Lefschetz Fibrations on Knot Surgery 4-Manifolds Via Stallings Twist

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ABSTRACT. In this paper, we construct a family of simply connected minimal symplectic 4-manifolds that admit arbitrarily many nonisomorphic Lefschetz fibration structures with the same genus fiber. We obtain these families by performing knot surgery on an elliptic surface $E(2)$ using connected sums of n copies of fibered knots, which in turn are obtained by Stallings twist from the square knot. Thus, all of these 4-manifolds are homotopy $E(2)$ surfaces. We show that they admit 2^n mutually nonisomorphic Lefschetz fibration structures of fiber genus $(4n + 1)$ by comparing their monodromy groups that are induced from the corresponding monodromy factorizations.

1. Introduction

Since it was known that any closed symplectic 4-manifold admits a Lefschetz pencil [Don99] and that a Lefschetz fibration structure can be obtained from a Lefschetz pencil by blowing-up the base loci, the study of Lefschetz fibrations has become an important research theme for understanding symplectic 4-manifolds topologically. In fact, Lefschetz pencils and Lefschetz fibrations have long been studied extensively by algebraic geometers and topologists in the complex category, and these notions can be extended to the symplectic category. It is also well known that an isomorphism class of Lefschetz fibrations over S^2 is completely characterized by *monodromy factorization*, an ordered sequence of right-handed Dehn twists whose product becomes the identity in the surface mapping class group corresponding to the generic fiber, up to Hurwitz equivalence and global conjugation equivalence. Note that the Hurwitz equivalence problem of monodromy factorizations provides a very interesting but challenging question in topology. For example, one aim for researchers in this field is to answer the following questions:

> Is the Hurwitz problem for mapping class group factorizations decidable? Do interesting criteria exist that can be used to conclude that two given factorizations are equivalent, or inequivalent, up to Hurwitz moves and global conjugation? [Aur06]

On the other hand, since the introduction of gauge theory, in particular Seiberg–Witten theory, topologists and geometers working on 4-manifolds have developed various techniques, and many fruitful and remarkable results have been obtained regarding the topology of 4-manifolds over the last 30 years. Among these, a knot surgery technique introduced by R. Fintushel and R. Stern turned

out to be one of most effective techniques for modifying smooth structures without changing the topological type of a given 4-manifold. Fintushel–Stern's knot surgery 4-manifold X_K is defined as follows [FS98]. Suppose that X is a simply connected smooth 4-manifold containing an embedded torus T of square 0 and $\pi_1(X \setminus T) = 1$. Then, for any knot $K \subset S^3$, we can construct a new 4-manifold, called *a knot surgery* 4-*manifold*, given by

$$X_K = X \natural_{T=T_m} S^1 \times M_K = [X \setminus (T \times D^2)] \cup [S^1 \times (S^3 \setminus N(K))].$$

This is constructed by taking a fiber sum along a torus T in X and $T_m = S^1 \times m$ in $S^1 \times M_K$, with the requirement that in the second expression the two pieces are glued together in such a way that the homology class $[pt \times \partial D^2]$ is identified with $[pt \times l]$, where M_K is the 3-manifold obtained by performing a 0-framed surgery along K, and m and l are the meridian and longitude of K, respectively. Then, Fintushel and Stern proved that under a mild condition on X and T, the knot surgery 4-manifold X_K is homeomorphic, but not diffeomorphic, to a given X. Furthermore, if X is a simply connected elliptic surface $E(2)$, T is a generic elliptic fiber, and K is a fibered knot in S^3, then it is known that the knot surgery 4-manifold $E(2)_K$ admits not only a symplectic structure, but also a genus $2g(K) + 1$ Lefschetz fibration structure [FS04]. Moreover, $E(2)_K$ is minimal [Sti99; Ush06].

In this paper, we continue to investigate inequivalent Lefschetz fibration structures on the knot surgery 4-manifold $E(2)_K$, and we answer the following question proposed by Smith [Smi98]:

> Does the diffeomorphism type of a smooth 4-manifold determine the equivalence class of a Lefschetz fibration by curves of some given genus?

Regarding this question, Smith first showed that $(T^2 \times \Sigma_2) \natural 9\overline{\mathbb{CP}}^2$ admits two nonisomorphic genus 9 Lefschetz fibrations by using the divisibility of the second integral homology class of the fiber [Smi98]. We have also studied Lefschetz fibration structures on $E(2)_K$ using several families of fibered knots K, such as 2-bridge knots and Kanenobu knots, and have obtained some fruitful results. For example, we have constructed a family of knot surgery 4-manifolds that admit two nonisomorphic Lefschetz fibration structures [PY09; PY11].

