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Mirror Theorem for Elliptic Quasimap Invariants of Local Calabi–Yau Varieties

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Abstract. The elliptic quasi-map potential function is explicitly calculated for Calabi–Yau complete intersections in projective spaces in [13]. We extend this result to local Calabi–Yau varieties. Using this and the wall crossing formula in [5], we can calculate the elliptic Gromov–Witten potential function.

1. Introduction

For a nonsingular variety X that has a GIT representation $W/\!\!/_\theta \mathbf{G}$, we can define the moduli spaces of ε-stable quasi-maps with genus g, k-markings to X with degree β, denoted by $Q^\varepsilon_{g,k}(X,\beta)$, for any g and k with $2g-2+k \geq 0$, $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^{\mathbf{G}}(W), \mathbb{Z})$ unless $2g-2+k=0$ and $\beta=0$. For each $Q^\varepsilon_{g,k}(X,\beta)$, we can define the canonical virtual fundamental class

$$[Q^\varepsilon_{g,k}(X,\beta)]^{\mathrm{vir}} \in A_*(Q^\varepsilon_{g,k}(X,\beta)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

of degree

$$c_1^{\mathbf{G}}(W).\beta + (\dim_{\mathbb{C}} X - 3)(1-g) + k.$$

See [7] for details.

Especially, for a Calabi–Yau variety X, since $c_1^{\mathbf{G}}(W) = 0$, every $[Q^\varepsilon_{1,0}(X,\beta)]^{\mathrm{vir}}$ for any $\beta \neq 0$ has degree 0. So, we can define the generating function

$$\langle\,\rangle^\varepsilon_{1,0} := \sum_{\beta \neq 0} q^\beta \deg[Q^\varepsilon_{1,0}(X,\beta)]^{\mathrm{vir}}$$

for each ε. In particular, when ε is small enough, that is, $\varepsilon = 0+$, it is called the elliptic quasi-map potential function of X.

Throughout this paper, let X be a total space of vector bundles

$$\mathcal{O}_{\mathbb{P}^{n-1}}(-l'_1)|_{X'} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-l'_2)|_{X'} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-l'_m)|_{X'}$$

over X', where X' is a complete intersection in $\mathbb{P}^{n-1}$ defined by deg l_i polynomials for $i = 1, 2, \ldots, r$ and $l_a, l'_b > 0$ for all a, b. We assume the Calabi–Yau condition

$$\sum_a l_a + \sum_b l'_b = n.$$

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Note that X has a natural GIT representation and is a Calabi–Yau variety. In this paper, we give an explicit formula of this elliptic quasi-map potential function for this X. Kim and Lho [13] already computed the elliptic quasi-map potential function in the case $m = 0$. Basically, we follow their idea to prove the main theorem, which we introduce, except for the computational part. It is as follows: By the quantum Lefschetz hyperplane section theorem [12], quasi-map invariants of X can be represented as twisted quasi-map invariants of $\mathbb{P}^{n-1}$, which also have natural GIT representations. Moreover, we apply the torus localization theorem for the latter since $\mathbb{P}^{n-1}$ has a natural torus action.

To state the main theorem, we first need some preparations. Givental [8] introduced the equivariant I-function for X, which is an $H^*_{\mathbf{T}}(\mathbb{P}^{n-1}) \otimes \mathbb{Q}(\lambda, \zeta)$-valued formal function in formal variables q, z, t_H:

$$
\begin{aligned}
&I^{\zeta}_{\mathbf{T}}(t, q) \\
&\quad := e^{t_H H/z} \sum_{d=0}^{\infty} q^d e^{t_H d} \\
&\qquad \times \frac{\prod_{j=1}^{r} \prod_{k=1}^{l_j d} (l_j H + kz) \prod_{j=1}^{m} \prod_{k=0}^{l'_j d - 1} (-l'_j H - kz + \zeta)}{\prod_{k=1}^{d} \prod_{j=1}^{n} (H - \lambda_j + kz)},
\end{aligned}
$$

where $\mathbf{T} = (\mathbb{C}^*)^n$ is the torus group acting on $\mathbb{P}^{n-1}$, $\lambda_1, \ldots, \lambda_n$ are the $\mathbf{T}$-equivariant parameters, and ζ is the $\mathbb{C}^*$-equivariant parameter for $\mathbb{C}^*$-action acting diagonally on the fiber of X over X'; $\mathbb{Q}(\lambda, \zeta)$ denotes the quotient field of the polynomial ring in $\lambda_1, \ldots, \lambda_n, \zeta$, H is the hyperplane class, and $t = t_H H$. Denote by $I_{\mathbf{T}}$ the specialization of $I^{\zeta}_{\mathbf{T}}$ with

$$\zeta = 0.$$

Denote by $\underline{I}_{\mathbf{T}}$ the specialization of $I_{\mathbf{T}}$ with

$$\lambda_i = \exp\left(2\pi i \sqrt{-1}/n\right), \quad i = 1, \ldots, n. \tag{1.1}$$

Let us define the formal functions $B_k(q, z) \in \mathbb{Q}[H]/(H^n - 1) \otimes_{\mathbb{Q}} \mathbb{Q}[[q, \frac{1}{z}]]$ and $C_k(q) \in \mathbb{Q}[[q]]$ for $k = 0, 1, \ldots, n-1$ inductively as follows. First, set $B_0 := \underline{I}_{\mathbf{T}}(0, q)$ and choose $C_0(q)$ by a coefficient of $1 = H^0$ in $B_0(q, z = \infty)$. Now, suppose $B_{k-1}(q, z)$ and $C_{k-1}(q)$ are defined. Then, define $B_k(q, z)$ by

$$B_k := \left(H + zq\frac{d}{dq}\right) \frac{B_{k-1}}{C_{k-1}(q)},$$

and $C_k(q)$ by the coefficient of H^k in $B_k(q, \infty)$. We can easily check that $C_k(q)$, $k = 0, 1, \ldots, n-1$, which are the so-called initial constants, are of the form $1 + O(q)$, and also that

$$C_k(q) H^k = B_k(q, \infty), \quad k = 0, 1, \ldots, n-1.$$

Note that $\mathbb{Q}[H]/(H^n - 1)$ is isomorphic to $H^*_{\mathbf{T}}(\mathbb{P}^{n-1})$ modulo (1.1).

Now we are ready to state the main theorem.

THEOREM 1.1.

$$\langle\,\rangle^{0+}_{1,0} = -\frac{3(n-1-r-m)^2+n-r+m-3}{48}\log\Big(1-q\prod_{a=1}^{r} l_a^{l_a}\prod_{b=1}^{m}(-l'_b)^{l'_b}\Big) - \frac{1}{2}\sum_{k=m}^{n-r-2}\binom{n-r-k}{2}\log C_k(q).$$

Define I_0^ζ and I_1^ζ by the $1/z$-expansion of

$$I^\zeta_{\mathbf{T}}|_{t=0} = I_0^\zeta + I_1^\zeta/z + O(1/z^2).$$

Denote by I_0 and I_1 the specializations of I_0^ζ and I_1^ζ with $\zeta = 0$. It is easy to check that $I_0^\zeta = I_0 = C_0$. Ciocan-Fontanine and Kim [5] proved the wall-crossing formula.

THEOREM 1.2 ([5]).

$$\langle\,\rangle^{0+}_{1,0} = -\frac{1}{24}\chi_{\mathrm{top}}(X)\log I_0 - \frac{1}{24}\int_X \frac{I_1^\zeta}{I_0}c_{\dim X-1}(T_X) + \langle\,\rangle^{\infty}_{1,0}|_{q^d\mapsto q^d\exp(\int_{d[\mathrm{line}]} I_1/I_0)}.$$

Here, we consider $c_{\dim X-1}(T_X)$ as an equivariant Chern class to define integration on X by localization. Combining these two theorems, we get the following theorem.

THEOREM 1.3.

$$\begin{aligned}\langle\,\rangle^{\infty}_{1,0}|_{q^d\mapsto q^d\exp(\int_{d[\mathrm{line}]} I_1/I_0)} &= \frac{1}{24}\chi_{\mathrm{top}}(X)\log I_0 + \frac{1}{24}\int_X \frac{I_1^\zeta}{I_0}c_{\dim X-1}(T_X)\\ &\quad - \frac{3(n-1-r-m)^2+n-r+m-3}{48}\log\Big(1-q\prod_{a=1}^{r} l_a^{l_a}\prod_{b=1}^{m}(-l'_b)^{l'_b}\Big)\\ &\quad - \frac{1}{2}\sum_{k=m}^{n-r-2}\binom{n-r-k}{2}\log C_k(q).\end{aligned}$$

When $m = 0$, Theorem 1.3 gives another proof of the result in [13]. Also, when $r = 0$, it gives another proof of the result in [11]. If both are nonzero, then this gives a new result.

