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SO(n) Covariant Local Tensor Valuations on Polytopes

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Abstract. The Minkowski tensors are valuations on the space of convex bodies in $\mathbb{R}^n$ with values in a space of symmetric tensors, having additional covariance and continuity properties. They are extensions of the intrinsic volumes, and as these, they are the subject of classification theorems and admit localizations in the form of measure-valued valuations. For these local tensor valuations, restricted to convex polytopes, a classification theorem has been proved recently under the assumption of isometry covariance, but without any continuity assumption. This characterization result is extended here, replacing the covariance under orthogonal transformations by invariance under proper rotations only. This yields additional local tensor valuations on polytopes in dimensions two and three, but not in higher dimensions. In this paper, they are completely classified.

1. Introduction

A valuation on the space $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$ is a mapping φ from $\mathcal{K}^n$ into some Abelian group such that

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$$

whenever $K, L, K \cup L \in \mathcal{K}^n$. The best known examples are the intrinsic volumes or Minkowski functionals. They arise as the suitably normalized coefficients of the polynomial in ρ that expresses, for a given convex body K, the volume of the outer parallel body of K at distance $\rho \geq 0$. The celebrated characterization theorem of Hadwiger states that every rigid motion invariant continuous real-valued valuation on $\mathcal{K}^n$ is a linear combination of the intrinsic volumes. This theorem was the first culmination of a rich theory of valuations on convex bodies (for the older history, see the surveys [15] and [17]), which in the last two decades has again been widened and deepened considerably. For an introduction and for references, we refer to [24], in particular, Chapter 6 and Section 10.16. A survey on recent developments is given by Alesker [3].

A natural extension of the intrinsic volumes is obtained if the volume is replaced by a higher moment. If the integral

$$\int_K x \otimes \cdots \otimes x \, \mathrm{d}x,$$

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where the integrand is an r-fold tensor product ($r \in \mathbb{N}$), is evaluated for the outer parallel body of K at distance ρ, we again obtain a polynomial in ρ, and its coefficients can be expressed as sums of symmetric tensors, which are functions of K. Suitably normalized, these yield the so-called Minkowski tensors. They are tensor-valued valuations on $\mathcal{K}^n$ with additional continuity and isometry covariance properties. (For brief introductions, we refer to [24], Section 5.4.2, and to [10].) After some sporadic treatments (e.g., [19; 20]), a thorough investigation of the Minkowski tensors began with the work of McMullen [16], who studied them on polytopes. Alesker [2] (based on his work in [1]) extended Hadwiger's classification theorem, showing that the real vector space of continuous, isometry covariant tensor valuations on $\mathcal{K}^n$ of given rank is spanned by suitable Minkowski tensors, multiplied by powers of the metric tensor. Questions of linear independence, leading to the determination of dimensions and bases, were treated in [11]. Minkowski tensors were studied and used in integral geometry ([5; 12; 22; 25]), in stochastic geometry, stereology and image analysis ([7; 8; 14; 26]), and for lower dimensions and ranks, they were applied in physics ([4; 18; 27; 28; 29]). We also refer the reader to lecture notes [13].

Just as the intrinsic volumes have local versions, the support measures (with curvature and area measures as marginal measures), so the Minkowski tensors have local versions. They associate with every convex body a series of tensor-valued measures. The mappings defined in this way are valuations with the additional properties of weak continuity, isometry covariance, and local determination. A corresponding classification theorem was proved in [9], based on the previous investigation [23] concerning the case of polytopes. This approach has some interesting features. First, on polytopes, a complete classification is possible without any continuity assumption. In [9], it was determined which of the obtained local tensor valuation mappings have weakly continuous extensions to all convex bodies. Second, the valuation property need not be assumed, but is a consequence of the classification theorem.

The isometry covariance assumed in these characterization results has two components, translation covariance (a certain polynomial behavior under translations) and covariance with respect to the orthogonal group $\mathrm{O}(n)$. Covariance with respect to other groups is also of interest. Recently, Haberl and Parapatits [6] were able to classify all measurable $\mathrm{SL}(n)$ covariant symmetric tensor valuations on convex polytopes containing the origin in the interior. In the opposite direction (a smaller group than $\mathrm{O}(n)$), it was shown by Saienko [21], under continuity and smoothness assumptions, that the classification of the local tensor valuations does not change for $n \geq 4$ if $\mathrm{O}(n)$ covariance is replaced by $\mathrm{SO}(n)$ covariance. In the physically relevant dimensions two and three, however, he surprisingly discovered additional local tensor valuations. The purpose of this paper is to study $\mathrm{SO}(n)$ covariant local tensor valuations on polytopes without assuming any continuity property and also to obtain the valuation property as a consequence. Thus, the aim is to extend the results of [23] replacing the orthogonal group $\mathrm{O}(n)$ by the group $\mathrm{SO}(n)$ of proper rotations. The main result is Theorem 2. We will study elsewhere which of the newly found mappings have a weakly continuous extension to all convex bodies.

After collecting some notation in Section 2, we formulate our results in Section 3. The proof is prepared by some auxiliary results in Section 4 and the refinement of two lemmas from [23] in Section 5. The main result is then proved in Section 6.

2. Notation

We work in the n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$) with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Its unit sphere is $\mathbb{S}^{n-1}$, and we write $\Sigma^n := \mathbb{R}^n \times \mathbb{S}^{n-1}$ and equip this with the product topology. By $G(n,k)$ we denote the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^n$, $k \in \{0, \dots, n\}$. For $L \in G(n,k)$, we write $\mathbb{S}_L := \mathbb{S}^{n-1} \cap L$. The orthogonal complement of $L \in G(n,k)$ is denoted by $L^\perp$. By $\mathcal{H}^k$ we denote the k-dimensional Hausdorff measure on $\mathbb{R}^n$, and $\mathcal{H}^{n-1}(\mathbb{S}^{n-1})$ defines the constant $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$. For a topological space S, we denote by $\mathcal{B}(S)$ the σ-algebra of its Borel sets. For $S \subset \mathbb{R}^n$, the set of bounded Borel sets in S is denoted by $\mathcal{B}_b(S)$.

The *orthogonal group* O(n) of $\mathbb{R}^n$ is the group of all linear mappings of $\mathbb{R}^n$ into itself preserving the scalar product, and SO(n) is the subgroup of *rotations*, which also preserve the orientation.

By $\mathcal{P}^n$ we denote the set of (convex and nonempty) polytopes in $\mathbb{R}^n$. For $k \in \{0, \dots, n\}$, the set of k-dimensional faces of the polytope P is denoted by $\mathcal{F}_k(P)$. For $F \in \mathcal{F}_k(P)$, the subspace $L(F) \in G(n,k)$, the *direction space* of F, is the translate, passing through 0, of the affine hull of F. The set $\nu(P,F) \subset \mathbb{S}_{L(F)^\perp}$ is the set of outer unit normal vectors of P at its face F. The generalized normal bundle (or normal cycle) of P is the subset $\operatorname{Nor} P \subset \Sigma^n$ consisting of all pairs (x,u) such that x is a boundary point of P and u is an outer unit normal vector of P at x.

This paper rests heavily on the previous papers [23] and [9] on local tensor valuations and uses much of their terminology. We recall here briefly the underlying conventions on tensors. For $p \in \mathbb{N}_0$, we denote by $\mathbb{T}^p$ the real vector space of symmetric tensors of rank p (or symmetric p-tensors for short) on $\mathbb{R}^n$. The scalar product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^n$ is used to identify $\mathbb{R}^n$ with its dual space, so that each vector $a \in \mathbb{R}^n$ is identified with the linear functional $x \mapsto \langle a, x \rangle$, $x \in \mathbb{R}^n$. Thus, $\mathbb{T}^1$ is identified with $\mathbb{R}^n$ (and $\mathbb{T}^0$ with $\mathbb{R}$), and for $p \geq 1$, each tensor $T \in \mathbb{T}^p$ is a symmetric p-linear functional on $\mathbb{R}^n$. The symmetric tensor product $a \odot b$ is always abbreviated by ab, and for $x \in \mathbb{R}^n$, the r-fold symmetric tensor product $x \odot \cdots \odot x$ is denoted by x^r.

The *metric tensor* Q on $\mathbb{R}^n$ is defined by $Q(x,y) := \langle x, y \rangle$ for $x, y \in \mathbb{R}^n$. For a subspace $L \in G(n,k)$, we denote by $\mathbb{T}^p(L)$ the space of symmetric p-tensors on L. We must distinguish between $Q_{(L)}$, the metric tensor on L with $Q_{(L)}(a,b) := \langle a, b \rangle$ for $a, b \in L$, and the tensor Q_L defined by

$$Q_L(a,b) := \langle \pi_L a, \pi_L b \rangle \quad \text{for } a, b \in \mathbb{R}^n,$$

where $\pi_L : \mathbb{R}^n \to L$ denotes the orthogonal projection. The mapping $\pi_L^* : \bigcup_{p\in\mathbb{N}_0} \mathbb{T}^p(L) \to \bigcup_{p\in\mathbb{N}_0} \mathbb{T}^p$ is defined by $(\pi_L^* T)(a_1, \dots, a_p) := T(\pi_L a_1, \dots, \pi_L a_p)$, $a_1, \dots, a_p \in \mathbb{R}^n$, for $T \in \mathbb{T}^p(L)$. In particular, $\pi_L^* Q_{(L)} = Q_L$. (This notation is different from that used in [23].)