In this paper, we show that for each integer $n > 0$, some instances of $E(2)_K$ admit (at least) 2^n nonisomorphic Lefschetz fibration structures, which is a significant extension of our previous result mentioned. To find such examples, we first perform a knot surgery on $E(2)$ using a connected sum of n copies of genus 2 fibered knots, which are obtained by Stallings twist from the square knot $3_1 \natural 3_1^*$. Then, we consider the corresponding monodromy factorizations and monodromy groups. Because two Hurwitz equivalent monodromy factorizations yield the same monodromy group, we prove that these monodromy groups (and therefore the corresponding monodromy factorizations) are mutually distinct by using a graph method developed by Humphries [Hum79], which is a key step in the proof. We finally conclude that the corresponding Lefschetz fibration structures are mutually nonisomorphic to each other. The main result of this paper is the following.
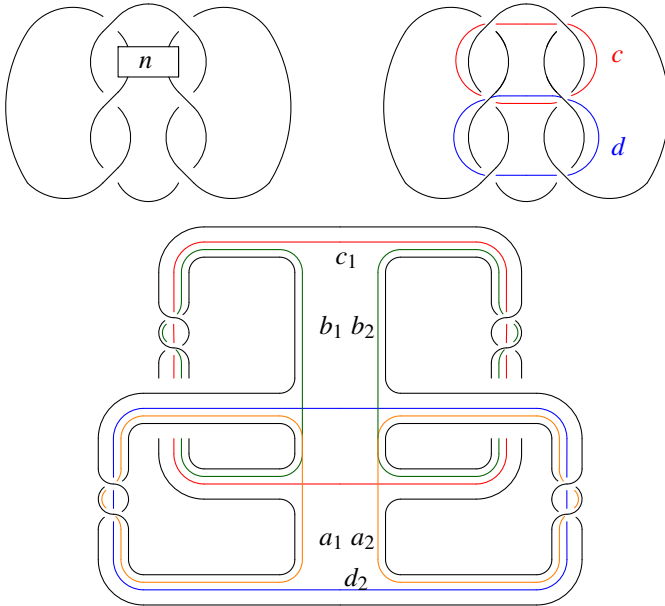
Figure 1 Square knot $K_0 = 3_1 \sharp 3_1^*$ and Stallings twist knot K_n

THEOREM 1.1. *For each integer $n > 0$ and $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, a knot surgery 4-manifold*

$$E(2)_{K_{m_1} \sharp K_{m_2} \sharp \cdots \sharp K_{m_n}}$$

admits (at least) 2^n nonisomorphic genus $(4n + 1)$ Lefschetz fibrations over S^2. Here, K_{m_i} $(1 \leq i \leq n)$ denotes a fibered knot obtained by performing $|m_i|$ left/right handed full twists on the square knot, as in Figure 1.

REMARK 1.2. Recently, Baykur [Bay14] obtained a similar result for nonminimal cases, that is, he proved that for any closed symplectic 4-manifold X that is not a rational or ruled surface and any integer $n > 0$, there are n nonisomorphic Lefschetz pencils of the same genus on a blowup of X.

The remainder of this paper is organized as follows. We first review generalities such as some definitions and basic facts regarding Lefschetz fibrations and Humphries' graph method. In Section 3, we explain how to construct a family of knot surgery 4-manifolds, and then we show that each $E(2)_{K_n}$ admits (at least) two distinct corresponding monodromy groups (Proposition 3.2). In addition, we prove by induction that $E(2)_{K_{m_1} \sharp K_{m_2} \sharp \cdots \sharp K_{m_n}}$ admits (at least) 2^n mutually distinct corresponding monodromy groups (Theorem 3.3).

2. Preliminaries

In this section, we briefly review some basic facts about Lefschetz fibrations, monodromy factorizations, and Humphries' graph method on the mapping class group of surfaces.

2.1. *Lefschetz Fibration*

DEFINITION 2.1. Let X be a compact, oriented smooth 4-manifold. A Lefschetz fibration is a proper smooth map $\pi : X \to B$, where B is a compact connected oriented surface and $\pi^{-1}(\partial B) = \partial X$, such that

(1) the set of critical points $C = \{p_1, p_2, \ldots, p_n\}$ of π is nonempty and lies in int(X), and π is injective on C;
(2) for each p_i and $b_i := \pi(p_i)$, there are local complex coordinate charts agreeing with the orientations of X and B such that π can be expressed as $\pi(z_1, z_2) = z_1^2 + z_2^2$.

Because each singular point in a Lefschetz fibration is related to a right-handed Dehn twist, it follows that if X is a Lefschetz fibration over S^2 with a generic fiber F of genus g, then it gives a sequence of right-handed Dehn twists whose product becomes the identity element in the mapping class group $\mathcal{M}_g$ of F. This ordered sequence of right-handed Dehn twists is called a *monodromy factorization* of the Lefschetz fibration. Note that a monodromy factorization is well defined up to Hurwitz equivalence and global conjugation equivalence [Kas80; Mat96; GS99].

DEFINITION 2.2. Two monodromy factorizations W_1 and W_2 are called *Hurwitz equivalent* if W_1 can be changed to W_2 in finitely many steps using the following two operations:

(1) *Hurwitz move:* $t_{c_n} \cdot \cdots \cdot t_{c_{i+1}} \cdot t_{c_i} \cdot \cdots \cdot t_{c_1} \sim t_{c_n} \cdot \cdots \cdot t_{c_{i+1}}(t_{c_i}) \cdot t_{c_{i+1}} \cdot \cdots \cdot t_{c_1}$;
(2) *inverse Hurwitz move:* $t_{c_n} \cdot \cdots \cdot t_{c_{i+1}} \cdot t_{c_i} \cdot \cdots \cdot t_{c_1} \sim t_{c_n} \cdot \cdots \cdot t_{c_i} \cdot t_{c_i}^{-1}(t_{c_{i+1}}) \cdot \cdots \cdot t_{c_1}$.

