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A Geometric Reverse to the Plus Construction and Some Examples of Pseudocollars on High-Dimensional Manifolds

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Abstract. In this paper, we develop a geometric procedure for producing a “reverse” to Quillen’s plus construction, a construction called a 1-*sided h-cobordism* or *semi-h-cobordism.* We then use this reverse to the plus construction to produce uncountably many distinct ends of manifolds called *pseudocollars,* which are stackings of 1-sided h-cobordisms. Each of our pseudocollars has the same boundary and prohomology systems at infinity and similar group-theoretic properties for their profundamental group systems at infinity. In particular, the kernel group of each group extension for each 1-sided h-cobordism in the pseudocollars is the same group. Nevertheless, the profundamental group systems at infinity are all distinct. A good deal of combinatorial group theory is needed to verify this fact, including an application of Thompson’s group V.

The notion of pseudocollars originated in Hilbert cube manifold theory, where it was part of a necessary and sufficient condition for placing a $\mathcal{Z}$-set as the boundary of an open Hilbert cube manifold.

1. Introduction and Main Results

In this paper, we develop a geometric procedure for producing a “reverse” to Quillen’s plus construction, a construction called a 1-*sided h-cobordism* or *semi-h-cobordism.* We then use this reverse to the plus construction to produce uncountably many distinct ends of manifolds called *pseudocollars,* which are stackings of 1-sided h-cobordisms. Each of our pseudocollars has the same boundary and prohomology systems at infinity and similar group-theoretic properties for their profundamental group systems at infinity. In particular, the kernel group of each group extension for each 1-sided h-cobordism in the pseudocollars is the same group. Nevertheless, the profundamental group systems at infinity are all distinct. A good deal of combinatorial group theory is needed to verify this fact, including an application of Thompson’s group V.

The notion of pseudocollars originated in Hilbert cube manifold theory, where it was part of a necessary and sufficient condition for placing a $\mathcal{Z}$-set as the boundary of an open Hilbert cube manifold.

We work in the category of smooth manifolds, but all our results apply equally well to the categories of PL and topological manifolds. The manifold version

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of Quillen's plus construction provides a way of taking a closed smooth manifold M of dimension $n \geq 5$ whose fundamental group $G = \pi_1(M)$ contains a perfect normal subgroup P that is the normal closure of a finite number of elements and produces a compact cobordism (W, M, M^+) to a manifold M^+ whose fundamental group is isomorphic to $Q = G/P$ and for which $M^+ \hookrightarrow W$ is a simple homotopy equivalence. By duality, the map $f : M \to M^+$ given by including M into W and then retracting onto M^+ induces an isomorphism $f_* : H_*(M; \mathbb{Z}Q) \to H_*(M^+; \mathbb{Z}Q)$ of homology with twisted coefficients. By a clever application of the s-cobordism theorem, such a cobordism is uniquely determined by M and P (see [8, p. 197]).

In "Manifolds with Non-stable Fundamental Group at Infinity I" [10], Guilbault outlines a structure to put on the ends of an open smooth manifold N with finitely many ends called a *pseudocollar,* which generalizes the notion of a collar on the end of a manifold introduced in Siebenmann's dissertation [25]. A pseudocollar is defined as follows. Recall that a manifold U^n with compact boundary is an open collar if $U^n \approx \partial U^n \times [0, \infty)$; it is a homotopy collar if the inclusion $\partial U^n \hookrightarrow U^n$ is a homotopy equivalence. If U^n is a homotopy collar which contains arbitrarily small homotopy collar neighborhoods of infinity, then we call U^n a *pseudo-collar.* We say that an open n-manifold N^n is collarable if it contains an open collar neighborhood of infinity and that N^n is *pseudocollarable* if it contains a pseudocollar neighborhood of infinity.

Each pseudocollar admits a natural decomposition as a sequence of compact cobordisms (W, M, M_-), where W is a 1-sided h-cobordism (see Definition 1 further). If a 1-sided h-cobordism is actually an s-cobordism (again, see Definition 1), then it follows that the cobordism (W, M_-, M) is a a plus cobordism. (This somewhat justifies the use of the symbol "M_-" for the right-hand boundary of a 1-sided h-cobordism, a play on the traditional use of M^+ for the right-hand boundary of a plus cobordism.)

The general problem of a reverse to Quillen's plus construction in the high-dimensional manifold category is as follows.

PROBLEM 1 (Reverse Plus Problem). Suppose G and Q are finitely presented groups and $\Phi : G \twoheadrightarrow Q$ is an onto homomorphism with $\ker(\Phi)$ perfect. Let M^n $(n \geq 5)$ be a closed smooth manifold with $\pi_1(M) \cong Q$.

Does there exist a compact cobordism (W^{n+1}, M, M_-) with

$$1 \longrightarrow \ker(\iota_\#) \longrightarrow \pi_1(M_-) \xrightarrow{\iota_\#} \pi_1(W) \longrightarrow 1$$

equivalent to

$$1 \longrightarrow \ker(\Phi) \longrightarrow G \xrightarrow{\Phi} Q \longrightarrow 1$$

and $M \hookrightarrow W$ a (simple) homotopy equivalence?

Notes:

- The fact that G and Q are finitely presented forces $\ker(\Phi)$ to be the normal closure of a finite number of elements. (See, e.g., [10] or [25].)

- Closed manifolds M^n ($n \geq 5$) in the various categories with $\pi_1(M)$ isomorphic to a given finitely presented group Q always exist. In the smooth category, we take a finite presentation for Q, take a disk $\mathbb{D}^{n+1}$, attach one trivially attached 1-handle for each generator of Q, attach one trivially attached 2-handle for each relator of Q to form a manifold N^{n+1}, and then let $M = \partial N$. Similar procedures exist in the other categories.

The following terminology was first introduced in [17].

Definition 1. Let N^n be a compact smooth manifold. A 1*-sided* h*-cobordism* (W, N, M) is a cobordism if either $N \hookrightarrow W$ or $M \hookrightarrow W$ is a homotopy equivalence (if it is a simple homotopy equivalence, then we call (W, N, M) a 1*-sided* s*-cobordism*). [A 1-sided h-cobordism (W, N, M) is presumably so named because it is "one side of an h-cobordism"].

We want to know under what circumstances 1-sided h-cobordisms exist and, if they do, how many are there. Also, we are interested in controlling the torsion and when it can be eliminated.

There are some cases in which 1-sided h-cobordisms are known not to exist. For instance, if P is finitely presented and perfect but not superperfect, $Q = \langle e \rangle$, and $M = \mathbb{S}^n$, then a solution to the reverse plus problem produces M_- that is a homology sphere. However, it is a standard fact that a manifold homology sphere must have a superperfect fundamental group! (See, e.g., [18].) (The definition of superperfect will be given in Definition 2.) The key point is that the solvability of the reverse plus problem depends not just upon the group data, but also upon the manifold M with which one begins.

Here is a statement of our main results.

Theorem 1.1 (Existence of 1-sided s-cobordisms). *Given* $1 \to S \to G \to Q \to 1$ *where* S *is a finitely presented superperfect group,* G *is a semidirect product of* Q *by* S, N *is any* n*-manifold with* $n \geq 6$, *and* $\pi_1(M) \cong Q$, *there exists a solution* (W, N, N_-) *to the reverse plus problem for which* $N \hookrightarrow W$ *is a simple homotopy equivalence.*

One of the primary motivations for Theorem 1.1 is that it provides a "machine" for constructing interesting pseudocollars. As an application, we use it to prove the following:

Theorem 1.2 (Uncountably many pseudocollars on closed manifolds with the same boundary and similar Pro-π_1). *Let* M^n *be a closed smooth manifold* ($n \geq 6$) *with* $\pi_1(M) \cong \mathbb{Z}$, *and let* S *be the finitely presented group* $V * V$, *which is the free product of two copies of Thompson's group* V. *Then there exists an uncountable collection of pseudocollars* $\{N^{n+1}_\omega \mid \omega \in \Omega\}$, *no two of which are homeomorphic at infinity, and each of which begins with* $\partial N^{n+1}_\omega = M^n$ *and is obtained by creating a group extension* $1 \to S \to G_{i+1} \to G_i \to 1$ *and then using Theorem 1.1 to create a* 1*-sided* h*-cobordism* (W_i, M_i, M_{i+1}) *with* $\pi_1(M_i) = G_i$ *and* $\pi_1(M_{i+1}) = G_{i+1}$

countably many times, using the same kernel group S each time. In particular, each N_i has a fundamental group at infinity that may be represented by an inverse sequence

$$\mathbb{Z} \overset{\alpha_1}{\twoheadleftarrow} G_1 \overset{\alpha_2}{\twoheadleftarrow} G_2 \overset{\alpha_3}{\twoheadleftarrow} G_3 \overset{\alpha_4}{\twoheadleftarrow} \cdots$$

with $\ker(\alpha_i) = S$ *for all* i.

An underlying goal of papers [10], [13], and [14] is to understand when noncompact manifolds with compact (possibly empty) boundary admit $\mathcal{Z}$-compactifications. In [5], it is shown that a Hilbert cube manifold admits a $\mathcal{Z}$-compactification if and only if it is pseudocollarable and the Whitehead torsion of the end can be controlled. In [11], Guilbault asks whether the universal cover of a closed aspherical manifold ($n \geq 6$) is always pseudocollarable. He further asks if pseudocollarbility plus control of the Whitehead torsion of the end is enough for finite-dimensional manifolds ($n \geq 6$) to admit a $\mathcal{Z}$-compactification. Still further, he shows that any two $\mathcal{Z}$-boundaries of an ANR must be shape equivalent. Finally, he and Ancel show in [1] that if two closed contractible manifolds M^n and N^n ($n \geq 6$) admit homeomorphic boundaries, then M is homeomorphic to N. This is most interesting when the contractible manifolds are universal covers of closed aspherical manifolds. In that case, these questions may be viewed as an approach to the famous Borel conjecture, which asks whether two aspherical manifolds with isomorphic fundamental group are necessarily homeomorphic.

2. A Handlebody-Theoretic Reverse to the Plus Construction

In this section, we describe our partial solution to the reverse plus problem. Our solution only applies to superperfect (defined in Definition 2), finitely presented kernel groups where the total group G of the group extension $1 \to K \to G \to Q \to 1$ is a semidirect product (defined in Definition 3).

However, we believe that our special case is easy to use and easy to understand. For example, when M and S are fixed, we are able to analyze various solutions to the reverse plus problem by studying the algebraic problem of computing semidirect products of Q by S.

DEFINITION 2. A group G is said to be *superperfect* if its first two homology groups are 0, that is, if $H_1(G) = H_2(G) = 0$. (Recall that a group is *perfect* if its first homology group is 0.)

EXAMPLE 1. A perfect group is superperfect if it admits a finite *balanced* presentation, that is, a finite presentation with the same number of generators as relators. (The converse is false.)