2. Elliptic Quasimap Potential Function of X

In this section, we simplify the elliptic quasi-map potential function of X. We closely follow the notation in [13] and state the results in [13] without proof in this and next sections.

2.1. Quantum Lefschetz Theorem and Divisor Axiom

We will write elliptic quasi-map potential function as a generating function of quasi-map invariants of $\mathbb{P}^{n-1}$, which is much easier to deal with. Consider $Q^{0+}_{g,k}(\mathbb{P}^{n-1}, d)$. Here and further, since the degree of a quasi-map to $\mathbb{P}^{n-1}$ can be regarded as a nonnegative integer, we used the notation d instead of β. Denote by f the universal map from the universal curve $\mathcal{C}$ of $Q^{0+}_{g,k}(\mathbb{P}^{n-1}, d)$ to the stack quotient $[\mathbb{C}^n/\mathbb{C}^*]$:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & [\mathbb{C}^n/\mathbb{C}^*] . \\ \downarrow{\scriptstyle \pi} & & \\ Q^{0+}_{g,k}(\mathbb{P}^{n-1}, d) & & \end{array}$$

Since the domain curves of objects in $Q^{0+}_{g,k}(\mathbb{P}^{n-1}, d)$ have no rational tails for any g and k, every irreducible component with genus 1 in the domain curves of objects in $Q^{0+}_{1,0}(\mathbb{P}^{n-1}, d)$ must have a positive degree if it exists. So, we can apply the quantum Lefschetz hyperplane section theorem in [12] to get following formula:

$$(j_{g,k,d})_*[Q^{0+}_{g,k}(X, d)]^{\mathrm{vir}} = \mathrm{e}(E_{g,k,d} \oplus E'_{g,k,d}) \cap [Q^{0+}_{g,k}(\mathbb{P}^{n-1}, d)]^{\mathrm{vir}}$$

for $g = 0$, $k = 2, 3, \dots$ and $g = 1$, $k = 0$, $d > 0$. Here,

$$j_{g,k,d} : Q^{0+}_{g,k}(X, d) \cong Q^{0+}_{g,k}(X', d) \hookrightarrow Q^{0+}_{g,k}(\mathbb{P}^{n-1}, d)$$

if $d > 0$ and

$$j_{0,k,0} : Q^{0+}_{0,k}(X, 0) \cong X \times \overline{M}_{0,k} \to \mathbb{P}^{n-1} \times \overline{M}_{0,k} \cong Q^{0+}_{0,k}(\mathbb{P}^{n-1}, 0),$$

where $\overline{M}_{0,k}$ is the moduli stack of stable curves with genus 0 and k-markings. Also,

$$E_{g,k,d} = R^0\pi_* f^*[(E \times \mathbb{C}^n)/\mathbb{C}^*],\ E'_{g,k,d} = R^1\pi_* f^*[(E' \times \mathbb{C}^n)/\mathbb{C}^*],$$

where $E = \bigoplus_a E_a$ (resp. $E' = \bigoplus_b E'_b$), E_a (resp. E'_b) is the one-dimensional $\mathbb{C}^*$-representation space with weight $l_a\theta$ (resp. $-l'_b\theta$), where $\theta : \mathbb{C}^* \to \mathbb{C}^*$ is the identity map, and e stands for the Euler class. So, we can rewrite the potential function as

$$\langle\,\rangle^{0+}_{1,0} = \sum_{d \neq 0} q^d \deg(\mathrm{e}(E_{1,0,d} \oplus E'_{1,0,d}) \cap [Q^{0+}_{1,0}(\mathbb{P}^{n-1}, d)]^{\mathrm{vir}}). \tag{2.1}$$

On the other hand, denote by

$$Q^{0+,0+}_{g,k|s}(\mathbb{P}^{n-1}, d)$$

the moduli space of genus g, degree class d stable quasi-maps to $\mathbb{P}^{n-1}$ with ordinary k pointed markings and infinitesimally weighted s pointed markings. We can also define their natural virtual fundamental classes

$$[Q^{0+,0+}_{g,k|s}(\mathbb{P}^{n-1}, d)]^{\mathrm{vir}}$$

(see §2 of [4]). Using this, we define the invariants

$$\begin{aligned}&\langle \gamma_1\psi^{a_1},\ldots,\gamma_k\psi^{a_k};\delta_1,\ldots,\delta_s\rangle^{0+,0+}_{g,k|s,d}\\&\quad=\int_{\mathrm{e}(E_{g,k|s,d}\oplus E'_{g,k|s,d})\cap[Q^{0+,0+}_{g,k|s}(\mathbb{P}^{n-1},d)]^{\mathrm{vir}}}\prod_i ev_i^*(\gamma_i)\psi_i^{a_i}\prod_j \hat{ev}_j^*(\delta_j)\end{aligned}$$

for $\gamma_i\in H^*(\mathbb{P}^{n-1})\otimes\mathbb{Q}(\lambda)$, $\delta_j\in H^*([\mathbb{C}^n/\mathbb{C}^*],\mathbb{Q})$, and for $g=0, k=2,3,\ldots$ and $g=1$, $k=0$, $d>0$, where ψ_i is the psi-class associated with the ith marking, and ev_i (resp. $\hat{ev}_j$) is the evaluation map to $\mathbb{P}^{n-1}$ (resp. $[\mathbb{C}^n/\mathbb{C}^*]$) at the ith (resp. jth) marking (resp. infinitesimally weighted marking); $E_{g,k|s,d}=R^0\pi_*f^*[(E\times\mathbb{C}^n)/\mathbb{C}^*]$, $E'_{g,k|s,d}=R^1\pi_*f^*[(E'\times\mathbb{C}^n)/\mathbb{C}^*]$ with f and π defined as

$$\begin{array}{ccc}\mathcal{C} & \xrightarrow{\ f\ } & [\mathbb{C}^n/\mathbb{C}^*]\\ {\scriptstyle\pi}\big\downarrow & & \\ Q^{0+,0+}_{g,k|s}(\mathbb{P}^{n-1},d) & & \end{array}$$

where $\mathcal{C}$ is the universal curve. Note that in the definition of invariants, there is a constraint on g and k because the quantum Lefschetz theorem holds only in this case. So, in this case, we can interpret these invariants as invariants for X that are basically defined without constraint on g and k.

Here, we are focusing only on $Q^{0+,0+}_{1,0|1}(\mathbb{P}^{n-1},d)$, which is isomorphic to the universal curve of $Q^{0+}_{1,0}(\mathbb{P}^{n-1},d)$. Define the generating function

$$\langle\tilde{H}\rangle^{0+}_{1,0|1}:=\sum_{d=1}^{\infty}q^d\langle;\tilde{H}\rangle^{0+,0+}_{1,0|1,d},$$

where $\tilde{H}\in H^2([\mathbb{C}^n/\mathbb{C}^*],\mathbb{Q})$ is the hyperplane class. Then, by the divisor axiom we have

$$q\frac{d}{dq}\langle\,\rangle^{0+}_{1,0}=\langle\tilde{H}\rangle^{0+}_{1,0|1}\tag{2.2}$$

2.2. *Localization*

Now, we will calculate it by using **T**-equivariant quasi-map theory. Recall that $\mathbf{T}=(\mathbb{C}^*)^n$ is the n-dimensional torus acting on $\mathbb{P}^{n-1}$ in a standard way. Let $\{p_i\}_i$ be the set of **T**-fixed points of $\mathbb{P}^{n-1}$. The **T**-fixed loci of $Q^{0+,0+}_{1,0|1}(\mathbb{P}^{n-1},d)$ can be divided into two types according to whether the reduced image is a point in $\mathbb{P}^{n-1}$ or not. A quasi-map is called a vertex type over p_i if its regularization map is constant over p_i. For the definition of regularization map, see [7]. Otherwise, the quasi-map in $Q^{0+,0+}_{1,0|1}(\mathbb{P}^{n-1},d)^{\mathbf{T}}$ is said to be of loop type. A loop-type quasi-map is said to be of loop type over p_i if the infinitesimally weighted marking of the quasi-map maps to p_i.