This relation results from the choice of a Hurwitz system, a set of arcs that connect the base point b_0 to b_i.

Furthermore, a choice of a generic fiber also provides an additional equivalence relation. Two monodromy factorizations W_1 and W_2 are called *global conjugation equivalent* if $W_2 = f(W_1)$ for some $f \in \mathcal{M}_g$.

DEFINITION 2.3. Two Lefschetz fibrations $f_1 : X_1 \to B_1$ and $f_2 : X_2 \to B_2$ are called *isomorphic* if there exist orientation-preserving diffeomorphisms $H : X_1 \to X_2$ and $h : B_1 \to B_2$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ H\ } & X_2 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\
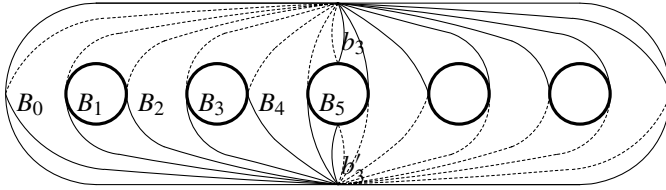B_1 & \xrightarrow{\ h\ } & B_2
\end{array}
\qquad (2.1)
$$

Figure 2 Vanishing cycles on $\eta_{1,g}$ for $g = 2$

DEFINITION 2.4. Let $\pi : X \to S^2$ be a Lefschetz fibration, and let F be a fixed generic fiber of the Lefschetz fibration that is a closed surface of genus g. Suppose that $W = w_n \cdot \cdots \cdot w_2 \cdot w_1$ is a monodromy factorization of the Lefschetz fibration corresponding to F. Then, the *monodromy group* $G_F(W) \subset \mathcal{M}_g$ is defined to be a subgroup of $\mathcal{M}_g$ generated by $\{w_1, w_2, \ldots, w_n\}$.

LEMMA 2.5 [Yun08]. *If two monodromy factorizations W_1 and W_2 give iso-morphic Lefschetz fibrations over S^2 with respect to chosen generic fibers F_1 and F_2, respectively, that are homeomorphic to F, then the monodromy groups $G_{F_1}(W_1)$ and $G_{F_2}(W_2)$ are conjugate to each other as subgroups of the map-ping class group $\mathcal{M}_g$. Moreover, if a generic fiber $F = F_1 = F_2$ is fixed, then $G_F(W_1) = G_F(W_2)$.*

REMARK 2.6. If a generic fiber $F = F_1 = F_2$ is fixed, then we can also prove that $G_F(W_1) = G_F(W_2)$ using Humphries' graph method (Corollary 2.14).

Note that a monodromy factorization of $E(2)_K$ was originally studied by Fin-tushel and Stern [FS04], and the second author also found an explicit monodromy factorization of $E(2)_K$ [Yun08] using a factorization of the identity element in the mapping class group discovered by Matsumoto [Mat96], Cadavid [Cad98], Korkmaz [Kor01], and Gurtas [Gur04].

LEMMA 2.7 [Kor01]. *Let $M(2, g)$ be the desingularization of a double cover of $\Sigma_g \times S^2$ branched over $4(\{pt.\} \times S^2) \cup 2(\Sigma_g \times \{pt.\})$. Then, $M(2, g)$ admits a monodromy factorization $\eta_{1,g}^2$, with*

$$\eta_{1,g} = t_{B_0} \cdot t_{B_1} \cdot t_{B_2} \cdot \cdots \cdot t_{B_{2g}} \cdot t_{B_{2g+1}} \cdot t_{b_{g+1}}^2 \cdot t_{b'_{g+1}}^2,$$

where B_j, b_{g+1}, and b'_{g+1} are simple closed curves on Σ_{2g+1}, as in Figure 2.

THEOREM 2.8 [FS04]. *Let $K \subset S^3$ be a fibered knot of genus g. Then, $E(2)_K$ admits a monodromy factorization of the form*

$$\Phi_K(\eta_{1,g}^2) \cdot \eta_{1,g}^2,$$

where $\eta_{1,g}$ is as in Lemma 2.7, and

$$\Phi_K = \phi_K \oplus id \oplus id : \Sigma_g \sharp \Sigma_1 \sharp \Sigma_g \to \Sigma_g \sharp \Sigma_1 \sharp \Sigma_g$$

is an extension of a monodromy ϕ_K of the fibered knot K such that

$$S^3 \setminus N(K) = (\Sigma_g^1 \times [0,1])/((x,1) \sim (\phi_K(x),0)),$$

where Σ_g^1 is an oriented surface of genus $g = g(K)$ with one boundary component.

Note that Fintushel and Stern's construction of a Lefschetz fibration structure on $E(2)_K$ gives an explicit method for obtaining the monodromy factorization of $E(2)_K$ using a monodromy map of the corresponding fibered knot K.

