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Gromov–Witten Theory of Target Curves and the Tautological Ring

Felix Janda

Abstract. In the Gromov–Witten theory of a target curve, we consider descendent integrals against the virtual fundamental class relative to the forgetful morphism to the moduli space of curves. We show that cohomology classes obtained in this way lie in the tautological ring.

0. Introduction

Let X be a nonsingular projective variety over $\mathbb{C}$, let $\overline{M}_{g,n}(X, \beta)$ be the moduli space of stable maps to X of class β, and let

$$\pi : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n}$$

be the forgetful map to the moduli space of stable curves.[1] The moduli space $\overline{M}_{g,n}(X, \beta)$ possesses a perfect obstruction theory defining a virtual fundamental class

$$[\overline{M}_{g,n}(X, \beta)]^{\mathrm{vir}} \in H_*(\overline{M}_{g,n}(X, \beta), \mathbb{Q})$$

of expected dimension [1; 8].

The tautological rings $RH^*(\overline{M}_{g,n})$ of $\overline{M}_{g,n}$ are most compactly defined (see [2]) as the smallest system of subrings of $H^*(\overline{M}_{g,n})$[2] stable under pushforward and pullback by the maps

- $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$ forgetting one of the markings,
- $\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g_1+g_2,n_1+n_2}$ gluing two curves at a point,
- $\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ gluing together two markings of a curve.

Although this definition seems restrictive, many geometric classes lie in the tautological ring.

Question 1 (Pandharipande [2]). Does

$$\pi_*[\overline{M}_{g,n}(X, \beta)]^{\mathrm{vir}} \in H_*(\overline{M}_{g,n}) \cong H^*(\overline{M}_{g,n})$$

lie inside $RH^*(\overline{M}_{g,n})$ when X is defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers?

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[1] See [3] for an introduction to the moduli space of stable maps.

[2] We will generally work with $\mathbb{Q}$-valued homology and cohomology.

This question can be answered affirmatively for toric varieties by the method of virtual localization [4]. We show in this article that the answer is also "yes" in the case where X is a curve. For this, it is convenient to generalize the question.

In the first direction of generalization, we also allow arbitrary descendent insertions: For each marking, there is an evaluation map $\mathrm{ev}_i : \overline{M}_{g,n}(X,\beta) \to X$. Furthermore, for each marking, let ψ_i be the first Chern class of the cotangent line bundle at marking i. For any choice of n cohomology classes $\gamma_1,\ldots,\gamma_n \in H^*(X)$ and nonnegative integers $k_1,\ldots,k_n$, more general cohomology classes can be defined by

$$\pi_*\left(\prod_{i=1}^n \psi_i^{k_i}\,\mathrm{ev}_i^*(\gamma_i) \cap [\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}\right) \in H^*(\overline{M}_{g,n}).$$

We will call such a class a "GW-class". Integrating a GW-class gives the corresponding usual Gromov–Witten descendent invariant. We can again ask whether GW-classes lie inside $RH^*(\overline{M}_{g,n})$.

Another direction of generalization is possible via relative Gromov–Witten theory [6; 7; 5], where for a smooth variety X together with a smooth divisor D, the moduli space $\overline{M}_{g,n}(X,\beta,\mu_1,\ldots,\mu_m)$ of relative stable maps is considered. This moduli space is a compactification of the space of stable maps to X such that the preimage of D is finite and the cohomology-valued partitions $\mu_1,\ldots,\mu_m$ specify for each connected component D' of D the ramification profile over D' and the class of the source curve in D'. We follow the convention that the preimages of D are marked, so that a projection map

$$\pi : \overline{M}_{g,n}(X,\beta,\eta_1,\ldots,\eta_m) \to \overline{M}_{g,n+\ell(\eta)}$$

can be defined, where $\ell(\eta)$ is the sum of the lengths of the partitions $\eta_1,\ldots,\eta_m$.

QUESTION 2. Does

$$\pi_*\left(\prod_{i=1}^n \psi_i^{k_i}\,\mathrm{ev}_i^*(\gamma_i) \cap [\overline{M}_{g,n}(X,\beta,\mu_1,\ldots,\mu_m)]^{\mathrm{vir}}\right) \in H^*(\overline{M}_{g,n+\ell(\eta)})$$

lie inside $RH^*(\overline{M}_{g,n})$ when X and D are defined over $\overline{\mathbb{Q}}$?

The main result of this paper is the following theorem.

THEOREM 1. *If X is an algebraic curve and D a collection of pairwise distinct points on X, all (relative) GW-classes lie in the tautological ring $RH_*(\overline{M}_{g,n})$.*

To say more about this result, we specialize the discussion to the case where X is a curve.

Recall that the cohomology of an algebraic curve X of genus h over $\mathbb{C}$ has a basis

$$\{1,\alpha_1,\ldots,\alpha_h,\beta_1,\ldots,\beta_h,\omega\}$$

such that 1 is the identity of the cup product, ω is the Poincaré dual of a point, and the $\alpha_i \in H^{1,0}(X,\mathbb{C})$ and $\beta_i \in H^{0,1}(X,\mathbb{C})$ form a symplectic basis of $H^1(X,\mathbb{C})$, that is, $\alpha_i \cup \beta_i = \omega$ and $\beta_i \cup \alpha_i = -\omega$ for all i, and all other cup products vanish.

The particular case of Theorem 1 where $X = \mathbb{P}^1$ was proven in [2]. A large part of our proof is a reduction to this particular case. We also build on the series of articles [10; 9; 11], which gives an effective way to calculate all relative Gromov–Witten invariants of X.

We now give an overview of the proof of Theorem 1. If the GW-class has only even cohomology insertions, then we can use the degeneration formula to calculate the GW-class in terms of GW-classes of $\mathbb{P}^1$ relative to a point, and by the results of [2] these are also tautological. This is done in Section 1.