3. Formulation of Results

The *Minkowski tensors* of a convex body $K \in \mathcal{K}^n$ are given by

$$\Phi_k^{r,s}(K) = \frac{1}{r!s!}\frac{\omega_{n-k}}{\omega_{n-k+s}} \int_{\Sigma^n} x^r u^s \Lambda_k(K, \mathrm{d}(x,u))$$

for $k = 0, \dots, n-1$ and $r, s \in \mathbb{N}_0$. We refer to [24], Section 4.2, for the support measures $\Lambda_0(K, \cdot), \dots, \Lambda_{n-1}(K, \cdot)$ appearing here and to [24], Section 5.4, for a brief introduction to the Minkowski tensors. The *local Minkowski tensors* are defined by

$$\phi_k^{r,s}(K, \eta) = \frac{1}{r!s!}\frac{\omega_{n-k}}{\omega_{n-k+s}} \int_{\eta} x^r u^s \Lambda_k(K, \mathrm{d}(x,u))$$

for $\eta \in \mathcal{B}(\Sigma^n)$. If $P \in \mathcal{P}^n$ is a polytope, the special form of the support measures yields a more explicit expression, namely

$$\phi_k^{r,s}(P, \eta) = C_{n,k}^{r,s} \sum_{F\in\mathcal{F}_k(P)} \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x,u) x^r u^s \mathcal{H}^{n-k-1}(\mathrm{d}u)\mathcal{H}^k(\mathrm{d}x),$$

where we now use the abbreviation

$$C_{n,k}^{r,s} := (r!s!\omega_{n-k+s})^{-1},$$

and where the function $\mathbf{1}_\eta$ is the characteristic function of η. The attempt to characterize these local tensor valuations on polytopes by their basic properties revealed in [23] that these properties are also shared by the *generalized local Minkowski tensors*. For a polytope $P \in \mathcal{P}^n$, these are defined by

$$\phi_k^{r,s,j}(P, \eta) := C_{n,k}^{r,s} \sum_{F\in\mathcal{F}_k(P)} Q_{L(F)}^j \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x,u) x^r u^s \mathcal{H}^{n-k-1}(\mathrm{d}u)\mathcal{H}^k(\mathrm{d}x) \quad (1)$$

for $\eta \in \mathcal{B}(\Sigma^n)$, $k \in \{0, \dots, n-1\}$, $r, s \in \mathbb{N}_0$, and for $j \in \mathbb{N}_0$ if $k > 0$ but only for $j = 0$ if $k = 0$. Recall that $x^r u^s$ in (1) denotes a symmetric tensor product and that also the product of $Q_{L(F)}^j$ with the subsequent tensor-valued integral is a symmetric tensor product.

For fixed k, r, s, j and with $p := 2j + r + s$, the tensor $\phi_k^{r,s,j}$ defines a mapping $\Gamma : \mathcal{P}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$. For such a mapping Γ, the following properties are of interest. It is called *translation covariant* of degree $q \le p$ if

$$\Gamma(P + t, \eta + t) = \sum_{j=0}^{q} \Gamma_{p-j}(P, \eta)\frac{t^j}{j!} \quad (2)$$

with tensors $\Gamma_{p-j}(P,\eta) \in \mathbb{T}^{p-j}$ for $P \in \mathcal{P}^n$, $\eta \in \mathcal{B}(\Sigma^n)$, and $t \in \mathbb{R}^n$. Here $\eta + t := \{(x+t,u) : (x,u) \in \eta\}$, and $\Gamma_p = \Gamma$. If Γ is translation covariant of degree zero, it is called *translation invariant*, and Γ is just called *translation covariant* if it is translation covariant of some degree $q \le p$. The mapping Γ is called SO(n) *covariant* if $\Gamma(\vartheta P, \vartheta\eta) = \vartheta\Gamma(P,\eta)$ for $P \in \mathcal{P}^n$, $\eta \in \mathcal{B}(\Sigma^n)$, and $\vartheta \in \mathrm{SO}(n)$, where $\vartheta\eta := \{(\vartheta x, \vartheta u) : (x,u) \in \eta\}$. Here the operation of SO(n) on $\mathbb{T}^p$ is defined by $(\vartheta T)(x_1,\dots,x_p) := T(\vartheta^{-1}x_1,\dots,\vartheta^{-1}x_p)$ for $x_1,\dots,x_p \in \mathbb{R}^n$ and $\vartheta \in \mathrm{SO}(n)$. Similarly, O($n$) covariance is defined. Finally, the mapping Γ is *locally defined* if $\eta \cap \mathrm{Nor}\, P = \eta' \cap \mathrm{Nor}\, P'$ with $P, P' \in \mathcal{P}^n$ and $\eta, \eta' \in \mathcal{B}(\Sigma^n)$ implies $\Gamma(P,\eta) = \Gamma(P',\eta')$.

The mapping defined by $\Gamma(P,\eta) := \phi_k^{r,s,j}(P,\eta)$ for fixed k, r, s, j has the following properties. For each $P \in \mathcal{P}^n$, $\Gamma(P,\cdot)$ is a $\mathbb{T}^p$-valued measure with $p = 2j + r + s$. The mapping Γ is translation covariant, O(n) covariant, and locally defined. These properties do not change (except that the rank must be adjusted) if Γ is multiplied (symmetrically) by a power of the metric tensor.

The following theorem was essentially proved in [23] with some simplifications and supplements provided in [9].

THEOREM 1. *For $p \in \mathbb{N}_0$, let $T_p(\mathcal{P}^n)$ denote the real vector space of all mappings $\Gamma : \mathcal{P}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ with the following properties*:

(a) $\Gamma(P,\cdot)$ *is a $\mathbb{T}^p$-valued measure for each $P \in \mathcal{P}^n$*,
(b) Γ *is translation covariant and* O(n) *covariant*, *and*
(c) Γ *is locally defined.*

Then a basis of $T_p(\mathcal{P}^n)$ is given by the mappings $Q^m \phi_k^{r,s,j}$, where $m, r, s, j \in \mathbb{N}_0$ satisfy $2m + 2j + r + s = p$, $k \in \{0,\dots,n-1\}$, and $j = 0$ if $k \in \{0, n-1\}$.

The purpose of the following is to extend this characterization of local tensor valuations on polytopes from O(n) to SO(n) covariance. It was discovered by Saienko [21] that, under this weaker assumption, there are additional tensor valuations in dimensions two and three.

In the following, the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are endowed with fixed orientations. Let $P \in \mathcal{P}^3$. For each edge $F \in \mathcal{F}_1(P)$, we choose a unit vector $v_F \in L(F)$. For $u \in \mathbb{S}_{L(F)^\perp}$, let $v_F \times u =: \overline{u}$ denote the vector product of v_F and u in $\mathbb{R}^3$; thus $\overline{u}$ is the unique unit vector such that $(v_F, u, \overline{u})$ is a positively oriented orthonormal basis of $\mathbb{R}^3$. We define

$$\begin{aligned}
&\widetilde{\phi}^{r,s,j}(P,\eta) \\
&\quad := \sum_{F \in \mathcal{F}_1(P)} Q^j_{L(F)} v_F \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x,u)\, x^r (v_F \times u) u^s \,\mathcal{H}^1(\mathrm{d}u)\,\mathcal{H}^1(\mathrm{d}x) \qquad (3)\\
&\quad = \sum_{F \in \mathcal{F}_1(P)} v_F^{2j+1} \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x,u)\, x^r (v_F \times u) u^s \,\mathcal{H}^1(\mathrm{d}u)\,\mathcal{H}^1(\mathrm{d}x) \qquad (4)
\end{aligned}$$

for $\eta \in \mathcal{B}(\Sigma^3)$ and $r, s, j \in \mathbb{N}_0$. Here we have used that $Q_{L(F)} = v_F^2$ since $\dim F = 1$. The tensor $\widetilde{\phi}^{r,s,j}(P,\eta)$ is well defined since it does not change if the vector v_F is replaced by $-v_F$. Since

$$\widetilde{\phi}^{r,s,j}(P+t,\eta+t)=\sum_{i=0}^{r}\binom{r}{i}\widetilde{\phi}^{r-i,s,j}(P,\eta)t^i \tag{5}$$

for $t\in\mathbb{R}^3$, the mapping $\widetilde{\phi}^{r,s,j}$ is translation covariant. It is also SO(3) covariant since $\vartheta v_F\times\vartheta u=\vartheta(v_F\times u)$ for $\vartheta\in\mathrm{SO}(3)$.

Now let $n=2$. For $u\in\mathbb{S}^1$, let $\overline{u}\in\mathbb{S}^1$ be the unique vector for which $(u,\overline{u})$ is a positively oriented orthonormal basis of $\mathbb{R}^2$. For $P\in\mathcal{P}^2$, $k\in\{0,1\}$, and $\eta\in\mathcal{B}(\Sigma^2)$, we define

$$\widetilde{\phi}_k^{r,s}(P,\eta):=\sum_{F\in\mathcal{F}_k(P)}\int_F\int_{\nu(P,F)}\mathbf{1}_\eta(x,u)x^r\overline{u}u^s\mathcal{H}^{1-k}(\mathrm{d}u)\mathcal{H}^k(\mathrm{d}x). \tag{6}$$

Of course, if $\dim P=2$ and $F\in\mathcal{F}_1(P)$, then $\nu(P,F)=\{u_F\}$ with a unique vector u_F, and we have

$$\widetilde{\phi}_1^{r,s}(P,\eta)=\sum_{F\in\mathcal{F}_1(P)}\overline{u_F}u_F^s\int_F\mathbf{1}_\eta(x,u_F)x^r\mathcal{H}^1(\mathrm{d}x).$$

If $\dim P=1$ and $F\in\mathcal{F}_1(P)$, then $P=F$ and $\nu(P,F)=\{\pm u_F\}$, and therefore

$$\widetilde{\phi}_1^{r,s}(P,\eta)=\int_F[\mathbf{1}_\eta(x,u_F)\overline{u_F}u_F^s+\mathbf{1}_\eta(x,-u_F)(-\overline{u_F})(-u_F)^s]x^r\mathcal{H}^1(\mathrm{d}x).$$

For the case $k=0$, we note that, for $F\in\mathcal{F}_0(P)$, we have $F=\{x_F\}$, and hence

$$\widetilde{\phi}_0^{r,s}(P,\eta)=\sum_{F\in\mathcal{F}_0(P)}x_F^r\int_{\nu(P,F)}\mathbf{1}_\eta(x_F,u)\overline{u}u^s\mathcal{H}^1(\mathrm{d}u).$$

The translation covariance and SO(2) covariance of $\widetilde{\phi}_k^{r,s}$ are easy to check.