Now, we will explain how to obtain a monodromy map explicitly. Let $L \subset S^3$ be a fibered link. Then it admits a fibration structure over S^1 given by

$$S^3 \setminus N(L) \approx (\Sigma_L \times [0,1])/_{(x,1)\sim(\phi_L(x),0)}.$$

We denote this fibration structure by (Σ_L, ϕ_L).

LEMMA 2.9 [Har82; Bon83; GK90]. *Every fibered link in S^3 is related to the unknot by a sequence of Hopf plumbings, deplumbings, and twistings.*

(0) *The left-handed Hopf band has monodromy map t_c, and the right-handed Hopf band has monodromy map t_c^{-1}, where c is the core simple closed curve of each Hopf band.*

(1) *If (Σ_L, ϕ_L) is a fibration structure of a fibered link L and L' is the link that is obtained by plumbing the positive (left-handed) Hopf band H^+ to Σ_L, then $L' = \partial \Sigma_{L'}$ has a fibration structure $(\Sigma_{L'}, t_c \circ \phi_L)$, where c is the core circle of the Hopf band. If we perform negative (right-handed) Hopf band plumbing, then the monodromy map becomes $t_c^{-1} \circ \phi_L$.*

(2) *Let (Σ_L, ϕ_L) be a fibration structure of a fibered link L and suppose that there is an embedded circle c in Σ_L that is unknotted in S^3. Furthermore, suppose that $lk(c, c^+) = 0$, where c^+ is a push off of c in the chosen normal direction of Σ_L. Then, a (± 1)-Dehn surgery along c^+ yields a new fibration structure $(\Sigma_{L'}, t_c^{\pm 1} \circ \phi_L)$. This operation is called Stallings twist.*

REMARK 2.10. Note that the notation used in Lemma 2.9 is different from that of Harer, who employed the left-handed Dehn twist as the standard Dehn twist. In addition, there are more complicated conditions included in (2) in Harer's article, but we only require the stated condition in our construction.

2.2. Humphries' Graph Method

Let Σ_g^b be an oriented compact surface with genus g and b boundary components. When $b = 0$, we simply denote this as Σ_g.

DEFINITION 2.11 [Hum79]. Suppose that $\{\gamma_1, \gamma_2, \ldots, \gamma_{2g}\}$ is a set of simple closed curves on Σ_g that generates $H_1(\Sigma_g; \mathbb{Z}_2)$. Let us define the graph $\Gamma(\gamma_1, \gamma_2, \ldots, \gamma_{2g})$ as follows:

- a vertex for each simple closed curve γ_i ($1 \leq i \leq 2g$);
- an edge between γ_i and γ_j if $i_2(\gamma_i, \gamma_j) = 1$, where $i_2(\gamma_i, \gamma_j)$ is the modulo 2 intersection number between two curves γ_i and γ_j;
- we assume that there is no intersection between any two edges.

Then, for any simple closed curve γ on Σ_g, we can express the homology class of $\gamma = \sum_{i=1}^{2g} \varepsilon_i \gamma_i$ ($\varepsilon_i = 0, 1$) as an element of $H_1(\Sigma_g; \mathbb{Z}_2)$. By denoting $\overline{\gamma} := \bigcup_{\varepsilon_i=1} \overline{\gamma_i}$, where $\overline{\gamma_i}$ is the union of the closures of all half-edges with one end vertex γ_i, we define $\chi_\Gamma(\gamma) := \chi_\Gamma(\overline{\gamma})$ as the modulo 2 Euler number.

Let $\mathcal{M}_g^b$ be the mapping class group of an oriented surface Σ_g^b. For a simple closed curve c on Σ_g^b, t_c denotes the right-handed Dehn twist along c, and $t_{c_1} \cdot t_{c_2}$ means $t_{c_1} \circ t_{c_2}$. We also use the notation $f(t_c) = f \circ t_c \circ f^{-1}$ for $f \in \mathcal{M}_g^b$, which is equal to $t_{f(c)}$.

Humphries [Hum79] showed that the minimal number of Dehn twist generators of the mapping class group $\mathcal{M}_g$ or $\mathcal{M}_g^1$ with $g \geq 2$ is $2g + 1$, using symplectic transvection and the modulo 2 Euler number of a graph. Furthermore, he proved the following.

LEMMA 2.12 [Hum79; PY11]. *Assume that $g \geq 2$. Let $\Gamma = \Gamma(\gamma_1, \gamma_2, \ldots, \gamma_{2g})$ be a graph corresponding to a set of simple closed curves $\{\gamma_1, \gamma_2, \ldots, \gamma_{2g}\}$ that generates $H_1(\Sigma_g; \mathbb{Z}_2)$. Suppose that $\mathcal{G}_\Gamma$ is a subgroup of $\mathcal{M}_g$ generated by*

$$\{t_\alpha | \alpha \text{ is a nonseparating simple closed curve on } \Sigma_g \text{ such that } \chi_\Gamma(\alpha) = 1\}.$$

Then, $\mathcal{G}_\Gamma$ is a nontrivial proper subgroup of $\mathcal{M}_g$. Moreover, if β is a nonseparating simple closed curve on Σ_g with $\chi_\Gamma(\beta) = 0$, then $t_\beta \notin \mathcal{G}_\Gamma$.