LEMMA 2.1. *Let S be a superperfect group. Let K be a cell complex that has a fundamental group isomorphic to S. Then all elements of $H_2(K)$ can be killed by attaching* 3*-cells.*

Proof. By Proposition 7.1.5 in [9] there is a $K(S,1)$ that is formed from K by attaching cells of dimension 3 and higher. Let L be such $K(S,1)$. Then L^3 is formed from K^2 by attaching only 3-cells, and $H_2(L^3) \cong H_2(L)$, as L is formed from L^3 by attaching cells of dimension 4 and higher, which cannot affect H_2. But $H_2(L) \cong H_2(S)$ by definition, and $H_2(S) \cong 0$ by hypothesis. Thus, all elements of $H_2(K)$ can be killed by attaching 3-cells. □

Definition 3. A group extension

$$1 \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\sigma} Q \longrightarrow 1$$

is a *semidirect product* if there is a left-inverse τ (which is a homomorphism) to σ.

Note that in this case,

- there are "slide relators" $qk = \phi(q)(k)q$, where ϕ is the outer action of Q on K, which "represent the price of sliding k across q";
- every word $k_1q_1k_2q_2 \cdot \ldots \cdot k_nq_n$ admits a normal form $k'q'$ where all elements from K come first on the left and all elements of Q come last on the right;
- there is a presentation for G in terms of the presentations for K and Q and the slide relators; to wit:

Claim 1.

$$\begin{aligned}\langle \alpha_1, \ldots, \alpha_{k_1}, \beta_1, \ldots, \beta_{k_2} \mid r_1, \ldots, r_{l_1}, s_1, \ldots, s_{l_2},\\ \beta_1\alpha_1(\phi(\beta_1)(\alpha_1)\beta_1)^{-1}, \ldots, \beta_{k_2}\alpha_{k_1}(\phi(\beta_{k_2})(\alpha_{k_1})\beta_{k_2})^{-1}\rangle\end{aligned} \tag{1}$$

is a presentation for G. (The $\beta_j\alpha_i(\phi(\beta_j)(\alpha_i)\beta_j)^{-1}$ are "slide relators".)

Proof. Clearly, there is a homomorphism from the group presented above to G. From this it follows that $G = KQ$ and that $K \cap Q = \{1\}$. From this it follows that the kernel is trivial (in the finite case, just check orders). □

Lemma 2.2 (Equivariant Attaching of Handles). *Let M^n be a smooth manifold, $n \geq 5$, with M one boundary component of W with $\pi_1(M) \cong G$. Let $P \trianglelefteq G$ and $Q = G/P$. Let $\overline{M}$ be the cover of M with fundamental group P and give $H_*(\overline{M};\mathbb{Z})$ the structure of a $\mathbb{Z}Q$-module. Let $2k+1 \leq n$, and let S be a finite collection of elements of $H_k(M;\mathbb{Z})$ that all admit embedded spherical representatives having trivial tubular neighborhoods. If $k = 1$, then assume that all elements of S represent elements of P.*

Then we can equivariantly attach $(k+1)$-handles across S, *that is, if $\overline{S} = \{s_{j,q} \mid q \in Q\}$ is the collection of lifts of elements of S to $\overline{M}$, then we can attach $(k+1)$-handles across tubular neighborhoods of the $s_{j,q}$ so that each lift $s_{j,q}$ projects down via the covering map p to an element s_j of S and so that the covering map extends to send each $(k+1)$-handle $H_{j,q}$ attached across a* *tubular neighborhood of $s_{j,q}$ in $\overline{M}$ bijectively onto a handle attached across the projection via the covering map of the tubular neighborhood of the element s_j in M.*

Proof. $H_k(\overline{M};\mathbb{Z})$ has the structure of a $\mathbb{Z}Q$-module. The action of Q on $\overline{S}$ permutes the elements of S. For each embedded sphere s_j in S, lift it via its inverse images under the covering map to a pairwise disjoint collection of embedded spheres $s_{j,q}$. (This is possible since a point of intersection or self-intersection would have to project down to a point of intersection or self-intersection, respectively, by the evenly covered neighborhood property of covering spaces.) The $s_{j,q}$ all have trivial tubular neighborhoods. Attach a $(k+1)$-handle across the tubular neighborhood of the elements s_j of the S. For all $j \in \{1, \dots, |S|\}$ and $q \in Q$, attach a $(k+1)$-handle across the spherical representative $s_{j,q}$; extend the covering projection so it projects down in a bijective fashion from the handle attached along $s_{j,q}$ onto the handle we attached along s_j. □

LEMMA 2.3. *Let A, B, and C be R-modules with a free R-module B (on the basis S), and let $\Theta : A \oplus B \to C$ be an R-module homomorphism. Suppose $\Theta|_A$ is onto. Then* $\ker(\Theta) \cong \ker(\Theta|_A) \oplus B$.

Proof. Define $\phi : \ker(\Theta|_A) \oplus B \to \ker(\Theta)$ as follows. For each $s \in S$, where S is a basis for B, choose $\alpha(s) \in A$ with $\Theta(\alpha(s), 0) = \Theta(0, s)$, as $\Theta|_A$ is onto. Extend α to a homomorphism from B to A and note that α has the same property for all $b \in B$. Then set $\phi(x, b) = (x - \alpha(b), b)$.

(Well-defined) Let $x \in \ker(\Theta|_A)$ and $b \in B$. Then $\Theta(\phi(x, b)) = \Theta(x - \alpha(b), b) = \Theta(x, 0) + \Theta(-\alpha(b), 0) + \Theta(0, b) = 0 + -\Theta(\alpha(b), 0) + \Theta(0, b) = 0 + -\Theta(0, b) + \Theta(0, b) = 0$. So, ϕ is well-defined.

Define $\psi : \ker(\Theta) \to \ker(\Theta|_A) \oplus B$ by $\psi(z) = (\pi_1(z) + \alpha(\pi_2(z)), \pi_2(z))$, where $\pi_1 : A \oplus B \to A$ and $\pi_2 : A \oplus B \to B$ are the canonical projections.

(Well-defined) Let $z \in \ker(\Theta)$. It is clear that $\pi_2(z) \in B$, so it remains to prove that $\pi_1(z) + \alpha(\pi_2(z)) \in \ker(\Theta|_A)$. [Note that $\Theta(z) = \Theta|_A(\pi_1(z)) + \Theta|_B(\pi_2(z)) \Rightarrow \Theta|_A(\pi_1(z)) = -\Theta|_B(\pi_2(z))$. Note also that, by the definition of α, $\Theta(\alpha(\pi_2(z))) = \Theta(0, \pi_2(z))$.] We compute $\Theta|_A(\pi_1(z) + \alpha(\pi_2(z))) = \Theta|_A(\pi_1(z)) + \Theta(\alpha(\pi_2(z)), 0) = -\Theta|_B(\pi_2(z)) + \Theta(0, \pi_2(z)) = -\Theta(0, \pi_2(z)) + \Theta(0, \pi_2(z)) = 0$. So, ψ is well-defined.

(Homomorphism) Clear.

(Inverses) Let $(x, b) \in \ker(\Theta|_A) \oplus B$. Then $\psi(\phi(x, b)) = \psi(x - \alpha(b), b) = (\pi_1(x - \alpha(b), b) + \alpha(\pi_2(x - \alpha(b), b)), \pi_2(x - \alpha(b), b)) = (x - \alpha(b) + \alpha(b), b) = (x, b)$.

Let $z \in \ker(\Theta)$. Then $\phi(\psi(z)) = \phi(\pi_1(z) + \alpha(\pi_2(z)), \pi_2(z)) = (\pi_1(z) + \alpha(\pi_2(z)) - \alpha(\pi_2(z)), \pi_2(z)) = (\pi_1(z), \pi_2(z)) = z$.

So, ϕ and ψ are inverses of each other, and the lemma is proven. □

DEFINITION 4. A k-handle is said to be *trivially attached* if it is possible to attach a canceling $(k+1)$-handle.

Here is our solution to the reverse plus problem in the high-dimensional manifold category.

THEOREM 1.1 (An Existence Theorem for Semi-s-Cobordisms). *Given* $1 \to S \to G \to Q \to 1$ *where* S *is a finitely presented superperfect group*, G *is a semi-direct product of* Q *by* S, *and any closed* n*-manifold* N *with* $n \geq 6$ *and* $\pi_1(N) \cong Q$, *there exists a solution* (W, N, N_-) *to the reverse plus problem for which* $N \hookrightarrow W$ *is a simple homotopy equivalence.*

Proof. Start by taking N and crossing it with $\mathbb{I}$. Let $Q \cong \langle \alpha_1, \dots, \alpha_{k_1} \mid r_1, \dots, r_{l_1} \rangle$ be a presentation for Q. Let $S \cong \langle \beta_1, \dots, \beta_{k_2} \mid s_1, \dots, s_{l_2} \rangle$ be a presentation for S. Take a small n-disk D inside of $N \times \{1\}$. Attach a trivial 1-handle h_i^1 for each β_i in this disk D. Note that because they are trivially attached, there are canceling 2-handles k_i^2, which may also be attached inside the disk together with the 1-handles $D \cup \{h_i^1\}$. We identify these 2-handles now, but do not attach them yet. They will be used later.

Attach a 2-handle h_j^2 across each of the relators s_j of the presentation for S in the disk together with the 1-handles $D \cup \{h_i^1\}$, choosing the framing so that it is trivially attached in the manifold that results from attaching h_i^1 and k_i^2 (although we have not yet attached the handles k_i^2). Note that because they are trivially attached, there are canceling 3-handles k_j^3, which may also be attached in the portion of the manifold consisting of the disk D together with the 1-handles $\{h_i^1\}$ and the 2-handles $\{k_i^2\}$. We identify these 3-handles now, but do not attach them yet. They will be used later.

Attach a 2-handle $f_{i,j}^2$ for each relator $\beta_j \alpha_i \beta_j^{-1} [\phi(\beta_j)(\alpha_i)]^{-1}$, choosing the framing so that it is trivially attached in the result of attaching the h_i^1, k_i^2, h_2^j, and k_j^3. This is possible since each of the relators becomes trivial when the k_i^2s and k_i^3s are attached. Note that because the $f_{i,j}^2$ are trivially attached, there are canceling 3-handles $g_{i,j}^3$. We identify these 3-handles now, but do not attach them yet. They will be used later. Call the resulting cobordism with only h_i^1s, h_j^2s, and $f_{i,j}^2$s attached (W', N, M') and call the right-hand boundary M'.

Note that we now have $\pi_1(N) \cong Q$, $\pi_1(W') \cong G$, and $\iota_\# : \pi_1(M') \to \pi_1(W)$ an isomorphism because, by inverting the handlebody decomposition, we are starting with M' and adding $(n-1)$- and $(n-2)$-handles, which do not affect π_1 as $n \geq 6$.

Consider the cover $\overline{W'}$ of W' corresponding to S. Then the right-hand boundary $\overline{M'}$ of this cover also has fundamental group isomorphic to S by covering space theory. Also, the left-hand boundary $\widetilde{N}$ of this cover has a trivial fundamental group.