In the presence of odd insertions, new phenomena can occur. For example, we might obtain odd classes in $H^*(\overline{M}_{g,n})$. These can only be tautological if they vanish, since by definition tautological classes are algebraic. More generally, we might obtain classes inside a piece $H^{i,j}(\overline{M}_{g,n})$ of the Hodge diamond with $i \neq j$. We call such classes *unbalanced.*

COROLLARY. *All unbalanced GW-classes of curves vanish.*

In fact, we will first prove this corollary in Section 4 and use it as an input for the proof of Theorem 1.

The remaining GW-classes are *balanced.* In Section 5 we will give an algorithm to calculate such GW-classes in the presence of odd cohomology in terms of GW-classes with only even insertions. It is a straightforward generalization of the algorithm given in [11].

If there are odd insertions, then we cannot use a degeneration formula to reduce to the case of $\mathbb{P}^1$. Still, it is possible to deform X into a chain of elliptic curves to reduce to the genus 0 and genus 1 cases. This is done in Section 2. Therefore, starting from Section 3, we will assume X to be of genus one.

As in [11], we will use the following properties of Gromov–Witten theory to relate GW-classes with odd insertions to those with only even insertions:

- algebraicity of the virtual fundamental classes,
- invariance under monodromy transformations of X,
- degeneration formulae,
- vanishing relations from the group structure on an elliptic curve.

We study relations coming from the monodromy invariance of Gromov–Witten theory and the group structure of an elliptic curve in Sections 3.1 and 3.2, respectively.

For the proof of the corollary, we will only need the results from Sections 2, 3.1, and 4. It is even possible to adapt the proof so that the use the reduction to genus 1 is not necessary. Its proof is the main new part of this article.

We have tried to apply the analogous methods in the case where X is a quintic surface, but they do not seem to suffice in this case.

Notation and Conventions

Because of our extensive use of the degeneration formula, we will always allow our source curves to be disconnected. So we will always work with disconnected

Gromov–Witten invariants, GW-classes, and a tautological ring of not necessarily connected curves. These can however be related to their connected counterparts in a purely combinatorial fashion.

The only possible curve classes of X are multiples of the class of X. We write $\overline{M}_{g,n}(X,d)$ for the space of stable maps of curve class $d[X]$ or, in other words, of degree d.

We use the notation

$$\langle \tau_{k_1}(\gamma_1)\dots\tau_{k_n}(\gamma_n) \mid \eta\rangle_{g,d}^X := \int_{[\overline{M}_{g,n}(X,\eta)]^{\mathrm{vir}}} \prod_{i=1}^n \psi_i^{k_i}\,\mathrm{ev}_i^*(\gamma_i)$$

for relative Gromov–Witten invariants, where $\eta = (\eta_1,\dots,\eta_m)$ is a collection of partitions. For GW-classes, we use the analogous but nonstandard notation

$$[\tau_{k_1}(\gamma_1)\dots\tau_{k_n}(\gamma_n) \mid \eta]_r^X$$
$$:= \pi_*\Bigl(\prod_{i=1}^n \psi_i^{k_i}\,\mathrm{ev}_i^*(\gamma_i) \cap [\overline{M}_{g,n}(X,\eta)]^{\mathrm{vir}}\Bigr) \in H_{2r}(\overline{M}_{g,n+\ell(\eta)}).$$

We have left out the degree d since it is the size of any of the usual partitions $\eta_1,\dots,\eta_m$. The notation also does not specify the genus g of the source curve since it is determined by the formula

$$r = 2g-2+n+d(2-2h) - \sum_{i=1}^n (k_i + \mathrm{codim}(\gamma_i)) - \sum_{i=1}^m (d-\ell(\eta_i)).$$

If this would lead to a half-integer value of g, we define the GW-class to be zero.

1. Even Classes

We consider the computation of GW-classes of a curve X with only even insertions.

There is a nonsingular family X_t of curves of genus $h>0$ over $\mathbb{C}$ such that $X_t \cong X$ for $t \neq 0$ and X_0 is an irreducible curve of geometric genus $h-1$ with a node. The degeneration formula relates the GW-classes of X to the GW-classes of the normalization $\tilde{X}_0$ of X_0 relative to the two preimages of the marked point. It is important to note that the even classes of X can be lifted to X_t. All of this discussion generalizes to the situation of X relative to marked points $q_1,\dots,q_m$.

Let

$$M = \prod_{h\in H} \tau_{o_h}(1) \prod_{h'\in H'} \tau_{o'_{h'}}(\omega)$$

be a monomial in insertions of even classes, and let $\eta_1,\dots,\eta_m$ be choices of splittings at the relative points. Since the target curve is irreducible, the degeneration formula [7] in this case says that

$$[M \mid \eta_1,\dots,\eta_m]_r^X = \sum_{|\mu|=d} \mathfrak{z}(\mu)\iota_*[M \mid \eta_1,\dots,\eta_m,\mu,\mu]_r^{\tilde{X}_0},$$

where the sum is over partitions $\mu = (\mu_1, \ldots, \mu_{\ell(\mu)})$, the automorphism factor $\mathfrak{z}(\mu)$ is defined by

$$\mathfrak{z}(\mu) = |\operatorname{Aut}(\mu)| \prod_{i=1}^{\ell(\mu)} \mu_i,$$

and ι is the map gluing together the last two markings.

Using this formula repeatedly, we can reduce the genus h until we arrive at the case of $X = \mathbb{P}^1$ relative to $q_1, \ldots, q_n$, which has been studied in [2]. This implies that Theorem 1 is true in the case that all γ_i are even classes.