The mappings $\widetilde{\phi}^{r,s,j}(\cdot,\eta)$ and $\widetilde{\phi}_k^{r,s}(\cdot,\eta)$ $(k=0,1)$ defined on polytopes in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, are valuations. This is proved as it was done for the mappings $\phi_k^{r,s,j}(\cdot,\eta)$ in [9, Theorem 3.3].

The following result is the counterpart to Theorem 1 with the rotation group $\mathrm{SO}(n)$ instead of the orthogonal group $\mathrm{O}(n)$.

THEOREM 2. *For $p\in\mathbb{N}_0$, let $\widetilde{T}_p(\mathcal{P}^n)$ denote the real vector space of all mappings $\Gamma:\mathcal{P}^n\times\mathcal{B}(\Sigma^n)\to\mathbb{T}^p$ with the following properties*:

(a) *$\Gamma(P,\cdot)$ is a $\mathbb{T}^p$-valued measure for each $P\in\mathcal{P}^n$*,
(b) *Γ is translation covariant and* $\mathrm{SO}(n)$ *covariant, and*
(c) *Γ is locally defined.*

Then a basis of $\widetilde{T}_p(\mathcal{P}^n)$ is given by the mappings $Q^m\phi_k^{r,s,j}$, where $m,r,s,j\in\mathbb{N}_0$ satisfy $2m+2j+r+s=p$, $k\in\{0,\dots,n-1\}$, and $j=0$ if $k\in\{0,n-1\}$, together with

- *if $n\geq 4$, no more mappings,*
- *if $n=3$, the mappings $Q^m\widetilde{\phi}^{r,s,j}$, where $m,r,s,j\in\mathbb{N}_0$ satisfy $2m+2j+r+s+2=p$,*

- *if* $n = 2$, *the mappings* $Q^m \widetilde{\phi}_k^{r,s}$, *where* $m, r, s \in \mathbb{N}_0$ *satisfy* $2m + r + s + 1 = p$ *and* $k \in \{0, 1\}$.

As in [9] (and similarly earlier in Alesker's work [2]), this general result follows from its particular case where Γ is translation invariant. Therefore, we formulate this case separately, deleting the assertion of linear independence, which we discuss in the next section.

THEOREM 3. *Let* $p \in \mathbb{N}_0$. *Let* $\Gamma : \mathcal{P}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ *be a mapping with the following properties*:

(a) $\Gamma(P, \cdot)$ *is a* $\mathbb{T}^p$*-valued measure for each* $P \in \mathcal{P}^n$,
(b) Γ *is translation invariant and* SO(n) *covariant*, *and*
(c) Γ *is locally defined*.

Then Γ *is a linear combination*, *with constant coefficients*, *of the mappings* $Q^m \phi_k^{0,s,j}$, *where* $m, s, j \in \mathbb{N}_0$ *satisfy* $2m + 2j + s = p$, $k \in \{0, \dots, n-1\}$, *and* $j = 0$ *if* $k \in \{0, n-1\}$, *together with*

- *if* $n \geq 4$, *no more mappings*,
- *if* $n = 3$, *the mappings* $Q^m \widetilde{\phi}^{0,s,j}$, *where* $m, s, j \in \mathbb{N}_0$ *satisfy* $2m + 2j + s + 2 = p$,
- *if* $n = 2$, *the mappings* $Q^m \widetilde{\phi}_k^{0,s}$, *where* $m, s \in \mathbb{N}_0$ *satisfy* $2m + s + 1 = p$ *and* $k \in \{0, 1\}$.

In the next section, we prove the linear independence result contained in Theorem 2 and show how Theorem 2 follows from Theorem 3 (and Proposition 1). In Section 5, we extend two lemmas of [23] from O(n) covariance to SO(n) covariance. The proof of Theorem 3 then follows in Section 6.

4. Auxiliary Results

First, we explain how Theorem 2 is deduced from Theorem 3 and Proposition 1. Each of $\phi_k^{r,s,j}$ if $n \geq 2$, each of $\widetilde{\phi}^{r,s,j}$ if $n = 3$, and each of $\widetilde{\phi}_k^{r,s}$ if $n = 2$ is a mapping $\Gamma : \mathcal{P}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ (for suitable p) that has the following properties:

(a) $\Gamma(P, \cdot)$ is a $\mathbb{T}^p$-valued measure for each $P \in \mathcal{P}^n$,
(b) Γ is translation covariant of some degree $q \leq p$ and SO(n) covariant, and
(c) Γ is locally defined.

It follows from [9, Lemmas 3.1, 3.2] that each Γ_{p-j} appearing in (2) satisfies

$$\Gamma_{p-j}(P + t, \eta + t) = \sum_{r=0}^{q-j} \Gamma_{p-j-r}(P, \eta) \frac{t^r}{r!}$$

for $j = 0, \dots, q$ and that Γ_{p-j} has again the properties (a), (b), (c). In particular, the choice $j = q$ yields that Γ_{p-q} is translation invariant. It is now clear that the procedure described in [9, pp. 1534–1535] allows us to deduce Theorem 2 from Theorem 3 (and Proposition 1).

We turn to linear independence.

PROPOSITION 1. *Let $p \in \mathbb{N}_0$. The local tensor valuations $Q^m \phi_k^{r,s,j}$ with $m, r, s, j \in \mathbb{N}_0$, $2m + 2j + r + s = p$, $k \in \{0, \dots, n-1\}$, and $j = 0$ if $k \in \{0, n-1\}$, together with*

- *if $n = 3$, the local tensor valuations $Q^m \widetilde{\phi}^{r,s,j}$ with $m, r, s, j \in \mathbb{N}_0$, $2m + 2j + r + s + 2 = p$,*
- *if $n = 2$, the local tensor valuations $Q^m \widetilde{\phi}_k^{r,s}$ with $m, r, s \in \mathbb{N}_0$, $2m + r + s + 1 = p$, $k \in \{0, 1\}$,*

are linearly independent.

Proof. For $n \geq 4$, the assertion is covered by [9, Thm. 3.1]. For the remaining cases $n \in \{2, 3\}$, we extend the proof of that theorem.

Case 1: $n = 3$.

Assume the linear relation

$$\sum_{\substack{m,r,s,j,k \\ 2m+2j+r+s=p}} a_{kmrsj} Q^m \phi_k^{r,s,j} + \sum_{\substack{m,r,s,j \\ 2m+2j+r+s+2=p}} b_{mrsj} Q^m \widetilde{\phi}^{r,s,j} = 0 \tag{7}$$

with $a_{kmrsj}, b_{mrsj} \in \mathbb{R}$ and with $a_{0mrsj} = a_{2mrsj} = 0$ for $j \neq 0$. Evaluating this at $(\vartheta P, \vartheta \eta)$, with arbitrary $P \in \mathcal{P}^3$, $\eta \in \mathcal{B}(\Sigma^3)$, and $\vartheta \in \mathrm{O}(3)$, we obtain

$$\vartheta \Bigg[\sum_{\substack{m,r,s,j,k \\ 2m+2j+r+s=p}} a_{kmrsj} Q^m \phi_k^{r,s,j}(P, \eta) + (\det \vartheta) \sum_{\substack{m,r,s,j \\ 2m+2j+r+s+2=p}} b_{mrsj} Q^m \widetilde{\phi}^{r,s,j}(P, \eta) \Bigg] = 0$$

by the covariance properties of $\phi_k^{r,s,j}$ and $\widetilde{\phi}^{r,s,j}$. This shows that the tensor in brackets is zero, and since we can choose $\det \vartheta = 1$ and $\det \vartheta = -1$, we conclude that

$$\sum_{\substack{m,r,s,j,k \\ 2m+2j+r+s=p}} a_{kmrsj} Q^m \phi_k^{r,s,j} = 0 \tag{8}$$

and

$$\sum_{\substack{m,r,s,j \\ 2m+2j+r+s+2=p}} b_{mrsj} Q^m \widetilde{\phi}^{r,s,j} = 0. \tag{9}$$

The proof of [9, Thm. 3.1] shows that (8) implies that all coefficients a_{kmrsj} are zero. Hence, in the following, we need only deal with relation (9).