Consider the handlebody chain complex $C_*(\overline{W'}, \widetilde{N}; \mathbb{Z})$. This is naturally a $\mathbb{Z}Q$-module complex. It looks like

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & C_3(\overline{W'},\widetilde{N};\mathbb{Z}) \arrow[r] \arrow[d, "="] & C_2(\overline{W'},\widetilde{N};\mathbb{Z}) \arrow[r, "\partial_2"] \arrow[d, "\cong"] & C_1(\overline{W'},\widetilde{N};\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & C_0(\overline{W'},\widetilde{N};\mathbb{Z}) \arrow[r] \arrow[d, "="] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & \bigoplus_{i=1}^{l_2}\mathbb{Z}Q \oplus \bigoplus_{j=1}^{k_1 * k_2}\mathbb{Z}Q \arrow[r, "\partial_2"] & \bigoplus_{i=1}^{k_2}\mathbb{Z}Q \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

where $C_2(\overline{W'},\widetilde{N};\mathbb{Z})$ decomposes as $A = \bigoplus_{i=1}^{l_2}\mathbb{Z}Q$, which has a $\mathbb{Z}Q$-basis obtained by arbitrarily choosing one lift of the 2-handles for each of the h_j^2, and $B = \bigoplus_{j=1}^{k_1 \cdot k_2}\mathbb{Z}Q$, which has a $\mathbb{Z}Q$-basis obtained by arbitrarily choosing one lift of the 2-handles for each of the $f_{i,j}^2$. Set $C = C_1(\overline{W'},\widetilde{N};\mathbb{Z}) \cong \bigoplus_{i=1}^{k_2}\mathbb{Z}Q$ (as $\mathbb{Z}Q$-modules). Choose a preferred basepoint $\overline{*}$ and a preferred lift of the the disk D to a disk $\overline{D}$ in $\overline{M}$. Decompose ∂_2 as $\partial_{2,1} = \partial_2|_A$ and $\partial_{2,2} = \partial_2|_B$

Since S is perfect, we must have $l_2 \geq k_2$.

We examine the contribution of $\partial_{2,1}$ to $H_2(\overline{W'},\widetilde{N};\mathbb{Z})$. It will be useful to first look downstairs at the $\mathbb{Z}$-chain complex for (W', N). Let A' be the submodule of $C_2(W', N;\mathbb{Z})$ determined by the h_j^2, and let C' be $C_1(W', N;\mathbb{Z})$, which is generated by the h_i^1. Then A' is a finitely generated free Abelian group, so the kernel K' of $\partial'_{2,1} : A' \to C'$ is a subgroup of a finitely generated free Abelian group, and thus K' is a finitely generated free Abelian group, say on the basis $\{\kappa_1, \dots, \kappa_a\}$.

CLAIM 2. $\ker(\partial_{2,1})$ *is a free* $\mathbb{Z}Q$*-module on a generating set of cardinality* $|a|$.

Proof. The lifts of D to $\overline{M}$ are discrete. The group Q acts as deck transformations on $\overline{M}$, transitively permuting the lifts of D as the cover $\overline{M}$ is a regular cover. A preferred basepoint $\overline{*}$ and a preferred lift of the the disk D to a disk $\overline{D}$ in $\overline{M}$ have already been chosen for the identification of $C_*(\overline{W'},\widetilde{N};\mathbb{Z})$ with the $\mathbb{Z}Q$-module $C_*(W', N;\mathbb{Z}Q)$. Let the handles attached inside the preferred lift $\overline{D}$ be our preferred lifts $\overline{h_i^1}$, and let the lifts of the h_j^2 that attach to $\overline{D} \cup (\bigcup \overline{h}_i^1)$ be our preferred 2-handles $\overline{h}_j^2$.

$$\begin{aligned}
&\text{Since the lifts of } D \text{ are discrete, if } q_1 \neq q_2 \in Q,\\
&\partial_{2,1}(q_1\overline{c_1} + q_2\overline{c_2}) = 0 \in \mathbb{Z}Q \text{ if and only if}\\
&\partial_{2,1}(\overline{c_1}) = \partial_{2,1}(\overline{c_2}) = 0 \in \mathbb{Z}.
\end{aligned} \tag{‡}$$

With this in mind, let $\overline{\kappa_i}$ be a lift of the chain κ_i in a generating set for K' in the disk D to $\overline{D}$. Then $\partial_{2,1}(\overline{\kappa_i}) = 0$. Moreover, Q transitively permutes each $\overline{\kappa_i}$ with the other lifts of κ_i to the other lifts of D. Now, suppose $\partial_{2,1}(c) = 0$ with an element c of $C_2(W', N;\mathbb{Z}Q)$. Write c as $\sum_{t=1}^{m} n_t q_t a_t$. By (‡) we must have all the q_ts equal (for all nonzero a_t); call this single element q. Consider $q^{-1}c$. We have that $\partial_{2,1}(q^{-1}c) = q^{-1}\partial_{2,1}(c) = q^{-1}0 = 0$, so $q^{-1}c$ is an element of $\ker(\partial'_{2,1})$, and so $q^{-1}c$ is a linear combination of $\{\kappa_1, \dots, \kappa_a\}$, as K' is a generating set for the kernel of $\partial'_{2,1}$. This proves that $\overline{\kappa_t}$ generate $\ker(\partial_{2,1})$. This proves the claim. □

Now, we have $\partial_2 : A \oplus B \to C$. Recall that S is a finitely presented superperfect group and $\partial_2|_A$ is onto. By Lemma 2.3, we have that $\ker(\partial_2) \cong \ker(\partial_2|_A) \oplus B$. By Claim 2, $\ker(\partial_2|_A)$ is a free and finitely generated $\mathbb{Z}Q$-module. Clearly, B is a free and finitely generated $\mathbb{Z}Q$-module. Thus, $\ker(\partial_2) \cong H_2(\overline{W'}, \widetilde{N}; \mathbb{Z})$ is a free and finitely generated $\mathbb{Z}Q$-module.

By Lemma 2.1, we may choose spherical representatives for all elements of $H_2(\overline{W'}; \mathbb{Z})$. Since $\widetilde{N}$ is simply connected, $H_1(\widetilde{N}; \mathbb{Z}) = 0$, and so by the long exact sequence in homology, $H_2(\overline{W'}; \mathbb{Z}) \to H_2(\overline{W'}, \widetilde{N}; \mathbb{Z})$ is onto, so any element of $H_2(\widetilde{W'}, \overline{N}; \mathbb{Z})$ also admits a spherical representative.

So, we may choose spherical representatives for any element of $H_2(W', N; \mathbb{Z}Q)$. Let $\{s_l\}$ be a collection of embedded pairwise disjoint 2-spheres that form a free finite $\mathbb{Z}Q$-basis for $H_2(W', N; \mathbb{Z}Q)$.

Note that the $\{s_l\}$ can be arranged to live in the right-hand boundary M' of W' by moving them off the cores of the 1- and 2-handles by transversality.

If we add the k_i^2, h_j^3, and $g_{i,j}^3$ to W', and similarly make sure that k_i^2, k_j^3, and $g_{i,j}^3$ do not intersect the $\{s_l\}$, and call the resulting cobordism W'', then we can think of the $\{s_l\}$ as living in the right-hand boundary of (W'', N, M''). Note that W'' is diffeomorphic to $N \times \mathbb{I}$.

We wish to attach 3-handles along the collection $\{s_l\}$ and, later, 4-handles complimentary to those 3-handles. A priori, this may be impossible; for instance, there is a framing issue. To make this possible, we borrow a trick from [15] to alter the 2-spheres to a usable collection without changing the elements of $H_2(W', N; \mathbb{Z}Q)$ they represent.

CLAIM 3. *For each s_l, we may choose a second embedded 2-sphere t_l such that*

- *t_l represents the same element of $\pi_2(M'')$ as s_l (as elements of $\pi_2(W')$, they will be different),*
- *each t_l misses the attaching regions of all the $\{h_i^1\}$, $\{k_i^2\}$, $\{h_j^2\}$, $\{k_j^3\}$, $\{f_{i,j}^2\}$, and $\{g_{i,j}^3\}$,*
- *in the collection $\{t_k\}$, the elements are pairwise disjoint, and it is disjoint from the entire collection $\{s_l\}$.*

Proof. Note that each canceling $(2,3)$-handle pairs h_j^2 and k_j^3 or $f_{i,j}^2$ and $g_{i,j}^3$ forms an $(n+1)$-disk attached along an n-disk, which is a regular neighborhood of a 2-disk filling the attaching sphere of the 2-handle. Also, each canceling $(1,2)$-handle h_i^1 and k_i^2 forms an $(n+1)$-disk in $N \times \{1\}$ attached along an n-disk, which is a regular neighborhood of a 1-disk filling the attaching sphere of the 1-handle. We may push a given s_l off the $(2,3)$-handle pairs and then off the $(1,2)$-handle pairs, making sure not to pass back into the $(2,3)$-handle pairs. Let t_l be the end result of the pushes. Make the collection $\{t_l\}$ pairwise disjoint and disjoint from the $\{s_l\}$ by transversality, making sure not to pass back into the $(1,2)$- or $(2,3)$-handle pairs. □

Replace each s_l with $s_l\#(-t_l)$, an embedded connected sum of s_l with a copy of t_l with its orientation reversed.

Since the t_ls miss *all* the handles attached to the original collar $N \times [0,1]$, they can be pushed into the right-hand copy of N. Thus, s_l and $s_l\#(-t_l)$ represent the same element of $H_2(W', N; \mathbb{Z}Q)$. Hence, the collection $\{s_l\#(-t_l)\}$ is still a free basis for $H_2(W', N; \mathbb{Z}Q)$. Furthermore, each $s_l\#(-t_l)$ bounds an embedded 3-disk in the boundary of W''. This means that each $s_l\#(-t_l)$ has a product neighborhood structure, and we may use it as the attaching region for a 3-handle h_l^3. Choose the framing of h_l^3 so that it is a trivially attached 3-handle with respect to W'' and choose a canceling 4-handle k_l^4. We identify these 4-handles now, but do not attach them yet. They will be used later. Call the resulting cobordism with the h_i^1, h_j^2, $f_{i,j}^2$, and h_l^3 attached (W''', N, M). Let $W^{(iv)}$ be $M \times [0,1]$ with the k_i^2, k_j^3, and k_l^4 attached. Then $W''' \cup_M W^{(iv)}$ has all canceling handles and so is diffeomorphic to $N \times [0,1]$. Clearly, $W''' \cup_M W^{(iv)}$ strong deformation retracts onto the right-hand boundary N. We will see that $(W^{(iv}, N, M)$ (modulo torsion) satisfies the conclusion of the theorem.