2. Reduction to Genus One

Recall that we have chosen a symplectic basis $\alpha_i, \beta_i \in H^1(X, \mathbb{C})$. There is a deformation $Y \to \mathbb{P}^1$ of X into $\tilde{X} = E \cup X'$, a curve of genus one and a curve of genus $h-1$ connected at a node p. Moreover, the symplectic basis of $H^1(X, \mathbb{C})$ can be lifted to Y such that over $\tilde{X}$ the classes α_1, β_1 give a symplectic basis of $H^1(E, \mathbb{C})$ and the other α_i and β_i give a symplectic basis of $H^1(X', \mathbb{C})$. Furthermore, the deformation can be chosen such that ω deforms to the Poincaré dual class of a point on the genus 1 curve. Similarly, in the relative theory, the deformations of the relative points $q_1, \ldots, q_m$ can be assumed to lie on the genus 1 component.

The degeneration formula is slightly more complicated to write down in this case since there is a choice for the splitting of the domain curve into two parts, one for each component of $\tilde{X}$, and a choice of splitting μ at p. For each partition $g = g_1 + g_2 + \ell(\mu) - 1$ of g, there is a gluing map

$$\iota\colon \overline{M}_{g_1, n_1+\ell(\eta)+\ell(\mu)} \times \overline{M}_{g_2, n_2+\ell(\mu)} \to \overline{M}_{g, n_1+n_2+\ell(\eta)},$$

gluing two curves along the last $\ell(\mu)$ markings.

Let M_ω, M_1, and M_2 be monomials in insertions of elements in $\{\omega\}$, $\{\alpha_1, \beta_1\}$, and $\{\alpha_i, \beta_i \mid i \neq 1\}$, respectively. Furthermore, let $M := \tau_{o_H}(1)$ where

$$\tau_{o_H}(1) := \prod_{h \in H} \tau_{o_h}(1)$$

is a monomial in insertions of the identity. After a change of sign, a general GW-class we wish to calculate is of the form

$$[M M_\omega M_1 M_2 \mid \eta_1, \ldots, \eta_m]_r^X.$$

By the degeneration formula this equals

$$\sum_{\substack{r_1+r_2=r,\\ |\mu|=d, I \subset H}} \mathfrak{z}(\mu) \iota_*([\tau_{o_I}(1) M_\omega M_1 \mid \eta_1, \ldots, \eta_m, \mu]_{r_1}^E, [\tau_{o_{H \setminus I}}(1) M_2 \mid \mu]_{r_2}^{X'}).$$

Since the tautological rings are compatible with ι, we can induct on the genus of X to reduce to the case where X is of genus 1. Let us fix a symplectic basis α, β of $H^1(X, \mathbb{C})$ for this and the following sections. In this case, we can use a different degeneration to simplify the problem further. Namely, X can be degenerated to X

with a rational tail. This can be used to move the ω insertions and all but one relative point to the rational tail.

We have therefore reduced the proof of Theorem 1 to showing the following statements.

THEOREM 2. *Let X be a curve of genus 1 relative to a point p with symplectic basis $\alpha, \beta \in H^1(X, \mathbb{C})$. Then, for every partition η of d and any monomial M in insertions of identity classes α and β, the classes*

$$[M \mid \eta]_r^X$$

lie in the tautological ring $RH^*(\overline{M}_{g,n+\ell(\eta)})$. In particular, if the number of insertions of α does not equal the number of insertions of β, then the class is zero.

3. Relations

In this section, we introduce two suitably generalized methods of [11] to produce relations between relative GW-classes of genus one targets.

3.1. Relations from Monodromy

By choosing a suitable loop in the moduli space $\overline{M}_{1,1}$ starting at the point corresponding to (X, p) around the point corresponding to the nodal elliptic curve, we obtain a deformation of X to itself that leaves the even cohomology invariant while it acts on $H^1(X, \mathbb{C})$ via

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \mapsto \phi\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right) := \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ \alpha+\beta\end{pmatrix}.$$

In fact, the complete monodromy group acts trivially on the even cohomology and via the standard $\mathrm{SL}_2(\mathbb{Z})$-representation on $H^1(X, \mathbb{C}) \cong \mathbb{C}^2$.

Because of the deformation invariance of Gromov–Witten theory, applying this transformation to *all* the descendent insertions leaves the GW-class invariant. This gives a relation between GW-classes.

We will use only these relations to establish the vanishing of unbalanced classes in Section 4.

For the proof of Theorem 2, we consider certain linear combinations of these relations, which have a nice form if we assume that the vanishing of GW-classes of unbalanced classes has already been shown. Let I and J be index sets of the same order, and

$$\mathbf{n}\colon I \to \Psi_{\mathbb{Q}} \quad \text{and} \quad \mathbf{m}\colon J \to \Psi_{\mathbb{Q}}$$

be refined descendent assignments. Here, a refined descendent assignment is a formal $\mathbb{Q}$-linear combination of usual descendent assignments. Monomials of descendents with such assignments are just expanded multilinearly. Refined descendent assignments only serve as a formal tool here. We consider the resulting GW-classes to lie in the $\mathbb{Q}$-vector space

$$\bigoplus_{g\geq 0} H^\star(\overline{M}_{g,n+\ell(\eta)}).$$

Generalizing the definition of the map ι_* suitably, we can apply the degeneration formula also to GW-classes involving refined descendent assignments.

