can further assume that w is indecomposable, that is, cannot be expressed as a product of $w_1 w_2$ with w_1 and w_2 sections of strictly positive symmetric powers of V. In this case the order of the discriminant of w along H is at least $r-1$, that is, $r-1 \leq \delta$. Since $r \geq 2$, we also have that $r/2 \leq \delta$.

Notice that the fundamental group of $U - H \simeq \mathbb{D}^{n-1} \times \mathbb{D}^*$ is $\mathbb{Z}$. We can thus choose generators e_1, e_2 of V and write

$$w = \prod_{i=1}^{r} (a_i(x, y^{1/r})e_1 + b_i(x, y^{1/r})e_2)$$

for suitable holomorphic functions a_i, b_i. Set $w_i = a_i(x, y^{1/r})e_1 + b_i(x, y^{1/r})e_2$. We claim that

$$\mathrm{ord}_H(\sigma_W(w_i)) \geq (1 + 2 + \cdots + (k-1))\left(\frac{1}{r} - 1\right) = \frac{k(k-1)}{2}\left(\frac{1}{r} - 1\right).$$

Indeed, for the two first rows of the matrix used to compute $\sigma(w_i)$, the order along H is nonnegative. The chain rule shows that on the third row, a monomial $y^{1/r-1}$ will appear. By the product rule the fourth row will be a linear combination of the third row multiplied by $y^{1/r-2}$ and another expression involving the monomial $y^{2(1/r-1)}$. The multiple of the third row will be disregarded when taking determinants. The claim follows by induction.

Therefore the order of $\prod(\sigma_W(w_i))$ along H is at least

$$r\frac{k(k-1)}{2}\left(\frac{1}{r} - 1\right) = \frac{k(k-1)}{2}(1-r) \geq -\frac{k(k-1)}{2}\delta. \tag{3.1}$$

The lemma follows. □

Remark 3.5. Let $\mathcal{W}_2$ be a 2-web on $(\mathbb{C}^2, 0)$ having reduced discriminant equal to $C = \{y = 0\}$. If C is not invariant by $\mathcal{W}_2$, then in suitable coordinates, $\mathcal{W}_2$ is defined by $v_2 = \frac{\partial}{\partial y}^2 - y\frac{\partial}{\partial x}^2$; see [13, Lemma 2.1]. If we consider the linear system $|W|$ of dimension k locally generated by $1, y, x, y^2, xy, x^2, \dots, x^{s-2}y^2, x^{s-1}y$, x^s when $k+1 = 3s$, then the polar divisor of $\sigma_W(v_2)$ has order exactly $k(k-1)/2$ over C. In fact, we have the local decomposition $v_2 = u \cdot v$, where $u = \frac{\partial}{\partial y} - \sqrt{y}\frac{\partial}{\partial x}$ and $v = \frac{\partial}{\partial y} + \sqrt{y}\frac{\partial}{\partial x}$, so over a general solution of v, say $f(t) = (2t^3/3, t^2)$, we see that $f^*(v) = \frac{1}{t}\frac{\partial}{\partial t}$ and that f^*W is generated by $1, t^2, t^3, \dots, t^{3s-1}, t^{3s}$. A simple computation shows that $\sigma_W(f)$ has a pole at $t = 0$ of order exactly $k(k-1)/2$. Since f ramifies over C, we see that v contributes to the order of $\sigma_W(v_2)$ along C with $-k(k-1)/4$. Since the other factor of v_2 also contributes with $-k(k-1)/4$, we deduce that $\sigma_W(v_2)$ has a pole of order exactly $k(k-1)/2$ along C. For the cases $k+1 = 3s-1$ or $3s-2$, we consider the linear systems $1, y, x, y^2, xy, x^2, \dots, x^{s-2}y^2, x^{s-1}y$ and $1, y, x, y^2, xy, x^2, \dots, x^{s-2}y^2$, respectively.