Let $F \in \mathcal{P}^3$ be a one-dimensional polytope and consider sets of the form $\eta = \beta \times \omega$ with Borel sets $\beta \subset \operatorname{relint} F$ and $\omega \subset \mathbb{S}_{L(F)^\perp}$. Expression (4) yields

$$\widetilde{\phi}^{r,s,j}(F, \beta \times \omega) = v_F^{2j+1} \int_\beta x^r \mathcal{H}^1(\mathrm{d}x) \int_\omega (v_F \times u) u^s \mathcal{H}^1(\mathrm{d}u),$$

and hence (9) gives

$$\sum_{\substack{m,r,s,j \\ 2m+2j+r+s+2=p}} b_{mrsj} Q^m v_F^{2j+1} \int_\beta x^r \mathcal{H}^1(\mathrm{d}x) \int_\omega (v_F \times u) u^s \mathcal{H}^1(\mathrm{d}u) = 0.$$

Since this holds for all F, β, ω as specified, we can argue as in the proof of [9, Thm. 3.1] and conclude that, for each fixed r, we have

$$\sum_{\substack{m,s,j\\2m+2j+s+2=p-r}} b_{mrsj}\, Q^m v_F^{2j+1}(v_F \times u)u^s = 0 \tag{10}$$

for all $u \in \mathbb{S}_{L(F)^\perp}$.

Let (e_1, e_2, e_3) be a positively oriented orthonormal basis of $\mathbb{R}^3$ such that $e_1 = v_F$. We apply (10) to the $(p-r)$-tuple

$$(\underbrace{x, \dots, x}_{p-r}) \quad \text{with } x = x_1 e_1 + x_2 e_2 + x_3 e_3 \in \mathbb{R}^3.$$

This gives

$$\sum_{\substack{m,s,j\\2m+2j+s+2=p-r}} b_{mrsj}(x_1^2 + x_2^2 + x_3^2)^m x_1^{2j+1}(-u_3 x_2 + u_2 x_3)(u_2 x_2 + u_3 x_3)^s = 0$$

for all $u_2, u_3 \in \mathbb{R}$ such that $u_2 e_2 + u_3 e_3 \in \mathbb{S}^2$. Denoting by θ the angle from $u_2 e_2 + u_3 e_3$ to $x_2 e_2 + x_3 e_3$, we can write the last equation as

$$\sum_{s \ge 0} \beta_s (x_2^2 + x_3^2)^{\frac{s+1}{2}} \sin\theta \cos^s\theta = 0 \tag{11}$$

with

$$\beta_s = \sum_{\substack{m,j\\2m+2j+2=p-r-s}} b_{mrsj}(x_1^2 + x_2^2 + x_3^2)^m x_1^{2j+1}.$$

Since (11) holds for all $\theta \in \mathbb{R}$, it follows that $\beta_s = 0$ for all s. Now the proof of [9, Thm. 3.1] shows that all coefficients b_{mrsj} are zero.

Case 2: $n = 2$.

Then $\phi_k^{r,s,j} \neq 0$ only for $k \in \{0, 1\}$ and hence also only for $j = 0$. Therefore, we assume the linear relation

$$\sum_{\substack{m,r,s,k\\2m+r+s=p}} a_{kmrs}\, Q^m \phi_k^{r,s,0} + \sum_{\substack{m,r,s,k\\2m+r+s+1=p}} b_{kmrs}\, Q^m \widetilde{\phi}_k^{r,s} = 0 \tag{12}$$

with $a_{kmrs}, b_{kmrs} \in \mathbb{R}$ and $k \in \{0, 1\}$. Similarly as in Case 1, we obtain that all coefficients a_{kmrs} are zero, and hence we need only deal with the relation

$$\sum_{\substack{m,r,s,k\\2m+r+s+1=p}} b_{kmrs}\, Q^m \widetilde{\phi}_k^{r,s} = 0. \tag{13}$$

Let $F \in \mathcal{P}^2$ be a d-dimensional polytope, $d \in \{0, 1\}$, and consider sets of the form $\eta = \beta \times \omega$ with Borel sets $\beta \subset \operatorname{relint} F$ and $\omega \subset \mathbb{S}_{L(F)^\perp}$.

Subcase 2a: $d = 0$. Then $\widetilde{\phi}_1^{r,s}(F, \beta \times \omega) = 0$. It follows from (13) that

$$\sum_{\substack{m,r,s\\2m+r+s+1=p}} b_{0mrs}\, Q^m \int_\beta x^r \mathcal{H}^0(\mathrm{d}x) \int_\omega \overline{u} u^s \mathcal{H}^1(\mathrm{d}u) = 0.$$

Since this holds for all F, β, ω as specified, we obtain for each fixed r that

$$\sum_{\substack{m,s\\2m+s+1=p-r}} b_{0mrs}\, Q^m \overline{u} u^s = 0 \tag{14}$$

for all $u \in \mathbb{S}^1$. We now choose the orthogonal basis (e_1, e_2) of $\mathbb{R}^2$ such that $e_1 = u$ and $e_2 = \overline{u}$. Applying (14) to the $(p-r)$-tuple

$$(\underbrace{x, \ldots, x}_{p-r}) \qquad \text{with } x = x_1 e_1 + x_2 e_2 \in \mathbb{R}^2,$$

we obtain

$$\sum_{m=0}^{\lfloor (p-r-1)/2 \rfloor} d_m (x_1^2 + x_2^2)^m x_2 x_1^{p-r-2m-1} = 0$$

with $d_m = b_{0mr(p-r-2m-1)}$. This yields that all coefficients d_m are zero, and hence all coefficients in (12) with $k = 0$ are zero.

Subcase 2b: $d = 1$. Then $\widetilde{\phi}_0^{r,s}(F, \beta \times \omega) = 0$. We choose $\omega = \{u_F\}$, where u_F is one of the two unit normal vectors of F. Then from (13) applied to $(F, \beta \times \omega)$ we obtain

$$\sum_{\substack{m,r,s\\2m+r+s+1=p}} b_{1mrs}\, Q^m \overline{u_F} u_F^s \int_\beta x^r \mathcal{H}^1(\mathrm{d}x) = 0.$$

As before, for each fixed r, this yields

$$\sum_{\substack{m,s\\2m+s+1=p-r}} b_{1mrs}\, Q^m \overline{u_F} u_F^s = 0. \tag{15}$$

We choose the orthogonal basis (e_1, e_2) of $\mathbb{R}^2$ such that $e_1 = u_F$ and $e_2 = \overline{u_F}$. Applying (15) to the $(p-r)$-tuple

$$(\underbrace{x, \ldots, x}_{p-r}) \qquad \text{with } x = x_1 e_1 + x_2 e_2 \in \mathbb{R}^2,$$

we obtain

$$\sum_{\substack{m,s\\2m+s+1=p-r}} b_{1mrs} (x_1^2 + x_2^2)^m x_2 x_1^s = 0.$$

Now we can conclude as before that all coefficients in (15) and hence all coefficients in (12) with $k = 1$ are zero. □

5. Some Refined Lemmas

In this section, we extend Lemmas 3 and 4 in [23], essentially from O(n) covariance to SO(n) covariance. (We remark that in Lemma 3 of [23], the group SO(n) should be replaced by O(n) since this is used in the proof. This does not affect the rest of the paper, where Lemma 3 is only applied with O(n).)

LEMMA 1. *Let $L \in G(n,k)$ with $k \in \{1,\dots,n-1\}$, and let $r \in \mathbb{N}_0$ and $T \in \mathbb{T}^r$.*

(a) *Let $k \geq 2$. If $\vartheta T = T$ for each $\vartheta \in \mathrm{SO}(n)$ that fixes $L^\perp$ pointwise, then*

$$T = \sum_{j=0}^{\lfloor r/2 \rfloor} Q_L^j \pi_{L^\perp}^* T^{(r-2j)} \tag{16}$$

with tensors $T^{(r-2j)} \in \mathbb{T}^{r-2j}(L^\perp)$, $j = 0,\dots,\lfloor r/2 \rfloor$.

(b) *Let $k = 1$. Let v_L be a unit vector spanning L. Then*

$$T = \sum_{j=0}^{r} v_L^j \pi_{L^\perp}^* T^{(r-j)}$$

with tensors $T^{(m)} \in \mathbb{T}^m(L^\perp)$.

Proof. Given an orthonormal basis $(e_1,\dots,e_n)$ of $\mathbb{R}^n$, we associate with $T \in \mathbb{T}^r$, represented in coordinates by

$$T = \sum_{1 \leq i_1 \leq \dots \leq i_r \leq n} t_{i_1 \dots i_r} e_{i_1} \cdots e_{i_r},$$

the polynomial on $\mathbb{R}^n$ defined by

$$p_T(y) = \sum_{1 \leq i_1 \leq \dots \leq i_r \leq n} t_{i_1 \dots i_r} y_{i_1} \cdots y_{i_r}, \qquad y = \sum_{i=1}^{n} y_i e_i. \tag{17}$$

The mapping $T \mapsto p_T$ is a vector space isomorphism between $\mathbb{T}^r$ and the vector space of homogeneous polynomials of degree r on $\mathbb{R}^n$. It is compatible with the operation of the orthogonal group, that is, it satisfies $p_{\vartheta T}(y) = p_T(\vartheta^{-1} y)$ for $y \in \mathbb{R}^n$ and $\vartheta \in \mathrm{O}(n)$.

We choose the orthonormal basis $(e_1,\dots,e_n)$ in such a way that $e_1,\dots,e_k$ span the subspace L and $e_{k+1},\dots,e_n$ span its orthogonal complement $L^\perp$.