For a subset $\delta \subset I$, let $S(\delta)$ be the set of all subsets of $I \sqcup J$ of cardinality $|I|$ containing δ. For any $D \subseteq I \sqcup J$, we may consider the class

$$\tau_{\mathbf{n},\mathbf{m}}(D) := \prod_{i \in I} \tau_{n_i}(\gamma_i^D) \prod_{j \in J} \tau_{m_j}(\gamma_j^D),$$

where

$$\gamma_k^D = \begin{cases} \alpha & \text{if } k \in D, \\ \beta & \text{otherwise.} \end{cases}$$

Finally, we consider a monomial

$$N = \prod_{h \in H} \tau_{o_h}(1) \prod_{h' \in H'} \tau_{o'_{h'}}(\omega)$$

in the monodromy invariant insertions.

Proposition 1. *The monodromy relation $R(N, \mathbf{n}, \mathbf{m}, \delta) = 0$ holds for any proper subset $\delta \subset I$. Here*

$$R(N, \mathbf{n}, \mathbf{m}, \delta) = \sum_{D \in S(\delta)} [N \tau_{\mathbf{n},\mathbf{m}}(D)]_d^X.$$

Proof. Consider the application of the monodromy transform ϕ to

$$\left[N \prod_{i \in I} \tau_{n_i}(\gamma_i^\delta) \prod_{j \in J} \tau_{m_j}(\beta) \right]_d^X.$$

This class vanishes since it is unbalanced because $\delta \subset I$ is a proper subset. After applying ϕ, all terms but those with exactly $|I|$ insertions of α vanish. The sum of these remaining terms is exactly $R(N, \mathbf{n}, \mathbf{m}, \delta)$. □

3.2. *Relations from the Elliptic Action*

Using the group structure of X induced by identifying X with its Jacobian via a point $0 \in X$ gives another set of relations.

Let the *small diagonal* of X^r be the subset

$$\{(x, \dots, x) : x \in X\} \subset X^r,$$

and $\Delta_r \in H^{r-1}(X^r, \mathbb{C})$ be its Poincaré dual. To obtain the relations, we will use the fact that Δ_r is invariant under the diagonal action of the elliptic curve X on X^r and the Künneth decomposition of Δ_r.

Let K and H be two ordered index sets, and P a set partition of K into subsets of size at least 2. For any part p of P, we have a product evaluation map

$$\phi_p : \overline{M}_{g,K \sqcup H}(X, d) \to X^{|p|}.$$

Let $\mathbf{l}\colon K \to \Psi$ be an assignment of descendents. Finally, let M be a monomial in insertions of the identity

$$M = \prod_{h \in H} \tau_{o_h}(1).$$

PROPOSITION 2. *The elliptic vanishing relation $V(M, P, \mathbf{l}) = 0$ holds. Here*

$$V(M, P, \mathbf{l}) := \pi_*\biggl(\prod_{h \in H} \psi_h^{o_h} \prod_{k \in K} \psi_k^{l_k} \prod_{p \in P} \phi_p^*(\Delta_{|p|}) \cap [\overline{M}_{g, K \sqcup H}(X, d)]^{\mathrm{vir}}\biggr). \quad (1)$$

Notice that no insertions of ω appear and that we do not work in the relative theory. There is a natural generalization to a more general assignment $\mathbf{l}\colon K \to \Psi_{\mathbb{Q}}$.

Proof of Proposition 2. The elliptic curve X acts on the moduli space $\overline{M}_{g, H \sqcup K}(X, d)$ by the action induced from the group operation $X \times X \to X$. The action can be used to fix the image in X of one marked point q. This gives an X-equivariant splitting

$$\overline{M}_{g, H \sqcup K}(X, d) \cong \mathrm{ev}_q^{-1}(0) \times X.$$

In particular, there exists an algebraic quotient

$$\overline{M}_{g, H \sqcup K}(X, d)/X \cong \mathrm{ev}_q^{-1}(0)$$

of $\overline{M}_{g, H \sqcup K}(X, d)$.

Notice that the products in (1) are pulled back via the projection map

$$\overline{M}_{g, H \sqcup K}(X, d) \to \overline{M}_{g, H \sqcup K}(X, d)/X$$

from an analogous class on the quotient space. Furthermore, the virtual fundamental class is also pulled back from the quotient. Thus, the push–pull formula applied to the projection map implies that the GW-class must vanish. $\square$

To apply these relations, we need to reformulate them as relations between GW-classes of X. To rewrite the ϕ_p-pullbacks as products of usual pullbacks via the evaluation maps, we Künneth-decompose the classes Δ_r. For example, for Δ_2 and Δ_3, we have

$$\begin{aligned}
\Delta_2 &= 1 \otimes \omega + \omega \otimes 1 - \alpha \otimes \beta + \beta \otimes \alpha, \\
\Delta_3 &= 1 \otimes \omega \otimes \omega + \omega \otimes 1 \otimes \omega + \omega \otimes \omega \otimes 1 - \omega \otimes \alpha \otimes \beta + \omega \otimes \beta \otimes \alpha \\
&\qquad - \alpha \otimes \omega \otimes \beta + \beta \otimes \omega \otimes \beta - \alpha \otimes \beta \otimes \omega + \beta \otimes \alpha \otimes \omega.
\end{aligned}$$

In general, Δ_r is a sum $\Delta_r = \Delta_r^{\mathrm{even}} + \Delta_r^{\mathrm{odd}}$, where Δ_r^{even} is the sum of the r classes of the form

$$\omega \otimes \cdots \otimes \omega \otimes 1 \otimes \omega \cdots \otimes \omega,$$

and Δ_r^{odd} is the sum of the $\binom{r}{2}$ linear combinations of classes

$$\begin{aligned}
&-\omega \otimes \cdots \otimes \omega \otimes \alpha \otimes \omega \otimes \cdots \otimes \omega \otimes \beta \otimes \omega \otimes \cdots \otimes \omega \\
&+\omega \otimes \cdots \otimes \omega \otimes \beta \otimes \omega \otimes \cdots \otimes \omega \otimes \alpha \otimes \omega \otimes \cdots \otimes \omega.
\end{aligned}$$

We are mostly interested in the odd summand since the even summand is usually already known by an induction hypothesis.