Define $Q^{\mathbf{T}}_{\mathrm{vert},i,d}$ to be the substack of $Q^{0+,0+}_{1,0|1}(\mathbb{P}^{n-1},d)^{\mathbf{T}}$ consisting of vertex type over p_i. Define $Q^{\mathbf{T}}_{\mathrm{loop},i,d}$ to be the substack of $Q^{0+,0+}_{1,0|1}(\mathbb{P}^{n-1},d)^{\mathbf{T}}$ consisting of loop type over p_i.

By the virtual localization theorem, $\langle\tilde{H}\rangle^{0+}_{1,0|1}$ can be divided into the sum of the localization contribution $\mathbf{Vert}_i$ from all the vertex types over $p_i \in (\mathbb{P}^{n-1})^{\mathbf{T}}$ and the localization contribution $\mathbf{Loop}_i$ from all the loop types over $p_i \in (\mathbb{P}^{n-1})^{\mathbf{T}}$; that is,

$$\langle\tilde{H}\rangle^{0+}_{1,0|1} := \sum_i \mathbf{Vert}_i + \sum_i \mathbf{Loop}_i, \tag{2.3}$$

where

$$\mathbf{Vert}_i := \sum_{d\neq 0} q^d \int_{[Q^{\mathbf{T}}_{\mathrm{vert},i,d}]^{\mathrm{vir}}} \frac{\mathrm{e}^{\mathbf{T}}(E_{1,0|1,d}\oplus E'_{1,0|1,d})|_{Q^{\mathbf{T}}_{\mathrm{vert},i,d}}\hat{ev}_1^*(\tilde{H})}{\mathrm{e}^{\mathbf{T}}(N^{\mathrm{vir}}_{Q^{\mathbf{T}}_{\mathrm{vert},i,d}/Q^{0+,0+}_{1,0|1}(\mathbb{P}^{n-1},d)})},$$

$$\mathbf{Loop}_i := \sum_{d\neq 0} q^d \int_{[Q^{\mathbf{T}}_{\mathrm{loop},i,d}]^{\mathrm{vir}}} \frac{\mathrm{e}^{\mathbf{T}}(E_{1,0|1,d}\oplus E'_{1,0|1,d})|_{Q^{\mathbf{T}}_{\mathrm{loop},i,d}}\hat{ev}_1^*(\tilde{H})}{\mathrm{e}^{\mathbf{T}}(N^{\mathrm{vir}}_{Q^{\mathbf{T}}_{\mathrm{loop},i,d}/Q^{0+,0+}_{1,0|1}(\mathbb{P}^{n-1},d)})},$$

where $\mathrm{e}^{\mathbf{T}}$ stands for the $\mathbf{T}$-equivariant Euler class, and $N^{\mathrm{vir}}_{Q^{\mathbf{T}}_{\mathrm{vert},i,d}/Q^{0+,0+}_{1,0|1}(\mathbb{P}^{n-1},d)}$ (resp. $N^{\mathrm{vir}}_{Q^{\mathbf{T}}_{\mathrm{loop},i,d}/Q^{0+,0+}_{1,0|1}(\mathbb{P}^{n-1},d)}$) is the virtual normal bundle of $Q^{\mathbf{T}}_{\mathrm{vert},i,d}$ (resp. $Q^{\mathbf{T}}_{\mathrm{loop},i,d}$) into $Q^{0+,0+}_{1,0|1}(\mathbb{P}^{n-1},d)$. Here, we regard the hyperplane class $\tilde{H}$ as a $\mathbf{T}$-equivariant class in $H^2_{\mathbf{T}}([\mathbb{C}^n/\mathbb{C}^*],\mathbb{Q})$.

2.2.1. Vertex Term. Let

$$Q^{0+}_{1,0}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}$$

be the $\mathbf{T}$-fixed part of $Q^{0+}_{1,0}(\mathbb{P}^{n-1},d)$ the elements of which have domain components only over p_i under the regularization map. Then, $Q^{\mathbf{T}}_{\mathrm{vert},i,d}$ is isomorphic to the universal curve of $Q^{0+}_{1,0}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}$. So, by the divisor axiom,

$$\mathbf{Vert}_i = q\frac{d}{dq}\sum_{d\neq 0} q^d \int_{[Q^{0+}_{1,0}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}]^{\mathrm{vir}}} \frac{\mathrm{e}^{\mathbf{T}}(E_{1,0,d}\oplus E'_{1,0,d})|_{Q^{0+}_{1,0}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}}}{\mathrm{e}^{\mathbf{T}}(N^{\mathrm{vir}}_{Q^{0+}_{1,0}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}/Q^{0+}_{1,0}(\mathbb{P}^{n-1},d)})}.$$

On the other hand,

$$\begin{aligned}
&\mathrm{e}^{\mathbf{T}}(E_{1,0,d})|_{Q^{0+}_{1,0}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}}\\
&\quad= \mathrm{e}^{\mathbf{T}}\Big(\prod_a \pi_*\mathcal{O}_{\mathcal{C}}(l_a\hat{\mathbf{x}})\otimes E_a\Big)\\
&\quad= \mathrm{e}^{\mathbf{T}}\Big(\prod_a \pi_*\mathcal{O}_{l_a\hat{\mathbf{x}}}(l_a\hat{\mathbf{x}})\otimes E_a\Big)\mathrm{e}^{\mathbf{T}}\Big(\prod_a R\pi_*\mathcal{O}_{\mathcal{C}}\otimes E_a\Big)\\
&\quad= \mathrm{e}^{\mathbf{T}}\Big(\prod_a \pi_*\mathcal{O}_{l_a\hat{\mathbf{x}}}(l_a\hat{\mathbf{x}})\otimes E_a\Big)\frac{\mathrm{e}^{\mathbf{T}}(\prod_a \pi_*\mathcal{O}_{\mathcal{C}}\otimes E_a)}{\mathrm{e}^{\mathbf{T}}(\prod_a R^1\pi_*\mathcal{O}_{\mathcal{C}}\otimes E_a)},
\end{aligned}$$

where $\hat{\mathbf{x}}$ is base loci on a universal curve $\mathcal{C}$, and π is a projection from $\mathcal{C}$ to $Q^{0+}_{1,0}(\mathbb{P}^{n-1}, d)^{\mathbf{T}, p_i}$. The first equality comes from the idea in [6], and the second equality comes from the long exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(l_a \hat{\mathbf{x}}) \to \mathcal{O}_{l_a \hat{\mathbf{x}}}(l_a \hat{\mathbf{x}}) \to 0.$$

Similarly, we can show that

$$\mathrm{e}^{\mathbf{T}}(E'_{1,0,d})|_{Q^{0+}_{1,0}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}} = \mathrm{e}^{\mathbf{T}}\left(\prod_b \pi_* \mathcal{O}_{l'_b \hat{\mathbf{x}}} \otimes E'_b\right) \frac{\mathrm{e}^{\mathbf{T}}(\prod_b R^1 \pi_* \mathcal{O}_{\mathcal{C}} \otimes E'_b)}{\mathrm{e}^{\mathbf{T}}(\prod_b \pi_* \mathcal{O}_{\mathcal{C}} \otimes E'_b)}.$$

Also, we can see that

$$\begin{aligned} &\mathrm{e}^{\mathbf{T}}(N^{\mathrm{vir}}_{Q^{0+}_{1,0}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}/Q^{0+}_{1,0}(\mathbb{P}^{n-1},d)}) \\ &= \mathrm{e}^{\mathbf{T}}(R\pi_* \mathcal{O}_{\mathcal{C}}(\hat{\mathbf{x}}) \otimes T_{p_i}\mathbb{P}^{n-1}) \\ &= \mathrm{e}^{\mathbf{T}}(\pi_* \mathcal{O}_{\hat{\mathbf{x}}}(\hat{\mathbf{x}}) \otimes T_{p_i}\mathbb{P}^{n-1}) \frac{\mathrm{e}^{\mathbf{T}}(\pi_* \mathcal{O}_{\mathcal{C}} \otimes T_{p_i}\mathbb{P}^{n-1})}{\mathrm{e}^{\mathbf{T}}(R^1 \pi_* \mathcal{O}_{\mathcal{C}} \otimes T_{p_i}\mathbb{P}^{n-1})}. \end{aligned}$$