(a) Let assumption (a) be satisfied. Then the polynomial p_T defined by (17) satisfies $p_T(\vartheta^{-1} y) = p_{\vartheta T}(y) = p_T(y)$ for each $\vartheta \in \mathrm{SO}(n)$ fixing $L^\perp$ pointwise. For $\rho > 0$ and $\zeta_{k+1},\dots,\zeta_n \in \mathbb{R}$, the group of such rotations is transitive on the set

$$\{y = y_1 e_1 + \dots + y_n e_n \in \mathbb{R}^n : y_1^2 + \dots + y_k^2 = \rho^2, y_{k+1} = \zeta_{k+1},\dots, y_n = \zeta_n\}.$$

(Here it is used that $k \geq 2$.) Therefore, the proof of Lemma 3 in [23] yields the assertion.

(b) Now let $k = 1$. Then we can assume that $e_1 = v_L$ and write

$$\begin{aligned} p_T(y) &= \sum_{1 \leq i_1 \leq \dots \leq i_r \leq n} t_{i_1 \dots i_r} y_{i_1} \cdots y_{i_r} \\ &= \sum_{j=0}^{r} y_1^j \sum_{2 \leq i_{j+1} \leq \dots \leq i_r \leq n} t_{1 \dots 1 i_{j+1} \dots i_r} y_{i_{j+1}} \cdots y_{i_r}. \end{aligned}$$

We define the tensor $T^{(r-j)} \in \mathbb{T}^{r-j}(L^\perp)$ by

$$T^{(r-j)} := \sum_{2\le i_{j+1}\le\cdots\le i_r\le n} t_{1\dots 1 i_{j+1}\dots i_r} e_{i_{j+1}} \cdots e_{i_r}$$

and obtain assertion (b). □

For a $\mathbb{T}^r$-valued Borel measure F on $\mathbb{S}^{n-1}$, we say that it *intertwines orthogonal transformations* if $F(\theta B) = (\theta F)(B)$ for all $B \in \mathcal{B}(\mathbb{S}^{n-1})$ and all orthogonal transformations $\theta \in \mathrm{O}(n)$. We say that F *intertwines rotations* if $F(\vartheta B) = (\vartheta F)(B)$ for all $B \in \mathcal{B}(\mathbb{S}^{n-1})$ and all rotations $\vartheta \in \mathrm{SO}(n)$. (Note that this terminology differs from that in [23].)

We recall Lemma 4 from [23].

LEMMA 2. *Let $n \in \mathbb{N}$, $r \in \mathbb{N}_0$, and let $F : \mathcal{B}(\mathbb{S}^{n-1}) \to \mathbb{T}^r$ be a $\mathbb{T}^r$-valued measure that intertwines orthogonal transformations. Then*

$$F(B) = \sum_{j=0}^{\lfloor r/2\rfloor} a_j Q^j \int_B u^{r-2j} \mathcal{H}^{n-1}(\mathrm{d}u) \tag{18}$$

for $B \in \mathcal{B}(\mathbb{S}^{n-1})$ with real constants a_j, $j = 0, \dots, \lfloor r/2\rfloor$.

In [23], this lemma was proved for $n \ge 2$. If $n = 1$, then $\mathbb{S}^0 = \{e, -e\}$ and $F(\{e\}) = a(e)e^r$, $F(\{-e\}) = a(-e)e^r$ with real constants $a(e)$, $a(-e)$. If $\theta \in \mathrm{O}(1)$ satisfies $\theta e = -e$, then $F(\theta\{e\}) = \theta F(\{e\})$ yields $a(-e) = (-1)^r a(e)$. With $a_0 = a(e)$ and $a_j = 0$ for $j > 0$, F can also be written in the form (18).

The following lemma concerns rotations only.

LEMMA 3. *Let $r \in \mathbb{N}_0$, and let $F : \mathcal{B}(\mathbb{S}^{n-1}) \to \mathbb{T}^r$ be a $\mathbb{T}^r$-valued measure that intertwines rotations.*

(a) *If $n \ge 3$, then*

$$F(B) = \sum_{j=0}^{\lfloor r/2\rfloor} a_j Q^j \int_B u^{r-2j} \mathcal{H}^{n-1}(\mathrm{d}u) \tag{19}$$

for $B \in \mathcal{B}(\mathbb{S}^{n-1})$ with real constants a_j, $j = 0, \dots, \lfloor r/2\rfloor$.

(b) *Let $n = 2$. Fix an orientation of $\mathbb{R}^2$, and for $u \in \mathbb{S}^1$, let $\overline{u}$ be the unit vector such that $(u, \overline{u})$ is a positively oriented orthonormal basis of $\mathbb{R}^2$. Then*

$$F(B) = \sum_{j=0}^{r} a_j \int_B \overline{u}^j u^{r-j} \mathcal{H}^1(\mathrm{d}u)$$

for $B \in \mathcal{B}(\mathbb{S}^1)$ with real constants a_j.

Proof. We modify the argumentation in the proof of [23, Lemma 4], replacing the group $\mathrm{O}(n)$ by $\mathrm{SO}(n)$. We fix a vector $u \in \mathbb{S}^{n-1}$ and denote by $B_{u,\rho}$ the spherical cap with center u and spherical radius $\rho \in (0, \pi/2)$. Let $T := F(B_{u,\rho})$. Then

$\vartheta T = T$ for all rotations $\vartheta \in \mathrm{SO}(n)$ fixing u. We choose the orthonormal basis $(e_1, \dots, e_n)$ such that $e_n = u$.

(a) If $n \geq 3$, then $\dim u^\perp \geq 2$. Therefore, the proof of [23, Lemma 4] goes through if we apply in $L = u^\perp$ part (a) of the present Lemma 1 (instead of [23, Lemma 3]).

(b) Now we assume that $n = 2$. (Note that $Q = u^2 + \overline{u}^2$ in this case, so that no factor Q^j is required.) We apply Lemma 1(b) with $L = u^\perp$ and $v_L = \overline{u}$. This gives

$$T = \sum_{j=0}^{r} \overline{u}^j \pi^*_{L^\perp} T^{(r-j)}$$

with tensors $T^{(r-j)} \in \mathbb{T}^{r-j}(\mathrm{lin}\{u\})$. Since every tensor in $\mathbb{T}^{r-j}(\mathrm{lin}\{u\})$ is of the form $b_j u^{r-j}$ with a real constant b_j (and since the tensor $T^{(r-j)}$ depends on u and ρ), we obtain that

$$F(B_{u,\rho}) = \sum_{j=0}^{r} b_j(u,\rho)\overline{u}^j u^{r-j}.$$

This holds for all $u \in \mathbb{S}^1$ and does not depend on the choice of the basis. Since F intertwines rotations, we have $\vartheta F(B_{u,\rho}) = F(\vartheta B_{u,\rho}) = F(B_{\vartheta u,\rho})$ for $\vartheta \in \mathrm{SO}(2)$. This can be written as

$$\sum_{j=0}^{r} b_j(u,\rho)(\vartheta\overline{u})^j(\vartheta u)^{r-j} = \sum_{j=0}^{r} b_j(\vartheta u,\rho)(\vartheta\overline{u})^j(\vartheta u)^{r-j}.$$

The tensors $(\vartheta\overline{u})^j(\vartheta u)^{r-j}$, $j = 0, \dots, r$, are linearly independent, and hence $b_j(u,\rho) =: b_j(\rho)$ does not depend on u.

For given $u \in \mathbb{S}^1$, we can choose $e_2 = u$ and then obtain, for $m \in \{0, \dots, r\}$,

$$\binom{r}{m} F(B_{u,\rho})(\underbrace{-e_1, \dots, -e_1}_{m}, \underbrace{e_2, \dots, e_2}_{r-m}) = b_m(\rho).$$

Now we have all the ingredients to finish the proof in the same way as [23, Lemma 4] was proved. □

6. Proof of Theorem 3

To prove Theorem 3, we assume that $\Gamma : \mathcal{P}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ is a mapping that has the following properties:

(a) $\Gamma(P, \cdot)$ is a $\mathbb{T}^p$-valued measure for each $P \in \mathcal{P}^n$,
(b) Γ is translation invariant and SO(n) covariant, and
(c) Γ is locally defined.

We reduce the proof of Theorem 3 to the classification of a simpler type of tensor-valued mappings. Let $k \in \{0, \dots, n-1\}$, and let $L \in G(n,k)$. Let $A \in \mathcal{B}_b(L)$ and $B \in \mathcal{B}(\mathbb{S}^{n-1})$. Let $P \subset L$ be a polytope with $A \subset P$. Then $A \times (B \cap L^\perp) \subset \mathrm{Nor}\, P$, and since Γ is locally defined, $\Gamma(P, A \times (B \cap L^\perp)) =: \varphi(A, B)$ does not depend on P. Since each coordinate of $\varphi(\cdot, B)$ with respect to some basis is a locally finite Borel measure that is invariant under translations of L into itself, it follows that $\varphi(A, B) = \mathcal{H}^k(A)\Delta_k(L, B)$ with a tensor $\Delta_k(L, B)$. This defines a mapping

$$\Delta_k : G(n, k) \times \mathcal{B}(\mathbb{S}^{n-1}) \to \mathbb{T}^p.$$

From the properties of Γ it follows that this mapping has the following properties:

(a′) $\Delta_k(L, \cdot)$ is a $\mathbb{T}^p$-valued measure for each $L \in G(n, k)$,
(b′) Δ_k is SO(n) covariant in the sense that

$$\Delta_k(\vartheta L, \vartheta B) = \vartheta \Delta_k(L, B) \quad \text{for } \vartheta \in \mathrm{SO}(n), \tag{20}$$

(c′) $\Delta_k(L, B) = \Delta_k(L, B \cap L^\perp)$ for $L \in G(n, k)$ and $B \in \mathcal{B}(\mathbb{S}^{n-1})$.