To prove that $(W^{(iv)}, M, N)$ satisfies the desired properties, we must study W''' more carefully. Since $\pi_1(\overline{W'''}, \widetilde{N}) \cong S$ by construction and S is perfect, $H_1(W''', N; \mathbb{Z}Q) = 0$. Note that since $\ker(\partial_2)$ is a free finitely generated $\mathbb{Z}Q$-module and $\{h_l^3\}$ is a set whose attaching spheres are a free $\mathbb{Z}Q$-basis for $\ker(\partial_2)$, $\partial_3 : C_3(W''', N; \mathbb{Z}Q) \to C_2(W''', N; \mathbb{Z}Q)$ is onto and has no kernel. This means that $H_m(W''', N; \mathbb{Z}Q) \cong 0$ for $m = 2, 3$. Clearly, $H_*(W''', N; \mathbb{Z}Q) \cong 0$ for $* \geq 4$ as $C_*(W''', N; \mathbb{Z}Q) \cong 0$ for $* \geq 4$.

$$\text{Thus, } H_*(\overline{W'''}, \widetilde{N}; \mathbb{Z}) \cong 0, \text{ that is, } H_*(W''', N; \mathbb{Z}Q) \cong 0. \qquad (*)$$

However, this is not the only homology complex we wish to prove acyclic; we also wish to show that $H_*(W''', M; \mathbb{Z}Q) \cong 0$. Using noncompact Poincaré duality, we can do this by showing that the relative cohomology with compact supports is 0, that is, $H_c^*(\overline{W'''}, \widetilde{N}; \mathbb{Z}) \cong 0$.

By the cohomology with compact supports, we mean to take the chain complex that has linear functions $f : C_i(\overline{W'''}, \widetilde{N}; \mathbb{Z}) \to \mathbb{Z}$ to $\mathbb{Z}$ relative to $\widetilde{N}$ that are nonzero on only finitely many of the relative handles. Duality is well known in the setting where $C_*(\overline{W'''}, \widetilde{N}; \mathbb{Z})$ is locally finite, which in turn depends on the fact that $\overline{W'''}$ is a covering space of a compact manifold with finitely many handles attached.The coboundary map δ_* sends a cochain f in $C_c^i(\overline{W'''}, \widetilde{N}; \mathbb{Z})$ to the cochain g in $C_c^{i+1}(\overline{W'''}, \widetilde{N}; \mathbb{Z})$, where $g(\partial(n_j q_i h_j))$ is $\delta(f)(n_j q_i h_j)$.

Clearly, $\delta_1 : C_c^0(\overline{W'''}, \widetilde{N}; \mathbb{Z}) \to C_c^1(\overline{W'''}, \widetilde{N}; \mathbb{Z})$ and $\delta_4 : C_c^3(\overline{W'''}, \widetilde{N}; \mathbb{Z}) \to C_c^4(\overline{W'''}, \widetilde{N}; \mathbb{Z})$ are the zero maps. This means we must show that $\ker(\delta_2) = 0$, that is, δ_2 is 1–1, and $\operatorname{im}(\delta_3) = C_3$, that is, δ_3 is onto. Finally, we must show exactness at C_c^2, that is, we must show that $\operatorname{im}(\delta_2) = \ker(\delta_3)$.

Consider the acyclic complex

$$0 \longrightarrow C_3(\overline{W'''}, \widetilde{N}; \mathbb{Z}) \xrightarrow{\partial_3} C_2(\overline{W'''}, \widetilde{N}; \mathbb{Z}) \xrightarrow{\partial_2} C_1(\overline{W'''}, \widetilde{N}; \mathbb{Z}) \longrightarrow 0.$$

($\ker(\delta_2) = 0$) Let $f \in C_c^1(\overline{W'''}, \widetilde{N}; \mathbb{Z})$ be nonzero, that is, let $f : C_1(\overline{W'''}, \widetilde{N}; \mathbb{Z}) \to 0$ have compact support and there be $c_1 \in C_1(\overline{W'''}, \widetilde{N}; \mathbb{Z})$ such that $c_1 \neq 0$ and

$f(c_1) \neq 0$. As ∂_2 is onto, choose $c_2 \in C_2(\overline{W'''}, \widetilde{N}; \mathbb{Z})$ with $c_2 \neq 0$ and $\partial_2(c_2) = c_1$. Then $\delta_2(f)(c_2) = f(\partial_2(c_2)) = f(c_1) \neq 0$, and $\delta_2(f)$ is not the zero cochain.

($\mathrm{im}(\delta_3) = C^3$) Let $g \in C_c^3(\overline{W'''}, \widetilde{N}; \mathbb{Z})$ be a basis element with $g(qh_i^3) = 1$ and the remaining $g(q'h_{i'}^3) = 0$. We must show that there is $f \in C_C^2(\overline{W'''}, \widetilde{N}; \mathbb{Z})$ with $\delta_3(f) = g$. Consider $\partial_3(qh_i^3)$. This is a basis element for $C_2(\overline{W'''}, \widetilde{N}; \mathbb{Z})$.

Choose $f \in C_c^2(\overline{W'''}, \widetilde{N}; \mathbb{Z})$ to have $f(\partial_3(qh_i^3)) = 1$ and 0 otherwise. Then $\delta_3(f)(q_i h_j^3) = f(\partial_3(q_i h_j^3)) = 1 = g(qh_i^3)$.

This proves that $\delta_3(f) = g$ and δ_3 is onto.

($\mathrm{im}(\delta_2) = \ker(\delta_3)$) Clearly, if $f \in \mathrm{im}(\delta_2)$, then $\delta_3(f) = 0$ as δ is a chain map.

Suppose that $\delta_3(f) = 0$ but $f \neq 0$. Let qh_i^1 be a basis element for $C_2(\overline{W'''}, \widetilde{N}; \mathbb{Z})$. Choose $c_{2,i} \in C_2(\overline{W'''}, \widetilde{N}; \mathbb{Z})$ with $\partial_2(qh_i^1) = c_{2,i}$ as ∂_2 is onto.

Set $g(qh_i^1) = f(c_{2,i})$.

Then $\delta_2(g)(c_{2,i}) = g(\partial_2(c_{2,i})) = g(qh_i^1) = f(c_{2,i})$, and thus f is in the image of δ_2.

So, $H_C^*(\overline{W'''}, \widetilde{N}; \mathbb{Z}) \cong 0$, and thus $H_*(\overline{W'''}, \overline{M}; \mathbb{Z}) \cong 0$ by Theorem 3.35 in [16] and $H_*(W''', M; \mathbb{Z}) \cong 0$.

Note that we again have $\pi_1(N) \cong Q$, $\pi_1(W''') \cong G$, and $\iota_\# : \pi_1(M) \cong \pi_1(W''')$ an isomorphism, as attaching 3-handles does not affect π_1, and, dually, attaching $(n-3)$-handles does not affect π_1 for $n \geq 6$.

We read $W^{(iv)}$ as $(W^{(iv)}, N, M)$. This is (almost) the cobordism we desire. (We will further need to deal with torsion issues.) Moreover, in this form, $W^{(iv)}$ is $N \times [0, 1]$ with $[(n+1)-4]$-, $[(n+1)-3]$-, and $[(n+1)-2]$-handles attached to the right-hand boundary. Since $n \geq 6$, adding these handles does not affect $\pi_1(W^{(iv)})$. Thus, $\iota_\# : \pi_1(N) \to \pi_1(W^{(iv)})$ is an isomorphism; as was previously noted, $\pi_1(M) \cong G$.

Let $H : W''' \cup_M W^{(iv)} \to W''' \cup_M W^{(iv)}$ be a strong deformation retraction onto the right-hand boundary N. We will produce a retraction $r : W''' \cup_M W^{(iv)} \to W^{(iv)}$. Then $r \circ H$ will restrict to a strong deformation retraction of $W^{(iv)}$ onto its right-hand boundary N. This, in turn, will yield a strong deformation retraction of $W^{(iv)}$ read right to left onto its left-hand boundary N.

Note that $H_C^*(\overline{W'''}, \widetilde{N}; \mathbb{Z}) \cong 0$ by ($*$). By Theorem 3.35 in [16], we have that $H_*(\overline{W'''}, \overline{M}; \mathbb{Z}) \cong 0$, and $H_*(W''', M; \mathbb{Z}Q) \cong 0$, respectively, by the natural $\mathbb{Z}Q$ structure on $C_*(\overline{W'''}; \mathbb{Z})$.

To get the retraction r, we will use the following proposition from [12].

Proposition 2.4. *Let (X, A) be a CW pair for which $A \hookrightarrow X$ induces a π_1 isomorphism. Suppose also that $L \trianglelefteq \pi_1(A)$ and $A \hookrightarrow X$ induces $\mathbb{Z}[\pi_1(A)/L]$-homology isomorphisms in all dimensions. Next, suppose $\alpha_1, \ldots, \alpha_k$ is a collection of loops in A that normally generates L. Let X' be the complex obtained by attaching a 2-cell along each α_l, and let A' be the resulting subcomplex. Then $A' \hookrightarrow X'$ is a homotopy equivalence. (Note: In this situation, we call $A \hookrightarrow X$ a* mod L homotopy equivalence.)

Since $H_*(W''', M; \mathbb{Z}Q) = 0$, we have that by Proposition 2.4, W''' union the 2-handles f_j^2 strong deformation retracts onto M union the 2-handles f_j^2. We may now extend via the identity to get a strong deformation retraction

$$r : W''' \cup_{M \cup \text{2-handles}} W^{(iv)} \to W^{(iv)}.$$

Now $r \circ H$ is the desired strong deformation retraction of both $W^{(iv)}$ onto its right-hand boundary N and $W^{(iv)}$ read dually onto its left-hand boundary N.

Now, suppose that, for the cobordism $(W^{(iv)}, N, M)$, we have $\tau(W^{(iv)}, N) = A \neq 0$. As the epimorphism $\eta : G \to Q$ admits a left inverse $\zeta : Q \to G$, by the functoriality of Whitehead torsion, we have that $\text{Wh}(\eta) : \text{Wh}(G) \to \text{Wh}(Q)$ is onto and admits a left inverse $\text{Wh}(\zeta) : \text{Wh}(Q) \to \text{Wh}(G)$. Choose B so that $A + B = 0$ in $\text{Wh}(Q)$ and set $B' = \text{Wh}(\zeta)(B)$. The realization theorem from [24] provides us a cobordism (R, M, N_-) with $\tau(R, M) = B'$. If $W = (W^{(iv)} \cup_M R)$, the by Theorem 20.2 in [6], $\tau(W, N) = \tau(W^{(iv)}, N) + \tau(W, W^{(iv)})$. By Theorem 20.3 in [6], $\tau(W, W^{(iv)}) = \text{Wh}(\eta)(\tau(R, M))$. So, $\tau(W^{(iv)}, N) + \text{Wh}(\eta)(\tau(R, M) = A + \text{Wh}(\eta)(B') = A + B = 0$, and (W, N, N_-) is a 1-sided s-cobordism. $\square$

3. Some Preliminaries to Creating Pseudocollarable High-Dimensional Manifolds

Our goal in this section is to display the usefulness of 1-sided s-cobordisms by using them to create large numbers of topologically distinct pseudocollars (to be defined further), all with similar group-theoretic properties.

We start with some basic definitions and facts concerning pseudocollars.

DEFINITION 5. Let W^{n+1} be a 1-ended manifold with compact boundary M^n. We say that W is *inward tame* if W admits a cofinal sequence of "clean" neighborhoods of infinity (N_i) such that each N_i is finitely dominated. [A *neighborhood of infinity* is a subspace of the closure whose complement is compact. A neighborhood of infinity N is *clean* if (1) N is a closed subset of W, (2) $N \cap \partial W = \emptyset$, and (3) N is a codimension-0 submanifold with bicollared boundary.]