On the other hand,

$$Q^{0+}_{1,0}(\mathbb{P}^{n-1}, d)^{\mathbf{T}, p_i} \cong \overline{M}_{1,0|d}/S_d,$$

where S_d is the symmetric group of degree d acting on $\overline{M}_{1,0|d}$ by a permutation of infinitesimally weighted markings. Furthermore, $\overline{M}_{1,0|d}$ is smooth, and

$$[Q^{0+}_{1,0}(\mathbb{P}^{n-1}, d)^{\mathbf{T}, p_i}]^{\mathrm{vir}} = \frac{1}{d!}[\overline{M}_{1,0|d}]$$

under the isomorphism. Here, $[\overline{M}_{1,0|d}]$ is the fundamental class of $\overline{M}_{1,0|d}$. Therefore,

$$\mathbf{Vert}_i = q\frac{d}{dq} \sum_{d \neq 0} \frac{q^d}{d!} \int_{\overline{M}_{1,0|d}} (1 + c_i(\lambda)\mathrm{e}(\mathbb{E})) F_{i,d},$$

where

$$F_{i,d} := \frac{\mathrm{e}^{\mathbf{T}}(\prod_a \pi_* \mathcal{O}_{l_a \hat{\mathbf{x}}}(l_a \hat{\mathbf{x}}) \otimes E_a) \mathrm{e}^{\mathbf{T}}(\prod_b \pi_* \mathcal{O}_{l'_b \hat{\mathbf{x}}} \otimes E'_b)}{\mathrm{e}^{\mathbf{T}}(\pi_* \mathcal{O}_{\hat{\mathbf{x}}}(\hat{\mathbf{x}}) \otimes T_{p_i}\mathbb{P}^{n-1})} \tag{2.4}$$

with $\hat{\mathbf{x}} := \sum_{j=1}^d \hat{x}_j$, the sum of loci of infinitesimally weighted markings in the universal curve; $\mathbb{E} := (R^1 \pi_* \mathcal{O}_{\mathcal{C}})^\vee$ is the Hodge bundle on $\overline{M}_{1,0|d}$, and $c_i(\lambda)$ is the element in $\mathbb{Q}(\lambda)$ uniquely determined by

$$1 + c_i(\lambda)\mathrm{e}(\mathbb{E}) = \frac{\mathrm{e}^{\mathbf{T}}(\mathbb{E}^\vee \otimes T_{p_i}\mathbb{P}^{n-1}) \mathrm{e}^{\mathbf{T}}(\mathcal{O}_{\overline{M}_{1,0|d}} \otimes E|_{p_i}) \mathrm{e}^{\mathbf{T}}(\mathbb{E}^\vee \otimes E'|_{p_i})}{\mathrm{e}^{\mathbf{T}}(\mathcal{O}_{\overline{M}_{1,0|d}} \otimes T_{p_i}\mathbb{P}^{n-1}) \mathrm{e}^{\mathbf{T}}(\mathbb{E}^\vee \otimes E|_{p_i}) \mathrm{e}^{\mathbf{T}}(\mathcal{O}_{\overline{M}_{1,0|d}} \otimes E'|_{p_i})}.$$

So, by a simple computation we can check that

$$c_i(\lambda) = \sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i} + \sum_a \frac{1}{l_a \lambda_i} + \sum_b \frac{1}{l'_b \lambda_i}.$$

Note that $c_i(\lambda)$ is independent of d and that $\mathrm{e}(\mathbb{E})^2 = 0$ because $\mathbb{E}$ comes from the Hodge bundle on $\overline{M}_{1,1}$.

By the same argument as in [13], we can relate the genus one invariants with the genus zero invariants.

PROPOSITION 2.1.

$$24\sum_{d\neq 0}\frac{q^d}{d!}\int_{\overline{M}_{1,0|d}} \mathrm{e}(\mathbb{E})F_{i,d} = \sum_{d\neq 0}\frac{q^d}{d!}\int_{\overline{M}_{0,2|d}} F_{i,d},$$

$$e^{24\sum_{d\neq 0} q^d/d!\int_{\overline{M}_{1,0|d}} F_{i,d}} = \sum_{d\neq 0}\frac{q^d}{d!}\int_{\overline{M}_{0,3|d}} F_{i,d},$$

where the classes $F_{i,d}$ on $\overline{M}_{0,2|d}$ and $\overline{M}_{0,3|d}$ are defined in the same way as in (2.4).

In conclusion, we have

$$\mathbf{Vert}_i = \frac{q}{24}\frac{d}{dq}\left(c_i(\lambda)\sum_{d\neq 0}\frac{q^d}{d!}\int_{\overline{M}_{0,2|d}} F_{i,d} + \log\left(\sum_{d\neq 0}\frac{q^d}{d!}\int_{\overline{M}_{0,3|d}} F_{i,d}\right)\right). \tag{2.5}$$

3. Localized Invariants

3.1. Localized Generating Functions in Genus Zero Theory

In order to do equivariant quasi-map theory of $\mathbb{P}^{n-1}$ instead of that of X, we need to use $E \times E'$-twisted Poincaré metric on $H^*_{\mathbf{T}}(\mathbb{P}^{n-1}) \otimes \mathbb{Q}(\lambda)$, that is, for $a, b \in H^*_{\mathbf{T}}(\mathbb{P}^{n-1}) \otimes \mathbb{Q}(\lambda)$,

$$\langle a, b\rangle^{E\times E'} = \int_{\mathbb{P}^{n-1}} \frac{\mathrm{e}^{\mathbf{T}}(E)\cup a\cup b}{\mathrm{e}^{\mathbf{T}}(E')},$$

where $\mathrm{e}^{\mathbf{T}}(E)$ (resp. $\mathrm{e}^{\mathbf{T}}(E')$) is the $\mathbf{T}$-equivariant Euler class of E (resp. E'). Here, we used the notation E (resp. E') instead of $[(E \times (\mathbb{C}^n\backslash\{0\}))/\mathbb{C}^*] \cong \bigoplus_a \mathcal{O}_{\mathbb{P}^{n-1}}(l_a)$ (resp. $[(E' \times (\mathbb{C}^n\backslash\{0\}))/\mathbb{C}^*] \cong \bigoplus_b \mathcal{O}_{\mathbb{P}^{n-1}}(-l'_b)$) by avoiding abuse of notation. Let ϕ_i be the basis of $H^*_{\mathbf{T}}(\mathbb{P}^{n-1}) \otimes \mathbb{Q}(\lambda)$ such that

$$\phi_i|_{p_j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and let ϕ^i be its dual basis with respect to $E \times E'$-twisted Poincaré metric.

Also, as in [13], we need to use the twisted virtual fundamental class

$$\mathrm{e}^{\mathbf{T}}(E_{0,k,d} \oplus E'_{0,k,d}) \cap [Q^{0+}_{0,k}(\mathbb{P}^{n-1}, d)]^{\mathrm{vir}}$$

in genus zero quasi-map theory. By using this, we will define local correlators. Let

$$Q^{0+}_{0,k}(\mathbb{P}^{n-1}, d)^{\mathbf{T}, p_i}$$

be the **T**-fixed part of $Q^{0+}_{0,k}(\mathbb{P}^{n-1},d)$ with elements having domain components only over p_i. By using the twisted virtual fundamental class

$$\frac{\mathrm{e}^{\mathbf{T}}(E_{0,k,d}\oplus E'_{0,k,d})\cap[Q^{0+}_{0,k}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}]^{\mathrm{vir}}}{\mathrm{e}^{\mathbf{T}}(N^{\mathrm{vir}}_{Q^{0+}_{0,k}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}/Q^{0+}_{0,k}(\mathbb{P}^{n-1},d)})}$$

we define it as follows:

$$\begin{aligned}
&\langle\gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k}\rangle^{0+,p_i}_{0,k,d}\\
&\quad:=\int_{\frac{\mathrm{e}^{\mathbf{T}}(E_{0,k,d}\oplus E'_{0,k,d})\cap[Q^{0+}_{0,k}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}]^{\mathrm{vir}}}{\mathrm{e}^{\mathbf{T}}(N^{\mathrm{vir}}_{Q^{0+}_{0,k}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}/Q^{0+}_{0,k}(\mathbb{P}^{n-1},d)})}}\prod_i ev_i^*(\gamma_i)\psi_i^{a_i};\\
&\langle\langle\gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k}\rangle\rangle^{0+,p_i}_{0,k}\\
&\quad:=\sum_{s,d}\frac{q^d}{s!}\langle\gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k},t,\dots,t\rangle^{0+,p_i}_{0,k+s,d},\quad\text{for } t\in H^*_{\mathbf{T}}(\mathbb{P}^{n-1})\otimes\mathbb{Q}(\lambda),
\end{aligned}$$

where ψ_i is the psi-class associated with the ith marking, ev_i is the ith evaluation map, and q is a formal Novikov variable. Here, we used the notation $E_{0,k,d}$ (resp. $E'_{0,k,d}$) instead of $E_{0,k,d}|_{Q^{0+}_{0,k}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}}$ (resp. $E'_{0,k,d}|_{Q^{0+}_{0,k}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}}$) by avoiding abuse of notation.