Now let $P \in \mathcal{P}^n$, $A \in \mathcal{B}(\mathbb{R}^n)$, and $B \in \mathcal{B}(\mathbb{S}^{n-1})$. Since $\Gamma(P, \cdot)$ is concentrated on Nor P (see [9, Lemma 3.3], whose proof does not use O(n) covariance) and

$$(A \times B) \cap \operatorname{Nor} P = \bigcup_{k=0}^{n-1} \bigcup_{F \in \mathcal{F}_k(P)} (A \cap \operatorname{relint} F) \times (B \cap \nu(P, F))$$

is a disjoint union, we have

$$\begin{aligned}\Gamma(P, A \times B) &= \Gamma(P, (A \times B) \cap \operatorname{Nor} P) \\ &= \sum_{k=0}^{n-1} \sum_{F \in \mathcal{F}_k(P)} \Gamma(P, (A \cap \operatorname{relint} F) \times (B \cap \nu(P, F))) \\ &= \sum_{k=0}^{n-1} \sum_{F \in \mathcal{F}_k(P)} \mathcal{H}^k(A \cap \operatorname{relint} F)\Delta_k(L(F), B \cap \nu(P, F)). \end{aligned} \tag{21}$$

This is the reason why we want to determine $\Delta_k(L, B)$.

To classify these mappings, let $k \in \{0, \dots, n-1\}$, $L \in G(n, k)$, and $B \in \mathcal{B}(\mathbb{S}^{n-1})$.

Case 1: $k = 0$. Then $L^\perp = \mathbb{R}^n$, and $\Delta_0(\{0\}, \cdot) : \mathcal{B}(\mathbb{S}^{n-1}) \to \mathbb{T}^p$ is a $\mathbb{T}^p$-valued measure that, by (20), intertwines rotations.

Subcase 1a: $n \geq 3$. Lemma 3(a) gives

$$\Delta_0(\{0\}, B) = \sum_{j=0}^{\lfloor p/2 \rfloor} a_j Q^j \int_B u^{p-2j} \mathcal{H}^{n-1}(\mathrm{d}u) \tag{22}$$

with real constants a_j.

Subcase 1b: $n = 2$. Lemma 3(b) gives

$$\Delta_0(\{0\}, B) = \sum_{j=0}^{p} a_j \int_B \overline{u}^j u^{p-j} \mathcal{H}^1(\mathrm{d}u) \tag{23}$$

with real constants a_j.

Case 2: $k \geq 2$. If $\vartheta \in \mathrm{SO}(n)$ fixes $L^\perp$ pointwise, then $\vartheta L = L$, and it follows from (20) (together with (c′)) that $T := \Delta_k(L, B)$ satisfies $\vartheta T = T$. Therefore, we infer from Lemma 1(a) that

$$\Delta_k(L, B) = \sum_{j=0}^{\lfloor p/2 \rfloor} Q_L^j \pi_{L^\perp}^* T^{(p-2j)}(L, B) \tag{24}$$

with tensors $T^{(p-2j)}(L, B) \in \mathbb{T}^{p-2j}(L^\perp)$, $j = 0, \ldots, \lfloor p/2 \rfloor$.

Let $y \in L \cap \mathbb{S}^{n-1}$ and $x_1, \ldots, x_p \in L^\perp$. For $q \in \{0, \ldots, \lfloor p/2 \rfloor\}$, we apply both sides of (24) to the p-tuple $(y, \ldots, y, x_1, \ldots, x_{p-2q})$ and obtain

$$\begin{aligned} &\Delta_k(L, B)(y, \ldots, y, x_1, \ldots, x_{p-2q}) \\ &\quad = \binom{p}{2q}^{-1} T^{(p-2q)}(L, B)(x_1, \ldots, x_{p-2q}). \end{aligned} \tag{25}$$

Let $\vartheta \in \mathrm{SO}(n)$ and $B \in \mathcal{B}(\mathbb{S}_{L^\perp})$. Let $y \in L \cap \mathbb{S}^{n-1}$, $x_1, \ldots, x_{p-2j} \in L^\perp$, and $j \in \{0, \ldots, \lfloor \frac{p}{2} \rfloor\}$. Then, using (25), (20), the definition of the operation of ϑ on tensors, and then again (25), we get

$$\begin{aligned} &T^{(p-2j)}(\vartheta L, \vartheta B)(\vartheta x_1, \ldots, \vartheta x_{p-2j}) \\ &\quad = \binom{p}{2j} \Delta_k(\vartheta L, \vartheta B)(\vartheta y, \ldots, \vartheta y, \vartheta x_1, \ldots, \vartheta x_{p-2j}) \\ &\quad = \binom{p}{2j} [\vartheta \Delta_k(L, B)](\vartheta y, \ldots, \vartheta y, \vartheta x_1, \ldots, \vartheta x_{p-2j}) \\ &\quad = \binom{p}{2j} \Delta_k(L, B)(y, \ldots, y, x_1, \ldots, x_{p-2j}) \\ &\quad = T^{(p-2j)}(L, B)(x_1, \ldots, x_{p-2j}). \end{aligned} \tag{26}$$

Let $i_L : L \to \mathbb{R}^n$ be the inclusion map. Later, we have to observe that

$$i_{\vartheta L^\perp}^* \vartheta \pi_{L^\perp}^* Q_{(L^\perp)} = i_{\vartheta L^\perp}^* \vartheta Q_{L^\perp} = i_{\vartheta L^\perp}^* Q_{\vartheta L^\perp} = Q_{(\vartheta L^\perp)}.$$

Since $\vartheta x_i \in \vartheta L^\perp$ for $i \in \{1, \ldots, p-2j\}$, we have

$$\begin{aligned} &[i_{\vartheta L^\perp}^* \vartheta \pi_{L^\perp}^* T^{(p-2j)}(L, B)](\vartheta x_1, \ldots, \vartheta x_{p-2j}) \\ &\quad = [\vartheta \pi_{L^\perp}^* T^{(p-2j)}(L, B)](\vartheta x_1, \ldots, \vartheta x_{p-2j}) \\ &\quad = [\pi_{L^\perp}^* T^{(p-2j)}(L, B)](x_1, \ldots, x_{p-2j}) \\ &\quad = T^{(p-2j)}(L, B)(x_1, \ldots, x_{p-2j}). \end{aligned}$$

Thus, we finally get

$$T^{(p-2j)}(\vartheta L, \vartheta B) = i_{\vartheta L^\perp}^* \vartheta \pi_{L^\perp}^* T^{(p-2j)}(L, B), \tag{27}$$

where both sides are considered as tensors in $\mathbb{T}^{p-2j}(\vartheta L^\perp)$. (Of course, the effect of $i_{\vartheta L^\perp}^*$ and $\pi_{L^\perp}^*$ on the right side of (27) is trivial if the appropriate domain is considered in each case.)

Let $\theta \in \mathrm{O}(L^\perp)$ (the orthogonal group of $L^\perp$). We can choose a rotation $\vartheta \in \mathrm{SO}(n)$ such that the restriction of ϑ to $L^\perp$ coincides with θ and $\vartheta L = L$. Then (26) (or (27)) implies that

$$T^{(p-2j)}(L, \theta B) = \theta T^{(p-2j)}(L, B), \tag{28}$$

were again both sides are considered as tensors in $\mathbb{T}^{p-2j}(L^\perp)$.

Because of (28), it follows from Lemma 2 (applied in $L^\perp$) that

$$T^{(p-2j)}(L, B) = \sum_{i=0}^{\lfloor p/2\rfloor - j} \alpha_{ipj}(L) Q^i_{(L^\perp)} \int_B u^{p-2j-2i} \mathcal{H}^{n-k-1}(\mathrm{d}u) \tag{29}$$

with real constants $\alpha_{ipj}(L)$ (recall that $B \in \mathcal{B}(\mathbb{S}_{L^\perp})$).

To show that the coefficients $\alpha_{ipj}(L)$ in (29) are independent of L, we fix a k-dimensional linear subspace L_0 and put $\alpha_{ipj}(L_0) =: \alpha_{kipj}$. For a given k-dimensional subspace L, there is a rotation $\vartheta \in \mathrm{SO}(n)$ with $L = \vartheta L_0$. From (27) and (29) we obtain, for $B \in \mathcal{B}(\mathbb{S}_{L^\perp})$ and $B_0 = \vartheta^{-1} B \in \mathcal{B}(\mathbb{S}_{L_0^\perp})$,

$$\begin{aligned}
T^{(p-2j)}(L, B) &= T^{(p-2j)}(\vartheta L_0, \vartheta B_0) = i^*_{\vartheta L_0^\perp} \vartheta \pi^*_{L_0^\perp} T^{(p-2j)}(L_0, B_0) \\
&= i^*_{\vartheta L_0^\perp} \vartheta \sum_{i=0}^{\lfloor p/2\rfloor - j} \alpha_{ipj}(L_0) Q^i_{L_0^\perp} \int_{B_0} u^{p-2j-2i} \mathcal{H}^{n-k-1}(\mathrm{d}u) \\
&= \sum_{i=0}^{\lfloor p/2\rfloor - j} \alpha_{kipj} Q^i_{(\vartheta L_0^\perp)} \int_{\vartheta B_0} u^{p-2j-2i} \mathcal{H}^{n-k-1}(\mathrm{d}u) \\
&= \sum_{i=0}^{\lfloor p/2\rfloor - j} \alpha_{kipj} Q^i_{(L^\perp)} \int_{B} u^{p-2j-2i} \mathcal{H}^{n-k-1}(\mathrm{d}u).
\end{aligned} \tag{30}$$

Relations (24) and (30) now yield

$$\Delta_k(L, B) = \sum_{j=0}^{\lfloor p/2\rfloor} Q^j_L \sum_{i=0}^{\lfloor p/2\rfloor - j} \alpha_{kipj} Q^i_{L^\perp} \int_B u^{p-2j-2i} \mathcal{H}^{n-k-1}(\mathrm{d}u).$$

Inserting $Q_{L^\perp} = Q - Q_L$, expanding, and regrouping, we see that

$$\Delta_k(L, B) = \sum_{a=0}^{\lfloor p/2\rfloor} \sum_{b=a}^{\lfloor p/2\rfloor} c_{pkab} Q^a Q_L^{b-a} \int_B u^{p-2b} \mathcal{H}^{n-k-1}(\mathrm{d}u) \tag{31}$$

with real constants c_{pkab}.