COROLLARY 2.13 [Hum79; PY11]. *For any nonseparating simple closed curves c and γ on Σ_g, we have that*

(1) *if $\chi_\Gamma(c) = 1$, then $\chi_\Gamma(t_c(\gamma)) \equiv \chi_\Gamma(\gamma)$ (mod 2);*
(2) *if $\chi_\Gamma(c) = 0$, then $\chi_\Gamma(t_c(\gamma)) \equiv \chi_\Gamma(\gamma) + i_2(c, \gamma)$ (mod 2).*

COROLLARY 2.14. *Assume that two genus g Lefschetz fibrations over S^2 have monodromy factorizations ξ_1 and ξ_2 with respect to a fixed generic fiber F. If there exists a graph Γ as in Definition 2.11 such that $G_F(\xi_1) \leq \mathcal{G}_\Gamma$ and $G_F(\xi_2) \nleq \mathcal{G}_\Gamma$, then ξ_1 is not isomorphic to ξ_2 as a genus g Lefschetz fibration over S^2.*

Proof. Note that the monodromy group is an invariant subgroup of the mapping class group under Hurwitz equivalences. Therefore, we must consider the role of global conjugation equivalences. Because we consider a Lefschetz fibration with a fixed generic fiber F, its monodromy factorization is completely determined by the Hurwitz system. However, even in such a case with fixed generic fiber and a fixed given Hurwitz system, there exists the possibility of global conjugation resulting from a cyclic permutation of monodromy factorization.

Let ξ be a monodromy factorization of a Lefschetz fibration with respect to a fixed generic fiber F and a fixed Hurwitz system. Then, its monodromy factorization can be determined up to a global conjugation by using a map $\psi \in G_F(\xi)$, and

this global conjugation can be realized as a sequence of Hurwitz moves, inverse Hurwitz moves, and cyclic permutation. Even though a monodromy factorization is determined up to global conjugations by using the elements in $G_F(\xi)$, its monodromy group is well defined, and it does not depend on $\psi \in G_F(\xi)$ by Definition 2.4 and Corollary 2.13.

More precisely, $G_F(\xi_1) \leq \mathcal{G}_\Gamma$ implies that

$$\{\chi_\Gamma(c)|c \text{ is a nonseparating s.c.c. on } F \text{ such that } t_c \in G_F(\xi_1)\} = \{1\}.$$

This implies that for any $\psi \in G_F(\xi_1)$,

$$\{\chi_\Gamma(c)|c \text{ is a nonseparating s.c.c. on } F \text{ such that } t_c \in G_F(\psi(\xi_1))\} = \{1\}$$

by Corollary 2.13. On the other hand, $G_F(\xi_2) \nleq \mathcal{G}_\Gamma$ implies that

$$\{\chi_\Gamma(c)|c \text{ is a nonseparating s.c.c. on } F \text{ such that } t_c \in G_F(\xi_2)\} = \{0, 1\}.$$

Therefore, we conclude that $\psi(\xi_1) \ncong \xi_2$ for any $\psi \in G_F(\xi_1)$.   □

3. Nonisomorphic Lefschetz Fibrations

As mentioned in the Introduction, we have previously studied nonisomorphic Lefschetz fibration structures on knot surgery 4-manifolds $E(n)_K$ for the 2-bridge knot case [PY09] and Kanenobu knot case [PY11], and we proved that these admit at least two nonisomorphic Lefschetz fibration structures in both cases. Recently, Baykur [Bay14] obtained a similar result regarding Lefschetz fibration structures on nonminimal symplectic 4-manifolds. In this section, we construct a family of simply connected minimal symplectic 4-manifolds $E(2)_K$, each of which admits arbitrarily many nonisomorphic Lefschetz fibration structures with the same genus fiber. To obtain these families, we first construct a family of connected sums of fibered knots obtained by Stallings twist from the square knot $3_1 \sharp 3_1^*$ as follows.

3.1. Square Knot as a Building Block

Let K_0 be the square knot $3_1 \sharp 3_1^*$, which is a connected sum of a right-handed trefoil knot and a left-handed trefoil knot, and let K_n be the knot obtained by Stallings twist from K_0 as shown in Figure 1. Here, the n in the box indicates n left-handed full twists when n is a positive integer and $|n|$ right-handed full twists when n is a negative integer. Then, it is well known that K_n is a genus 2 fibered knot and has the Alexander polynomial $\Delta_{K_n} = (t^2 - t + 1)^2$ [Sta78].