Let z be a formal variable. We define the following **T**-local generating functions:

$$\begin{aligned}
D_i&:=e_i\langle\langle 1,1,1\rangle\rangle^{0+,p_i}_{0,3}=1+O(q),\\
u_i&:=e_i\langle\langle 1,1\rangle\rangle^{0+,p_i}_{0,2}=t|_{p_i}+O(q),\\
S^{0+,p_i}_t(\gamma)&:=e_i\left\langle\!\!\left\langle\frac{1}{z-\psi},\gamma\right\rangle\!\!\right\rangle^{0+,p_i}_{0,2}\\
&=e^t\gamma|_{p_i}+O(q)\quad\text{for }\gamma\in H^*_{\mathbf{T}}(\mathbb{P}^{n-1})\otimes\mathbb{Q}(\lambda)[[q]],\\
J^{0+,p_i}&:=e_i\left\langle\!\!\left\langle\frac{1}{z(z-\psi)}\right\rangle\!\!\right\rangle^{0+,p_i}_{0,1}=e^t|_{p_i}+O(q),
\end{aligned}$$

where the unstable terms of $S^{0+,p_i}_t(\gamma)$ and J^{0+,p_i} are defined by using the quasimap graph spaces $QG^{0+}_{0,0,d}(\mathbb{P}^{n-1})$ or $QG^{0+}_{0,1,0}(\mathbb{P}^{n-1})$ as in [2; 3]. Also, the unstable term of u_i (this is the only case of $m=d=0$) is defined to be 0. So, in particular,

$$J^{0+,p_i}|_{t=0}=I_{\mathbf{T}}|_{t=0,p_i}. \tag{3.1}$$

Here the front terms e_i are defined by the formulas $\phi^i=e_i\phi_i$. The parameter z naturally appears as the $\mathbb{C}^*$-equivariant parameter in the graph construction (see §4 of [3]). It is originated from the $\mathbb{C}^*$-action on $\mathbb{P}^1$.

On the other hand, it is easy to check that

$$\sum_{d\neq 0}\frac{q^d}{d!}\int_{\overline{M}_{0,2|d}} F_{i,d} = u_i|_{t=0}, \qquad \sum_{d\neq 0}\frac{q^d}{d!}\int_{\overline{M}_{0,3|d}} F_{i,d} = D_i|_{t=0}.$$

So, applying it to (2.5), we have

$$\mathbf{Vert}_i = \frac{q}{24}\frac{d}{dq}\bigg(\Big(\sum_{j\neq i}\frac{1}{\lambda_j-\lambda_i}+\sum_a \frac{1}{l_a\lambda_i}+\sum_b\frac{1}{l'_b\lambda_i}\Big)u_i|_{t=0} + \log D_i|_{t=0}\bigg). \tag{3.2}$$

To describe this by using I-function for X, we need more generating functions. Denote by $QG^{0+}_{0,k,d}(\mathbb{P}^{n-1})$ the quasi-map graph spaces (see [3]) and by

$$QG^{0+}_{0,k,d}(\mathbb{P}^{n-1})^{\mathbf{T},p_i}$$

the $\mathbf{T}$-fixed part of $QG^{0+}_{0,k,d}(\mathbb{P}^{n-1})$ with elements having domain components only over p_i. As in [13], we define the invariants and generating functions on the graph spaces: for $\gamma_i \in H^*_{\mathbf{T}}(\mathbb{P}^{n-1})\otimes H^*_{\mathbb{C}^*}(\mathbb{P}^1)\otimes\mathbb{Q}(\lambda)$,

$$\begin{aligned}
&\langle \gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k}\rangle^{QG^{0+},p_i}_{k,d}\\
&\quad = \int_{\frac{\mathrm{e}^{\mathbf{T}}(R^0\pi_*f^*E\oplus R^1\pi_*f^*E')\cap[QG^{0+}_{0,k,d}(\mathbb{P}^{n-1})^{\mathbf{T},p_i}]^{\mathrm{vir}}}{\mathrm{e}^{\mathbf{T}}(N^{\mathrm{vir}}_{QG^{0+}_{0,k,d}(\mathbb{P}^{n-1})^{\mathbf{T},p_i}/QG^{0+}_{0,k,d}(\mathbb{P}^{n-1})})}}\prod_i ev_i^*(\gamma_i)\psi_i^{a_i};\\
&\langle\langle \gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k}\rangle\rangle^{QG^{0+},p_i}_{k}\\
&\quad = \sum_{s,d}\frac{q^d}{s!}\langle\gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k},t,\dots,t\rangle^{QG^{0+},p_i}_{k+s,d}, \quad \text{for } t\in H^*_{\mathbf{T}}(\mathbb{P}^{n-1})\otimes\mathbb{Q}(\lambda).
\end{aligned}$$

Here we denote by ev_i the ith evaluation map to $\mathbb{P}^{n-1}\times\mathbb{P}^1$ from the quasi-map graph spaces and regard t as the element $t\otimes 1$ in $H^*_{\mathbf{T}}(\mathbb{P}^{n-1})\otimes H^*_{\mathbb{C}^*}(\mathbb{P}^1)\otimes\mathbb{Q}(\lambda)$. We used the notation E (resp. E') instead of $[(E\times(\mathbb{C}^n))/\mathbb{C}^*]$ (resp. $[(E'\times(\mathbb{C}^n))/\mathbb{C}^*]$) by avoiding abuse of notation. The maps f and π are defined as follows:

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ f\ } & [\mathbb{C}^n/\mathbb{C}^*]\ . \\
\Big\downarrow{\scriptstyle\pi} & & \\
QG^{0+}_{0,k,d}(\mathbb{P}^{n-1})^{\mathbf{T},p_i} & &
\end{array}$$

Here, $\mathcal{C}$ is the universal curve.

Let $\mathbf{p}_\infty$ be the equivariant cohomology class in $H^*_{\mathbb{C}^*}(\mathbb{P}^1)$ defined by

$$\mathbf{p}_\infty|_0 = 0, \qquad \mathbf{p}_\infty|_\infty = -z.$$

Exactly as in [13], we can have the following factorization.

PROPOSITION 3.1.

$$J^{0+,p_i} = S_t^{0+,p_i}(P^{0+,p_i}),$$

where

$$P^{0+,p_i} := e_i \langle\langle 1 \otimes \mathbf{p}_\infty \rangle\rangle_1^{QG^{0+,p_i}}.$$

By the uniqueness lemma in §7.7 of [3] we have

$$S_t^{0+,p_i}(\gamma) = e^{u_i/z}\gamma|_{p_i}.$$

Hence Proposition 3.1 gives the expression

$$J^{0+,p_i} = e^{u_i/z}(r_{i,0} + O(z)), \tag{3.3}$$

where $r_{i,0} \in \mathbb{Q}(\lambda)[\![t,q]\!]$ is the constant term of P^{0+,p_i} in z. By the following result we can easily see that expression (3.3) is unique.

COROLLARY 3.2. *The equality*

$$\log J^{0+,p_i} = u_i/z + \log r_{i,0} + O(z) \in \mathbb{Q}(\lambda)((z))[\![t,q]\!]$$

holds as Laurent series of z over the coefficient ring $\mathbb{Q}(\lambda)$ in each power expansion of t and q, after regarding t as a formal element.

Also, as in [13], we have the following result.

COROLLARY 3.3.

$$D_i|_{t=0} = \frac{1}{r_{i,0}|_{t=0}}.$$

In conclusion, applying these to (3.2), we have

$$\begin{aligned}\mathbf{Vert}_i = \frac{q}{24}\frac{d}{dq}\Bigg(\Bigg(\sum_{j\neq i}\frac{1}{\lambda_j - \lambda_i} + \sum_a \frac{1}{l_a\lambda_i} + \sum_b \frac{1}{l'_b\lambda_i}\Bigg)u_i|_{t=0} \\ - \log(r_{i,0}|_{t=0})\Bigg),\end{aligned} \tag{3.4}$$

where u_i and $r_{i,0}$ are defined in terms of factors in J^{0+,p_i} as in (3.3). Also, $u_i|_{t=0}$ and $r_{i,0}|_{t=0}$ are related to the factors in $I_{\mathbf{T}}$ by (3.1).