DEFINITION 6. A manifold N^n with compact boundary is a *homotopy collar* if $\partial N^n \hookrightarrow N^n$ is a homotopy equivalence.

DEFINITION 7. A manifold is a *pseudocollar* if it is a homotopy collar that contains arbitrarily small homotopy collar neighborhoods of infinity. A manifold is *pseudocollarable* if it contains a pseudocollar neighborhood of infinity.

Pseudocollars naturally break up as 1-sided h-cobordisms, that is, if $N_1 \subseteq \text{int}(N_2)$ are homotopy collar neighborhoods of infinity of an end of a pseudocollarable manifold, then the $cl(N_2 \backslash N_1)$ is a cobordism (W, M, M_-), where $M \hookrightarrow W$ is a homotopy equivalence. Taking an decreasing chain of homotopy collar neighborhoods of infinity yields a decomposition of a pseudocollar as a "stack" of 1-sided h-cobordisms.

Conversely, if we start with a closed manifold M and use the techniques of Section 2 to produce a 1-sided h-cobordism (W_1, M, M_-), then take M_- and again use the techniques of Section 2 to produce a 1-sided h-cobordisms (W_2, M_-, M_{--}), and so on ad infinitum, and then we glue $W_1 \cup W_2 \cup \cdots$ together to produce an $(n+1)$-dimensional manifold N^{n+1}, then N is a pseudocollar.

So, 1-sided h-cobordisms are an appropriate tool to use when constructing pseudo-collars.

Definition 8. The *fundamental group system at* ∞, $\pi_1(\epsilon(X), r)$, of an end $\epsilon(X)$ of a noncompact topological space X, is defined by taking a cofinal sequence of neighborhoods of ∞ of the end of X, $N_1 \supseteq N_2 \supseteq N_3 \supseteq \ldots$, a proper ray $r : [0, \infty) \to X$, and looking at its related inverse sequence of fundamental groups $\pi_1(N_1, p_1) \leftarrow \pi_1(N_2, p_2) \leftarrow \pi_1(N_3, p_3) \leftarrow \ldots$ (where the bonding maps are induced by inclusion and the basepoint change isomorphism, induced by the ray r).

Such a fundamental group system at infinity has a well-defined associated pro-fundamental group system at infinity, given by its equivalence class inside the category of inverse sequences of groups under the following equivalence relation.

Definition 9. Two inverse sequences of groups (G_i, α_i) and (H_i, β_i) are said to be *proisomorphic* if there exist subsequences of each that may be fit into a commuting ladder diagram as follows:

$$\begin{tikzcd}
G_{i_1} & & G_{i_2} \arrow[ll, "\alpha_{i_1}"'] \arrow[dl, "g_{i_2}"] & & G_{i_3} \arrow[ll, "\alpha_{i_2}"'] \arrow[dl, "g_{i_3}"] & & G_{i_4} \arrow[ll, "\alpha_{i_3}"'] \arrow[dl, "g_{i_4}"] & & \cdots \arrow[ll, "\alpha_{j_4}"'] \\
& H_{j_1} \arrow[ul, "f_{j_1}"] & & H_{j_2} \arrow[ll, "\beta_{j_1}"'] \arrow[ul, "f_{j_2}"] & & H_{j_3} \arrow[ll, "\beta_{j_2}"'] \arrow[ul, "f_{j_3}"] & & H_{j_4} \arrow[ll, "\beta_{j_3}"'] \arrow[ul, "f_{j_4}"] & & \cdots \arrow[ll, "\beta_{j_4}"']
\end{tikzcd}$$

A more detailed introduction to fundamental group systems at infinity can be found in [9] or [11].

Definition 10. An inverse sequence of groups is *stable* if it is proisomorphic to a constant sequence $G \leftarrow G \leftarrow G \leftarrow G \ldots$ with the identity for bonding maps.

The following is a theorem of Brown from [4].

Theorem 3.1. *The boundary of a manifold M is collared, that is, there is a neighborhood N of ∂M in M such that $N \approx \partial M \times \mathbb{I}$.*

The following is from Siebenmann's thesis [25].

Theorem 3.2. *An open manifold W^{n+1} $(n \geq 5)$ admits a compactification as an $n+1$-dimensional manifold with an n-dimensional boundary manifold M^n if*

(1) *W is inward tame,*
(2) *$\pi_1(\epsilon(W))$ is stable for each end $\epsilon(W)$ of W, and*
(3) *$\sigma_\infty(\epsilon(W)) \in \widetilde{K}_0[\mathbb{Z}\pi_1(\epsilon(W))]$ vanishes for each end $\epsilon(W)$ of W.*

DEFINITION 11. An inverse sequence of groups is *semistable* or *Mittag-Leffler* if is it proisomorphic to a sequence $G_1 \twoheadleftarrow G_2 \twoheadleftarrow G_3 \twoheadleftarrow G_4 \ldots$ with epic bonding maps.

DEFINITION 12. An inverse sequence of finitely presented groups is *perfectly semistable* if and only if it is proisomorphic to a sequence $G_1 \twoheadleftarrow G_2 \twoheadleftarrow G_3 \twoheadleftarrow G_4 \ldots$ with epic bonding maps and perfect kernels.

The following two lemmas are well known and show that optimally chosen perfectly semistable inverse sequences behave well under passage to subsequences.

LEMMA 3.1. *Let*

$$1 \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\sigma} Q \longrightarrow 1$$

be a short exact sequence of groups with perfect K and Q. Then G is perfect.

LEMMA 3.2. *If $\alpha : A \to B$ and $\beta : B \to C$ are both onto and have perfect kernels, then $(\beta \circ \alpha) : A \to C$ is onto and has a perfect kernel.*

The following is a result from [14].

THEOREM 3.3 (Guilbault–Tinsley). *A noncompact manifold W^{n+1} with compact (possibly empty) boundary $\partial W = M$ is pseudocollarable if and only if*

(1) *W is inward tame,*
(2) *$\pi_1(\epsilon(W))$ is perfectly semistable for each end $\epsilon(W)$ of W, and*
(3) *$\sigma_\infty(\epsilon(W)) \in \widetilde{K}_0[\mathbb{Z}\pi_1(\epsilon(W))]$ vanishes for each end $\epsilon(W)$ of W.*

So, the profundamental group system at infinity of a pseudocollar is perfectly semistable. As is outlined in Chapter 4 of [11], the profundamental group system at infinity is independent of base ray for ends with semistable profundamental group at infinity, and hence for 1-ended pseudocollars.

THEOREM 1.2 (Uncountably many pseudocollars on closed manifolds with the same boundary and similar Pro-π_1). *Let M^n be a closed smooth manifold ($n \geq 6$) with $\pi_1(M) \cong \mathbb{Z}$, and let S be the finitely presented group $V * V$ that is the free product of two copies of Thompson's group V. Then there exists an uncountable collection of pseudocollars $\{N^{n+1}_\omega \mid \omega \in \Omega\}$, no two of which are homeomorphic at infinity, and each of which begins with $\partial N^{n+1}_\omega = M^n$ and is obtained by creating a group extension $1 \to S \to G_{i+1} \to G_i \to 1$ and then using Theorem 1.1 to create a 1-sided h-cobordism (W_i, M_i, M_{i+1}) with $\pi_1(M_i) = G_i$ and $\pi_1(M_{i+1}) = G_{i+1}$ countably many times, using the same kernel group S each time. In particular, each N_i has a fundamental group at infinity that may be represented by an inverse sequence*

$$\mathbb{Z} \overset{\alpha_1}{\twoheadleftarrow} G_1 \overset{\alpha_2}{\twoheadleftarrow} G_2 \overset{\alpha_3}{\twoheadleftarrow} G_3 \overset{\alpha_4}{\twoheadleftarrow} \cdots$$

with $\ker(\alpha_i) = S$ *for all i.*

We give a brief overview of our strategy. For convenience, we will start with the manifold $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, which has a fundamental group $\mathbb{Z}$. We let S be the free product of two copies of Thompson's group V, which is a finitely presented superperfect group for which $\mathrm{Out}(S)$ has torsion elements of all orders (see [21]). Then we will create a collection of group extensions, each with kernel group S, starting with $\mathbb{Z}$, then continuing to semidirect products $G_{p_1}, G_{p_2}, G_{p_3}, \ldots$ in infinitely many different ways using different automorphisms, each with order a prime number p_i strictly greater than the prime order p_{i-1} used in the last step, from the infinite group $\mathrm{Out}(S)$ as the outer actions. We will then use the theorem of the last chapter to create a collection of 1-sided s-cobordisms $(W_{p_1}, \mathbb{S}^1 \times \mathbb{S}^{n-1}, M_{p_1}), (W_{p_2}, M_{p_1}, M_{p_2}), (W_{p_3}, M_{p_2}, M_{p_3}), \ldots$ with $\pi_1(\mathbb{S}^1 \times \mathbb{S}^{n-1}) = \mathbb{Z}, \pi_1(M_{p_1}) = G_{p_1}, \pi_1(M_{p_2}) = G_{p_2}, \pi_1(M_{p_3}) = G_{p_3}, \ldots$.

Continuing inductively, we will obtain increasing sequences ω of prime numbers describing each sequence of 1-sided s-cobordisms. We will then glue together all the semi-s-cobordisms at each stage for each unique increasing sequence of prime numbers ω, creating for each an $(n+1)$-manifold N_ω^{n+1}, and show that there are uncountably many such pseudocollared $(n+1)$-manifolds N_ω, one for each increasing sequence of prime numbers ω, all with the same boundary $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ and all being the result of creating a semidirect product group extension $1 \to S \to G \to \mathbb{Z} \to 1$ with $\mathbb{Z}$ as the quotient group and with two copies of the same superperfect group P, $S = P * P$, as the kernel group at each stage. The fact that no two of these pseudocollars are homeomorphic at infinity will follow from the fact that no two of the inverse sequences of groups are proisomorphic. Much of the algebra in this chapter is aimed at proving that delicate result.

REMARK 1. There is an alternate strategy of creating an inverse sequence of group extensions $1 \to G_{i-1} \to G_i \to S_i \to 1$ the fundamental group G_i at each stage by the free product $G_i * S_i$; using a countable collection of freely indecomposable kernel groups $\{S_i\}$ would then allow us to create an uncountable collection of pseudocollars. An algebraic argument like that found in [26] or [7] would then complete the proof. However, they would not have the nice kernel properties that our construction has.

It seems likely that groups other than Thompson's group V would work for creating uncountably many pseudocollars, all with similar group-theoretic properties, from sequences of 1-sided s-cobordisms. However, for our purposes, V possesses the ideal set of properties.

4. Some Algebraic Lemmas, Part 1

In this section, we go over the main algebraic lemmas necessary to do our strategy of creating a semidirect product group extension $1 \to S \to G \to \mathbb{Z} \to 1$ with $\mathbb{Z}$ as the quotient group and with two copies of the same superperfect group P, $S = P * P$, as the kernel group at each stage.