Case 3: $k = 1$. Again, we assume that $B \in \mathcal{B}(\mathbb{S}_{L^\perp})$. Instead of (24), we can only infer from Lemma 1(b) that, after choosing a unit vector v_L spanning L, we have

$$\Delta_1(L, B) = \sum_{j=0}^{p} v_L^j \pi^*_{L^\perp} T^{(p-j)}(L, B) \tag{32}$$

with tensors $T^{(p-j)}(L,B)\in\mathbb{T}^{p-j}(L^\perp)$, $j=0,\dots,p$. Let $x_1,\dots,x_p\in L^\perp$. For $q\in\{0,\dots,p\}$, we apply both sides of (32) to the p-tuple

$$(\underbrace{v_L,\dots,v_L}_{q},x_1,\dots,x_{p-q}) \tag{33}$$

and obtain

$$\begin{aligned}\Delta_1(L,B)&(v_L,\dots,v_L,x_1,\dots,x_{p-q})\\&=\binom{p}{q}^{-1}T^{(p-q)}(L,B)(x_1,\dots,x_{p-q}).\end{aligned} \tag{34}$$

Again, $T^{(p-q)}(L,B)$ is a $\mathbb{T}^{p-q}(L^\perp)$-valued measure on $\mathbb{S}_{L^\perp}$. It intertwines rotations of $L^\perp$.

Subcase 3a: $n\geq 4$. Then $\dim L^\perp\geq 3$. Hence, we can apply Lemma 3(a) in $L^\perp$ and obtain that

$$T^{(p-q)}(L,B)=\sum_{i=0}^{\lfloor\frac{p-q}{2}\rfloor}\beta_{pqi}(L)Q^i_{(L^\perp)}\int_B u^{p-q-2i}\mathcal{H}^{n-2}(\mathrm{d}u). \tag{35}$$

In the same way as (30) was deduced, we conclude that

$$T^{(p-q)}(L,B)=\sum_{i=0}^{\lfloor\frac{p-q}{2}\rfloor}\beta_{pqi}Q^i_{(L^\perp)}\int_B u^{p-q-2i}\mathcal{H}^{n-2}(\mathrm{d}u) \tag{36}$$

with constants β_{pqi}. Relations (32) and (36) yield

$$\Delta_1(L,B)=\sum_{j=0}^{p}v_L^j\sum_{i=0}^{\lfloor\frac{p-j}{2}\rfloor}\beta_{pji}Q^i_{L^\perp}\int_B u^{p-j-2i}\mathcal{H}^{n-2}(\mathrm{d}u). \tag{37}$$

Since $v_L^2=Q_L$, we distinguish whether j is even or odd and write (37) as

$$\Delta_1(L,B)=\Delta_1^{(0)}(L,B)+\Delta_1^{(1)}(L,B)$$

with

$$\begin{aligned}\Delta_1^{(0)}(L,B)&=\sum_{a=0}^{\lfloor p/2\rfloor}Q_L^a\sum_{i=0}^{\lfloor p/2\rfloor-a}\beta_{p(2a)i}Q^i_{L^\perp}\int_B u^{p-2a-2i}\mathcal{H}^{n-2}(\mathrm{d}u),\\ \Delta_1^{(1)}(L,B)&=\sum_{b=0}^{\lfloor\frac{p-1}{2}\rfloor}Q_L^b v_L\sum_{i=0}^{\lfloor\frac{p-1}{2}\rfloor-b}\beta_{p(2b+1)i}Q^i_{L^\perp}\\&\quad\times\int_B u^{p-2b-1-2i}\mathcal{H}^{n-2}(\mathrm{d}u).\end{aligned} \tag{38}$$

We can choose a rotation $\vartheta\in\mathrm{SO}(n)$ such that $\vartheta v_L=-v_L$ and that the restriction of ϑ to $L^\perp$ is a reflection of $L^\perp$ into itself. Moreover, we specialize B to B' such that $\vartheta B'=B'$. Then the last equation yields $\vartheta\Delta_1^{(1)}(L,B')=-\Delta_1^{(1)}(L,B')$, whereas the rotation covariance of Δ_1 and of $\Delta_1^{(0)}$ yields $\vartheta\Delta_1^{(1)}(L,B')=$

$\Delta_1^{(1)}(L,B')$. Thus, we obtain $\Delta_1^{(1)}(L,B')=0$ for all $B'\in\mathcal{B}(\mathbb{S}_{L^\perp})$ with $\vartheta B'=B'$. Inserting (33), with various q, into (38) for B' (for which it is zero), we deduce that

$$\sum_{i=0}^{\lfloor\frac{p-1}{2}\rfloor-b}\beta_{p(2b+1)i}\,Q_{L^\perp}^i\int_{B'}u^{p-2b-1-2i}\mathcal{H}^{n-2}(\mathrm{d}u)=0$$

for $b=0,\dots,\lfloor\frac{p-1}{2}\rfloor$. Here B' can be any Borel set in $\mathbb{S}_{L^\perp}$ which is invariant under some reflection of $\mathbb{S}_{L^\perp}$. Therefore, we can deduce that

$$\sum_{i=0}^{\lfloor\frac{p-1}{2}\rfloor-b}\beta_{p(2b+1)i}\,Q_{L^\perp}^i u^{p-2b-1-2i}=0$$

for all $u\in\mathbb{S}_{L^\perp}$. As in the proof of Proposition 1, we conclude that all coefficients $\beta_{p(2b+1)i}$ are zero. It follows that $\Delta_1^{(1)}(L,B)=0$ for all B, and therefore $\Delta_1(L,B)=\Delta_1^{(0)}(L,B)$. Since $Q_{L^\perp}=Q-Q_L$, we obtain

$$\Delta_1(L,B)=\sum_{a=0}^{\lfloor p/2\rfloor}\sum_{b=a}^{\lfloor p/2\rfloor}c_{p1ab}\,Q^aQ_L^{b-a}\int_B u^{p-2b}\mathcal{H}^{n-2}(\mathrm{d}u) \tag{39}$$

with real constants c_{p1ab}.

Subcase 3b: $n=3$. The choice of the unit vector $v_L\in L$ in Case 3 determines (together with the given orientation of $\mathbb{R}^3$) an orientation of $L^\perp$. For a given unit vector $u\in L^\perp$, let $\overline{u}\in L^\perp$ be the unique unit vector such that $(v_L,u,\overline{u})$ is an orthonormal basis of $\mathbb{R}^3$. In other words, $\overline{u}=v_L\times u$, where $\times$ means the vector product.

Lemma 3(b), applied in $L^\perp$, yields

$$T^{(p-q)}(L,B)=\sum_{i=0}^{p-q}a_i(L)\int_B\overline{u}^i u^{p-q-i}\mathcal{H}^1(\mathrm{d}u)$$

with constants $a_i(L)$. Arguments as used previously in Case 2 show that $a_i(L)=a_i$ is independent of L. With this and (32), we get

$$\Delta_1(L,B)=\sum_{q=0}^{p}v_L^q\sum_{i=0}^{p-q}a_i\int_B(v_L\times u)^i u^{p-q-i}\mathcal{H}^1(\mathrm{d}u). \tag{40}$$

We write

$$\Delta_1=\Delta_1^{(00)}+\Delta_1^{(10)}+\Delta_1^{(01)}+\Delta_1^{(11)},$$

where $\Delta_1^{(10)}=\Delta_1^{(01)}=0$ if $p=0$, $\Delta_1^{(11)}=0$ if $p\le 1$, and otherwise

$$\Delta_1^{(\alpha\beta)}(L,B):=\sum_{\substack{q=0\\ q\equiv\alpha\bmod 2}}^{p}v_L^q\sum_{\substack{i=0\\ i\equiv\beta\bmod 2}}^{p-q}a_i\int_B(v_L\times u)^i u^{p-q-i}\mathcal{H}^1(\mathrm{d}u)$$

for $\alpha, \beta \in \{0, 1\}$. Using that $v_L^2 = Q_L$, $(v_L \times u)^2 = Q_{L^\perp} - u^2$, and $Q = Q_L + Q_{L^\perp}$, we get

$$\Delta_1^{(00)}(L, B) = \sum_{j=0}^{\lfloor p/2 \rfloor} Q_L^j \sum_{m=0}^{\lfloor p/2 \rfloor - j} b_m \int_B (Q - Q_L - u^2)^m u^{p-2j-2m} \mathcal{H}^1(\mathrm{d}u).$$

After expanding and regrouping, this can be written as

$$\Delta_1^{(00)}(L, B) = \sum_{a=0}^{\lfloor p/2 \rfloor} \sum_{b=a}^{\lfloor p/2 \rfloor} a_{pab} Q^a Q_L^{b-a} \int_B u^{p-2b} \mathcal{H}^1(\mathrm{d}u). \tag{41}$$