To describe the vertex term more concretely, denote by

$$Q_{g,k|s}^{0+,0+}(\mathbb{P}^{n-1}, d)^{\mathbf{T},p_i}$$

the $\mathbf{T}$-fixed part of $Q_{g,k|s}^{0+,0+}(\mathbb{P}^{n-1}, d)$, the domain components of which are only over p_i.

For $\gamma_i \in H_{\mathbf{T}}^*(\mathbb{P}^{n-1}) \otimes \mathbb{Q}(\lambda)$, $\tilde{t}, \delta_j \in H_{\mathbf{T}}^*([\mathbb{C}^n/\mathbb{C}^*], \mathbb{Q})$, denote

$$\begin{aligned}&\langle \gamma_1\psi^{a_1}, \ldots, \gamma_k\psi^{a_k}; \delta_1, \ldots, \delta_s\rangle_{0,k|s,d}^{0+,0+} \\ &\quad = \int_{e^{\mathbf{T}}(E_{0,k|s,d}\oplus E'_{0,k|s,d})\cap[Q_{0,k|s}^{0+,0+}(\mathbb{P}^{n-1},d)]^{\text{vir}}} \prod_i ev_i^*(\gamma_i)\psi_i^{a_i} \prod_j \hat{ev}_j^*(\delta_j);\end{aligned}$$

$$\langle\langle \gamma_1\psi^{a_1}, \ldots, \gamma_k\psi^{a_k}\rangle\rangle_{0,k}^{0+,0+}$$

$$
\begin{aligned}
&= \sum_{s,d} \frac{q^d}{s!} \langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k}; \tilde{t}, \dots, \tilde{t} \rangle_{0,k|s,d}^{0+,0+}; \\
&\langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k}; \delta_1, \dots, \delta_m \rangle_{0,k|s,d}^{0+,0+,p_i} \\
&\quad = \int_{\frac{\mathrm{e}^{\mathbf{T}}(E_{0,k|s,d} \oplus E'_{0,k|s,d}) \cap [Q_{0,k|s}^{0+,0+}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}]^{\mathrm{vir}}}{\mathrm{e}^{\mathbf{T}}(N^{\mathrm{vir}}_{Q_{0,k|s}^{0+,0+}(\mathbb{P}^{n-1},d)^{\mathbf{T},p_i}/Q_{0,k|s}^{0+,0+}(\mathbb{P}^{n-1},d)})}} \prod_i ev_i^*(\gamma_i)\psi_i^{a_i} \prod_j \hat{ev}_j^*(\delta_j); \\
&\langle\langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k} \rangle\rangle_{0,k}^{0+,0+,p_i} \\
&\quad = \sum_{s,d} \frac{q^d}{s!} \langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k}; \tilde{t}, \dots, \tilde{t} \rangle_{0,k|s,d}^{0+,0+,p_i}.
\end{aligned}
$$

Consider

$$
\begin{aligned}
\mathbb{S}(\gamma) &:= \sum_i \phi^i \left\langle\!\!\left\langle \frac{\phi_i}{z-\psi}, \gamma \right\rangle\!\!\right\rangle_{0,2}^{0+,0+}; \\
\mathbb{V}_{ii}(x,y) &:= \left\langle\!\!\left\langle \frac{\phi_i}{x-\psi}, \frac{\phi_i}{y-\psi} \right\rangle\!\!\right\rangle_{0,2}^{0+,0+} = \frac{1}{e_i(x+y)} + O(q); \\
\mathbb{U}_i &:= e_i \langle\langle 1, 1 \rangle\rangle_{0,2}^{0+,0+,p_i} = \tilde{t}|_{p_i} + O(q); \\
\mathbb{S}_i^{0+,p_i}(\gamma) &:= e_i \left\langle\!\!\left\langle \frac{1}{z-\psi}, \gamma \right\rangle\!\!\right\rangle_{0,2}^{0+,0+,p_i} = e^{\tilde{t}/z}\gamma|_{p_i} + O(q); \\
\mathbb{J}^{0+,p_i} &:= e_i \left\langle\!\!\left\langle \frac{1}{z(z-\psi)} \right\rangle\!\!\right\rangle_{0,1}^{0+,0+,p_i} = e^{\tilde{t}}|_{p_i} + O(q) = J^{0+,p_i}|_{t=0} + O(\tilde{t}).
\end{aligned}
$$

As before,

$$
\begin{aligned}
\mathbb{S}_i^{0+,p_i}(\gamma) &= e^{\mathbb{U}_i/z}\gamma|_{p_i}; \\
\mathbb{J}^{0+,p_i} &= e^{\mathbb{U}_i/z}\left(\sum_{k=0}^{m} \mathbb{R}_{i,k} z^k + O(z^{m+1})\right)
\end{aligned}
\tag{3.5}
$$

for some $\mathbb{R}_{i,k} \in \mathbb{Q}(\lambda)[[\tilde{t}, q]]$ (after regarding $\tilde{t}$ as a formal element).

Denote by $\mathbb{I}$ the infinitesimal I-function $\mathbb{J}^{0+,0+}$ defined and calculated explicitly in [4]:

$$
\mathbb{I}(\tilde{t}) = \left(\exp\left(\sum_{i=0}^{n-1} \frac{t_i}{z}\left(zq\frac{d}{dq} + H\right)^i\right)\right) I_{\mathbf{T}}|_{t=0},
$$

By (3.5) and the fact that $\mathbb{I}|_{p_i} = \mathbb{J}^{0+,p_i}$ we have

$$
\mathbb{I}|_{p_i} = e^{\mathbb{U}_i/z}\left(\sum_{k=0}^{m} \mathbb{R}_{i,k} z^k + O(z^{m+1})\right).
$$

Hence,

$$\mathbb{I}|_{\tilde{t}=0,\,p_i} = e^{\mathbb{U}_i|_{\tilde{t}=0}/z}\left(\sum_{k=0}^{m} \mathbb{R}_{i,k}|_{\tilde{t}=0} z^k + O(z^{m+1})\right).$$

On the other hand, since $\mathbb{I}|_{\tilde{t}=0} = I_{\mathbf{T}}|_{t=0} \in H^*_{\mathbf{T}}(\mathbb{P}^{n-1})[[q, \frac{1}{z}]]$ and $I_{\mathbf{T}}|_{t=0}$ is homogeneous of degree 0 if we set

$$\deg H = \deg \lambda_i = \deg z = 1 \quad \text{and} \quad \deg q = 0,$$

then after the specialization

$$\lambda_i = \lambda_0 \exp\left(2\pi i \sqrt{-1}/n\right), \quad i = 1, \ldots, n, \deg \lambda_0 = 1, \tag{3.6}$$

$\mathbb{I}|_{\tilde{t}=0} \in \mathbb{Q}[H]/(H^n - \lambda_0^n)[[q, \frac{1}{z}]]$ is also homogeneous of degree 0. Since $\mathbb{I}|_{\tilde{t}=0,\,p_i} = \mathbb{I}|_{\tilde{t}=0,\,H=\lambda_i}$ and $\lambda_i^n = \lambda_0^n$, $\mathbb{I}|_{\tilde{t}=0,\,p_i}$ modulo (3.6) is a series in $\mathbb{Q}[[q, \lambda_i/z, \frac{z}{\lambda}]]$. So, we obtain

$$\mathbb{I}|_{\tilde{t}=0,\,p_i} \equiv e^{\mu(q)\lambda_i/z}\left(\sum_{k=0}^{\infty} R_k(q)(z/\lambda_i)^k\right) \tag{3.7}$$

for some $\mu(q) \in \mathbb{Q}[[q]]$ and $R_k(q) \in \mathbb{Q}[[q]]$. Here, $\equiv$ means modulo (3.6). Even if we put $\lambda_0 = 1$ to match the specialization (3.6) with (1.1), then (3.7) still holds modulo (1.1). Since $\mu(q)\lambda_i = \mathbb{U}_i|_{\tilde{t}=0}$ modulo (3.6), $\mu(q) \in q\mathbb{Q}[[q]]$. Note that $\mu(q)$ and $R_k(q)$ are independent of i. Hence,

$$\begin{aligned}
\mathbb{I}|_{\tilde{t}=t_H\tilde{H},\,p_i} &= e^{\lambda_i t_H/z}(I_{\mathbf{T}}|_{t=0,\,q\mapsto qe^{t_H},\,p_i}) \\
&\equiv e^{\lambda_i t_H/z} e^{\mu(qe^{t_H})\lambda_i/z}\left(\sum_{k=0}^{\infty} R_k(qe^{t_H})(z/\lambda_i)^k\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{U}_i|_{\tilde{t}=t_H\tilde{H}} &\equiv \lambda_i(t_H + \mu(qe^{t_H})) \quad \text{and} \\
\mathbb{R}_{i,k}|_{\tilde{t}=t_H\tilde{H}} &\equiv R_k(qe^{t_H})/(\lambda_i)^k.
\end{aligned} \tag{3.8}$$