4. Unbalanced Classes

In this section, let us fix a monomial M in insertions of even classes of X and an index set I for the odd insertions. Our aim is to show that the classes

$$C(S) := \left[M \cdot \prod_{i \in I} \tau_{n_i}(\gamma_i^{I \setminus S}) \mid \eta \right]_r^X,$$

where

$$\gamma_i^{I \setminus S} := \begin{cases} \alpha & \text{if } i \notin S, \\ \beta & \text{if } i \in S, \end{cases}$$

vanish in the unbalanced case, that is, where $|I| \neq 2|S|$.

By symmetry, for the proof of the vanishing $C(S) = 0$, we can assume that $2|S| < |I|$. We then proceed by induction on $|S|$, starting with the empty case $|S| < 0$.

Choose a subset $J \subset I$ of size $2|S| + 1$ containing S. For any subset $T \subset J$ of size $|S| + 1$, consider the monodromy relation

$$C(T) = \left[M \cdot \prod_{i \in I} \tau_{n_i}(\phi(\gamma_i^{I \setminus T})) \mid \eta \right]_r^X = \sum_{S' \subset T} C(S').$$

After subtracting the left-hand side from the right-hand side and using the induction hypothesis, we obtain the relation

$$0 = R(T) := \sum_{\substack{S' \subset T \\ |S'| = |S|}} C(S'). \tag{2}$$

Relation (2) can be inverted as

$$C(S') = \sum_{i=0}^{|S|} (-1)^{i+|S|} c_i^{-1} \sum_{\substack{T \subset J \\ |T| = |S|+1, |T \cap S'| = i}} R(T),$$

where

$$c_i := (|S| + 1) \binom{|S|}{i} \neq 0.$$

Therefore, $C(S') = 0$ for any $S' \subset J$ of size $|S|$. In particular, we have established the induction step $C(S) = 0$.

5. Balanced Classes

In this section, we finish the proof of Theorem 2 in the remaining case of balanced classes, therefore giving a proof of Theorem 1. We follow the discussion of [11, Section 5.5] and try to keep the notation as similar as possible. Compared to [11], there is one additional induction on the codimension.

The following lemma will be used to determine relative GW-classes from a set of related absolute GW-classes. Before stating the lemma, we need to introduce a special refined descendent assignment.

Let $P(d)$ be the set of partitions of d, and $\mathbb{Q}^{P(d)}$ the $\mathbb{Q}$-vector space of functions from $P(d)$ to $\mathbb{Q}$. Let

$$\tilde{\tau}(\omega) = \sum_{q=0}^{\infty} c_q \tau_q(\omega)$$

be a refined descendent of ω. The Gromov–Witten theory of $\mathbb{P}^1$ relative to a point gives, for each $v \geq 0$, a function

$$\gamma_v \colon P(d) \to \mathbb{Q}, \qquad \eta \mapsto \langle \tilde{\tau}(\omega)^v \mid \eta \rangle^{\mathbb{P}^1}.$$

FACT. *There exists a $\mathbb{Q}$-linear combination $\tilde{\tau}(\omega)$ depending on d such that the set of functions*

$$\{\gamma_0, \gamma_1, \ldots\}$$

spans $\mathbb{Q}^{P(d)}$.

Proof. This is Lemma 5.6 in [11]. Its proof uses the Gromov–Witten Hurwitz correspondence [10]. □

We fix such a refined descendent assignment $\tilde{\tau}(\omega)$. Let us define

$$\tilde{\psi} = \sum_{q=0}^{\infty} c_q \psi^q$$

so that formally $\tilde{\tau}(\omega) = \tau_{\tilde{\psi}}(\omega)$.

LEMMA 1. *Let M, L, A, and B be monomials in insertions of* 1, ω, α, *and* β, *respectively*:

$$M = \prod_{h \in H} \tau_{o_h}(1), \qquad L = \prod_{h' \in H'} \tau_{o'_{h'}}(\omega),$$
$$A = \prod_{i \in I} \tau_{n_i}(\alpha), \qquad B = \prod_{j \in J} \tau_{m_j}(\beta),$$

and let $\eta \in P(d)$ be a splitting. Then the GW-classes

$$[MAB \mid \eta]_r^X, \qquad [MLAB]_{r,d}^X$$

are tautological if the classes

$$[M' \tilde{\tau}(\omega)^v AB]_{r,d}^X, \qquad [M'AB \mid \mu]_{r'}^X$$

are tautological for arbitrary $v \geq 0$, $r' \leq r$, $\mu \in P(d)$, and divisors M' of M except (*possibly*) *in the case $r' = r$, $M' = M$.*

Proof. We first study the case $M = 1$, $r = 0$. There is a degeneration of X into $X \cup_{\mathbf{pt}} \mathbb{P}^1$ we have already studied in Section 2. The corresponding degeneration formula spells here

$$[\tilde{\tau}(\omega)^v AB]^X_{0,d} = \sum_{|\eta|=d} \mathfrak{z}(\eta)\iota_*([AB \mid \eta]^X_0, [\tilde{\tau}(\omega)^v \mid \eta]^{\mathbb{P}^1}_0).$$

By Fact, letting v vary this determines $[AB \mid \eta]^X_0$ for all η. The degeneration formula

$$[LAB]^X_{0,d} = \sum_{|\eta|=d} \mathfrak{z}(\eta)\iota_\star([LAB \mid \eta]^X_0, [\tilde{\tau}(\omega)^v \mid \eta]^{\mathbb{P}^1}_0)$$

then determines the second kind of GW-class if $M = 1$, $r = 0$.

