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Extensions of Some Classical Local Moves on Knot Diagrams

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Abstract. We consider local moves on classical and welded diagrams: (self-)crossing change, (self-)virtualization, virtual conjugation, Delta, fused, band-pass, and welded band-pass moves. Interrelationships between these moves are discussed, and, for each of these moves, we provide an algebraic classification. We address the question of relevant welded extensions for classical moves in the sense that the classical quotient of classical object embeds into the welded quotient of welded objects. As a byproduct, we obtain that all of the local moves mentioned are unknotting operations for welded (long) knots. We also mention some topological interpretations for these combinatorial quotients.

Introduction

Although knot and link theory has its roots and foundations in the topology of embedded circles in 3-space, its study was early turned combinatorial by considering generic projections, which can be seen as decorated 4-valent planar graphs. This opened a new way to think the topology in terms of combinatorial local moves. First, ambient isotopies were proved to correspond to Reidemeister moves [32]. Other equivalence relations were then interpreted as the quotient under some additional local moves (see Figure 1): for general homotopy, we should authorize crossing changes (CC); for link-homotopy, only self-crossing changes (SC) [25]; and link-homology, introduced by Murakami and Nakanishi [27] and Matveev [23], corresponds to Delta moves (Δ). Other local moves were also investigated, still within some topological perspectives, such as the band-pass move (BP), which is motivated by the crossing of two bands, but also from more algebraic or even purely combinatorial considerations. These notions straightforwardly extend to other kinds of knotted objects in dimension 3.

Forgetting the planarity assumption for the decorated 4-valent graphs gives rise to the notion of *virtual links,* introduced from the diagram point of view by Kauffman [19] and from the Gauss diagram point of view by Goussarov, Polyak, and Viro [16]. In this virtual context, two forbidden local moves emerged, the over- and under-commute moves. The *welded* theory, first introduced in the braid

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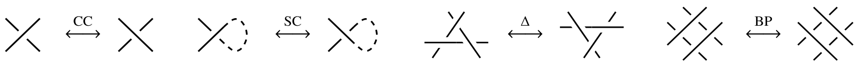


Figure 1 Classical local moves

context by Fenn, Rimányi, and Rourke [13], is defined by allowing one of them, and the *fused* theory, already mentioned by Kauffman [18], by allowing both.

Welded knots and links provide a sensible extension of usual knot theory in the sense that two classical links are equivalent as welded objects if and only if they are classically equivalent. In 2000, Satoh provided another topological motivation for welded knotted objects by generalizing a construction—given forty years earlier, in the classical case, by Yajima [38]—that inflates diagrams into embedded tori in 4-space which bound immersed solid tori with only ribbon singularities. The resulting map, the so-called Tube map, is surjective, but its injectivity remains an intriguing question: false for welded links [37; 2], true for welded braids [8], and undetermined for welded string links.

In [3], the authors used the Tube map to classify ribbon tubes and ribbon 2-torus links—which are a two–dimensional analogue of string links and links—up to link-homotopy. Along the paper, several phenomena emerged:

(i) link-homotopy among ribbon objects is generated by the image through the Tube map of a single local move, namely the self-virtualization (SV), and up to this move, the Tube map is one-to-one;
(ii) as in the classical case, every welded string link is link-homotopic to a welded braid, that is, the map from welded pure braids to welded string links up to self-virtualization is surjective;
(iii) the given classification of welded string links up to self-virtualization is a natural extension of the classification of classical string links up to link-homotopy given by Habegger and Lin [17]. As such, it suggests that self-virtualization is a natural welded extension of the classical self-crossing change in the sense that the embedding of planar 4-valent graphs into general 4-valent graphs induces an embedding of classical string links up to self-crossing change into welded string links up to self-virtualization.

Point (ii) has been developed in [4]. Point (i) is raised at the end of the present introduction, but the paper essentially pushes further the analysis of point (iii) by discussing the welded extensions of the classical Δ and BP moves. In doing so, we define several candidates for such extensions and compare them, carrying on a work initiated in the classical case by Murakami and Nakanishi [27] and Aida [1]. An unexpected outcome is that a given classical local move may admit several distinct welded extensions (see e.g. Proposition 3.15). Specifically, we consider in Figure 2 nonclassical local moves (see Section 2 for details).

We provide in Theorems 2.9 and 2.10 an ordering between the classical and nonclassical local moves. Notice that these results hold for all types of welded knotted objects. Moreover, in the case of links and string links, we provide a complete classification under these moves, as stated further below.

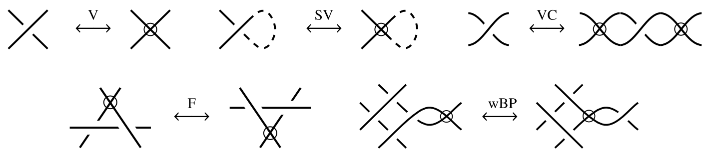


Figure 2 Nonclassical local moves

Recall that links up to Δ moves are classified by the linking numbers [27, Thm. 1.1], whereas links up to BP are classified by the modulo 2 reduction of $\sum_{1\leq k\neq i\leq n} \mathrm{lk}_{ik}$ [27; 22]. The main results of this paper can be summarized as follows.

THEOREM 1.

- *Welded links up to F are classified by the virtual linking numbers.*
- *Welded links up to VC are classified by the $(vlk_{ij} + vlk_{ji})$'s.*
- *Welded links up to CC are classified by the $(vlk_{ij} - vlk_{ji})$'s.*
- *Welded links up to wBP are classified by the vlk^{mod}_{i*}s and the $(vlk^{\mathrm{mod}}_{ij} + vlk^{\mathrm{mod}}_{ji})$'s.*

Here, the *virtual linking number* vlk_{ij} is the welded link invariant that counts, with signs, the crossings where the ith component overpasses the jth component, $\mathrm{vlk}^{\mathrm{mod}}_{ij}$ denotes its modulo 2 reduction, and $\mathrm{vlk}^{\mathrm{mod}}_{i*}$ denotes the modulo 2 reduction of $\sum_{1\leq k\neq i\leq n} \mathrm{vlk}_{ik}$.

As a consequence, we obtain that VC, Δ, F, BP, and wBP are all unknotting operations for welded knots, which recovers and extends a result recently proved by Satoh [36] using a different approach. We actually show the stronger result that these are all unknotting operations for welded long knots. Another consequence is the following extension result.

THEOREM 2.

- *Links up to Δ embed into welded links up to F.*
- *Links up to VC embed into welded links up to F.*
- *Links up to BP embed into welded links up to wBP.*

Note that the classification of welded links up to F has been independently proved in [31] with a completely different and algebraic approach. This completes a previous result of Fish and Keyman [15, Thm. 2] (see also [7, Thm. 3.7] for a shorter proof) stating that fused links with only classical crossings are classified by linking numbers.

To prove these results, we provide algebraic classifications of all the considered local moves for string links. For each of them, we give an explicit group isomorphism between the quotient space of (welded) string links and a power of $\mathbb{Z}$ or $\mathbb{Z}_2$. Our main tool will be the theory of Gauss diagrams, mentioned earlier, which is an even more combinatorial alternative to describe virtual diagrams. Whereas the virtual diagrams are a pleasant tool to picture local operations, Gauss diagrams appear to be more efficient to handle global manipulations. In the present paper, we adopt and use both points of view in parallel.

Let us conclude this introduction with a few comments returning back to topology.

Murakami and Nakanishi [27] introduced a notion of *link-homology,* which can be rephrased as the quotient where two elements are identified whenever the image of each strand in the homology groups of the complement of the other strands are the same. They noted that the classification of links up to Δ by the linking numbers implies that the link-homology is generated by the Δ move or, equivalently, that (string) links up to Δ describe (string) links up to link-homology. Similarly, string links up to self-crossing changes were studied in [17] as the group of string links up to link-homotopy, that is, the topological quotient where each connected component is allowed to cross itself.

As already mentioned, welded (string) links also have a topological interpretation, via Satoh's Tube map [35]. This topological interpretation is however partial since the Tube map is surjective but not injective. Indeed, performing SR, a local move depicted in Figure 7, on each classical crossing and reversing, the orientation on a given link diagram produces another diagram with same image through Tube; see [37, Thm. 3.3] or [2, Prop. 2.7]. It is moreover still unknown whether this move generates all the kernels of Tube for welded links.

In [3], the authors applied the Tube map to welded string links, producing *ribbon tubes.* It is shown in [3, Prop. 3.16] and [5] that SV generates link-homotopy on ribbon tubes and that the Tube map is injective on the quotient, thus producing a full topological interpretation for welded string links up to SV. Furthermore, the virtual linking number vlk_{ij} corresponds to the evaluation of any longitude—that is, any path from one boundary component to the other—of the jth tube in $\mathbb{Z}$ seen as the first homology group of the complement of the ith tube. Since longitudes are the only subspaces of a ribbon tube component that may have a nontrivial image in the homology groups of the complement of the other components, it follows from Proposition 3.6 that welded string links up to F describe faithfully ribbon tubes up to (the natural extension of Murakami and Nakanishi's notion of) link-homology.

Let us mention here that the term "link-homology" is also used in the literature as a synonym for bordance, which is a weakening of the concordance obtained by allowing any cobordism. It appears within the framework of usual knot theory (see [33; 34; 10] for a classification result), but also in the context of virtual knot theory seen as links in thickened surfaces up to isotopy and (de)stabilization. Carter, Kamada, and Saito [9] proved that virtual links up to virtual link-homology are classified by virtual linking numbers. As a byproduct, we obtain that the two forbidden moves generate virtual link-homology.