3.2. Main Construction

Since two Hurwitz equivalent monodromy factorizations yield the same monodromy group, the monodromy group sometimes provides sufficient information to distinguish some pairs of Lefschetz fibration structures. For example, we successfully distinguished some Lefschetz fibration structures on $E(2)_K$ for a family of Kanenobu knots, up to parity [PY11]. The main tool used in the proof was Humphries' graph method. In this paper, we use Humphries' graph method again
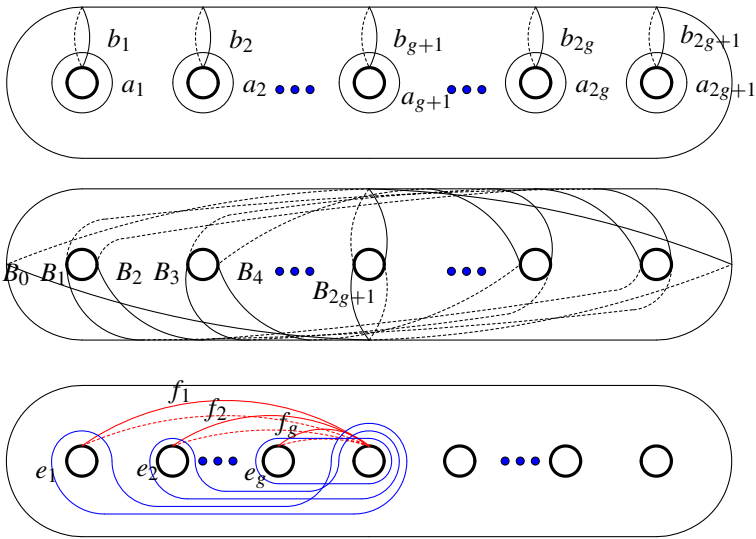
Figure 3 Curves for homology basis of $H_1(\Sigma_{2g+1}; \mathbb{Z}_2)$

to show that the corresponding monodromy groups appearing in the main theorem are mutually distinct. For this, we start with the following lemma.

LEMMA 3.1. *Let $\mathcal{B}$ be a subset of $H_1(\Sigma_{2g+1}; \mathbb{Z}_2)$ such that*
- $\{B_1, B_2, \ldots, B_{2g}, a_{g+1}, b_{g+1}\} \subset \mathcal{B}$, *and*
- *for every $1 \leq i \leq g$, one of $\{a_i, e_i\}$ is in $\mathcal{B}$, and one of $\{b_i, f_i\}$ is also in $\mathcal{B}$, where e_i is a simple closed curve representing the homology class of $a_i + a_{g+1}$, and f_i is a simple closed curve representing the homology class of $b_i + b_{g+1}$, as in Figure 3.*

Then, $\mathcal{B}$ is a basis of $H_1(\Sigma_{2g+1}; \mathbb{Z}_2)$.

Proof. For every $1 \leq i \leq g$, we have that either $a_i \in \mathcal{B}$ or $a_i = e_i - a_{g+1} \in$ Span($\mathcal{B}$). Similarly, either $b_i \in \mathcal{B}$ or $b_i = f_i - b_{g+1} \in$ Span($\mathcal{B}$). In $H_1(\Sigma_{2g+1}; \mathbb{Z}_2)$, we also have that

$$B_{2i-1} = B_{2i} + a_i + a_{2g+2-i}$$

and

$$B_{2i-1} = a_i + a_{i+1} + \cdots + a_{2g+2-i} + b_i + b_{2g+2-i}$$

for $1 \leq i \leq g$. Therefore,

$$\{a_1, b_1, a_2, b_2, \ldots, a_{2g+1}, b_{2g+1}\} \subset \text{Span}(\mathcal{B}).$$

Because $|\mathcal{B}| = 4g + 2 = \dim H_1(\Sigma_{2g+1}; \mathbb{Z}_2)$, $\mathcal{B}$ is a basis of $H_1(\Sigma_{2g+1}; \mathbb{Z}_2)$. $\qquad\square$
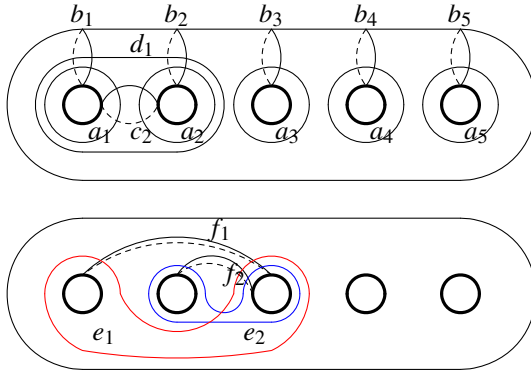
Figure 4   Simple closed curves on Σ_{2g+1} for $g = 2$

Now, we are ready to derive our main result. We use a basis $\mathcal{B}$ such as in Lemma 3.1 to find appropriate graphs Γ, which is a key step for applying Humphries' graph method.

PROPOSITION 3.2. *For each integer $n \in \mathbb{Z}$, the knot surgery 4-manifold $E(2)_{K_n}$ admits (at least) two nonisomorphic genus 5 Lefschetz fibrations over S^2.*

Proof. For $(p, q) \in \mathbb{Z}^2$, let $K_{p,q}$ be a family of knots obtained from K_0 by performing $\text{sign}(p)$ Stallings twists $|p|$ times along c_2 and $\text{sign}(q)$ Stallings twists $|q|$ times along d_1 in Figure 4 (corresponding to c and d in Figure 1). Then, we have that

$$K_{p,q} \sim K_{p+q},$$

which can be seen by isotoping the twists on the two strands in Figure 1, and its monodromy

$$\phi_{K_{p,q}} = t_{d_1}^q \circ t_{c_2}^p \circ t_{a_2}^{-1} \circ t_{b_2}^{-1} \circ t_{a_1} \circ t_{b_1}$$

can be easily obtained by Lemma 2.9.