In the same way, we obtain the representations

$$\Delta_1^{(10)}(L, B) = \sum_{a=0}^{\lfloor \frac{p-1}{2} \rfloor} \sum_{b=a}^{\lfloor \frac{p-1}{2} \rfloor} b_{pab} Q^a Q_L^{b-a} v_L \int_B u^{p-2b-1} \mathcal{H}^1(\mathrm{d}u) \quad \text{if } p \geq 1,$$

$$\Delta_1^{(01)}(L, B) = \sum_{a=0}^{\lfloor \frac{p-1}{2} \rfloor} \sum_{b=a}^{\lfloor \frac{p-1}{2} \rfloor} c_{pab} Q^a Q_L^{b-a} \int_B (v_L \times u) u^{p-2b-1} \mathcal{H}^1(\mathrm{d}u) \quad \text{if } p \geq 1,$$

$$\Delta_1^{(11)}(L, B) = \sum_{a=0}^{\lfloor \frac{p-2}{2} \rfloor} \sum_{b=a}^{\lfloor \frac{p-2}{2} \rfloor} d_{pab} Q^a Q_L^{b-a} v_L \int_B (v_L \times u) u^{p-2b-2} \mathcal{H}^1(\mathrm{d}u) \quad \text{if } p \geq 2.$$

Arguing as in Subcase 3a, we can show that $\Delta_1^{(10)}(L, B) + \Delta_1^{(01)}(L, B) = 0$. It follows that

$$\begin{aligned} \Delta_1(L, B) = & \sum_{a=0}^{\lfloor \frac{p}{2} \rfloor} \sum_{b=a}^{\lfloor \frac{p}{2} \rfloor} a_{pab} Q^a Q_L^{b-a} \int_B u^{p-2b} \mathcal{H}^1(\mathrm{d}u) \\ & + \sum_{a=0}^{\lfloor \frac{p}{2} \rfloor - 1} \sum_{b=a}^{\lfloor \frac{p}{2} \rfloor - 1} d_{pab} Q^a Q_L^{b-a} v_L \int_B (v_L \times u) u^{p-2b-2} \mathcal{H}^1(\mathrm{d}u). \end{aligned} \tag{42}$$

Subcase 3c: $n = 2$. By (32),

$$\Delta_1(L, B) = \sum_{q=0}^{p} v_L^q \pi_{L^\perp}^* T^{(p-q)}(L, B) \tag{43}$$

with $T^{(p-q)}(L, B) \in \mathbb{T}^{p-q}(L^\perp)$. We can assume that $v_L = \overline{u_L}$, where u_L is one of the two unit normal vectors of L. Then $B \subset \{u_L, -u_L\}$, and

$$T^{(p-q)}(L, B) = c_{p-q}(L, B) u_L^{p-q}.$$

As before, we have

$$T^{(p-q)}(\vartheta L, \vartheta B) = \vartheta T^{(p-q)}(L, B) \quad \text{for } \vartheta \in \mathrm{SO}(2). \tag{44}$$

First, suppose that $B = \{u_L\}$. Using (44) for the rotation ϑ with $\vartheta u_L = -u_L$, we see that $c_{p-q}(L, \{-u_L\}) = c_{p-q}(L, \{u_L\}) =: c_{p-q}(L)$. Thus, for this B, we can write (43) in the form

$$\Delta_1(L,B)=\sum_{q=0}^{p} c_{p-q}(L)\int_B \overline{u}^q u^{p-q}\mathcal{H}^0(\mathrm{d}u).$$

This also holds for $B=\{-u_L\}$, and since $\Delta_1(L,\{u_L,-u_L\})=\Delta_1(L,\{u_L\})+\Delta_1(L,\{-u_L\})$, it holds for arbitrary $B\in\mathcal{B}(\mathbb{S}_{L^\perp})$. Now we can deduce as in Case 2 that $c_{p-q}(L):=c_{p-q}$ is also independent of L. Since $\overline{u}^2+u^2=Q$, we obtain

$$\begin{aligned}\Delta_1(L,B)=&\sum_{a=0}^{\lfloor p/2\rfloor}\alpha_a Q^a\int_B u^{p-2a}\mathcal{H}^0(\mathrm{d}u)\\&+\sum_{a=0}^{\lfloor\frac{p-1}{2}\rfloor}\beta_a Q^a\int_B \overline{u}u^{p-2a-1}\mathcal{H}^0(\mathrm{d}u).\end{aligned}\tag{45}$$

The representations (22), (23), (31), (39), (42), (45) obtained for Δ_k now allow us to evaluate (21).

Let $P\in\mathcal{P}^n$, $A\in\mathcal{B}(\mathbb{R}^n)$, and $B\in\mathcal{B}(\Sigma^n)$. We first consider the case $n=3$. Using (22) for $k=0$, (42) for $k=1$, and (31) for $k=2$, we can write (21) in the form

$$\begin{aligned}&\Gamma(P,A\times B)\\&\quad=\sum_{F\in\mathcal{F}_0(P)}\mathcal{H}^0(A\cap\operatorname{relint}F)\\&\qquad\times\sum_{j=0}^{\lfloor\frac{p}{2}\rfloor}a_jQ^j\int_{B\cap\nu(P,F)}u^{p-2j}\mathcal{H}^2(\mathrm{d}u)\\&\qquad+\sum_{F\in\mathcal{F}_1(P)}\mathcal{H}^1(A\cap\operatorname{relint}F)\\&\qquad\times\left\{\sum_{a=0}^{\lfloor\frac{p}{2}\rfloor}\sum_{b=a}^{\lfloor\frac{p}{2}\rfloor}a_{pab}Q^aQ_{L(F)}^{b-a}\int_{B\cap\nu(P,F)}u^{p-2b}\mathcal{H}^1(\mathrm{d}u)\right.\\&\qquad+\sum_{a=0}^{\lfloor\frac{p}{2}\rfloor-1}\sum_{b=a}^{\lfloor\frac{p}{2}\rfloor-1}d_{pab}Q^aQ_{L(F)}^{b-a}v_{L(F)}\\&\qquad\left.\times\int_{B\cap\nu(P,F)}(v_{L(F)}\times u)u^{p-2b-2}\mathcal{H}^1(\mathrm{d}u)\right\}\\&\qquad+\sum_{F\in\mathcal{F}_2(P)}\mathcal{H}^2(A\cap\operatorname{relint}F)\sum_{a=0}^{\lfloor\frac{p}{2}\rfloor}\sum_{b=a}^{\lfloor\frac{p}{2}\rfloor}c_{p2ab}Q^aQ_{L(F)}^{b-a}\\&\qquad\times\int_{B\cap\nu(P,F)}u^{p-2b}\mathcal{H}^0(\mathrm{d}u).\end{aligned}$$

From (1) and (3) we have

$$\sum_{f\in\mathcal{F}_0(P)} \mathcal{H}^0(A\cap \operatorname{relint} F)Q^j \int_{B\cap\nu(P,F)} u^{p-2j}\mathcal{H}^2(\mathrm{d}u)$$
$$= \mathrm{const}\cdot Q^j\phi_0^{0,p-2j,0}(P, A\times B),$$
$$\sum_{F\in\mathcal{F}_1(P)} \mathcal{H}^1(A\cap \operatorname{relint} F)Q^a Q_{L(F)}^{b-a} \int_{B\cap\nu(P,F)} u^{p-2b}\mathcal{H}^1(\mathrm{d}u)$$
$$= \mathrm{const}\cdot Q^a\phi_1^{0,p-2b,b-a}(P, A\times B),$$
$$\sum_{F\in\mathcal{F}_1(P)} \mathcal{H}^1(A\cap \operatorname{relint} F)Q^a Q_{L(F)}^{b-a} v_{L(F)} \int_{B\cap\nu(P,F)} (v_{L(F)}\times u)u^{p-2b-2}\mathcal{H}^1(\mathrm{d}u)$$
$$= \mathrm{const}\cdot Q^a\widetilde{\phi}^{0,p-2b-2,b-a}(P, A\times B),$$
$$\sum_{F\in\mathcal{F}_2(P)} \mathcal{H}^2(A\cap \operatorname{relint} F)Q^a Q_{L(F)}^{b-a} \int_{B\cap\nu(P,F)} u^{p-2b}\mathcal{H}^0(\mathrm{d}u)$$
$$= \mathrm{const}\cdot Q^a\phi_2^{0,p-2b,b-a}(P, A\times B).$$

This shows that $\Gamma(P, A\times B)$ is a linear combination of expressions $Q^m\phi_k^{0,s,j}(P, A\times B)$ and $Q^m\widetilde{\phi}^{0,s,j}(P, A\times B)$ with coefficients independent of P, A, B and with indices as specified in Theorem 3. Since $\Gamma(P,\cdot)$, $Q^m\phi_k^{0,s,j}(P,\cdot)$, and $Q^m\widetilde{\phi}^{0,s,j}(P,\cdot)$ are measures on $\mathcal{B}(\Sigma^n)$, the relations still hold if $A\times B$ is replaced by a general $\eta\in\mathcal{B}(\Sigma^n)$. This proves Theorem 3 in the case $n=3$.

The proof for $n\geq 4$ is analogous using (22) for $k=0$, (39) for $k=1$, and (31) for $k\geq 2$. Also, the proof for $n=2$ is analogous, where (23) is used for $k=0$ and (45) for $k=1$. This finishes the proof of Theorem 3.

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