The paper is organized as follows. In Section 1, the central objects of the paper, classical/welded string links and the notion of local move, are defined from both the virtual and the Gauss diagrams points of view. In Section 2, we introduce all the considered local moves (see Figures 6 and 7) and study their interrelationships. In Section 3, we use virtual linking numbers to provide a complete algebraic classification of welded string links up to each considered local move. These algebraic identifications are then used to discuss welded extensions for classical (self)-crossing changes, Δ and BP moves. Section 4 is built on the string link case to address other kinds of knotted objects such as links and pure braids.

GLOSSARY. Throughout the paper, the various local moves studied in this paper will be denoted by the notation introduced in their defining figures. For the reader's convenience, we furthermore list these various acronyms, their meaning, and the references to their definition:

$Ri;\ i = 1, 2, 3$	Reidemeister move i	Figure 3
$vRi;\ i = 1, 2, 3$	virtual Reidemeister move i	Figure 3
OC	Overcommute move	Figure 3
UC	Undercommute move	Proposition 2.5
CC	Crossing Change	Figures 1 and 6
SC	Self-crossing Change	Figures 1 and 6
Δ	Delta move	Figures 1 and 6
BP	unoriented band-pass move	Figures 1 and 6
V	Virtualization move	Figures 2 and 7
SV	Self-virtualization move	Figures 2 and 7
VC	Virtual conjugation move	Figures 2 and 7
SR	Sign reversal move	Figure 7
F	Fused move	Figures 2 and 7
wBP	welded band-pass move	Figures 2 and 7

We also note here, as a point of convention, that the same acronym will often be used when referring to the equivalence relation on diagrams generated by the corresponding local move.

1. Classical and Welded String Links

We first introduce, in two different but equivalent ways, the main objects of this paper. All along the text, n will be a positive integer.

1.1. Virtual Diagrams

Fix n real numbers $0 < p_1 < \cdots < p_n < 1$.

DEFINITION 1.1. A *virtual string link diagram* is an immersion of n oriented intervals $\bigsqcup_{i\in\{1,\dots,n\}} I_i$ in $I \times I$, called *strands,* such that

- for each $i \in \{1, \dots, n\}$, the strand I_i has boundary $\partial I_i = \{p_i\} \times \{0, 1\}$ and is oriented from $\{p_i\} \times \{0\}$ to $\{p_i\} \times \{1\}$;
- the singular set is a finite number of transverse double points;

- each double point is labeled, either as a *positive crossing,* as a *negative crossing,* or as a *virtual crossing.* Positive and negative crossings are also called *classical crossings.*

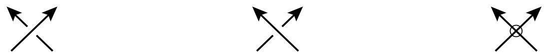
Strands are naturally ordered by the order of their endpoints on either $I \times \{0\}$ or $I \times \{1\}$.

A virtual string link diagram that has no virtual crossing is said to be *classical.*

Up to ambient isotopy and reparameterization, the set of virtual string link diagrams is naturally endowed with a structure of monoid by the stacking product, where the unit element is the trivial diagram $\bigcup_{i\in\{1,\dots,n\}}\{p_i\} \times I$; we denote this monoid by vSLD_n and its submonoid made of classical string link diagrams by SLD_n. We denote by $\iota\colon \mathrm{SLD}_n \hookrightarrow \mathrm{vSLD}_n$ the natural injection.

Crossings where the two preimages belong to the same strand are called *self-crossings.*

We will use the usual drawing convention for crossings:



positive crossing negative crossing virtual crossing

DEFINITION 1.2. A *local move* is a transformation that changes a diagram only inside a disk. It is specified by the contents of the disk, before and after the move. In our context, the contents will be pieces of strands, without any specified orientation, which may classically and virtually cross themselves. If the disk does not contain any virtual crossing neither before nor after the move, then we say that the local move is *classical.*

To represent a local move, we will draw only the disk where the move occurs. Examples are given in Figures 1 and 2.

DEFINITION 1.3. A *string link* is an equivalence class of SLD_n under the three classical Reidemeister moves. We denote by $\mathcal{SL}_n$ the set of string links; it is a monoid with composition induced by the stacking product.

A *welded string link* is an equivalence class of vSLD_n under the welded Reidemeister moves, which are the classical and the virtual Reidemeister moves, together with the mixed and overcommute (OC) moves given in Figure 3. We denote by $\mathrm{w}\mathcal{SL}_n$ the set of welded string links; it is a monoid with composition induced by the stacking product.

Elements of $\mathcal{SL}_1$ and $\mathrm{w}\mathcal{SL}_1$ are also called, respectively, long knots and welded long knots.

String-links can be seen as an intermediate object between braids and links. For convenience, we give short definitions of these objects:

- Compared with string links and welded string links, *pure braids* and *welded pure braids* are defined by requesting, in addition, that the immersed intervals are monotone with respect to the second coordinate. Ambient isotopies are then

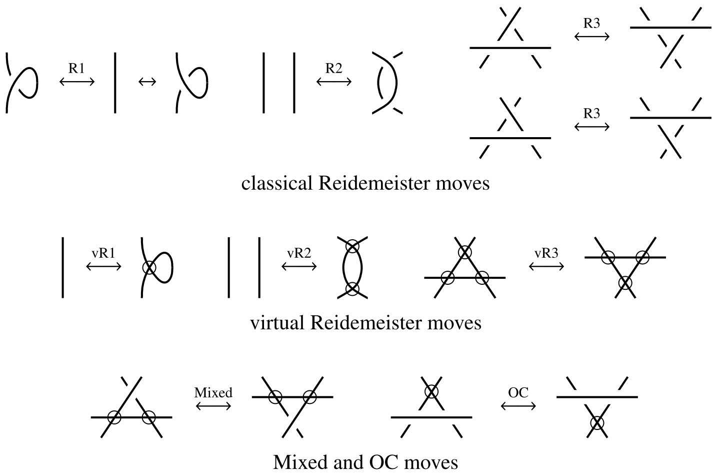


Figure 3 Welded Reidemeister move

also requested to respect this monotony. The stacking product then induces group structures, which we denote, respectively, by P_n and wP_n.

- *Links* and *welded links* are defined by replacing in Definition 1.1 the disjoint union of oriented intervals $\bigsqcup_{i\in\{1,\dots,n\}} I_i$ by a disjoint union of oriented circles $\bigsqcup_{i\in\{1,\dots,n\}} S^1_i$ (ignoring the points p_i). Note that (welded) links are thus implicitly equipped with an enumeration of its connected components. We denote by $\mathcal{L}_n$ and $\mathrm{w}\mathcal{L}_n$ the sets of links and welded links. Elements of $\mathcal{L}_1$ and $\mathrm{w}\mathcal{L}_1$ are also called, respectively, knots and welded knots.

1.2. Gauss Diagrams

Welded string links can by definition be represented by virtual string link diagrams, which can, in turn, be alternatively described in terms of *Gauss diagrams.*

Definition 1.4. A *Gauss diagram* is defined over n ordered and oriented intervals, called *strands,* as a finite set of triplets (t, h, σ), called *arrows,* where t and h, called respectively the *tail* and the *head* of the arrow, are elements of the strands, and $\sigma \in \{\pm 1\}$ is a sign. Tails and heads, also called *endpoints* or *ends,* are all distinct and considered up to orientation-preserving homeomorphisms of the strands.

The strands are represented by parallel upward thick intervals arranged in increasing order, and each arrow by an actual thin arrow, going from its tail to its head, labeled by its sign.

Arrows having both ends on the same strand are called *self-arrows.*

See the right-hand side of Figure 4 for an example.

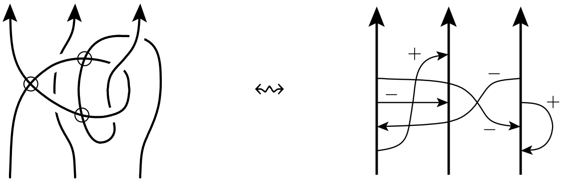


Figure 4 A virtual diagram and the corresponding Gauss diagram

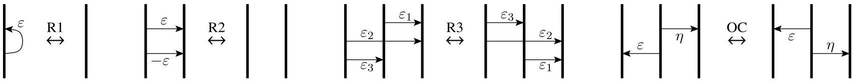


Figure 5 Welded Reidemeister moves on Gauss diagrams. There is a sign condition for applying move R3, namely that $\varepsilon_i \delta_i = \varepsilon_j \delta_j$, where $\delta_k = 1$ if the the kth strand, read from left to right, is oriented upward and -1 otherwise

DEFINITION 1.5. A *local move* is a transformation that changes arrows only on a given finite union of portions of strands. It is specified by the portions with their arrows, before and after the move, and it is assumed that no other arrow has an endpoint on these portions.

To represent a local move, we will draw only the portions where the move occurs by parallel thick intervals, without any specified orientation or ordering. The reader should be aware that when the local move is applied to a given Gauss diagram, the portions should be reordered—and even possibly put one above others if they are part of a same strand—and possibly reversed to get upward; (s)he should also keep in mind that there is a nonrepresented part, which is identical on each side of the move, but that no arrow can connect the nonrepresented part to the represented one. Examples are given in Figure 5 and on the right-hand sides of Figures 6 and 7.

There is a one-to-one correspondence between Gauss diagrams up to ambient isotopy and virtual diagrams up to virtual Reidemeister and mixed moves. It associates a Gauss diagram with any virtual diagram so that the set of positive and negative crossings in the virtual diagram are in one-to-one correspondence with, respectively, the set of $+1$-labeled and -1-labeled arrows in the Gauss diagram. This procedure is, for example, described in [3, Sec. 4.5] and is illustrated in Figure 4. Local moves on virtual diagrams have Gauss diagrams counterparts. Figure 5 gives the Gauss diagram versions of the classical Reidemeister moves. Note that the Gauss diagram counterparts of the virtual Reidemeister and mixed moves are actually trivial since they do not affect any classical crossing. Throughout the paper, we will use indifferently one or the other description.