Because $K_n \sim K_{p,q}$ whenever $n = p + q$, it follows that

$$\phi_{K_{p,q}} = t_{d_1}^q \circ t_{c_2}^p \circ t_{a_2}^{-1} \circ t_{b_2}^{-1} \circ t_{a_1} \circ t_{b_1}$$

is the monodromy of K_n and

$$\Phi_{K_{p,q}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2$$

is the monodromy factorization of $E(2)_{K_n}$. In fact,

$$\begin{aligned}
\phi_{K_{p,q}} &= t_{d_1}^q \circ t_{c_2}^{p+q} \circ t_{c_2}^{-q} \circ t_{a_2}^{-1} \circ t_{b_2}^{-1} \circ t_{a_1} \circ t_{b_1} \\
&= t_{d_1}^q \circ (t_{c_2}^{p+q} \circ t_{a_2}^{-1} \circ t_{b_2}^{-1} \circ t_{a_1} \circ t_{b_1}) \circ t_{(t_{a_2}^{-1} \circ t_{b_2}^{-1} \circ t_{a_1} \circ t_{b_1})^{-1}(c_2)}^{-q} \\
&= t_{d_1}^q \circ (t_{c_2}^n \circ t_{a_2}^{-1} \circ t_{b_2}^{-1} \circ t_{a_1} \circ t_{b_1}) \circ t_{d_1}^{-q}
\end{aligned}$$

because $t_{b_1}^{-1} \circ t_{a_1}^{-1} \circ t_{b_2} \circ t_{a_2}(c_2) = d_1$. Therefore, we have confirmed that in fact $\phi_{K_{p,q}}$ is conjugate to $\phi_{K_{p+q}}$.

Let $\varepsilon_p \equiv p \pmod 2$ and $\varepsilon_q \equiv q \pmod 2$ with $\varepsilon_p, \varepsilon_q \in \{0, 1\}$. In order to use Humphries' graph method, we want to find a basis $\mathcal{B}_{\varepsilon_p + 2\varepsilon_q}$, indexed by $\varepsilon_p + 2\varepsilon_q$, of $H_1(\Sigma_5; \mathbb{Z}_2)$ such that

- $\{B_1, B_2, B_3, B_4, b_3, a_3\} \subset \mathcal{B}_{\varepsilon_p + 2\varepsilon_q}$ and
- $G_F(\Phi_{K_{p,q}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2) \le \mathcal{G}_{\Gamma(\mathcal{B}_{\varepsilon_p + 2\varepsilon_q})}$.

In $H_1(\Sigma_5; \mathbb{Z}_2)$, we obtain

$$\Phi_{K_{p,q}}(B_0) = B_0 + b_1 + b_2 + a_1 + a_2,$$
$$\Phi_{K_{p,q}}(B_1) = B_1 + b_1 + b_2 + a_2 + \varepsilon_p c_2 + \varepsilon_q d_1,$$
$$\Phi_{K_{p,q}}(B_2) = B_2 + b_2 + a_1 + a_2 + \varepsilon_p c_2,$$
$$\Phi_{K_{p,q}}(B_3) = B_3 + b_2 + \varepsilon_p c_2,$$
$$\Phi_{K_{p,q}}(B_4) = B_4 + a_2 + \varepsilon_p c_2 + \varepsilon_q d_1,$$

because for every $0 \le i \le 4$, have that

$$\begin{aligned}
\Phi_{K_{p,q}}(B_i) = {} & B_i + i_2(B_i, b_1)b_1 + i_2(t_{b_1}(B_i), a_1)a_1 \\
& + i_2(t_{a_1} \circ t_{b_1}(B_i), b_2)b_2 \\
& + i_2(t_{b_2}^{-1} \circ t_{a_1} \circ t_{b_1}(B_i), a_2)a_2 \\
& + \varepsilon_p i_2(t_{a_2}^{-1} \circ t_{b_2}^{-1} \circ t_{a_1} \circ t_{b_1}(B_i), c_2)c_2 \\
& + \varepsilon_q i_2(t_{c_2}^p \circ t_{a_2}^{-1} \circ t_{b_2}^{-1} \circ t_{a_1} \circ t_{b_1}(B_i), d_1)d_1.
\end{aligned}$$

Because $\Phi_{K_{p,q}}(t_{B_i}) \in G_F(\Phi_{K_{p,q}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2)$ for $0 \le i \le 4$, Corollary 2.13 implies that for each graph $\Gamma(\mathcal{B}_{\varepsilon_p + 2\varepsilon_q})$,

(a) an even number of elements in $\{b_1, b_2, a_1, a_2\}$ have $\chi_\Gamma = 0$,
(b) an even number of elements in $\{b_1, b_2, a_2, \varepsilon_p c_2, \varepsilon_q d_1\}$ have $\chi_\Gamma = 0$,
(c) an even number of elements in $\{b_2, a_1, a_2, \varepsilon_p c_2\}$ have $\chi_\Gamma = 0$,
(d) an even number of elements in $\{b_2, \varepsilon_p c_2\}$ have $\chi_\Gamma = 0$,
(e) an even number of elements in $\{a_2, \varepsilon_p c_2, \varepsilon_q d_1\}$ have $\chi_\Gamma = 0$.