Thompson's group V is finitely presented, superperfect, simple, and contains torsion elements of all orders. Note that simple implies that V is centerless, Hopfian, and freely indecomposable.

An introduction to some basic properties of Thompson's group V can be found in [21], There, it is shown that V is finitely presented and simple. It is also noted in [21] that V contains torsion elements of all orders, as V contains a copy of every symmetric group on n letters, and hence of every finite group. In [3], it is noted that V is superperfect.

LEMMA 4.1. *Every non-Abelian simple group is perfect.*

DEFINITION 13. A group G is *Hopfian* if every onto map from G to itself is an isomorphism. Equivalently, a group is Hopfian if it is not isomorphic to any of its proper quotients.

LEMMA 4.2. *Every simple group is Hopfian.*

Let $S = P_1 * P_2$ be the free product of two copies of V with itself. This is clearly finitely presented, perfect (by Meyer–Vietoris), and superperfect (again, by Meyer–Vietoris). Note that S is a free product of nontrivial groups, so S is centerless. In [20], it is noted that free products of Hofpian, finitely presented, and freely indecomposable groups are Hopfian, so $S = V * V$ is Hopfian; S (and not V itself) will be the superperfect group we use in our constructions.

We need a few lemmas.

LEMMA 4.3. *Let A, B, C, and D be nontrivial groups. Let $\phi : A \times B \to C * D$ be a surjective homomorphism. Then one of $\phi(A \times \{1\})$ and $\phi(\{1\} \times B)$ is trivial, and the other is all of $C * D$.*

Proof. Let $x \in \phi(A \times \{1\}) \cap \phi(\{1\} \times B)$. Then $x \in \phi(A \times \{1\})$, so x commutes with everything in $\phi(\{1\} \times B)$. But $x \in \phi(\{1\} \times B)$, so x commutes with everything in $\phi(A \times \{1\})$. As ϕ is onto, this implies $\phi(A \times \{1\}) \cap \phi(\{1\} \times B) \leq Z(C * D)$.

However, by a standard normal forms argument, the center of a free product is trivial! So, $\phi(A \times \{1\}) \cap \phi(\{1\} \times B) \leq Z(C * D) = 1$. This implies that $\phi(A \times \{1\}) \times \phi(\{1\} \times B) = C * D$. By a result in [2], a nontrivial direct product cannot be a nontrivial free product. (A proof using the Kurosh subgroup theorem can be found in many group theory texts, such as Theorem 6.3.10 of [23]. An alternate, much simpler proof due to P.M. Neumann can be found in [19] in the observation after Lemma IV.1.7.) Thus, $\phi(A \times \{1\}) = C * D$ or $\phi(\{1\} \times B) = C * D$, and the other is the trivial group. The result follows. □

COROLLARY 4.4. *Let $A_1, \ldots, A_n$ be nontrivial groups, and let $C * D$ be a free product of nontrivial groups. Let $\phi : A_1 \times \cdots \times A_n \to C * D$ be a surjective homomorphism.*

Then one of the $\phi(\{1\} \times \cdots \times A_i \times \cdots \times \{1\})$ is all of $C * D$, and the rest are all trivial.

Proof. Proof is by induction.

($n = 2$) This is Lemma 4.3.

(Inductive step) Suppose the result is true for $n-1$. Set B = $A_1 \times \cdots \times A_{n-1}$. By Lemma 4.3, either $\phi(B \times \{1\})$ is all of $C * D$ and $\phi(\{1\} \times A_n)$ is trivial, or $\phi(B \times \{1\})$ is trivial and $\phi(\{1\} \times A_n)$ is all of $C * D$.

If $\phi(B \times \{1\})$ is trivial and $\phi(\{1\} \times A_n)$ is all of $C * D$, we are done.

If $\phi(B \times \{1\})$ is all of $C * D$ and $\phi(\{1\} \times A_n)$ is trivial, then, by the inductive hypothesis, we are also done. $\square$

COROLLARY 4.5. *Let $S_1, S_2, \ldots, S_n$ all be copies of the same nontrivial free product, and let $\psi : S_1 \times S_2 \times \cdots \times S_n \to S_1 \times S_2 \times \cdots \times S_n$ be an isomorphism. Then ψ decomposes as a "matrix of maps" $\psi_{i,j}$, where each $\psi_{i,j} = \pi_{S_j} \circ \psi|_{S_i}$ (where π_{S_j} is the projection onto S_j), and there is a permutation σ on n indices such that each $\psi_{\sigma(j),j} : S_{\sigma(j)} \to S_j$ is an isomorphism, and all other $\psi_{i,j}$ are the zero map.*

Proof. By Corollary 4.4 applied to $\pi_{S_j} \circ \psi$, we clearly have a situation where each $\pi_{S_j} \circ \psi|_{S_i}$ is either trivial or onto. If we use a schematic diagram with an arrow from S_i to S_j to indicate nontriviality of a map $\psi_{i,j}$, we obtain a diagram like the following:

$$\begin{tikzcd}[column sep=small]
S_1 \arrow[dd] \arrow[ddrr] \arrow[ddrrrr] & \times & S_2 & \times & S_3 & \times & S_4 \arrow[dd] & \times & S_5 \arrow[dd] & \times & S_6 \arrow[ddrr] & \times & S_7 \arrow[ddll] & \times & \cdots & & S_n \arrow[dd] \arrow[ddl] \arrow[ddlll] \\
 & & & & & & & & & & & & & & \cdots & & \\
S_1 & \times & S_2 & \times & S_3 & \times & S_4 & \times & S_5 & \times & S_6 & \times & S_7 & \times & \cdots & & S_n
\end{tikzcd}$$

where a priori some of S_i in the domain may map onto multiple S_j in the target, and there are no arrows emanating from some of the S_i in the domain.

By the injectivity of ψ, there must be at least one arrow emanating from each S_i, whereas by surjectivity of ψ, there must be at least one arrow ending at each S_j. Corollary 4.4 prevents more than one arrow from ending in a given S_j. By the pigeonhole principle, the arrows determine a one-to-one correspondence between the factors in the domain and those in the range. A second application of injectivity now shows that each arrow represents an isomorphism. $\square$

Note that the $\psi_{i,j}$ form a matrix where each row and each column contain exactly one isomorphism, and the rest of the maps are trivial maps—what would be a permutation matrix (see, e.g., p. 100 in [22]) if the isomorphisms were replaced by 1s and the trivial maps were replaced by 0s.

COROLLARY 4.6. *Let $S_1, S_2, \ldots, S_n$ all be copies of the same nontrivial Hopfian free product, and let $\psi : S_1 \times S_2 \times \cdots \times S_n \to S_1 \times S_2 \times \cdots \times S_m$ be an epimorphism with $m < n$. Then ψ decomposes as a "matrix of maps" $\psi_{i,j} = \pi_{S_j} \circ \psi|_{S_i}$, and there is a* 1–1 *function σ from the set $\{1, \ldots, m\}$ to the set $\{1, \ldots, n\}$ such that $\psi_{\sigma(j),j} : S_{\sigma(j)} \to S_j$ is an isomorphism, and all other $\psi_{i,j}$ are the zero map.*

Proof. Begin with a schematic arrow diagram as we had in the previous lemma. By surjectivity and Lemma 4.4, each of the m factors in the range is at the end of exactly one arrow. From there, we may conclude that each arrow represents an epimorphism, and, hence, by Hopfian, an isomorphism.

To complete the proof, we must argue that at most one arrow can emanate from an S_i factor. Suppose to the contrary that two arrows emanate from a given S_i factor. Then we have an epimorphism of S_i onto a nontrivial direct product in which each coordinate function is a bijection. This is clearly impossible. $\square$

5. Some Algebraic Lemmas, Part 2

Let p_i denote the ith prime number. Let Ω be the uncountable set consisting of all increasing sequences of prime numbers $(p_{i_1}, p_{i_2}, p_{i_3}, \ldots)$. For $\omega \in \Omega$ and $n \in \mathbb{N}$, define (ω, n) to be the finite sequence consisting of the first n entries of ω.

For the group $S = P_1 * P_2$, where each P_i is Thompson's group V, choose $u_i \in P_1$ to have order$(u_i) = p_i$.

Recall that if K is a group, then $\mathrm{Aut}(K)$ is the automorphism group of K. Define $\mu : K \to \mathrm{Aut}(K)$ to be $\mu(k)(k') = kk'k^{-1}$. Then the image of μ in $\mathrm{Aut}(K)$ is called *the inner automorphism group of K,* $\mathrm{Inn}(K)$. The inner automorphism group of a group K is always normal in $\mathrm{Aut}(K)$. The quotient group $\mathrm{Aut}(K)/\mathrm{Inn}(K)$ is called the *outer automorphism group* $\mathrm{Out}(K)$. The kernel of μ is called the *center of K,* $Z(K)$; it is the set of all $k \in K$ such that $kk'k^{-1} = k'$ for all $k' \in K$. We have the exact sequence

$$1 \longrightarrow Z(K) \longrightarrow K \xrightarrow{\mu} \mathrm{Aut}(K) \xrightarrow{\alpha} \mathrm{Out}(K) \longrightarrow 1$$

Define the map $\Phi : P_1 \to \mathrm{Out}(P_1 * P_2)$ by $\Phi(u) = \phi_u$, where $\phi_u \in \mathrm{Out}(P_1 * P_2)$ is the outer automorphism defined by the automorphism

$$\phi_u(p) = \begin{cases} p & \text{if } p \in P_1, \\ upu^{-1} & \text{if } p \in P_2. \end{cases}$$

(ϕ_u is called a *partial conjugation.*)

CLAIM 4. $\Phi : P_1 \to \mathrm{Out}(P_1 * P_2)$ *is an embedding.*

Proof. Suppose $\Phi(u)$ is an inner automorphism for some u not e in P_1. Since $\Phi(u)$ acts on P_2 by conjugation by u, to be an inner automorphism, $\Phi(u)$ must also act on P_1 by conjugation by u. Now, $\Phi(u)$ acts on P_1 trivially for all $p \in P_1$, which implies that u is in the center of P_1. However, P_1 is centerless! Thus, no $\Phi(u)$ is an inner automorphism for any $u \in P_1$. $\square$

So, for each u_i with prime order p_i (the ith prime), ϕ_{u_i} has prime order p_i, as does every conjugate of ϕ_{u_i} in $\mathrm{Out}(P_1 * P_2)$, as Φ is an embedding.

LEMMA 5.1. *For any finite collection of groups* $A_1, A_2, \ldots, A_n$, $\prod_{i=1}^n \mathrm{Out}(A_i)$ *embeds in* $\mathrm{Out}(\prod_{i=1}^n A_i)$.