This correspondence yields a faithful representation of welded string links by Gauss diagrams up to the welded Reidemeister moves depicted in Figure 5.

Welded pure braids and welded links also enjoy Gauss diagram descriptions:

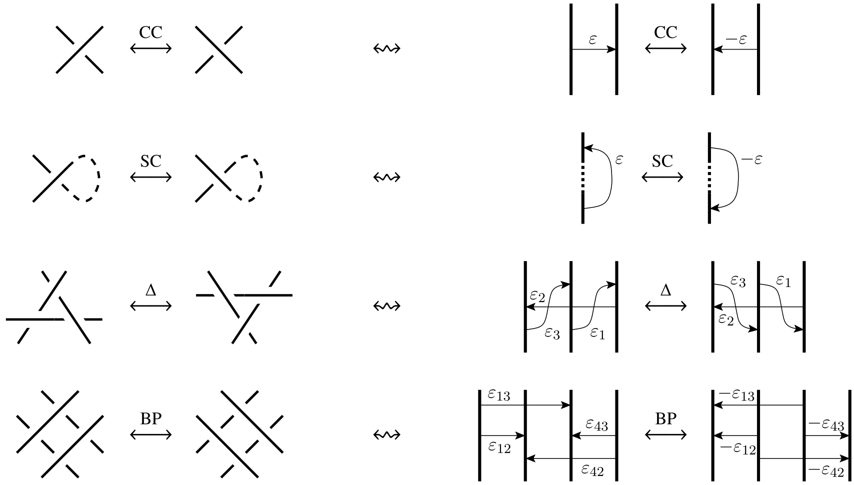


Figure 6 Classical local moves. The Gauss diagram version of move Δ, resp. of BP, is subject to the condition $\varepsilon_i \delta_i = \varepsilon_j \delta_j$, resp. $\varepsilon_{ij}\varepsilon_{kl} = \delta_i \delta_j \delta_k \delta_l$, where $\delta_k = 1$ if the kth strand, read from left to right, is oriented upward and -1 otherwise

- welded pure braids are faithfully represented, up to the welded Reidemeister moves, by Gauss diagrams with only horizontal arrows. This result, usually considered as folklore (see, e.g., [6]), is a consequence of a similar result on virtual braids [11, Prop. 2.24] (see [12] for a complete proof in the welded case);
- replacing intervals by oriented circles in Definition 1.4, we obtain a tool that, up to welded Reidemeister moves, faithfully represents welded links. Gauss diagrams were actually first defined over a single circle to describe knots; see, for instance, [16] or [14].

2. Local Moves and Their Relations

In this section, we introduce several local moves and study their interrelationships.

2.1. Local Moves

In Figures 6 and 7, we introduce the different local moves that we will study in detail. We first consider the classical local moves, which were already presented in Figure 1.

The crossing change CC is certainly the simplest and most natural local move in classical knot theory. Its refinement SC requires the additional self-connectedness condition that the two involved pieces of strand belong to the same strand. Note that, although the modification remains local, checking that the pieces are connected is not. This latter move was introduced by Milnor [25]

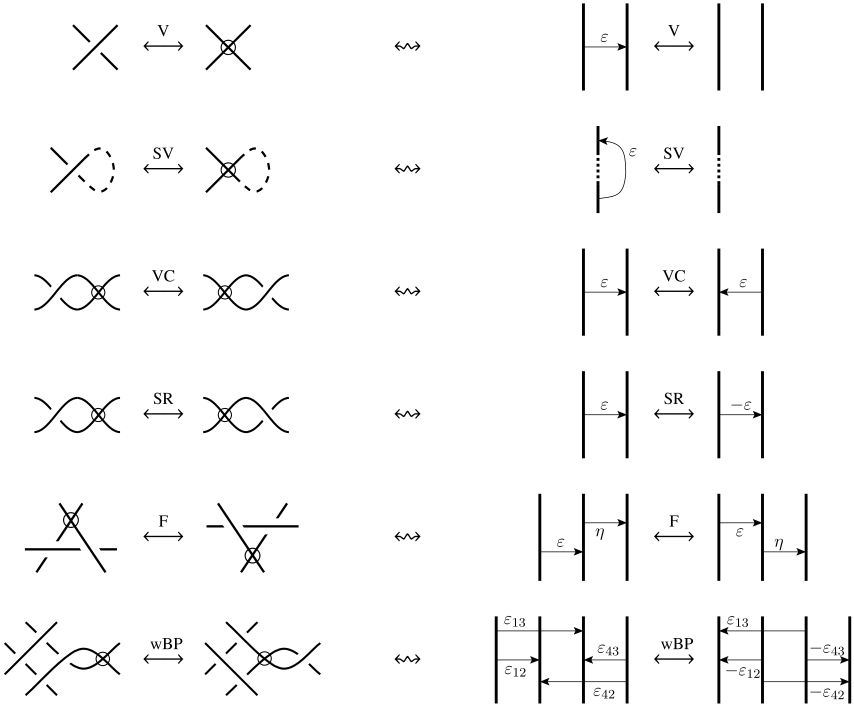


Figure 7 Nonclassical local moves

as a generating move for link-homotopy and further studied by Habegger and Lin [17].

The Δ move was introduced by Murakami and Nakanishi [27] and Matveev [23] as a local and combinatorial incarnation of the link-homology quotient of links. There exists another representation of this move, given by its mirror image, which is easily checked to be equivalent; see [27, Fig. 1.1(c)]. Similar observations hold for each of the local moves introduced in the rest of the paper, and we only give one formulation and freely use equivalent versions, leaving as an exercise to the reader to check that they are indeed equivalent.

The BP move is the unoriented counterpart of the band-pass move, introduced by Murakami [26] as an alternative unknotting operation for knots.

Let us now turn to nonclassical local moves.

The *virtualization* move V and its self-connected refinement SV simply replace a classical crossing and self-crossing by a virtual one or vice-versa. It is fairly obvious that V is an unknotting operation for welded knotted objects.

The *virtual conjugation* move VC is best known in the literature—where it is usually refered to as the virtualization move—under the form given in Figure 2. In this paper, it is convenient to use the equivalent reformulation given in Figure 7.

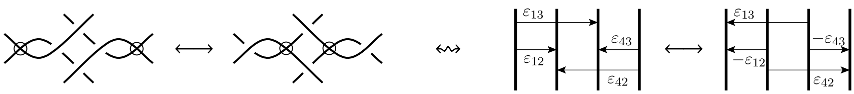
The *sign reversal* move SR is a composition of the VC and CC moves.

From the Gauss diagram point of view, the moves V, SV, VC, and SR are the simplest and most natural local moves, since they all involve a single arrow, which is modified by, respectively, being removed/added, having its orientation reversed or having its sign reversed.

Note that the overcommute move OC of Figure 3 can be interpreted as allowing adjacent tails to cross one another. Similarly, move F can be seen as a commutation between a tail and an adjacent head. Furthermore, as we will see later, this move is equivalent, up to OC, to the undercommute move UC, so that it actually allows any pair of adjacent endpoints to commute. In other words, F defines the *fused* quotient of welded objects, introduced by Kauffman and Lambropoulou [20; 21].

The move wBP can be seen as a welded analogue of the classical BP move. Note that its Gauss diagram incarnation should require some sign restrictions, but as we will prove in a diagrammatical way that SR can be realized using wBP, they can be released.

REMARK 2.1. The wBP move may appear asymmetric—and thus unnatural—to the reader but turns out to be the simplest candidate for a welded analogue of the BP move in the sense of Theorem 2.9 and Proposition 3.17. As a matter of fact, we can consider the following more symmetric version:



One can show that it is in fact equivalent to wBP; in practice, however, such a symmetrized version of wBP is less convenient, since it involves more crossings.

To conclude this section, we introduce some generic notation.

NOTATION 2.2. For any local move μ, we denote by $\mathrm{w}\mathcal{SL}_n^{\mu}$ the quotient of $\mathrm{w}\mathcal{SL}_n$ under the move μ.

If μ is classical, then we furthermore denote by $\mathcal{SL}_n^{\mu}$ the quotient of $\mathcal{SL}_n$ under the move μ.

We will use similar notation for classical and welded pure braids and links.

2.2. *Relation Between Local Moves*

DEFINITION 2.3. Let M_1 and M_2 be two local moves.

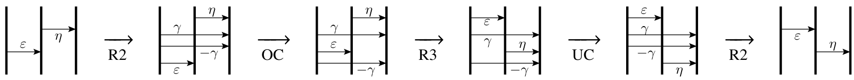
We say that M_2 *w-generates* M_1 if M_1 can be realized using M_2 and welded Reidemeister moves. We denote it by $M_2 \overset{w}{\Rightarrow} M_1$. If $M_2 \overset{w}{\Rightarrow} M_1$ and $M_1 \overset{w}{\Rightarrow} M_2$, then we say that M_1 and M_2 are *w-equivalent.*

If M_1 and M_2 are classical, then we say that M_2 *c-generates* M_1 if M_1 can be realized using M_2 and classical Reidemeister moves. We denote this by $M_2 \overset{c}{\Rightarrow} M_1$. If $M_2 \overset{c}{\Rightarrow} M_1$ and $M_1 \overset{c}{\Rightarrow} M_2$, then we say that M_1 and M_2 are *c-equivalent.*

REMARK 2.4. The inclusion $\iota\colon \mathrm{SLD}_n \hookrightarrow \mathrm{vSLD}_n$ induces a well-defined map $\iota_*\colon \mathcal{SL}_n^{\mathrm{M}_c} \longrightarrow \mathrm{w}\mathcal{SL}_n^{\mathrm{M}_w}$ whenever $\mathrm{M}_w \overset{w}{\Rightarrow} \mathrm{M}_c$ with M_c a classical local move. However, the induced map is, in general, not injective. This will be one of the main motivations for Definition 3.12.