DEFINITION 3.6. The extactic divisor of a web $\mathcal{W} = [v] \in \mathbb{P}H^0(X, \mathrm{Sym}^d TX \otimes \mathcal{M})$ with respect to a linear system $|W|$ of dimension k is the divisor $\mathcal{E}(\mathcal{W}, |W|)$ defined by the vanishing of

$$\Delta(v)^{k(k-1)/2} \cdot \sigma_W(v).$$

In the case $X = \mathbb{P}^2$, for $\mathcal{W} = [v] \in \mathbb{P}H^0(\mathbb{P}^2, \mathrm{Sym}^d T\mathbb{P}^2(r-d))$ a d-web of degree r, we define the n-extactic divisor of $\mathcal{W}$, denoted by $\mathcal{E}_n(\mathcal{W})$, as the extactic divisor of the web with respect to the linear system $|W| = \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))$.

THEOREM 3.7 (Theorem A of Introduction). *Let $\mathcal{W}$ be a d-web of degree r on $\mathbb{P}^2$. Then $\mathcal{E}_n(\mathcal{W})$, when different from $\mathbb{P}^2$, is a curve of degree*

$$\frac{n}{8} \cdot [(n+1)(n+2)(4d+(n+3)(r-d)) + (n+3)(n^2+3n-2)(d-1)(d+2r)].$$

Proof. The discriminant of the web is given by a section of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}((d-1)(d+2r)))$ according to Lemma 3.3. On the other hand, our linear system has dimension $k = n(n+3)/2$, and in particular the associated jet differential has degree $m = n(n+1)(n+2)(n+3)/8$. Therefore, $\sigma_W(v)$ is a meromorphic section of $\mathcal{O}_{\mathbb{P}^2}(n)^{\otimes(k+1)d} \otimes \mathcal{O}_{\mathbb{P}^2}(r-d)^{\otimes m}$. Finally, the extactic divisor $\mathcal{E}_n(\mathcal{W})$ is cut out by a section of $\mathcal{O}_{\mathbb{P}^2}(n)^{\otimes(k+1)d} \otimes \mathcal{O}_{\mathbb{P}^2}(r-d)^{\otimes m} \otimes \mathcal{O}_{\mathbb{P}^2}((d-1)(d+2r))^{k(k-1)/2}$. □

PROPOSITION 3.8. *Let C be an irreducible curve, and let k_C be the dimension of the restriction to C of a linear system $|W| \subset \mathbb{P}H^0(X, \mathcal{N})$ of dimension k. If C is invariant by a d-web $\mathcal{W}$ defined by $v \in H^0(X, \mathrm{Sym}^d TX \otimes \mathcal{L})$, then*

$$(k - k_C)C \leq \mathcal{E}(\mathcal{W}, |W|).$$

Proof. Suppose first that C is not contained in $\Delta(v)$. Note that $l := k - k_C$ is the number of linearly independent elements of $|W|$ containing C. In a local coordinate system, we can write $C = \{f = 0\}$, $v = v_1 \cdots v_d$, and $s_0, \dots, s_k$ a basis of W. If $l > 0$, we can choose this base such that $s_i = f.\hat{s}_i$ for $i = 0, \dots, l-1$. By hypothesis, we can also assume that $v_1(f) = L.f$ for some holomorphic function L. Therefore,

$$\sigma_W(v_1) = \det \begin{pmatrix} f.\hat{s}_0 & f.\hat{s}_1 & \cdots & f.\hat{s}_{l-1} & \cdots & s_k \\ f.L_{1,0} & f.L_{1,1} & \cdots & f.L_{1,l-1} & \cdots & v_1(s_k) \\ \vdots & & & & & \\ f.L_{k,0} & f.L_{k,1} & \cdots & f.L_{k,l-1} & \cdots & v_1^{(k)}(s_k) \end{pmatrix},$$

which has f^{k-k_C} as a factor. The case where C is contained in $\Delta(v)$ is analogous using the decomposition as in the proof of Lemma 3.4. □

PROPOSITION 3.9. *Let $v \in H^0(X, \mathrm{Sym}^d TX \otimes \mathcal{L})$ be a symmetric vector field with coefficients in $\mathcal{L}$ defining an indecomposable d-web $\mathcal{W}$. If $\sigma_W(v)$ vanishes identically, then every leaf of $\mathcal{W}$ is contained in an element of the linear system $|W|$.*