In general, there are additional sums in the degeneration formula, one for the distribution of the factors of M and one for the splitting of the domain curve. However, by the hypothesis of the lemma and the fact that we already have shown the tautologicalness of GW-classes of $\mathbb{P}^1$, only the summand corresponding to the distribution of all of M to X and all of r to X may be nontautological. But then we can mirror our argument in the simple case. $\square$

5.1. Simple Case

To illustrate the principle of the proof, we start with the GW-classes with only two odd insertions (one of each α and β). So for descendent assignments n and m, a monomial of identity insertions

$$M = \prod_{h\in H} \tau_{o_h}(1),$$

and the choice of splitting μ for the relative point, we wish to determine

$$[M\tau_n(\alpha)\tau_m(\beta) \mid \mu]^X_r$$

in terms of GW-classes with only even insertions. By induction on r and M, assume that this statement has already been proven for all $r' \le r$ and $M' \mid M$ except (possibly) in the case $r' = r$, $M' = M$.

Let K_v be an index set with $v + 2$ elements. We first look at the elliptic vanishing relation $V(M, \{K_v\}, \mathbf{l})$, where $\mathbf{l}$ assigns $\tilde{\psi}$ to every element of K_v. The relation contains $2\binom{v+2}{2}$ summands that contain odd classes, and in fact since the descendent assignment is identical for each element of K_v, each of them is equal to

$$-[M\tilde{\tau}(\omega)^v\tilde{\tau}(\alpha)\tilde{\tau}(\beta)]^X_{r,d},$$

which we thus have determined in terms of even GW-classes.

Lemma 1 and the induction hypothesis yield the determination of the classes

$$[M\tilde{\tau}(\alpha)\tilde{\tau}(\beta) \mid \eta]^X_r, \qquad [ML\tilde{\tau}(\alpha)\tilde{\tau}(\beta)]^X_{r,d} \tag{3}$$

for any monomial L in descendents of ω.

Next, we look at the elliptic vanishing relation $V(M, \{K_v\}, \mathbf{l})$, where this time the descendent assignment $\mathbf{l}$ takes the value $\tilde{\psi}$ at all but the first element of K_v, where it takes the value ψ^n. The even terms are still of no relevance, but now there are four kinds of odd summands. They are

$$\begin{gathered}
-(v+1)[M\tilde{\tau}(\omega)^v \tau_n(\alpha)\tilde{\tau}(\beta)]^X_{r,d} \\
+(v+1)[M\tilde{\tau}(\omega)^v \tau_n(\beta)\tilde{\tau}(\alpha)]^X_{r,d} \\
-\binom{v+1}{2}[M\tilde{\tau}(\omega)^{v-1} \tau_n(\omega)\tilde{\tau}(\alpha)\tilde{\tau}(\beta)]^X_{r,d} \\
+\binom{v+1}{2}[M\tilde{\tau}(\omega)^{v-1} \tau_n(\omega)\tilde{\tau}(\beta)\tilde{\tau}(\alpha)]^X_{r,d}.
\end{gathered}$$

We are only interested in the first pair of summands since the second two are determined by (3). Applying the relation $R(M\tilde{\tau}(\omega)^v, \{\psi^n\}, \{\tilde{\psi}\}, \emptyset)$, we see that the first two summands are equal. Therefore, we now know

$$[M\tilde{\tau}(\omega)^v \tau_n(\alpha)\tilde{\tau}(\beta)]^X_{r,d}$$

and by Lemma 1 also

$$[M\tau_n(\alpha)\tilde{\tau}(\beta) \mid \eta]^X_r, \qquad [ML\tau_n(\alpha)\tilde{\tau}(\beta)]^X_{r,d}. \tag{4}$$

Repeating this argument, we successively determine

$$[M\tilde{\tau}(\alpha)\tau_m(\beta) \mid \eta]^X_r, \qquad [ML\tilde{\tau}(\alpha)\tau_m(\beta)]^X_{r,d}, \tag{5}$$

$$[M\tau_n(\alpha)\tau_m(\beta) \mid \eta]^X_r. \tag{6}$$

For (5), we need the elliptic vanishing relation $V(M, \{K_v\}, \mathbf{l})$, where $\mathbf{l}$ takes the value $\tilde{\psi}$ on all but the last elements of K_v, where it is ψ^m. As before, two terms in this relation are not yet determined, and these are proportional to each other by the monodromy relation $R(M\tilde{\tau}(\omega)^v, \{\tilde{\psi}\}, \{\psi^m\}, \emptyset)$.

For (6), we use the relation $V(M, \{K_v\}, \mathbf{l})$ with $\mathbf{l}$ having the value $\tilde{\psi}$ on all but the first and the last element of K_v, where it takes the values n and m, respectively. To see that there is only a pair of not yet determined terms, we in particular need to use (4) and (5). We finish with the use of the relation $R(M\tilde{\tau}(\omega)^v, \{\psi^n\}, \{\psi^m\}, \emptyset)$.

5.2. *General Case*

Let I and J be two ordered index sets of the same size, and let

$$\mathbf{n}\colon I \to \Psi_{\mathbb{Q}}, \qquad \mathbf{m}\colon J \to \Psi_{\mathbb{Q}}$$

be general descendent assignments. To prove Theorem 2, for a monomial M in insertions of the identity, we need to calculate the GW-classes

$$\left[M \prod_{i\in I} \tau_{n_i}(\alpha) \prod_{j\in J} \tau_{m_j}(\beta) \mid \eta\right]^X_{r,d}$$

in terms of lower GW-classes. This follows from the following lemma.