PROPOSITION 2.5. *Move F is w-equivalent to the following undercommute move*

Proof. From the Gauss diagram point of view, F can be realized as



Note that the restrictions on signs requested to perform the R3 move can be fulfilled since we are free to choose the value of γ and free to choose the orientation of the piece of strand that supports the tail of the ε-labeled arrow.

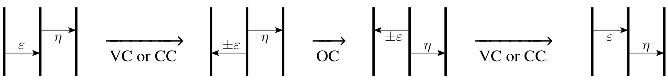
Conversely, UC can be similarly realized using F. □

It follows, in particular, that $\mathrm{w}\mathcal{SL}_n^{\mathrm{F}}$ is actually the fused quotient studied in [20; 21], where both "forbidden moves" are allowed. From the Gauss diagram point of view, it has also the following consequence.

COROLLARY 2.6. *Up to F, two arrow ends that are consecutive on a strand can be exchanged. Consequently, for Gauss diagram representatives of elements in* $w\mathcal{SL}_n^F$*, ends of arrows can be moved freely along a strand, so that the only relevant informations are the strands it starts from and goes to.*

PROPOSITION 2.7. *Both VC and CC w-generate F.*

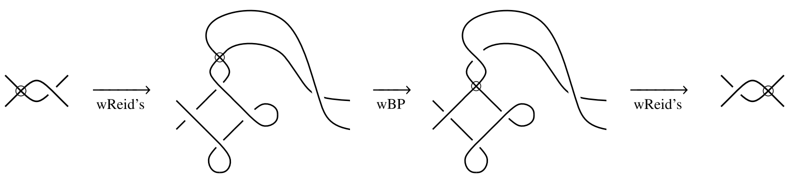
Proof. From the Gauss diagram point of view, F can be realized as



□

PROPOSITION 2.8. *The move wBP w-generates SR.*

Proof. From the virtual diagram point of view, SR can be realized as



□

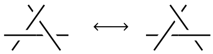
Now we state the first theorem of this section, which emphasizes the parallel between the classical and welded realms.

THEOREM 2.9. *The following diagram holds*:

$$\begin{array}{ccccccc} CC & \overset{c}{\Rightarrow} & BP & \overset{c}{\Rightarrow} & \Delta & \overset{c}{\Rightarrow} & SC \\ \Downarrow{\scriptstyle w} & & \Downarrow{\scriptstyle w} & & \Downarrow{\scriptstyle w} & & \Downarrow{\scriptstyle w} \\ V & \overset{w}{\Rightarrow} & wBP & \overset{w}{\Rightarrow} & F & \overset{w}{\Rightarrow} & SV \end{array}$$

Proof. The upper line gathers classical known facts:

- CC $\overset{c}{\Rightarrow}$ BP is obvious;
- $\Delta \overset{c}{\Rightarrow}$ SC is a classical fact, proved in [27, Lemma 1.1];
- BP $\overset{c}{\Rightarrow} \Delta$ is proved by combining [30, Prop. 1] and [1, Lemma 2]; more precisely, Lemma 2 of [1] shows (when ignoring orientations) that BP w-generates the following local move:



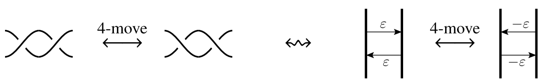
whereas [30, Prop. 1(5)] shows that this local move w-generates Δ.

The statements SV $\overset{w}{\Rightarrow}$ SC, V $\overset{w}{\Rightarrow}$ CC, and V $\overset{w}{\Rightarrow}$ wBP are direct consequences of the fact that virtualization V and SV moves allow us to remove any arrow and reinsert it with reversed sign.

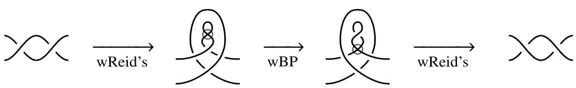
Since BP and wBP differ by one application of SR, the statement wBP $\overset{w}{\Rightarrow}$ BP is a corollary of Proposition 2.8.

From the Gauss diagram point of view, Δ modifies the relative positions of three arrows. It follows hence from Corollary 2.6 that F $\overset{w}{\Rightarrow} \Delta$. For the same reason, F $\overset{w}{\Rightarrow}$ SV since both ends of a self-arrow can be made adjacent and the self-arrow removed using R1.

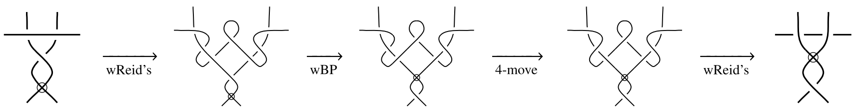
To prove the last statement wBP $\overset{w}{\Rightarrow}$ F, we first note that wBP w-generates the 4-*move:*[1]



Indeed, we have the following, which can be seen as a welded analogue of [26, Fig. 8]:



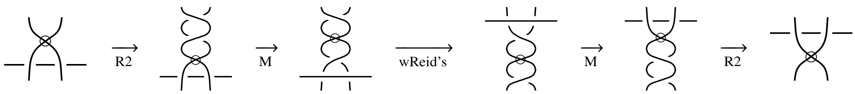
The sequence



[1]It is still an open problem, known as the 4-move conjecture and first posed by Nakanishi [29], whether this move is an unknotting operation on classical knots.

hence proves that wBP w-generates the following move:

The following sequence, together with Proposition 2.5, then concludes the proof:



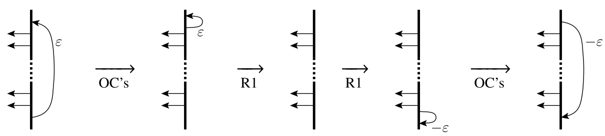
□

Note that if the statement $\mathrm{M}_2 \stackrel{c}{\Rightarrow} \mathrm{M}_1$, for some classical local moves M_1 and M_2, is proved by realizing locally M_1 using M_2, then it automatically follows that $\mathrm{M}_2 \stackrel{w}{\Rightarrow} \mathrm{M}_1$. For instance, it follows directly from the proofs in the classical case that $\mathrm{CC} \stackrel{w}{\Rightarrow} \mathrm{BP} \stackrel{w}{\Rightarrow} \Delta$. On the contrary, the proof that $\Delta \stackrel{c}{\Rightarrow} \mathrm{SC}$ is not local, and promoting it to the welded realm requires some attention.

THEOREM 2.10. *The following relations hold*: $CC \stackrel{w}{\Rightarrow} BP \stackrel{w}{\Rightarrow} \Delta \stackrel{w}{\Rightarrow} SC$.

Proof. As already noted, only the relation $\Delta \stackrel{w}{\Rightarrow} \mathrm{SC}$ needs to be proved. We adopt the Gauss diagram point of view. Let a be a self-arrow on which we want to realize a crossing change. We proceed by induction on the *width* of a, which is defined as the number of heads located on the portion of strand between the two endpoints of a.

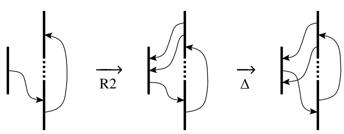
If a has width zero, then there is no head between the endpoints of a, and the crossing change can be realized as



Now, we assume that a has width $d \in \mathbb{N}^*$ and that the statement is proved for self-arrows having width smaller than d. We call an *interior* arrow any self-arrow that has both endpoints located in the portion of strand between the endpoints of a. There are two cases.

There is an interior arrow b: then we proceed in three steps.

Step 1: Remove b by pushing its tail next to its head as follows. Tails can be crossed using OC. Heads from non interior arrows can be crossed using the sequence



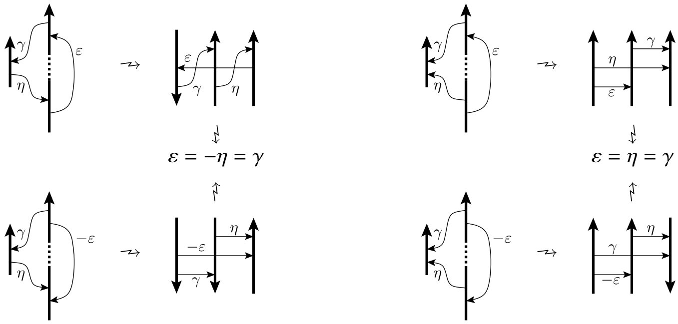


Figure 8 Correspondence between sign restrictions

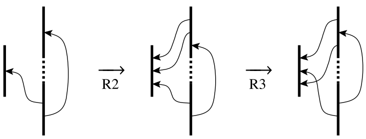
The restrictions on signs requested to perform the Δ move can be fulfilled since we are free to choose the signs of the arrows created with the R2 move and free to choose the orientation of the piece of strand that supports the tail of the non interior arrow. Heads from interior arrows can be crossed by using the induction hypothesis, which allows us to turn them into tails using self-crossing changes; an interior arrow has indeed a strictly smaller width than a. The arrow b can now be removed using R1.

Step 2: Since none of the operations of Step 1 has increased the number of head between its endpoints, a has now width $d-1$, and the induction hypothesis can be used to perform a self-crossing change on it.

Step 3: The arrow b can be placed back by performing Step 1 backward.

There is no interior arrow: then we also proceed in three steps.