Proof. Let C be an invariant curve not contained in $\Delta(v)$. Then, in a neighborhood of a generic point, we can decompose $v = v_1 \cdots v_d$. By assumption we have $\sigma_W(v_i) = 0$ for some i, but the transitivity of the monodromy of the web implies that $\sigma_W(v_j) = 0$ for $j = 1 \dots, d$, and therefore C is an element of the linear system. The case where C is contained in $\Delta(v)$ is analogous. □

REMARK 3.10. Propositions 3.8 and 3.9 provide a useful tool to bound the number of curves (in a given linear system) invariant by a given web. This was one of the original motivations to introduce the extactic divisors for foliations; see [15]. Another motivation was to be able to explicitly determine the invariant curves of a given foliation. Unfortunately, in the case of webs, our approach to define the extactic divisors does not provide explicit formulas for them.

4. Lines on Projective Surfaces

4.1. Second Fundamental Form

Let $X \subset \mathbb{P}^N$ be a submanifold. The second fundamental form of X (see [9, Section 1.b]) is a morphism of $\mathcal{O}_X$-modules

$$\mathrm{II} : \mathrm{Sym}^2 TX \longrightarrow NX.$$

If $v \in T_x X$, then $\mathrm{II}(v, v)$ is proportional to the projection on the normal bundle of X at x of the osculating plane of any curve through x with tangent space at x generated by v. Dualizing the morphism II and tensoring the result by NX, we obtain $\omega_{\mathrm{II}} \in H^0(X, \mathrm{Sym}^2 \Omega^1_X \otimes NX)$.

When $X \subset \mathbb{P}^3$ is a nondegenerate surface (i.e., not contained in a plane), the second fundamental form induces a 2-web $\mathcal{W}_{\mathrm{II}}$ on X defined by ω_{II}. Since $\mathrm{Sym}^2 \Omega^1_X \simeq K_X^{\otimes 2} \otimes \mathrm{Sym}^2 TX$, we obtain

$$v_{\mathrm{II}} \in H^0(X, \mathrm{Sym}^2 TX \otimes K_X^{\otimes 2} \otimes NX),$$

defining $\mathcal{W}_{\mathrm{II}}$ as well.

We collect in the next proposition a number of well-known properties of the second fundamental form of surfaces in $\mathbb{P}^3$, which will be useful in what follows.

PROPOSITION 4.1. *Let X be an irreducible surface contained in an open subset of $\mathbb{P}^3$. The following assertions hold.*

(1) *The second fundamental form vanishes identically on X if and only if X is an open subset of a $\mathbb{P}^2$ linearly embedded in $\mathbb{P}^3$.*
(2) *The discriminant $\Delta(v_{\mathrm{II}}) = \Delta(\omega_{\mathrm{II}})$ vanishes identically if and only X is a cone or X is the tangential surface of a curve $C \subset \mathbb{P}^3$.*
(3) *If $i : C \to X \cap \mathbb{P}^2$ is the inclusion of a planar curve satisfying $i^*\omega_{\mathrm{II}} = 0$, then C is a line, or C is contained in the discriminant of ω_{II}*

Proof. Item (1) is proved in [9, (1.51)]. For item (2), see (1.52) loc. cit. Let us prove item (3). Assume that $C \subset X \cap \mathbb{P}^2$ is not a line and satisfies $i^*\omega_{\mathrm{II}} = 0$. Therefore, for a general point of C, its osculating plane is tangent to X. It follows that the intersection of X and the $\mathbb{P}^2$ containing C is nonreduced and, consequently, C is contained in the discriminant of ω_{II}. $\square$

4.2. Salmon's Theorem

THEOREM 4.2. *Let $X \subset \mathbb{P}^3$ be an irreducible surface of degree d. If X is not a ruled surface, then there exists $s \in H^0(X, \mathcal{O}_X(11d-24))$ whose zero divisor is cutting out all lines contained in X. In particular, the number of lines contained in a smooth projective surface of degree $d \geq 3$ is at most $d(11d-24)$.*