Proof. The natural map from $\prod_{i=1}^n \mathrm{Aut}(A_i)$ to $\mathrm{Aut}(\prod_{i=1}^n A_i)$, which sends a Cartesian product of automorphisms individually in each factor to that product considered as an automorphism of the Cartesian product, is clearly an embedding. Now, $\mathrm{Inn}(A_1 \times \cdots \times A_n)$ is the image under this natural map of $\mathrm{Inn}(A_1) \times \cdots \times \mathrm{Inn}(A_n)$, because if $b_i \in A_i$, then $(b_1, \ldots, b_n)^{-1}(a_1, \ldots, a_n)(b_1, \ldots, b_n) = (b_1^{-1}a_1b_1, \ldots, b_n^{-1}a_nb_n)$. So, the induced map on quotient groups, from $\prod_{i=1}^n \mathrm{Out}(A_i)$ to $\mathrm{Out}(\prod_{i=1}^n A_i)$, is also a monomorphism. □

Now, because the quotient map $\Psi : \prod_{i=1}^n \mathrm{Out}(A_i) \to \mathrm{Out}(\prod_{i=1}^n A_i)$ is an embedding, $\mathrm{order}(\phi_1, \ldots, \phi_n)$ in $\mathrm{Out}(\prod_{i=1}^n A_i)$ is just $\mathrm{lcm}(\mathrm{order}(\phi_1), \ldots, \mathrm{order}(\phi_n))$, which is just its order in $\prod_{i=1}^n \mathrm{Out}(A_i)$. Moreover, each conjugate of $(\phi_1, \ldots, \phi_n)$ in $\mathrm{Out}(\prod_{i=1}^n A_i)$ has the same order $\mathrm{lcm}(\phi_1, \ldots, \phi_n)$. Finally, note that if each ϕ_i has prime order and each prime occurs only once, then $\mathrm{order}(\phi_1, \ldots, \phi_n) = \mathrm{order}(\phi_1) \times \cdots \times \mathrm{order}(\phi_n)$.

LEMMA 5.2. *Let K be a group and suppose $\Theta : K \rtimes_\phi \mathbb{Z} \to K \rtimes_\psi \mathbb{Z}$ is an isomorphism that restricts to an isomorphism $\overline{\Theta} : K \to K$. Then ϕ and ψ are conjugate as elements of* $\mathrm{Out}(K)$

Proof. We use the presentations $\langle \mathrm{gen}(K), a \mid \mathrm{rel}(K), aka^{-1} = \phi(k)\rangle$ and $\langle \mathrm{gen}(K), b \mid \mathrm{rel}(K), bkb^{-1} = \psi(k)\rangle$ of the domain and range, respectively, Since Θ induces an isomorphism on the infinite cyclic quotients by K, there exists $c \in K$ with $\Theta(a) = cb^{\pm 1}$. We assume that $\Theta(a) = cb$, with the case $\Theta(a) = cb^{-1}$ being similar.

For each $k \in K$, we have

$$\begin{aligned}\Theta(\phi(k)) &= \Theta(aka^{-1}) \\ &= \Theta(a)\Theta(k)\Theta(a)^{-1} \\ &= cb\Theta(k)b^{-1}c^{-1} \\ &= c\psi(\Theta(k))c^{-1}.\end{aligned}$$

Denoting by $\iota_c : K \to K$ the conjugation by c, we have $\overline{\Theta}\phi = \iota_c\psi\overline{\Theta}$ in $\mathrm{Aut}(K)$. Quotienting out by $\mathrm{Inn}(K)$ and abusing notation slightly, we have $\overline{\Theta}\phi = \psi\overline{\Theta}$ or $\overline{\Theta}\phi\overline{\Theta}^{-1} = \psi$ in $\mathrm{Out}(K)$. □

Once again, let S_i denote the free product of two copies of Thompson's group V.

LEMMA 5.3. *For any finite strictly increasing sequence of primes $(s_1, s_2, \ldots, s_n)$, define $\phi_{(s_1,\ldots,s_n)} : S_1 \times \cdots \times S_n \to S_1 \times \cdots \times S_n$ by $\phi_{(s_1,\ldots s_n)}(x_1, \ldots, x_n) = (\phi_{u_1}(x_1), \ldots, \phi_{u_n}(x_n))$, where ϕ_{u_i} is the partial conjugation outer automorphism associated above to the element u_i with prime order s_i. Let $(s_1, \ldots, s_n)$ and $(t_i, \ldots, t_n)$ be increasing sequences of prime numbers of length n. Let $G_{(s_1,\ldots,s_n)} = (S_1 \times \cdots \times S_n) \rtimes_{\phi_{(s_1,\ldots,s_n)}} \mathbb{Z}$ and $G_{(t_i,\ldots,t_n)} = (S_1 \times \cdots \times S_n) \rtimes_{\phi_{(t_1,\ldots,t_n)}} \mathbb{Z}$ be two semidirect products with such outer actions. Then $G_{(s_1,\ldots,s_n)}$ is isomorphic to $G_{(t_i,\ldots,t_n)}$ if and only if $\{s_1, \ldots, s_n\} = \{t_1, \ldots, t_n\}$ for the underlying sets.*

Proof. ($\Rightarrow$) Let $\theta : G_{(s_1,\ldots,s_n)} \to G_{(t_i,\ldots,t_n)}$ be an isomorphism. There are n factors of S in the kernel group of each of $G_{(\omega,n)}$ and $G_{(\eta,n)}$. Then θ must preserve the commutator subgroup, as the commutator subgroup is a characteristic subgroup and so induces an isomorphism of the perfect kernel group $K = S_1 \times S_2 \times \cdots \times S_n$, say $\overline{\theta}$. By Corollary 4.5, it must permute the factors of K, say via σ.

Now, the isomorphism θ must take the (infinite cyclic) abelianization $G_{(s_1,\dots,s_n)}/K_{(s_1,\dots,s_n)}$ of the one to the (infinite cyclic) abelianization $G_{(t_i,\dots,t_n)}/K_{(t_i,\dots,t_n)}$ of the other and hence takes a generator of $G_{(s_1,\dots,s_n)}/K_{(s_1,\dots,s_n)}$ (say $aK_{(s_1,\dots,s_n)}$) to a generator of $G_{(t_i,\dots,t_n)}/K_{(t_i,\dots,t_n)}$ (say $b^e K_{(t_i,\dots,t_n)}$, where $bK_{(t_i,\dots,t_n)}$ is a given generator of $G_{(t_i,\dots,t_n)}/K_{(t_i,\dots,t_n)}$ and $e = \pm 1$). Then since θ takes $K_{(s_1,\dots,s_n)} = [G_{(s_1,\dots,s_n)}, G_{(s_1,\dots,s_n)}]$ to $[G_{(t_i,\dots,t_n)}, G_{(t_i,\dots,t_n)}] = K_{(t_i,\dots,t_n)}$, it follows that θ takes a to a multiple of b^e, say $c^{-1}b^e$, where c lies in $K_{(t_i,\dots,t_n)}$, and $e = \pm 1$.

Now, by Lemma 5.2, $\phi_{(s_1,\dots,s_n)}$ is conjugate in $\mathrm{Out}(K)$ to $\phi_{(t_1,\dots,t_n)}$, $\overline{\theta}(\phi_{(s_1,\dots,s_n)})\overline{\theta}^{-1} = \phi_{(t_1,\dots,t_n)}$. However, Ψ is an embedding by Lemma 4! This shows that $\mathrm{order}(\phi_{(s_1,\dots,s_n)}) = \prod_{i=1}^{n} s_i$ and $\mathrm{order}(\phi_{(t_1,\dots,t_n)}) = \prod_{i=1}^{n} t_i$ are equal, so, as both s_i and t_i are prime and occur only once in each increasing sequence, $\{s_1, \dots, s_n\} = \{t_1, \dots, t_n\}$ by the fundamental theorem of arithmetic.

($\Leftarrow$) Clear. $\square$

LEMMA 5.4. *Let $(\omega, n) = (s_1, \dots, s_n)$ and $(\eta, m) = (t_1, \dots, t_m)$ be increasing sequences of prime numbers with $n > m$.*

Let $G_{(\omega,n)} = (S_1 \times \cdots \times S_n) \rtimes_{\phi_{(\omega,n)}} \mathbb{Z}$ and $G_{(\eta,m)} = (S_1 \times \cdots \times S_m) \rtimes_{\phi_{(\eta,m)}} \mathbb{Z}$ be two semidirect products. Then there is an epimorphism $g : G_{(\omega,n)} \to G_{(\eta,m)}$ if and only if $\{t_1, \dots, t_m\} \subseteq \{s_1, \dots, s_n\}$.

Proof. The proof in this case is similar to the case $n = m$, except that the epimorphism g must crush out $n - m$ factors of $K_{(\omega,n)} = S_1 \times \cdots \times S_n$ by Corollary 4.6 and the pigeonhole principle and thus is an isomorphism on the remaining factors.

($\Rightarrow$) Suppose there is an epimorphism $g : G_{(\omega,n)} \to G_{(\eta,m)}$. Then g must send the commutator subgroup of $G_{(\omega,n)}$ onto the commutator subgroup of $G_{(\eta,m)}$. By Corollary 4.6, g must send m factors of $K_{(\omega,n)} = S_1 \times \cdots \times S_n$ in the domain isomorphically onto the m factors of $K_{(\eta,m)} = S_1 \times \cdots \times S_m$ in the range and send the remaining $n - m$ factors of $K_{(\omega,n)}$ to the identity. Let $\{i_1, \dots, i_m\}$ be the indices in $\{1, \dots, n\}$ of factors in $K_{(\omega,n)}$, which are sent onto a factor in $K_{(\eta,m)}$, and let $\{j_1, \dots, j_{n-m}\}$ be the indices in $\{1, \dots, n\}$ of factors in $K_{(\omega,n)}$, which are sent to the identity in $K_{(\eta,m)}$. Then g induces an isomorphism between $S_{i_1} \times \cdots \times S_{i_m}$ and $K_{(\eta,m)}$. Set $L_m = S_{i_1} \times \cdots \times S_{i_m}$.

Also, by an argument similar to Lemmas 5.2 and 5.3, g sends the infinite cyclic group $G_{(\omega,n)}/K_{(\omega,n)}$ isomorphically onto the infinite cyclic quotient $G_{(\eta,m)}/K_{(\eta,m)}$.

Note that $L_m \rtimes_{\phi_{(s_{i_1},\dots,s_{i_m})}} \mathbb{Z}$ is a quotient group of $G_{(\omega,n)}$ by a quotient map that sends $S_{j_1} \times \cdots \times S_{j_{n-m}}$ to the identity. Consider the induced map $g' : L_m \rtimes_{\phi_{(s_{i_1},\dots,s_{i_m})}} \mathbb{Z} \to G_{(\eta,m)}$. Since g' maps L_m isomorphically onto $K_{(\eta,m)}$ and preserves the infinite cyclic quotients, we have that the kernel of g must equal exactly $S_{j_1} \times \cdots \times S_{j_{n-m}}$; thus, by the fundamental isomorphism theorem, we have that g' is an isomorphism.