Step 1: Push the tail of a toward its head until it has crossed one head. In doing so, the tail of a first crosses a number of tails (of noninterior arrows), and we request that these are not crossed using OC but using the sequence



The restrictions on signs requested to perform the R3 move can be fulfilled since we are free to choose the signs of the arrows created with the R2 move and free to choose the orientation of the piece of strand that supports the head of the non interior arrow. Finally, the first head met by the tail of a is crossed using the sequence given in Step 1 of the previous case.

Step 2: Since none of the operations of Step 1 has increased the number of head between its endpoints, a has now width $d-1$, and the induction hypothesis can be used to perform a self-crossing change on it.

Step 3: The tail of a can now be pushed back to its initial position by performing Step 1 backward. It is indeed illustrated in Figure 8 that the Δ and R3 moves performed in Step 1 and the corresponding move performed in this final step have sign restrictions that are simultaneously satisfied.[2] Some random orientations have been chosen for the strands in Figure 8, but changing it would merely add a sign on both sides. □

As observed in [4, Lemma 4.4], SC is an unknotting operation on w$\mathcal{SL}_1$. Indeed, by SC and OC moves, any Gauss diagram of a long knot can be turned into a diagram where all arrows have adjacent endpoints, which is clearly trivial by R1. It follows that Theorem 2.10 and Proposition 2.7 have the following corollary.

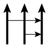
Corollary 2.11. *Moves* Δ, *BP*, *F*, *VC*, *and wBP are all unknotting operations on welded long knots and hence on welded knots.*

Remark 2.12. The statement of Corollary 2.11 on welded knots has been also recently proved in a different way by Satoh [36].

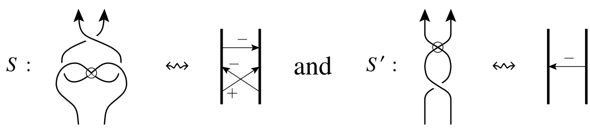
A consequence of Corollary 2.11 is that, on one strand, that is, for welded long knots, all the local moves considered in Theorem 2.9 are w-equivalent. In the next section, we provide classification results, which point out that, except in a few cases (see, e.g., Corollary 3.10), this is no longer true on more strands. However, w$\mathcal{SL}_n^{\Delta}$ is not considered there, and since it provides another example of w-equivalence on two strands, we address it now.

Proposition 2.13. *On two strands*, Δ *and SC are w-equivalent*, *that is*, *we have* $w\mathcal{SL}_2^{\Delta} = w\mathcal{SL}_2^{SC}$.

Proof. It has been proved in Theorem 2.10 that $\Delta \overset{w}{\Rightarrow}$ SC. Conversely, any Δ move involves at least two pieces of strand that belong to the same strand. By performing a self-crossing change on the corresponding crossing before and after, any Δ move can then be replaced by an R3. □

Remark 2.14. On three strands and more, the Milnor invariants $\mu^w_{i_1i_2i_3}$ defined in [3, Sec. 5.2]—which are also described, in terms of Gauss diagram formula, as ⟨ +  −  , −⟩, using the notation from [4, Sec. 3.2]—detects any Δ move but is invariant under SC.

Note also that, even on two strands, w$\mathcal{SL}_2^{\mathrm{F}}$ is a proper quotient of w$\mathcal{SL}_2^{\Delta}$ since the elements



are equal in the former but not in the latter. Indeed, the invariant Q_2, defined in the proof of [4, Lemma 4.10] as the invariant for welded string links up to SC

[2] This can also be trivially checked from the virtual diagram point of view.

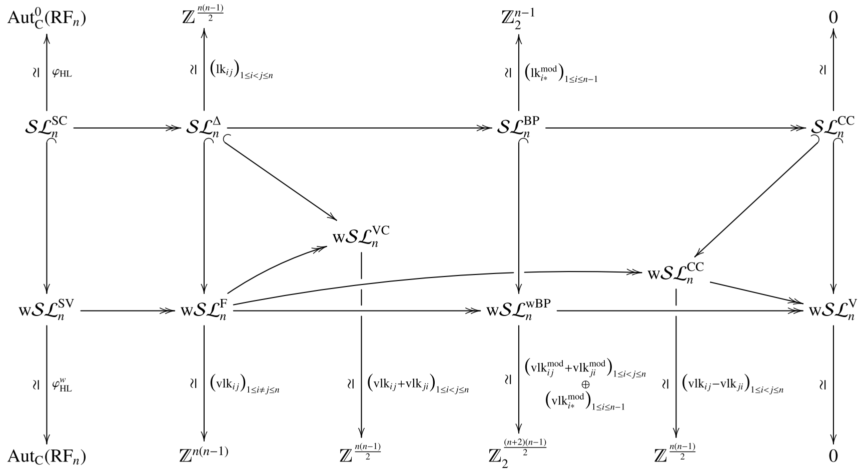
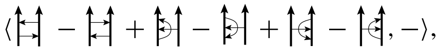


Figure 9 Summary of the classification and extension results

given by , is also invariant under Δ: if a Δ move involves three distinct strands, only one of the three arrows affected by this move can be involved in the computation of Q_2, so that the value of Q_2 is the same before and after the move; if a Δ move involves only one or two distinct strands, then, as noticed in the proof of Proposition 2.13, it can be replaced by two SC and one R3 moves. The invariant Q_2 is hence well defined on $\mathrm{w}\mathcal{SL}_2^{\Delta}$, and it is directly computed that $Q_2(S) = -1$ whereas $Q_2(S') = 0$.

3. Classifying Invariants

In this section, we classify string link diagrams modulo the main local moves studied before, and as a corollary, we discuss how some classical local moves can be extended to the welded case. A global description of the results is given in Figure 9.

3.1. Classifications

DEFINITION 3.1. Let μ be a local move, and let $\phi\colon \mathrm{vSLD}_n \longrightarrow A$, for some monoid A, be a morphism of monoids.

We say that ϕ *w-classifies* μ if ϕ is invariant under μ and the welded Reidemeister moves and that the induced map $\phi_*\colon \mathrm{w}\mathcal{SL}_n^{\mu} \longrightarrow A$ is an isomorphism.

If μ is classical, then we say that ϕ *c-classifies* μ if ϕ is invariant under μ and the classical Reidemeister moves and if the induced map $\phi_*\colon \mathcal{SL}_n^{\mu} \longrightarrow A$ is an isomorphism.

Now, we define a few welded invariants that will classify the moves introduced in the previous section.

DEFINITION 3.2. For every $i \neq j \in \{1, \dots, n\}$, we define the *virtual linking number* $\mathrm{vlk}_{ij} : \mathrm{vSLD} \longrightarrow \mathbb{Z}$ as the map that counts, with signs, the crossings where the ith component passes over the jth component. From the Gauss diagram point of view, it simply counts the signs of all the arrows going from the ith to jth strands.

The following lemma is considered as folklore. The first part is clear from the Gauss diagram point of view, and the second part can be proved using the virtual diagram point of view.

LEMMA 3.3. *For every $i \neq j \in \{1, \dots, n\}$, vlk_{ij} is invariant under welded Reidemeister moves, and if $D \in SLD_n$ is a classical diagram, then $vlk_{ij}(D) = vlk_{ji}(D)$.*

NOTATION 3.4. For every $i \neq j \in \{1, \dots, n\}$, we set

- the *linking number* $\mathrm{lk}_{ij} : \mathrm{SLD}_n \longrightarrow \mathbb{Z}$ as the restriction to classical diagrams of either (and equivalently) vlk_{ij}, vlk_{ji}, or $\frac{1}{2}(\mathrm{vlk}_{ij} + \mathrm{vlk}_{ji})$;
- $\mathrm{vlk}_{i*} := \sum_{\substack{1 \leq k \leq n \\ k \neq i}} \mathrm{vlk}_{ik}$ and $\mathrm{lk}_{i*} := \sum_{\substack{1 \leq k \leq n \\ k \neq i}} \mathrm{lk}_{ik}$;
- $\mathrm{vlk}_{ij}^{\mathrm{mod}}$, $\mathrm{vlk}_{i*}^{\mathrm{mod}}$, $\mathrm{lk}_{ij}^{\mathrm{mod}}$, and $\mathrm{lk}_{i*}^{\mathrm{mod}}$ as the modulo 2 reduction of vlk_{ij}, vlk_{i*}, lk_{ij}, and lk_{i*}, respectively.

Linking and virtual linking numbers can be similarly defined for classical or welded pure braids and links.

Habegger and Lin [17] defined a map $\varphi_{\mathrm{HL}} : \mathrm{SLD}_n \longrightarrow \mathrm{Aut}_{\mathrm{C}}^0(\mathrm{RF}_n)$, which was extended into a map $\varphi_{\mathrm{HL}}^w : \mathrm{vSLD}_n \longrightarrow \mathrm{Aut}_{\mathrm{C}}(\mathrm{RF}_n)$ in [3], in the sense that $\varphi_w \circ \iota = \varphi$. Here, RF_n denotes the largest quotient of the free group over $x_1, \dots, x_n$ such that each x_i commutes with all its conjugates, $\mathrm{Aut}_{\mathrm{C}}(\mathrm{RF}_n)$ is the group of automorphisms of RF_n mapping each x_i to a conjugate of itself, and $\mathrm{Aut}_{\mathrm{C}}^0(\mathrm{RF}_n)$ is the subgroup of such automorphisms fixing $x_1 \cdots x_n$.

The following classification results are known.