Proof. Let $p \in \mathbb{P}^3$ be a general point, and let $|W|$ be the linear system of the restriction to X of hyperplanes containing p. Consider the invariant jet differential associated with $|W|$, $\sigma_W \in H^0(X, \mathrm{E}_{2,3}(T^*X) \otimes \mathcal{O}_X(3))$. The action of σ_W on v_{II} gives us a meromorphic section of $\mathcal{O}_X(3)^{\otimes 2} \otimes (K_X^{\otimes 2} \otimes NX)^{\otimes 3}$, and then the divisor $\mathcal{E}(\mathcal{W}_{\mathrm{II}}, |W|)$ is given by a holomorphic section s of

$$\mathcal{O}_X(3)^{\otimes 2} \otimes (K_X^{\otimes 2} \otimes NX)^{\otimes 3} \otimes \underbrace{K_X^{\otimes 2} \otimes NX^{\otimes 2}}_{\Delta(v_{II})^{2(2-1)/2}} \simeq \mathcal{O}_X(13d-26).$$

Basic properties of the second fundamental form implies that every line contained in X is invariant by it. Proposition 3.8 implies that s vanishes on every line.

Since X is not uniruled by assumption, Proposition 4.1(2) implies that the discriminant of ω_{II} is not identically zero. If s vanishes identically, then Proposition 3.9 implies that through a general point of X there exists a planar curve invariant by $\mathcal{W}_{\mathrm{II}}$. But planar curves invariant by $\mathcal{W}_{\mathrm{II}}$ and not contained in the discriminant of ω_{II} are lines. This contradicts our assumption on the nonuniruledness of X, proving that s is not identically zero.

Consider now the linear projection $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with center at p. Its restriction to X, still denoted by π, has ramification divisor R cut out by $r \in H^0(X, \mathcal{O}_X(d-1))$. We claim that $(s)_0 \geq 2R$. Fix a general point x of R. At a neighborhood of it, write $v_{\mathrm{II}} = w_1 \cdot w_2$. The orbits of w_1, w_2 will have contact of order at least three with the element of $|W|$ tangent to X at x. This is sufficient to show that the rank at x of the matrix defining $\sigma_W(w_1)$ and $\sigma_W(w_2)$ is at most two. The claim follows.

To conclude the proof of Salmon's theorem, it suffices to divide s by r^2 to obtain a holomorphic section of $\mathcal{O}_X(11d-24)$ vanishing along all the lines contained in X. $\square$

The proof is not very different from Salmon's proof. The divisor defined by s coincides with the flecnodal divisor studied by Salmon. Indeed, according to [11, p. 138], every plane containing one of the null directions of II at a point $p \notin \Delta(\mathrm{II})$, except the tangent plane, intersects the surface at a a planar curve having an inflection at p. The vanishing of $\sigma(v_{\mathrm{II}})$ at p implies that, for one of the asymptotic curves through p, the order of contact at p with this planar curve is at least 3. It follows that p is also an inflection for the asymptotic curve. For a modern version of Salmon's argument, see [10, Section 8].

Darboux [4, p. 372] shows that through a general point there are exactly 27 conics (curves) that have abnormal contact with a surface $X \subset \mathbb{P}^3$. Thus there exists a 27-web that is tangent to every conic contained in X. Control on the normal bundle of this web (i.e., a formula linear in the degree of X) would give a bound on the number of conics on a surface.

5. Involutive Lines

5.1. Bound on the Number of Involutive Lines

Consider the projective space $\mathbb{P}^3$ endowed with a contact structure $\mathcal{C}$ induced by a constant symplectic form σ on $\mathbb{C}^4$. If $\sigma = \sum_{i,j=0}^{3} \lambda_{ij} dx_i \wedge dx_j$ is a symplectic form on $\mathbb{C}^4$, then the associated contact structure $\mathcal{C} = \mathcal{C}_\sigma$ is defined by $\omega \in H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(2))$, which in homogeneous coordinates can be written as

$$\omega = i_R \sigma,$$

where $R = \sum_{i=0}^{3} x_i \frac{\partial}{\partial x_i}$ is the radial (or Euler's) vector field, and i_R stands for the interior product with R. A similar construction endows a contact structure over the projective space $\mathbb{P}^{2m+1}$.