LEMMA 2. *For $s, t \geq 0$, the GW-classes*

$$\left[M \prod_{i \leq s} \tau_{n_i}(\alpha) \prod_{s<i\in I} \tilde{\tau}(\alpha) \prod_{J\ni j\leq t} \tau_{m_j}(\beta) \prod_{t<j} \tilde{\tau}(\beta) \mid \eta\right]_r^X,$$

$$\left[ML \prod_{i \leq s} \tau_{n_i}(\alpha) \prod_{s<i\in I} \tilde{\tau}(\alpha) \prod_{J\ni j\leq t} \tau_{m_j}(\beta) \prod_{t<j} \tilde{\tau}(\beta)\right]_{r,d}^X,$$

for an arbitrary monomial L in insertions of point classes ω are determined in terms of the GW-classes with strictly less insertions as well as

$$\begin{aligned}&\left[M' \prod_{i \leq s'} \tau_{n_i}(\alpha) \prod_{s'<i\in I} \tilde{\tau}(\alpha) \prod_{J\ni j\leq t'} \tau_{m_j}(\beta) \prod_{t'<j} \tilde{\tau}(\beta) \mid \eta\right]_{r'}^X,\\ &\left[M'L' \prod_{i \leq s'} \tau_{n_i}(\alpha) \prod_{s'<i\in I} \tilde{\tau}(\alpha) \prod_{J\ni j\leq t'} \tau_{m_j}(\beta) \prod_{t'<j} \tilde{\tau}(\beta)\right]_{r',d}^X,\end{aligned} \tag{7}$$

where L' is an arbitrary monomial in insertions of ω, and we have $(r', s', t', M') < (r, s, t, M)$. Here we have used the partial order defined by $(r', s', t', M') \leq (r, s, t, M)$ if and only if $r' \leq r$, $s' \leq s$, $t' \leq t$, and $M' \mid M$.

Proof. We need additional notation. For $v \geq 0$, let W be an index set of cardinality v. Define K_v by

$$K_v = I \sqcup W \sqcup J$$

with order implicit in the notation. Let $\mathbf{l}_{f[s]l[t]} \colon K_v \to \Psi_{\mathbb{Q}}$ be the descendent assignment with

$$\mathbf{l}_{f[s]l[t]}(k) = \begin{cases} n_k & \text{if } k \text{ is one of the first } s \text{ elements of } I, \\ m_k & \text{if } k \text{ is one of the first } t \text{ elements of } J, \\ \tilde{\psi} & \text{else.} \end{cases}$$

We call the s first elements of $I \subset K_v$ and the t first elements of $J \subset K_v$ *special* elements of K_v with respect to (s, t).

Let $\sigma \colon I \to J$ be a bijection, which we can, using the orders on I and J, also interpret as a permutation of I. Let P_σ be the set partition of K_v with the first part $\{1, \sigma(1)\} \cup W$ and pairs $\{i, \sigma(i)\}$ as the other parts.

Consider the relations $V(M, P_\sigma, \mathbf{l}_{f[s],l[t]})$ for varying σ. By the induction hypothesis we only need to consider the terms from the Künneth decomposition with exactly $|I| + |J|$ odd insertions. After expanding the product, there are $2 \cdot \binom{v+2}{2} \cdot 2^{|I|-1}$ terms of this kind. If we consider the odd part of the Künneth decomposition corresponding to the part $\{1, \sigma(1)\} \cup W$ of P in more detail, then we see that, depending on the s, t and $\sigma(1)$, still different kinds of terms might occur. We only need to take into account the terms with the least possible amount of point classes ω distributed to the special elements of K_v with respect to (s, t)

since all possible other terms are of the form (7) for

$$(s', t') \in \{(s-1, t), (s, t-1), (s-1, t-1)\}.$$

The remaining terms occur still with a combinatorial multiplicity C_σ depending on the number of special elements in $\{1, \sigma(1)\}$. These multiplicities are

$$C_\sigma = \begin{cases} 1 & \text{if } \{1, \sigma(1)\} \text{ contains 2 special elements,} \\ v+1 & \text{if } \{1, \sigma(1)\} \text{ contains 1 special elements,} \\ \binom{v+2}{2} & \text{if } \{1, \sigma(1)\} \text{ contains 0 special elements.} \end{cases}$$

The last case can only occur if $s = 0$.

Let V be the relation obtained by summing these relations over all permutations σ and weighting with C_σ^{-1} and a sign,

$$\sum_\sigma (-1)^{\binom{|I|}{2}} \operatorname{sign}(\sigma) C_\sigma^{-1} V(M, P_\sigma, \mathbf{1}_{f[s], l[t]}),$$

and removing terms determined by the induction hypothesis or of the form (7) for (s', t') as before. Using the notation from Section 3.1, we can write

$$V = \sum_{\delta \subset I} \sum_{D \in S^*(\delta)} (-1)^{|I|-|\delta|} |\delta|!(|I| - |\delta|)!$$
$$\left[M \tilde{\tau}(\omega)^v \prod_{i \le s} \tau_{n_i}(\gamma_i^D) \prod_{s < i \in I} \tilde{\tau}(\gamma_i^D) \prod_{J \ni j \le t} \tau_{m_j}(\gamma_j^D) \prod_{t < j} \tilde{\tau}(\gamma_j^D) \right]_{r,d}^X,$$

where $S^*(\delta)$ denotes the set of all subsets of $I \sqcup J$ such that $D \cap I = \delta$. Using the substitution

$$e_k = \sum_{|\delta| = k} \sum_{D \in S^*(\delta)}$$
$$\left[M \tilde{\tau}(\omega)^v \prod_{i \le s} \tau_{n_i}(\gamma_i^D) \prod_{s < i \in I} \tilde{\tau}(\gamma_i^D) \prod_{J \ni j \le t} \tau_{m_j}(\gamma_j^D) \prod_{t < j} \tilde{\tau}(\gamma_j^D) \right]_{r,d}^X,$$

we can write V more compactly as

$$V = \sum_{k=0}^{|I|} (-1)^{|I|-k} k!(|I| - k)! e_k.$$

We wish to eliminate $e_0, \dots, e_{|I|-1}$ from V to obtain a formula for

$$e_{|I|} = \left[M \tilde{\tau}(\omega)^v \prod_{i \le s} \tau_{n_i}(\alpha) \prod_{s < i \in I} \tilde{\tau}(\alpha) \prod_{J \ni j \le t} \tau_{m_j}(\beta) \prod_{t < j} \tilde{\tau}(\beta) \right]_{r,d}^X.$$