PROPOSITION 3.5. *In the classical case*:

- [27, *Thm. 1.1*][3] *The map $(lk_{ij})_{1 \leq i < j \leq n} : SLD_n \longrightarrow \mathbb{Z}^{(n(n-1))/2}$ c-classifies Δ.*
- [22, *Thm. 11.6.7*] *and* [27, *Thm. A.2*][3] *The map $(lk_{i*}^{\mathrm{mod}})_{1 \leq i \leq n-1} : SLD_n \longrightarrow \mathbb{Z}_2^{n-1}$ c-classifies BP.*
- [17, *Thm. 1.7*] *The map $\varphi_{HL} : SLD_n \longrightarrow \mathrm{Aut}_C^0(RF_n)$ c-classifies SC.*

In the welded case:

- [3, *Thm. 2.34*] *The map $\varphi_{HL}^w : vSLD_n \longrightarrow \mathrm{Aut}_C(RF_n)$ w-classifies SV.*

We now provide new classification results. To this end, we use the Gauss diagram point of view and define, for every $k \in \mathbb{N}$, $\varepsilon \in \{\pm 1\}$ and $i \neq j \in \{1, \dots, n\}$, $G_{i,j}^{\varepsilon k}$ as the Gauss diagram that has only k horizontal ε–labeled arrows from strand i to j.

[3] In the given references, the statements are for links rather than string links, but as discussed in Section 4.1, up to Δ or BP, these notions are the same.

PROPOSITION 3.6. *The map* $(vlk_{ij})_{1\neq i<j\leq n}\colon vSLD_n \longrightarrow \mathbb{Z}^{n(n-1)}$ *w-classifies F.*

Proof. It essentially follows from Corollary 2.6 that virtual linking numbers are invariant under F.

It can be noted that $G^{\varepsilon k}_{i,j}$ satisfies $\mathrm{vlk}_{i,j}(G^{\varepsilon k}_{i,j}) = \varepsilon k$ and $\mathrm{vlk}_{p,q}(G^{\varepsilon k}_{i,j}) = 0$ for $(p,q)\neq(i,j)$. By stacking such Gauss diagrams in lexicographical order of $i \neq j \in \{1,\dots,n\}$ we obtain normal forms realizing any configuration of the virtual linking numbers.

Now, given a Gauss diagram, all self-arrows can be removed using F and welded Reidemeister moves since $\mathrm{F} \overset{w}{\Rightarrow} \mathrm{SC}$, and then, using Corollary 2.6, arrow ends can be reorganized in order to obtain one of the normal forms. □

PROPOSITION 3.7. *The map* $(vlk_{ij} + vlk_{ji})_{1\leq i<j\leq n}\colon vSLD_n \longrightarrow \mathbb{Z}^{(n(n-1))/2}$ *w-classifies VC.*

Proof. Performing VC on a self-arrow does not affect any $\mathrm{vlk}_{ij} + \mathrm{vlk}_{ji}$. Performing it on an arrow between strands i and j adds ± 1 to vlk_{ij}, whereas it adds ∓ 1 to vlk_{ji}; the sum $\mathrm{vlk}_{ij} + \mathrm{vlk}_{ji}$ hence remains invariant.

The surjectivity of the induced map is achieved by considering the same normal forms as in the proof of Proposition 3.6, but restricted to $G^{\varepsilon k}_{i,j}$ with $i < j$.

Given a Gauss diagram, all vlk_{ij} with $i > j$ can be made to vanish by creating sufficiently many arrows from the ith to jth strand using R2 moves and performing VC on them. Then by Proposition 3.6 there is a sequence of F and welded Reidemeister moves to one of the normal forms considered, and this concludes the proof since $\mathrm{VC} \overset{w}{\Rightarrow} \mathrm{F}$. □

PROPOSITION 3.8. *The map* $(vlk_{ij} - vlk_{ji})_{1\leq i<j\leq n}\colon vSLD_n \longrightarrow \mathbb{Z}^{(n(n-1))/2}$ *w-classifies CC.*

Proof. The proof is totally similar to that of Proposition 3.7. □

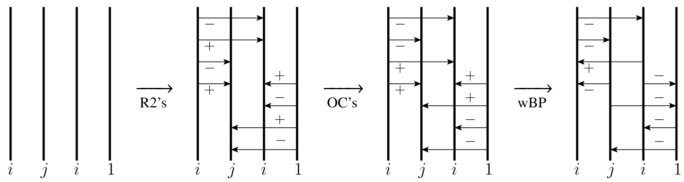
PROPOSITION 3.9. *The map* $(vlk^{\mathrm{mod}}_{ij} + vlk^{\mathrm{mod}}_{ji})_{1\leq i<j\leq n} \oplus (vlk^{\mathrm{mod}}_{i*})_{1\leq i\leq n-1}$: $vSLD_n \longrightarrow \mathbb{Z}_2^{(n(n-1))/2} \oplus \mathbb{Z}_2^{n-1} = \mathbb{Z}_2^{((n+2)(n-1))/2}$ *w-classifies wBP.*

Proof. A move wBP is a combination of CC and VC. As such, it modifies $\mathrm{vlk}_{ij} + \mathrm{vlk}_{ji}$ by a multiple of 2. Moreover, a move wBP changes the number of non-self-arrows with the tail on a given strand by 0 or 2: this is clear if none of the four involved pieces of strands are connected, and it can be case-by-case checked in the other situations. As a consequence, vlk_{i*} is also modified by a multiple of 2. In conclusion, the invariant is indeed invariant under wBP.

The surjectivity of the induced map is achieved by stacking (in any order) elements of the form $G^{+1}_{i,j}$ as follows: fix an element in $\mathbb{Z}_2^{((n+2)(n-1))/2}$, seen as some $(\mathrm{vlk}^{\mathrm{mod}}_{ij} + \mathrm{vlk}^{\mathrm{mod}}_{ji})_{1\leq i<j\leq n} \oplus (\mathrm{vlk}^{\mathrm{mod}}_{i*})_{1\leq i\leq n-1}$; start from the Gauss diagram with no arrow and, for each $i < j \in \{1,\dots,n\}$, add one $G^{+1}_{i,j}$ whenever $\mathrm{vlk}^{\mathrm{mod}}_{ij} + \mathrm{vlk}^{\mathrm{mod}}_{ji} \equiv 1$; next, for each $i \in \{1,\dots,n-1\}$, add $G^{+1}_{i,n}$ and $G^{+1}_{n,i}$ if $\mathrm{vlk}_{i*}(G) \neq$

vlk_{i*} and do nothing otherwise. The resulting Gauss diagram is the requested preimage.

The injectivity of the induced map is proved by induction on n. For $n = 1$, the result follows from Corollary 2.11. Now assume that $n > 1$ and that the result is true on $n - 1$ strands. It was shown in [3, Thm. 4.12] that every welded string link is related to a welded braid by welded Reidemeister and SV moves. Since every welded braid has an inverse and since wBP $\overset{w}{\Rightarrow}$ SV, it follows that $\mathrm{w}\mathcal{SL}_n^{\mathrm{wBP}}$ is a group, and it is thus sufficient to prove that the kernel of the induced map is trivial. Consider hence a Gauss diagram G that is in the kernel. In the following, we will apply some welded Reidemeister and wBP moves on G, but by abuse of notation, we will keep denoting it by G. First, we can modify G so that each $\mathrm{vlk}_{ij}(G)$ is either 0 or 1. Indeed, this is easily achieved using the SR move, and wBP w-generates SR by Proposition 2.8. Next, we show how to reduce all $\mathrm{vlk}_{1i}(G)$ and $\mathrm{vlk}_{i1}(G)$ to 0. If $\mathrm{vlk}_{1i}(G)$ is 1 for some $i \neq 1$, then there is $j \neq 1, i$ such that $\mathrm{vlk}_{1j}(G) = 1$, since otherwise $\mathrm{vlk}_{1*}^{\mathrm{mod}}(G)$ would be 1; moreover, we have that $\mathrm{vlk}_{i1}(G)$ is also 1, since otherwise $\mathrm{vlk}_{1i}^{\mathrm{mod}}(G) + \mathrm{vlk}_{i1}^{\mathrm{mod}}(G)$ would be 1, and likewise we have $\mathrm{vlk}_{j1}(G) = 1$. Then perform locally the following sequence anywhere on G (there, the indices at the bottom correspond to the labels of the strands to which the different pieces belong):



As a result, we have that $\mathrm{vlk}_{1i}(G) = \mathrm{vlk}_{i1}(G) = \mathrm{vlk}_{1j}(G) = \mathrm{vlk}_{j1}(G) = 0$. Repeat this operation until all $\mathrm{vlk}_{1i}(G)$ and $\mathrm{vlk}_{i1}(G)$ are 0, as desired. Using the normal form given in the proof of Proposition 3.6, there is a sequence of welded Reidemeister and F moves transforming G into a Gauss diagram with no arrow touching the first strand. Since wBP $\overset{w}{\Rightarrow}$ F, this sequence can be traded for a sequence of welded Reidemeister and wBP moves. By forgetting the first strand we obtain a welded string link on $n - 1$ strands that is in the kernel of the induced map. By induction hypothesis, G is hence trivial in $\mathrm{w}\mathcal{SL}_n^{\mathrm{wBP}}$. □

Classification results may be used to prove w-equivalence between local moves.