A reduced and irreducible curve $C \subset \mathbb{P}^3$ is called an involutive curve (with respect to a contact distribution $\mathcal{C}$) if $i^*\omega \in H^0(C_{sm}, \Omega^1_{C_{sm}}(2))$ vanishes identically. Here C_{sm} denotes the smooth locus of C, and $i : C_{sm} \to \mathbb{P}^3$ is the inclusion.

As in Introduction, let $\ell_i(d)$ be the number of involutive lines a degree d smooth surface in $\mathbb{P}^3$ can have.

THEOREM 5.1 (Theorem B of Introduction). *If $X \subset \mathbb{P}^3$ is a smooth surface of degree $d \geq 3$ in $\mathbb{P}^3$, then the number of involutive lines in X is at most $3d^2 - 4d$, that is, $\ell_i(d) \leq 3d^2 - 4d$. Moreover,*

$$\overline{\ell_i} = \limsup_{d \to \infty} \frac{\ell_i(d)}{d^2} \in [1, 3].$$

Proof. The restriction of the contact form to X gives a nonzero section ω of $\Omega^1_X \otimes \mathcal{O}_X(2)$. Every involutive line contained in X is invariant by the corresponding foliation $\mathcal{F}$.

The tangency locus between $\mathcal{F}$ and $\mathcal{W}_{\mathrm{II}}$ is cut out by a section of

$$K_X^{\otimes 2} \otimes \mathcal{O}_X(2)^{\otimes 2} \otimes NX = \mathcal{O}_X(3d - 4).$$

Since it must contain every involutive line inside X, the first part of the theorem follows. For the last part, take a surface X in $\mathbb{P}^3$ of degree $d \geq 3$ defined by a homogenous polynomial of the form $p(x_0, x_1) + q(x_2, x_3)$. Then X has at least d^2 involutive lines with respect to the contact form $i_R(dx_0 \wedge dx_1 + dx_2 \wedge dx_3)$. In fact, since p and q are binary forms, they can factored as a product of d linear forms, say $p(x_0, x_1) = \prod_{i=1}^{d} p_i(x_0, x_1)$ and $q(x_2, x_3) = \prod_{i=1}^{d} q_i(x_2, x_3)$. The lines $\{p_i(x_0, x_1) = q_j(x_2, x_3) = 0\}$ are all involutive and contained in X. $\square$

REMARK 5.2. As before, let $|W|$ be the linear system of the restriction to X of hyperplanes containing a general point $p \in \mathbb{P}^3$. Then the extatic divisor of $\mathcal{F}$ with respect to $|W|$ contains every involutive line and is given by a section of $\mathcal{O}_X(3) \otimes \mathcal{O}_X(d-2)^{\otimes 3} = \mathcal{O}_X(3d-3)$, which gives us a worst bound for the number of involutive lines.

5.2. Pairwise Disjoint Involutive m-Planes

THEOREM 5.3 (Theorem C of Introduction). *If $X \subset \mathbb{P}^{2m+1}$ is a smooth hypersurface of degree $d \geq 3$ in $\mathbb{P}^{2m+1}$, then the maximal number of pairwise disjoint involutive m-planes in X is at most $(d-1)^{m+1}+1$. Moreover, when $m=1$ and $d \geq 6$, the bound is sharp.*

To prove the theorem, we need the following lemma.

LEMMA 5.4. *If $X \subset \mathbb{P}^{2m+1}$ is a smooth hypersurface of degree at least 3, then pull-back of the contact form $\omega \in H^0(\mathbb{P}^{2m+1}, \Omega^1_{\mathbb{P}^{2m+1}}(2))$ to X vanishes exactly at a subscheme of dimension zero and length $((d-1)^{2m+2}-1)/(d-2)$ isolated singularities.*