Since

$$\begin{aligned}
r_{i,0}|_{t=0} &= \mathbb{R}_{i,0}|_{\tilde{t}=0}, \\
u_i|_{t=0} &= \mathbb{U}_i|_{\tilde{t}=0}, \quad \text{and} \\
c_i(\lambda) &= \left(\sum_{j\neq i} \frac{1}{\lambda_j - \lambda_i}\right) + \sum_a \frac{1}{l_a\lambda_i} + \sum_b \frac{1}{l'_b\lambda_i},
\end{aligned}$$

we conclude that

$$\begin{aligned}
\sum_i \mathbf{Vert}_i &= q\frac{q}{dq}\left(\sum_i \frac{-\log r_{i,0}|_{t=0}}{24} + \sum_i \frac{c_i(\lambda)u_i|_{t=0}}{24}\right) \\
&\equiv q\frac{q}{dq}\left(\frac{-n\log R_0(q)}{24} + \frac{1}{24}\left(\sum_a \frac{n}{l_a} + \sum_b \frac{n}{l'_b} - \binom{n}{2}\right)\mu(q)\right),
\end{aligned} \tag{3.9}$$

where μ and R_0 are defined in terms of factors in $I_{\mathbf{T}}$.

3.2. Loop Term

By the same argument as in [9], we can represent the loop term with genus zero invariants.

PROPOSITION 3.4.

$$\begin{aligned}\mathbf{Loop}_i &= \frac{1}{2}\left(\frac{d}{dt_H}\mathbb{U}_i|_{\tilde{t}=t_H\tilde{H}}\right)\Big|_{t_H=0}\\ &\quad\times \lim_{x,y\mapsto 0}\left(e^{-\mathbb{U}_i(1/x+1/y)}e_i\mathbb{V}_{ii}(x,y)-\frac{1}{x+y}\right)\Big|_{\tilde{t}=0}\\ &\equiv \frac{1}{2}\lambda_i(1+q\mu'(q))\\ &\quad\times \lim_{x,y\mapsto 0}\left(e^{-\lambda_i\mu(q)(1/x+1/y)}e_i\mathbb{V}_{ii}(x,y)|_{\tilde{t}=0}-\frac{1}{x+y}\right).\end{aligned} \tag{3.10}$$

The second equivalence comes from (3.8).

Now consider the equivariant cohomology basis

$$\{1, H := c_1^{\mathbf{T}}(\mathcal{O}(1)), \dots, H^{n-1}\}$$

of the **T**-equivariant cohomology ring

$$H^*_{\mathbf{T}}(\mathbb{P}^{n-1}) \cong \mathbb{Q}[\lambda_1,\dots,\lambda_n,h]/\left(\prod_{i=1}^n (h-\lambda_i)\right), \quad H\mapsto h.$$

There is the expression of V-correlators in terms of S-correlators by Theorem 3.2.1 of [6]:

$$e_i\mathbb{V}_{ii}(x,y)|_{\tilde{t}=0} = \frac{1}{e_i}\frac{\sum_j \mathbb{S}_{z=x,\tilde{t}=0}(H^j)|_{p_i}\mathbb{S}_{z=y,\tilde{t}=0}(H^{j^\vee})|_{p_i}}{x+y},$$

where $H^{j^\vee}$ is the dual of H^j with respect to $E\times E'$-twisted Poincaré metric g_{ij} modulo relations (3.6);

$$g_{ij} = \frac{\prod_{a=1}^r(l_a)}{\prod_{b=1}^m(-l'_b)}\sum_{k=0}^2 \lambda_0^{n(k-1)}\delta_{r-m+i+j,nk-1} \quad \text{for } 0\le i,j\le n-1.$$

We can calculate the $\mathbb{S}_{\tilde{t}=0}(H^k)$ in terms of $I_{\mathbf{T}}$ by the Birkhoff factorization method as in [13]. So, we can express $e_i\mathbb{V}_{ii}(x,y)|_{\tilde{t}=0}$ and $\mathbf{Loop}_i$ in terms of factors of $I_{\mathbf{T}}$.

Using the Birkhoff factorization method and the calculation in [14], we obtain

$$\begin{aligned}&\lim_{x,y\mapsto 0}\left(e^{-\lambda_i\mu(q)(1/x+1/y)}e_i\mathbb{V}_{ii}(x,y)|_{\tilde{t}=0}-\frac{1}{x+y}\right)\\ &\quad= \frac{\lambda_i^{n-2}}{\mathrm{e}^{\mathbf{T}}(T_{p_i}\mathbb{P}^{n-1})L(q)}q\frac{d}{dq}\mathrm{Loop}(q),\end{aligned}$$

where

$$L(q) = \left(1 - q\prod_a l_a^{l_a}\prod_b(-l'_b)^{l'_b}\right)^{-1/n},$$

$$\begin{aligned}\mathrm{Loop}(q) = &\frac{n}{24}\left(n-1-2\sum_{a=1}^{r}\frac{1}{l_a}-2\sum_{b=1}^{m}\frac{1}{l'_b}\right)\mu(q)\\ &-\frac{3(n-1-r-m)^2+(n-2)}{24}\log\left(1-q\prod_a l_a^{l_a}\prod_b(-l'_b)^{l'_b}\right)\\ &-\sum_{k=m}^{n-r-2}\binom{n-r-k}{2}\log C_k(q).\end{aligned}$$

Thus,

$$\mathbf{Loop}_i \equiv \frac{1}{2}\lambda_i(1+q\mu'(q))\frac{\lambda_i^{n-2}}{\mathrm{e}^{\mathbf{T}}(T_{p_i}\mathbb{P}^{n-1})L(q)}q\frac{d}{dq}\mathrm{Loop}(q). \tag{3.11}$$

Using the fact that equivariant $I_{\mathbf{T}}$-function satisfies the Picard–Fuchs equation $\mathrm{PF}I_{\mathbf{T}}|_{t=t_H\cdot H} = 0,$

$$\mathrm{PF} := \left(z\frac{d}{dt}\right)^n - 1 - q\prod_a\prod_{k=1}^{l_a}\left(l_a z\frac{d}{dt}+kz\right)\prod_b\prod_{k=0}^{l'_b-1}\left(-l'_b z\frac{d}{dt}-kz\right)$$

and the asymptotic form of

$$I_{\mathbf{T}}|_{t=t_H H, p_i},$$

we can calculate μ and R_0:

$$\mu(q) = \int_0^q \frac{L(x)-1}{x}\,dx, \qquad R_0(q) = L(q)^{(r-m+1)/2}.$$

For calculations, see [14]. Therefore,

$$\begin{aligned}\sum_i \mathbf{Loop}_i &\equiv \frac{1}{2}q\frac{d}{dq}\mathrm{Loop}(q)\sum_i\frac{\lambda_i^{n-1}}{\mathrm{e}^{\mathbf{T}}(T_{p_i}\mathbb{P}^{n-1})}\\ &= \frac{1}{2}q\frac{d}{dq}\mathrm{Loop}(q)\int_{\mathbb{P}^{n-1}}H^{n-1} = \frac{1}{2}q\frac{d}{dq}\mathrm{Loop}(q).\end{aligned} \tag{3.12}$$

3.3. *Proof of Main Theorem*

By combining (3.9), (3.12), and (2.2) we have

$$\frac{d}{dq}\left\{\langle\,\rangle_{1,0}^{0+} + \frac{3(n-1-r-m)^2+n-r+m-3}{48}\log\left(1-q\prod_{a=1}^{r}l_a^{l_a}\prod_{b=1}^{m}(-l'_b)^{l'_b}\right)\right.$$
$$\left. + \frac{1}{2}\sum_{k=m}^{n-r-2}\binom{n-r-k}{2}\log C_k(q)\right\} = 0$$

because of (2.3). Finally, since

$$\left\{\langle\,\rangle^{0+}_{1,0}+\frac{3(n-1-r-m)^2+n-r+m-3}{48}\log\left(1-q\prod_{a=1}^{r}l_a^{l_a}\prod_{b=1}^{m}(-l'_b)^{l'_b}\right)\right.$$
$$\left.+\frac{1}{2}\sum_{k=m}^{n-r-2}\binom{n-r-k}{2}\log C_k(q)\right\}\Bigg|_{q=0}=0,$$

we are done.