Let $R(\ell)$ be the sum

$$R(\ell) = \sum_{|\delta| = \ell} R(M \tilde{\tau}(\omega)^v, \mathbf{n}', \mathbf{m}', \delta),$$

where $\mathbf{n}'$ and $\mathbf{m}'$ are the restrictions of $\mathbf{l}_{f[s],l[t]}$ to I and J, respectively. Since unbalanced GW-classes vanish, we have the expansion

$$R(\ell) = \sum_{|\delta|\geq\ell} \sum_{D\in S^*(\delta)} \binom{|\delta|}{\ell}$$
$$\left[M\tilde{\tau}(\omega)^v \prod_{i\leq s} \tau_{n_i}(\gamma_i^D) \prod_{s<i\in I} \tilde{\tau}(\gamma_i^D) \prod_{J\ni j\leq t} \tau_{m_j}(\gamma_j^D) \prod_{t<j} \tilde{\tau}(\gamma_j^D) \right]^X_{r,d}$$
$$= \sum_{k\geq\ell} \binom{k}{\ell} e_k.$$

The following lemma in linear algebra gives us the formula for the desired $e_{|I|}$.

LEMMA 3. *Let $e_0, \ldots, e_n$ be a basis of the vector space $\mathbb{Q}^{n+1}$. Then the vectors*

$$V = \sum_{k=0}^{n} (-1)^{n-k} k!(n-k)! e_k$$

and

$$R(\ell) = \sum_{k\geq\ell} \binom{k}{\ell} e_k$$

for $0 \leq \ell < n$ form a basis of $\mathbb{Q}^{n+1}$.

Proof. Note that by formally extending the definition of $R(\ell)$ to $R(n)$ we obtain an $(n+1)\times(n+1)$ lower unitriangular matrix R with coefficients

$$R_{ab} = \binom{a}{b}.$$

The matrix R is therefore invertible, and the coefficients of its inverse R^{-1} are

$$(R^{-1})_{ab} = (-1)^{a+b} \binom{a}{b}.$$

In particular, the $R(0), \ldots, R(n-1)$ are linearly independent. To show that V is not a linear combination of these vectors, we expand V in terms of the basis corresponding to R,

$$V = \sum_{\ell=0}^{n} c_\ell R(\ell),$$

and check that the coefficient c_n is nonzero:

$$c_n = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} (-1)^{n-k} k!(n-k)! = (n+1)! \qquad \square$$

We next apply Lemma 1 to determine

$$\left[M \prod_{i\leq s} \tau_{n_i}(\alpha) \prod_{s<i\in I} \tilde{\tau}(\alpha) \prod_{J\ni j\leq t} \tau_{m_j}(\beta) \prod_{t<j} \tilde{\tau}(\beta) \mid \eta \right]^X_r$$

using the induction hypothesis for the r induction.

By a degeneration argument as in the simple case, we finally obtain a formula for

$$\left[ML \prod_{i \le s} \tau_{n_i}(\alpha) \prod_{s<i\in I} \tilde{\tau}(\alpha) \prod_{J \ni j \le t} \tau_{m_j}(\beta) \prod_{t<j} \tilde{\tau}(\beta)\right]^X_{r,d}. \qquad \square$$

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References

[1] K. Behrend and B. Fantechi, *The intrinsic normal cone,* Invent. Math. 128 (1997), no. 1, 45–88.

[2] F. Carel and R. Pandharipande, *Relative maps and tautological classes,* J. Eur. Math. Soc. (JEMS) 7 (2005), no. 1, 13–49.

[3] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology,* Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., 62, pp. 45–96, Amer. Math. Soc., Providence, RI, 1997.

[4] T. Graber and R. Pandharipande, *Localization of virtual classes,* Invent. Math. 135 (1999), no. 2, 487–518.

[5] E.-N. Ionel and T. H. Parker, *Relative Gromov–Witten invariants,* Ann. of Math. (2) 157 (2003), no. 1, 45–96.

[6] A.-M. Li and Y. Ruan, *Symplectic surgery and Gromov–Witten invariants of Calabi-Yau 3-folds,* Invent. Math. 145 (2001), no. 1, 151–218.

[7] J. Li, *A degeneration formula of GW-invariants,* J. Differential Geom. 60 (2002), no. 2, 199–293.

[8] J. Li and G. Tian, *Virtual moduli cycles and Gromov–Witten invariants of algebraic varieties,* J. Amer. Math. Soc. 11 (1998), no. 1, 119–174.

[9] A. Okounkov and R. Pandharipande, *The equivariant Gromov–Witten theory of* $\mathbf{P}^1$, Ann. of Math. (2) 163 (2006), no. 2, 561–605.

[10] ———, *Gromov–Witten theory, Hurwitz theory, and completed cycles,* Ann. of Math. (2) 163 (2006), no. 2, 517–560.

[11] ———, *Virasoro constraints for target curves,* Invent. Math. 163 (2006), no. 1, 47–108.

Department of Mathematics
University of Michigan
2074 East Hall, 530 Church Street
Ann Arbor, MI 48109
USA

janda@umich.edu