Finally, g' is an isomorphism of $L_m \rtimes_{\phi_{(s_{i_1},\dots,s_{i_m})}} \mathbb{Z}$ with $G_{(\omega,n)}$, which restricts to an isomorphism of L_m with $S_{t_1} \times \cdots \times S_{t_m}$, so, by Lemma 5.2, we have that $\phi_{(s_{i_1},\dots,s_{i_m})}$ is conjugate to $\phi_{(t_1,\dots,t_m)}$, so, in $\mathrm{Out}(\prod_{i=1}^n A_1)$, $\mathrm{order}(\phi_{(s_{i_1},\dots,s_{i_m})}) = \mathrm{order}(\phi_{(t_1,\dots,t_m)})$, and thus, as both s_i and t_i are prime and appear at most once, by an argument similar to Lemma 5.3, $\{t_1,\dots,t_m\} \subseteq \{s_1,\dots,s_n\}$ by the fundamental theorem of arithmetic.

($\Leftarrow$) Suppose $\{t_1,\dots,t_m\} \subseteq \{s_1,\dots,s_n\}$. Choose $a \in G_{(\omega,n)}$ with $aK_{(\omega,n)}$ generating the infinite cyclic quotient $G_{(\omega,n)}/K_{(\omega,n)}$ and choose $b \in G_{(\eta,m)}$ with $bK_{(\eta,m)}$ generating the infinite cyclic quotient $G_{(\eta,m)}/K_{(\eta,m)}$. Set $g(a) = b$.

Now send s_{i_j} to t_j (where S_{i_j} uses an element of order i_j in its semidirect product definition in the domain) under g. Send the elements of all other S_i to the identity.

We show that $g : G_{(\omega,n)} \to G_{(\eta,m)}$ is an epimorphism. Clearly, g is onto since $g(s_{i_j}) = t_j$. It remains to show that g respects the multiplication in each group.

Clearly, g respects the multiplication in each S_i and in $\mathbb{Z}$.

The proof reduces to showing that if $\alpha_{i_j} \in S_{i_j}$ and $a \in \mathbb{Z}$, then

$$g(a\alpha_{i_j}) = g(a)g(\alpha_{i_j}),$$
$$g(\phi_{s_{i_j}}(\alpha_i)a) = \phi_{t_j}(g(\alpha_{i_j}))g(a),$$

using the slide relators for each group and the fact that $s_{i_j} = t_j$, which implies $\phi_{s_{i_j}} = \phi_{t_j}$. So, g respects the multiplication in each group. This completes the proof. $\square$

6. Some Algebraic Lemmas, Part 3

Recall that Ω is an uncountable set consisting of increasing sequences of prime numbers $(p_{i_1}, p_{i_2}, p_{j_3}, \dots)$. For $\omega \in \Omega$ and $n \in \mathbb{N}$, recall that we have defined (ω, n) to be the finite sequence consisting of the first n entries of ω.

Recall also that p_i denotes the ith prime number, and for the group $S = P_1 * P_2$, where each P_i is Thompson's group V, we have chosen $u_i \in P_1$ to have $\mathrm{order}(u_i) = p_i$.

Finally, recall that we have defined a map $\Phi : P_1 \to \mathrm{Out}(P_1 * P_2)$ (where each P_j is a copy of Thompson's group V) by $\Phi(u) = \phi_u$, where $\phi_u \in \mathrm{Out}(P_1 * P_2)$ is the outer automorphism defined by the automorphism, called a *partial conjugation,*

$$\phi_u(p) = \begin{cases} p & \text{if } p \in P_1, \\ upu^{-1} & \text{if } p \in P_2. \end{cases}$$

Set $G_{(\omega,n)} = (S \times S \times \cdots \times S) \rtimes_{\phi_{(\omega,n)}} \mathbb{Z}$.

LEMMA 6.1. *$G_{(\omega,n)} \cong S \rtimes_{\phi_{w_{s_n}}} G_{(\omega,n-1)}$, where $\phi_{w_{s_n}}$ is a partial conjugation by u_{s_n}.*

Proof. Note that there is a presentation for $(S \times S \times \cdots \times S) \rtimes_{\phi_{(\omega,n)}} \mathbb{Z}$ that contains a presentation for $S \rtimes_{\phi_{w_{s_n}}} G_{(\omega,n-1)}$.

Generators: z, the generator of $\mathbb{Z}$, together with the generators of the first copy of S, the generators of the second copy of $S, \dots,$ and the generators of the nth copy of S.

Relators defining P_i: the relators for the copy of P_1 in the first copy of S, the relators for the copy of P_2 in the first copy of S, the relators for the copy of P_1 in the second copy of S, the relators for the copy of P_2 in the second copy of $S, \dots,$ and the relators for the copy of P_1 in the nth copy of S, the relators for the copy of P_2 in the nth copy of S.

Slide relators: The slide relators between z and the generators of P_2 in the first copy of S due to the semidirect product, the slide relators between z and the generators of P_2 in the second copy of S due to the semidirect product, $\dots$, the slide relators between z and the generators of P_1 in the nth copy of S due to the semidirect product, and the slide relators between z and the generators of P_2 in the nth copy of S due to the semidirect product. $\square$

Now, looking at $G_{(\omega,n)}$ as a semidirect product of S with $G_{(\omega,n-1)}$ yields an inverse sequence $(G_{(\omega,n)}, \alpha_n)$, which looks like

$$G_{(\omega,0)} \xleftarrow{\alpha_0} G_{(\omega,1)} \xleftarrow{\alpha_1} G_{(\omega,2)} \xleftarrow{\alpha_2} \cdots$$

with bonding maps $\alpha_i : G_{(\omega,i+1)} \to G_{(\omega,i)}$ that each crush out the most recently added copy of S.

A subsequence will look like

$$G_{(\omega,n_0)} \xleftarrow{\alpha_{n_0}} G_{(\omega,n_1)} \xleftarrow{\alpha_{n_1}} G_{(\omega,n_2)} \xleftarrow{\alpha_{n_2}} \cdots$$

with bonding maps $\alpha_{n_i} : G_{\omega,n_j)} \to G_{\omega,n_i)}$ that each crush out the most recently added $n_j - n_i$ copies of S.

LEMMA 6.2. *If, for inverse sequences $(G_{(\omega,n)}, \alpha_n)$, where $\alpha_n : G_{(\omega,n)} \to G_{(\omega,n-1)}$ is the bonding map crushing out the most recently added copy of S, ω does not equal η, then the two inverse sequences are not proisomorphic.*

Proof. Let $(G_{(\omega,n)}, \alpha_n)$ and $(G_{(\eta,m)}, \beta_m)$ be two inverse sequences of group extensions. Assume that there exists a commuting ladder diagram between subsequences of the two, as shown below. By discarding some terms if necessary, arrange that ω and η do not agree beyond the term n_0.

$$\begin{tikzcd}
G_{(\omega,n_0)} & & G_{(\omega,n_2)} \arrow[ll, "\alpha"'] \arrow[dl, "g_{n_2}"] & & G_{(\omega,n_4)} \arrow[ll, "\alpha"'] \arrow[dl, "g_{n_4}"] & & \cdots \arrow[ll, "\alpha"'] \\
& & & & & & \cdots \\
& G_{(\eta,m_1)} \arrow[uul, "f_{m_1}"] & & G_{(\eta,m_3)} \arrow[ll, "\beta"'] \arrow[uul, "f_{m_3}"] & & & \cdots \arrow[lll, "\beta"']
\end{tikzcd}$$

By the commutativity of the diagram, all fs and gs must be epimorphisms, as all the αs and βs are.

Now, it is possible that g_{n_2} is an epimorphism; by Lemma 5.4, (η, m_1) might be a subset of (ω, n_2) when considered as sets. However, f_{m_3} cannot also be an epimorphism, since (ω, n_2) cannot be a subset of (η, m_3) when considered as sets. Since the two sequences can only agree up to n_0, if (η, m_1) is a subset of (ω, n_2) when considered as sets, then there must be a prime p_i in (ω, n_2) in between some of the primes of (η, m_1). This prime p_i cannot be in (η, m_3) and is in (ω, n_2), so (ω, n_2) cannot be a subset of (η, m_3) when considered as sets, so f_{m_3} cannot be an epimorphism. □

7. Manifold Topology

We now begin an exposition of our example.

THEOREM 1.2 (Uncountably many pseudocollars on closed manifolds with the same boundary and similar Pro-π_1). *Let M^n be a closed smooth manifold $(n \geq 6)$ with $\pi_1(M) \cong \mathbb{Z}$, and let S be the finitely presented group $V * V$, which is the free product of two copies of Thompson's group V. Then there exists an uncountable collection of pseudocollars $\{N_\omega^{n+1} \mid \omega \in \Omega\}$, no two of which are homeomorphic at infinity, and each of which begins with $\partial N_\omega^{n+1} = M^n$ and is obtained by creating a group extension $1 \to S \to G_{i+1} \to G_i \to 1$ and then using Theorem 1.1 to create a 1-sided h-cobordism (W_i, M_i, M_{i+1}) with $\pi_1(M_i) = G_i$ and $\pi_1(M_{i+1}) = G_{i+1}$ countably many times, using the same kernel group S each time. In particular, each N_i has a fundamental group at infinity that may be represented by an inverse sequence*

$$\mathbb{Z} \overset{\alpha_1}{\twoheadleftarrow} G_1 \overset{\alpha_2}{\twoheadleftarrow} G_2 \overset{\alpha_3}{\twoheadleftarrow} G_3 \overset{\alpha_4}{\twoheadleftarrow} \cdots$$

with $\ker(\alpha_i) = S$ *for all* i.

Proof. For each element $\omega \in \Omega$, the set of all increasing sequences of prime numbers, we will construct a pseudocollar N_ω^{n+1} whose fundamental group at infinity is represented by the inverse sequence $(G_{(\omega,n)}, \alpha_{(\omega,n)})$. By Lemma 6.2, no two of these pseudocollars can be homeomorphic at infinity, and the theorem will follow.

To form one of the pseudocollars, start with $M = \mathbb{S}^1 \times \mathbb{S}^{n-1}$ with fundamental group $\mathbb{Z}$ and then create a 1-sided s-cobordism $(W_{(s_1)}, M, M_{(s_1)})$ corresponding to the group $G_{(s_1)}$ (s_1 a prime), using Theorem 1.1.

We then create another 1-sided s-cobordism $(W_{(s_1,s_2)}, M_{(s_1)}, M_{(s_1,s_2)})$ corresponding to the group $G_{(s_1,s_2)}$, again using Theorem 1.1 and Lemma 6.1.

We continue in the fashion ad infinitum.

The structure of the collection of all pseudocollars will be the set Ω described before.

We have shown that the profundamental group systems at infinity of each pseudocollar are non-pro-isomorphic in Lemma 6.2, so that all the ends are nondiffeomorphic (indeed, nonhomeomorphic).

This proves that we have uncountably many pseudocollars, each with boundary M, which have distinct ends. □

REMARK 2. The above argument should generalize to any manifold M^n with $n \geq 6$, where $\pi_1(M)$ is a finitely generated Abelian group of rank at least 1 and any finitely presented superperfect centerless freely indecomposable Hopfian group P with an infinite list of elements of different orders (the orders all being prime numbers was a convenient but inessential hypothesis). The quotient needs to be Abelian, so that the commutator subgroup will be the kernel group, which is necessarily superperfect; the quotient group must have rank at least 1, so that there is an element to send into the kernel group to act via the partial conjugation. The rest of the conditions should be self-explanatory.

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