3.4. Corollaries

First of all, if $m\geq 2$, then $I_0=1$ and $I_1^{\zeta}=0$. Thus, we have the following:

COROLLARY 3.5. *If $m\geq 2$, then*

$$\begin{aligned}\langle\,\rangle^{\infty}_{1,0}=\langle\,\rangle^{0+}_{1,0}\\ &=-\frac{3(n-1-r-m)^2+n-r+m-3}{48}\log\left(1-q\prod_{a=1}^{r}l_a^{l_a}\prod_{b=1}^{m}(-l'_b)^{l'_b}\right)\\ &\quad-\frac{1}{2}\sum_{k=m}^{n-r-2}\binom{n-r-k}{2}\log C_k(q).\end{aligned}$$

If $m=1$, then $I_0=1$ and

$$\int_X H\cup c_{\dim X-1}(T_X)=\binom{n}{2}-\sum_{a=1}^{r}\frac{n}{l_a}-\frac{n}{l'_1}.$$

Thus, we have

COROLLARY 3.6. *If $m=1$, then*

$$\begin{aligned}\langle\,\rangle^{\infty}_{1,0}|_{q\mapsto qe^{I_1(q)}}&=\frac{I_1(q)}{24}\left(\binom{n}{2}-\sum_a\frac{n}{l_a}-\frac{n}{l'_1}\right)\\ &\quad-\frac{3(n-r-2)^2+n-r-2}{48}\log\left(1-q\prod_{a=1}^{r}l_a^{l_a}\cdot(-l'_1)^{l'_1}\right)\\ &\quad-\frac{1}{2}\sum_{k=1}^{n-r-2}\binom{n-r-k}{2}\log C_k(q),\end{aligned}$$

where $I_1(q)\in\mathbb{Q}[\![q]\!]$ is the coefficient of H in I_1^{ζ}.

If $m=0$, then

$$\int_X H\cup c_{\dim X-1}(T_X)=\binom{n}{2}-\sum_{a=1}^{r}\frac{n}{l_a}.$$

Thus, we have the following:

COROLLARY 3.7. *If $m = 0$, then*

$$\langle\,\rangle^{\infty}_{1,0}|_{q\mapsto qe^{I_1(q)/I_0(q)}} = \frac{1}{24}\chi_{\mathrm{top}}(X)\log I_0 + \frac{1}{24}\frac{I_1(q)}{I_0(q)}\left(\binom{n}{2} - \sum_a \frac{n}{l_a}\right) - \frac{3(n-1-r)^2+n-r-3}{48}\log\left(1 - q\prod_{a=1}^{r} l_a^{l_a}\right) - \frac{1}{2}\sum_{k=0}^{n-r-2}\binom{n-r-k}{2}\log C_k(q),$$

where $I_1(q) \in \mathbb{Q}[[q]]$ is the coefficient of H in I_1^{ζ}.

4. Example

Let X be the total space of $\mathcal{O}_{\mathbb{P}^1}(-2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Then it can be obtained by pull-back of $\mathcal{O}_{\mathbb{P}^3}(-2)$ on $\mathbb{P}^3$ under Segre embedding $i : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. Note that the image of i is a quadric hypersurface in $\mathbb{P}^3$. Using identification $H_2(X,\mathbb{Z}) \cong H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$, define the Gromov–Witten invariants

$$N_d := \sum_{d_1+d_2=d} \deg[\overline{M}_{1,0}(X,(d_1,d_2))]^{\mathrm{vir}}$$

for positive integer d. In this section, we l calculate N_d, $d > 0$, explicitly by using main theorem.

First, we apply Theorem 1.1 to X by putting $n = 4$, $r = 1$, $m = 1$, and $l_1 = l_1' = 2$ and obtain

$$\langle\,\rangle^{0+}_{1,0} = -\frac{1}{12}\log(1-16q) - \frac{1}{2}\log\left(1 + q\frac{q}{dq}\frac{I_1(q)}{I_0(q)}\right), \tag{4.1}$$

where $I_0(q)$ and $I_1(q)$ are defined by

$$\sum_d q^d \frac{\prod_{k=1}^{2d}(2H+kz)\prod_{k=0}^{2d-1}(-2H-kz)}{\prod_{i=1}^{4}\prod_{k=1}^{d}(H-\lambda_i+kz)} \equiv I_0(q) + I_1(q)\frac{H}{z} + O\left(\frac{1}{z^2}\right). \tag{4.2}$$

Here, the left-hand side of (4.2) is $I_{\mathbf{T}}(0,q)$, and "≡" means modulo (1.1). Precisely,

$$I_0(q) = 1, \quad \text{and} \quad I_1(q) = \sum_{d>0}\frac{q^d}{d}\binom{2d}{d}^2. \tag{4.3}$$

By Theorem 1.3 and Corollary 3.6 we obtain

$$\langle\,\rangle^{\infty}_{1,0}|_{q^d\mapsto q^d\exp(dI_1(q))} = \frac{1}{12}I_1(q) + \langle\,\rangle^{0+}_{1,0}. \tag{4.4}$$

Let us define

$$Q := q\exp I_1(q) \quad \text{and} \quad T := \log Q.$$

Then, combining (4.1), (4.3), and (4.4), we have

$$\langle\,\rangle^{\infty}_{1,0}(Q) = \frac{T}{12} + \frac{1}{2}\log\left((1-16q)^{-1/6}q^{-7/6}\frac{dq}{dT}\right) \tag{4.5}$$

$$= -\frac{1}{3}q - \frac{11}{6}q^2 - \frac{124}{9}q^3 + O(q^4) \tag{4.6}$$

because

$$I_1(q) = T - \log q = 1 + 4q + 18q^2 + \frac{400}{3}q^3 + O(q^4)$$

and

$$1 + q\frac{d}{dq}I_1(q) = q\frac{dT}{dq} = 1 + 4q + 36q^2 + 400q^3 + O(q^4).$$

We have to mention is that in [10] it is proved that

$$-\frac{T}{12} + \langle\,\rangle^{\infty}_{1,0}(Q) = \frac{1}{2}\log\left((1-16q)^{-1/6}q^{-7/6}\frac{dq}{dT}\right),$$

which is exactly (4.5).

On the other hand, using

$$Q = q\exp I_1(q) = q + 4q^2 + 26q^3 + O(q^4),$$

we have

$$\langle\,\rangle^{\infty}_{1,0}(Q) = \sum_{d=1}^{\infty} N_d Q^d$$

$$= N_1 q + (4N_1 + N_2)q^2 + (26N_1 + 8N_2 + N_3)q^3 + O(q^4). \tag{4.7}$$

Then, comparing (4.6) and (4.7), we have

$$N_1 = -\frac{1}{3}, \qquad N_2 = -\frac{1}{2}, \qquad N_3 = -\frac{10}{9}, \qquad \dots.$$

Also, note that these numbers appeared in [1], where it is also shown that

$$\langle\,\rangle^{\infty}_{1,0}(Q) = \frac{T}{12} - \log\eta(\tau), \tag{4.8}$$

where η is the Dedekind eta function, and

$$Q = p^{1/2} - 4p + 6p^{3/2} + \cdots, \quad p = e^{2\pi i\tau},$$

that is,

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1 - e^{2n\pi i\tau}) = p^{1/24}\prod_{n=1}^{\infty}(1-p^n).$$

Indeed, mirror curves of X are a family of elliptic curves. If we regard τ as a parameter of a family of elliptic curves, which corresponds to

$$\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z}),$$

then they have shown (4.8) by modular properties and the behavior at the discriminant of the family of elliptic curves of partition function at genus 1, which is defined exactly in the same way as $-\frac{T}{12} + \langle\,\rangle^{\infty}_{1,0}(Q)$.

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