Proof. Let $i: X \to \mathbb{P}^{2m+1}$ be the inclusion and suppose $i^*\omega \in H^0(X, \Omega^1_X(2))$ has nonisolated singularities. If $F \in \mathbb{C}[x_0, \dots, x_{2m+1}]$ is a homogeneous polynomial of degree d defining X, then $dF_{|X}$ can be interpreted as sections of $\Omega^1_{\mathbb{P}^{2m+1}}(d)_{|X}$. If Z is a positive-dimensional irreducible component of the singular set of $i^*\omega$, then the restriction of ω to Z must be proportional to the restriction of dF to Z. Since ω has no singularities on $\mathbb{P}^{2m+1}$, we can write $dF_{|Z} = s \cdot \omega_{|Z}$ where $s \in H^0(Z, \mathcal{O}_Z(d-2))$. If $d > 2$, then s vanishes on hypersurface of Z, and the same holds for dF. It follows that X is singular along the zero locus of s, contrary to our assumptions.

It remains to determine the length of the zero scheme of $i^*\omega$, which can be done by computation of the top Chern class of $\Omega^1_X(2)$. If $h = c_1(\mathcal{O}_X(1))$, then the Chern polynomial of $\Omega^1_X(2)$ is given by

$$\frac{c(\Omega^1_{\mathbb{P}^{2m+1}|Y}(2))}{c(\mathcal{O}_X(2-d))} = \frac{(1+h)^{2m+2}}{(1+2h)(1-(d-2)h)},$$

and the top Chern class of $\Omega^1_X(2)$ is $d = h^{2m}$ times the coefficient of h^{2m}. Therefore

$$c_{2m}(\Omega^1_X(2)) = d\left(\sum_{i=0}^{2m}\sum_{j=0}^{2m-i}\binom{2m+2}{i}(-2)^j(d-2)^{2m-i-j}\right).$$

We can verify by induction that this last quantity is equal to $((d-1)^{2m+2} - 1)/(d-2)$, as wanted. □

Proof of Theorem C. Let $j: P \to X$ be a linear inclusion of an m-plane in X, and let $i: X \to \mathbb{P}^{2m+1}$ be the inclusion of X in $\mathbb{P}^{2m+1}$. Consider the exact sequence

$$0 \to N^*P(2) \to \Omega^1_{X|P}(2) \to \Omega^1_P(2) \to 0.$$

Since $j^*i^*\omega = 0$, it follows that $(i^*\omega)_{|P}$ is the image of a certain $\sigma \in H^0(P, N^*P(2))$. The intersection of the singular set of $i^*\omega$ with P coincides (set-theoretically) with the singular set of σ. Moreover, as a simple local computation shows, the length of zero scheme of σ is at least the length of the restriction of the zero scheme of $i^*\omega$ to any neighborhood of P. If N is the number of pairwise disjoint m-planes in X, then we can write

$$Nc_m(N^*P(2)) \le c_{2m}(\Omega^1_X(2)).$$

However, $c(N^*P(2)) = c(\Omega^1_{X|P}(2)) \cdot c(\Omega^1_P(2))^{-1}$, from which we deduce

$$c(N^*P(2)) = \frac{(1+h)^{2m+2}}{(1+2h)(1-(d-2)h)} \cdot \frac{1+2h}{(1+h)^{m+1}} = \frac{(1+h)^{m+1}}{(1-(d-2)h)},$$

where $h = c_1(\mathcal{O}_P(1))$. The coefficient of h^m is exactly $((d-1)^{m+1}-1)/(d-2)$. This computation, together with Lemma 5.4, implies

$$N \le (d-1)^{m+1} + 1,$$

as claimed. This concludes the first part of Theorem C. The following example from [17] will show that the bound is sharp when $m = 1$. Let us consider the surface

$$S_d = \{x_0^{d-1}x_1 + x_1^{d-1}x_2 + x_2^{d-1}x_3 + x_3^{d-1}x_0 = 0\}$$

with $d \ge 6$. Then S_d contains the $d(d-2)+2$ skew lines $(\alpha x : \beta y : x : y)$ for (α, β) satisfying $\alpha = -\beta^{d-1}$, $\beta^{(d-1)^2+1} = (-1)^d$. Moreover, these lines are involutive with respect to the contact form $i_R(dx_0 \wedge dx_2 + dx_1 \wedge dx_3)$. □

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