Corollary 3.10. *On two strands, SV and F are w-equivalent, but for $n \geq 3$, $w\mathcal{SL}_n^F$ is a proper quotient of $w\mathcal{SL}_n^{SV}$.*

Proof. The local move F is w-classified by virtual linking numbers, that is, $\mathrm{w}\mathcal{SL}_n^{\mathrm{F}}$ is isomorphic to $\mathbb{Z}^{n(n-1)}$. On the other hand, $\mathrm{w}\mathcal{SL}_n^{\mathrm{SV}}$ is isomorphic to $\mathrm{Aut}_{\mathrm{C}}(RF_n)$. For $n = 2$, it is easily seen that any element of $\mathrm{Aut}_C(\mathrm{RF}_2)$ can be written as φ_{n_1,n_2} for some integers $n_1, n_2 \in \mathbb{N}$, where $\varphi_{n_1,n_2}(x_1) = x_2^{n_1} x_1 x_2^{-n_1}$ and $\varphi_{n_1,n_2}(x_2) =$

$x_1^{n_2}x_2x_1^{-n_2}$, and that $\varphi_{n_1,n_2}\varphi_{n_3,n_4} = \varphi_{n_1+n_3,n_2+n_4}$. This implies that $\mathrm{Aut_C}(RF_2)$ is isomorphic $\mathbb{Z}^2$, whereas $\mathrm{Aut_C}(RF_n)$ is not abelian for $n > 2$. □

REMARK 3.11. The fact that, on two strands, SV $\overset{w}{\Rightarrow}$ F can also be seen as a corollary of [3, Prop. 4.11].

3.2. Welded Extensions

DEFINITION 3.12. Let M_c and M_w be two local moves such that M_c is classical and $\mathrm{M}_w \overset{w}{\Rightarrow} \mathrm{M}_c$. We say that M_w extends M_c if the map $\iota_*\colon \mathcal{SL}_n^{\mathrm{M}_c} \longrightarrow \mathrm{w}\mathcal{SL}_n^{\mathrm{M}_w}$ induced by the inclusion $\iota\colon \mathrm{SLD}_n \hookrightarrow \mathrm{vSLD}_n$ is injective.

This definition is motivated by the following direct consequence.

LEMMA 3.13. *Let M_c and M_w be two local moves such that M_c is classical and M_w extends M_c. If two classical diagrams are connected by a sequence of M_w and welded Reidemeister moves, then they are connected by a sequence of M_c and classical Reidemeister moves.*

Now, we can use the classification results of the previous section to obtain some extension results. In each case, it is sufficient to check that the target of the c-classifying map can be identified with a subset of the target of the w-classifying map and that, with this identification, the c-classifying map is in fact the composition of the w-classifying map with the injection $\iota\colon \mathrm{SLD}_n \hookrightarrow \mathrm{vSLD}_n$.

PROPOSITION 3.14 ([4, Thm. 4.3]). *The move SV extends SC.*

PROPOSITION 3.15. *Both F and VC extend* Δ.

This proposition, as well as Proposition 3.16, illustrates how a given classical local move can be extended in several different ways.

Proof of Proposition 3.15. To prove that F extends Δ, we use Propositions 3.5 and 3.6 and identify $\mathbb{Z}^{(n(n-1))/2}$ with the subset of $\mathbb{Z}^{n(n-1)}$ made of elements such that the ij and ji-summands are equal for every $i \neq j \in \{1,\dots,n\}$. The linking number lk_{ij} is then seen as simultaneously equal to vlk_{ij} and vlk_{ji}.

Similarly, to prove that VC extends Δ using Propositions 3.5 and 3.7, we identify $\mathbb{Z}^{(n(n-1))/2}$ with the subset $2\mathbb{Z}^{(n(n-1))/2}$ of even-valued elements in $\mathbb{Z}^{(n(n-1))/2}$. The linking number lk_{ij} should then rather be interpreted as $\frac{1}{2}(\mathrm{vlk}_{ij} + \mathrm{vlk}_{ji})$. □

PROPOSITION 3.16. *Both V and CC extend CC.*

Proof. In both situations, there is no ambiguity on how 0 is identified as a subset of 0 or of $\mathbb{Z}^{(n(n-1))/2}$. In the latter case, the result follows from the fact that $\mathrm{vlk}_{ij} = \mathrm{vlk}_{ji}$ on classical diagrams, so that $\mathrm{vlk}_{ij} - \mathrm{vlk}_{ji}$ vanishes. □

PROPOSITION 3.17. *The local move wBP extends BP.*

Proof. To prove the statement, $\mathbb{Z}_2^{n-1}$ should be identified with the $n-1$ last $(\mathrm{vlk}_{i*}^{\mathrm{mod}})$-summands of $\mathbb{Z}_2^{((n+2)(n-1))/2}$, the other being identically equal to 0. □

4. Braids and Links

We now investigate how the results given so far for string links can be transported to the more familiar context of braids and (possibly unordered) links. In particular, we prove the results stated in the Introduction.

4.1. Welded Links

Most of the results stated for classical and welded string links in this paper extend to the case of welded links. In particular, it can be noted that none of the proofs given in Section 2.2 uses the fact that we are dealing with intervals rather than with circles. It follows that all generation and equivalence results stated there hold the same for classical and welded links.

4.1.1. From String Links to Links. There is a natural way to associate a welded link with a welded string link, incarnated by the closure map $\mathrm{Cl}\colon \mathrm{w}\mathcal{SL}_n \longrightarrow \mathrm{w}\mathcal{L}_n$, which is defined, using the Gauss diagram point of view, by identifying pairwise the endpoints of each strand while keeping the order on the resulting circles. This map restricts to a well-defined map $\mathrm{Cl}\colon \mathcal{SL}_n \longrightarrow \mathcal{L}_n$. Cutting circles into intervals produces preimages, showing that these maps are surjective. However, this procedure does not provide a well-defined inverse for the closure map since it strongly depends on an arbitrary choice of cutting points; and, indeed, except for $\mathrm{Cl}\colon \mathcal{SL}_1 \longrightarrow \mathcal{L}_1$, the closure maps are not injective. A noteworthy consequence of Corollary 2.6 is that, up to F moves, and consequently up to most local moves considered in this paper, the procedure does provide a well-defined inverse, proving that the quotiented notions of welded string links and welded links coincide.

PROPOSITION 4.1. *The closure map induces one-to-one correspondences between* $\mathrm{w}\mathcal{SL}_n^\mu$ *and* $\mathrm{w}\mathcal{L}_n^\mu$ *for* $\mu = F$, *CC*, *VC*, *or wBP*.

Proof. It is sufficient to prove that, up to welded Reidemeister and the considered local moves, the opening procedure described above does not depend on the chosen cutting points. The resulting map $\mathrm{Op}\colon \mathrm{w}\mathcal{L}_n^\mu \longrightarrow \mathrm{w}\mathcal{SL}_n^\mu$ would then clearly satisfy $\mathrm{Op} \circ \mathrm{Cl} = \mathrm{Id}_{\mathrm{w}\mathcal{SL}_n^\mu}$ and $\mathrm{Cl} \circ \mathrm{Op} = \mathrm{Id}_{\mathrm{w}\mathcal{L}_n^\mu}$. To prove such an independence, it is sufficient, on the link side seen from the Gauss diagram point of view, to show that a cutting point can cross an arrow endpoint. On the string link side, it corresponds to moving the endpoint from one extremity of the strand to the other. Because of Corollary 2.6, this can be done freely using F or any other local move that w-generates F. □

Combined with the extension results given in the previous section, Proposition 4.1 induces similar statements for classical objects.

PROPOSITION 4.2. *The closure map induces one-to-one correspondences between $\mathcal{SL}_n^\mu$ and $\mathcal{L}_n^\mu$ for $\mu = \Delta$ or BP.*

Proof. Let D_1 and D_2 be two classical string link diagrams that have, through the closure map, the same image in $\mathcal{L}_n^\Delta$ and hence in $\mathrm{w}\mathcal{L}_n^{\mathrm{F}}$ since $\mathrm{F} \overset{w}{\Rightarrow} \Delta$. By Proposition 4.1 they are connected by a sequence of welded Reidemeister and F moves. However, by Proposition 3.15 and Lemma 3.13 they are hence connected by classical Reidemeister and Δ moves and thus represent the same element of $\mathcal{SL}_n^\Delta$.

The statement for BP is proved similarly. □

This statement can be independently proved using the fact that linking numbers simultaneously classify links [27, Thm. 1.1] and string links [24, Thm. 4.6] up to Δ.

It follows from Propositions 4.1 and 4.2 that most classification and extension results given in Section 3 hold the same for classical and welded links, as stated in Theorems 1 and 2.

4.1.2. Unordered Links. There is an obvious action of the symmetric group $\mathfrak{S}_n$ on classical and welded links, which simply permutes the order on the components. Classical and welded unordered links are the natural quotients under this action. For each classification given in Theorem 1, an action of $\mathfrak{S}_n$ can be defined on the target space, so that it results in a classification for the considered local move for unordered (welded) links. For instance, the target space of the classification of welded links up to F is $\mathbb{Z}^{n(n-1)} \cong \mathbb{Z}^{\{(i,j)|1\le i\neq j\le n\}}$. For all $\sigma \in \mathfrak{S}_n$ and $(a_{ij})_{1\le i\neq j\le n} \in \mathbb{Z}^{n(n-1)}$, we can set $\sigma \cdot (a_{ij})_{1\le i\neq j\le n} := (a_{\sigma(i)\sigma(j)})_{1\le i\neq j\le n}$; unordered welded links up to F are then classified by the symmetrized virtual linking numbers $\mathrm{vlk}_{ij} : \mathrm{vSLD}_n \longrightarrow \mathbb{Z}^{n(n-1)}/\mathfrak{S}_n$. More generally, we have the following:

PROPOSITION 4.3.

- *Unordered links up to Δ are classified by the symmetrized linking numbers.*
- *Unordered links up to BP are classified by the symmetrized lk_{i*}^{mod}s.*
- *Unordered welded links up to F are classified by the symmetrized virtual linking numbers.*
- *Unordered welded links up to VC are classified by the symmetrized $(vlk_{ij} + vlk_{ji})$s.*
- *Unordered welded links up to CC are classified by the symmetrized $(vlk_{ij} - vlk_{ji})$s.*
- *Unordered welded links up to wBP are classified by the symmetrized vlk_{i*}^{mod}'s and $(vlk_{ij}^{\mathrm{mod}} + vlk_{ji}^{\mathrm{mod}})$s.*

The target spaces of the classifications are, in general, not particularly nicely described. A notable exception is the case of classical links up to BP, which reduces to the number of i such that $\mathrm{lk}_{i*}^{\mathrm{mod}}$ is 1; see [22, Thm. 11.6.7] and [27, Thm. A.2].

These classifications can, in turn, be used to show extension results, that is, we have a strict analogue of Theorem 2 in the unordered case.

4.2. Welded Braids

It is well known that pure braids embed in string links (it was actually noticed by Artin; see, for instance, Chapter 6, Proposition 1.1 of [28] for a complete proof) and, similarly, welded pure braid embed in welded string links ([3, Rk 3.7]). However, this embedding is not, in general, preserved when we add other local moves. For instance, elements of $\mathrm{wP}_n^{\mathrm{F}}$, called *unrestricted pure virtual braids* in [7], do not embed in $\mathrm{w}\mathcal{SL}_n^{\mathrm{F}}$. Indeed, in [7, Thm. 2.7], $\mathrm{wP}_n^{\mathrm{F}}$ is proved to be non-abelian, whereas $\mathrm{w}\mathcal{SL}_n^{\mathrm{F}}$ is; it follows that there are adjacent endpoints of arrows that do not commute. This difference lies in the fact that, in the proof of Proposition 2.5, we allowed for the introduction of nonhorizontal arrows and, in particular, self-arrows. Indeed, in the sequence proving Proposition 2.5, the first and third strands may be part of the same component if the two initial arrows have endpoints on the same two strands. In this case, the R2 move creates two self-arrows.

In another direction, we may wonder if a given local move is strong enough to make surjective the natural map from (welded) pure braids to (welded) string links. In the classical case, the following is known.

PROPOSITION 4.4. [17, *p*. 399] *The natural embedding of P_n in $\mathcal{SL}_n$ induces an isomorphism between P_n^{SC} and $\mathcal{SL}_n^{SC}$.*

Note here that, since SC requires the presence of a self-crossing, which a pure braid cannot contain, it should be understood in the statement that P_n stands for its embedding in $\mathcal{SL}_n$. The same holds for the SV move in the welded settings, addressed in the next result.

PROPOSITION 4.5. *The natural embedding of wP_n in $w\mathcal{SL}$ induces*:

- *an isomorphism between wP_n^{SV} and $w\mathcal{SL}_n^{SV}$*;
- *an isomorphism between wP_n^{CC} and $w\mathcal{SL}_n^{CC}$*.

Proof. The first statement was proved in [3, Thm. 3.11]. The second follows from the fact that the quotient of wP_n by CC is isomorphic to the flat welded pure braid group, introduced in [7, Sec. 5.2], which is isomorphic to $\mathbb{Z}^{(n(n-1))/2}$; moreover, a straightforward verification shows that this isomorphism is realized by the $(\mathrm{vlk}_{ij} - \mathrm{vlk}_{ji})$'s. The statement follows then from the classification of $\mathrm{w}\mathcal{SL}_n^{\mathrm{CC}}$ given in Proposition 3.8. □

On the other hand, it is known that $\mathrm{wP}_n^{\mathrm{SC}}$ does not coincide with $\mathrm{w}\mathcal{SL}_n^{\mathrm{SC}}$ [4, Lemma 4.8]. It seems interesting to analyze the case of other local moves, but for this we should consider their oriented versions to have the right analogue for welded pure braids.

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References

[1] H. Aida, *The oriented Δ_{ij}-moves on links,* Kobe J. Math. 9 (1992), no. 2, 163–170.

[2] B. Audoux, *On the welded Tube map,* Contemp. Math. 670 (2016), 261–284.

[3] B. Audoux, P. Bellingeri, J.-B. Meilhan, and E. Wagner, *Homotopy classification of ribbon tubes and welded string links,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 17 (2017), no. 2, 713–761.

[4] ______, *On usual, virtual and welded knotted objects up to homotopy,* J. Math. Soc. Japan, 69 (2017), no. 3, 1079–1097.

[5] B. Audoux, J.-B. Meilhan, and E. Wagner, *On codimension two embeddings up to link-homotopy,* J. Topol., 10 (2017), no. 4, 1107–1123.

[6] D. Bar-Natan and Z. Dancso, *Finite-type invariants of w-knotted objects, I: w-knots and the Alexander polynomial,* Algebr. Geom. Topol. 16 (2016), no. 2, 1063–1133.

[7] V. G. Bardakov, P. Bellingeri, and C. Damiani, *Unrestricted virtual braids, fused links and other quotients of virtual braid groups,* J. Knot Theory Ramifications 24 (2015), no. 12, 1550063.

[8] T. E. Brendle and A. Hatcher, *Configuration spaces of rings and wickets,* Comment. Math. Helv. 88 (2013), no. 1, 131–162.

[9] J.S. Carter, S. Kamada, and M. Saito, *Stable equivalence of knots on surfaces and virtual knot cobordisms,* J. Knot Theory Ramifications 11 (2002), no. 3, 311–322; Knots 2000 Korea, Vol. 1 (Yongpyong).

[10] J.S. Carter, S. Kamada, M. Saito, and S. Satoh, *A theorem of Sanderson on link bordisms in dimension* 4, Algebr. Geom. Topol. 1 (2001), 299–310 (electronic).

[11] B. A. Cisneros De La Cruz, *Virtual braids from a topological viewpoint,* J. Knot Theory Ramifications 24 (2015), no. 6, 1550033.

[12] C. Damiani, *A journey through loop braid groups,* Expo. Math., 35, (2017), 252–285.

[13] R. Fenn, R. Rimányi, and C. Rourke, *The braid-permutation group,* Topology 36 (1997), no. 1, 123–135.

[14] T. Fiedler, *Gauss diagram invariants for knots and links,* Math. Appl., 532, Kluwer Academic Publishers, Dordrecht, 2001.

[15] A. Fish and E. Keyman, *Classifying links under fused isotopy,* J. Knot Theory Ramifications 25 (2016), no. 7, 1650042.

[16] M. Goussarov, M. Polyak, and O. Viro, *Finite-type invariants of classical and virtual knots,* Topology 39 (2000), no. 5, 1045–1068.

[17] N. Habegger and X.-S. Lin, *The classification of links up to link-homotopy,* J. Amer. Math. Soc. 3 (1990), 389–419.

[18] L. H. Kauffman, *Virtual knot theory,* European J. Combin. 20 (1999), no. 7, 663–690.

[19] ______, *A survey of virtual knot theory,* Knots in Hellas '98 (Delphi), Ser. Knots Everything, 24, pp. 143–202, World Sci. Publ., River Edge, NJ, 2000.

[20] L. H. Kauffman and S. Lambropoulou, *Virtual braids,* Fund. Math. 184 (2004), 159–186.

[21] ______, *Virtual braids and the L-move,* J. Knot Theory Ramifications 15 (2006), no. 6, 773–811.

[22] A. Kawauchi, *A survey of knot theory,* Birkhäuser, Basel, 1996.

[23] S. V. Matveev, *Generalized surgeries of three-dimensional manifolds and representations of homology spheres,* Mat. Zametki 42 (1987), no. 2, 268–278, 345.

[24] J.-B. Meilhan, *On Vassiliev invariants of order two for string links,* J. Knot Theory Ramifications 14 (2005), no. 5, 665–687.

[25] J. Milnor, *Link groups,* Ann. of Math. (2) 59 (1954), 177–195.

[26] H. Murakami, *Some metrics on classical knots,* Math. Ann. 270 (1985), 35–45.

[27] H. Murakami and Y. Nakanishi, *On a certain move generating link-homology,* Math. Ann. 284 (1989), no. 1, 75–89.
[28] K. Murasugi and B. I. Kurpita, *A study of braids,* Math. Appl., 484, Kluwer Academic Publishers, Dordrecht, 1999.
[29] Y. Nakanishi, *Fox's congruence modulo* $(2,1)$*,* Sûrikaisekikenkyûsho Kôkyûroku 813 (1985), 102–110.
[30] ———, *Replacements in the Conway third identity,* Tokyo J. Math. 14 (1991), no. 1, 197–203.
[31] T. Nasybullov, *The classification of fused links,* J. Knot Theory Ramifications 25 (2016), no. 21, 1650076.
[32] K. Reidemeister, *Unveränderter reprografischer, Einführung in die kombinatorische Topologie,* Nachdr. Ausg. Braunschw., Wissenschaftliche Buchgesellschaft, Darmstadt, 1972. Unveränderter reprografischer Nachdruck der Ausgabe Braunschweig 1951.
[33] B. J. Sanderson, *Bordism of links in codimension* 2*,* J. Lond. Math. Soc. (2) 35 (1987), no. 2, 367–376.
[34] ———, *Triple links in codimension* 2*,* Topology. Theory and applications, II (Pécs, 1989), Colloq. Math. Soc. János Bolyai, 55, pp. 457–471, North-Holland, Amsterdam, 1993.
[35] S. Satoh, *Virtual knot presentation of ribbon torus-knots,* J. Knot Theory Ramifications 9 (2000), no. 4, 531–542.
[36] ———, *Crossing changes, delta moves and sharp moves on welded knots,* Rocky Mountain J. Math. (2015, to appear), arXiv:1510.02554.
[37] B. Winter, *The classification of spun torus knots,* J. Knot Theory Ramifications 18 (2009), no. 9, 1287–1298.
[38] T. Yajima, *On the fundamental groups of knotted* 2*-manifolds in the* 4*-space,* J. Math., Osaka City Univ. 13 (1962), 63–71.

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