As a matter of convention, if $\varepsilon_p = 0$ or $\varepsilon_q = 0$, then the corresponding c_2 or d_1 is omitted from the set. We may assume that $\chi_\Gamma(c_2) = \chi_\Gamma(d_1) = 0$ for each graph $\Gamma(\mathcal{B}_{\varepsilon_p + 2\varepsilon_q})$ [PY11]. Then, (a)–(e) has a unique solution for each $(\varepsilon_p, \varepsilon_q) \in \{0, 1\} \times \{0, 1\}$ as follows:

- $t_{a_1}, t_{a_2}, t_{b_1}, t_{b_2} \in \mathcal{G}_{\Gamma(\mathcal{B}_0)}$,
- $t_{a_1}, t_{a_2}, t_{b_1}, t_{b_2} \notin \mathcal{G}_{\Gamma(\mathcal{B}_1)}$,
- $t_{b_1}, t_{b_2} \in \mathcal{G}_{\Gamma(\mathcal{B}_2)}$ and $t_{a_1}, t_{a_2} \notin \mathcal{G}_{\Gamma(\mathcal{B}_2)}$,
- $t_{a_1}, t_{a_2} \in \mathcal{G}_{\Gamma(\mathcal{B}_3)}$ and $t_{b_1}, t_{b_2} \notin \mathcal{G}_{\Gamma(\mathcal{B}_3)}$.

Let e_i be a simple closed curve on Σ_5 representing $a_i + a_3 \in H_1(\Sigma_5; \mathbb{Z}_2)$, and let f_i be a simple closed curve on Σ_5 representing $b_i + b_3 \in H_1(\Sigma_5; \mathbb{Z}_2)$ for $1 \le i \le 2$, as in Figure 4. Then, by Lemma 3.1 we can select a basis $\mathcal{B}_j$ as follows:

- $\mathcal{B}_0 = \{B_1, B_2, B_3, B_4, b_3, a_3, a_1, a_2, b_1, b_2\}$,
- $\mathcal{B}_1 = \{B_1, B_2, B_3, B_4, b_3, a_3, e_1, e_2, f_1, f_2\}$,

- $\mathcal{B}_2 = \{B_1, B_2, B_3, B_4, b_3, a_3, e_1, e_2, b_1, b_2\}$,
- $\mathcal{B}_3 = \{B_1, B_2, B_3, B_4, b_3, a_3, a_1, a_2, f_1, f_2\}$.

Now, we will show that

$$G_F(\Phi_{K_{p,q}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2) \leq \mathcal{G}_{\Gamma(\mathcal{B}_{\varepsilon_p + 2\varepsilon_q})}$$

for each $(p, q) \in \mathbb{Z}^2$. To do so, we only need to show that $B_0, B_5 \in \mathcal{G}_{\Gamma(\mathcal{B}_{\varepsilon_p + 2\varepsilon_q})}$ for each $(\varepsilon_p, \varepsilon_q) \in \{0, 1\} \times \{0, 1\}$. Because $B_5 = a_3 \in \mathcal{G}_{\Gamma(\mathcal{B}_{\varepsilon_p + 2\varepsilon_q})}$, it remains to show that

$$B_0 = B_1 + B_2 + B_3 + B_4 + a_3 \in \mathcal{G}_{\Gamma(\mathcal{B}_{\varepsilon_p + 2\varepsilon_q})}.$$

However, this is clear because $\overline{B_0}$ is isotopic to a disjoint union of five vertices

$$\overline{B_0} \simeq B_1 \cup B_2 \cup B_3 \cup B_4 \cup a_3$$

in each graph $\Gamma(\mathcal{B}_j)$ $(0 \leq j \leq 3)$, and therefore $\chi_{\Gamma(\mathcal{B}_j)}(B_0) = 1$.

Let us observe that

- $\chi_{\Gamma(\mathcal{B}_j)}(c_2) = 0 = \chi_{\Gamma(\mathcal{B}_j)}(d_1)$ for $0 \leq j \leq 3$,
- $i_2(\Phi_{K_{p,q}}(B_1), c_2) = 1$ and $i_2(\Phi_{K_{p,q}}(B_1), d_1) = 1$ for each $(p, q) \in \mathbb{Z}^2$,
- $i_2(\Phi_{K_{p,q}}(B_2), c_2) = 1$ and $i_2(\Phi_{K_{p,q}}(B_2), d_1) = 0$ for each $(p, q) \in \mathbb{Z}^2$.

This implies that

- $\Phi_{K_{p,q}}(t_{B_1}), \Phi_{K_{p,q}}(t_{B_2}) \in G_F(\Phi_{K_{p,q}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2) \leq \mathcal{G}_{\Gamma(\mathcal{B}_{\varepsilon_p + 2\varepsilon_q})}$,
- if $(p - r, q - s) \equiv (1, 0)$ or $(0, 1)$ (mod 2), then $\Phi_{K_{r,s}}(t_{B_1}) \notin \mathcal{G}_{\Gamma(\mathcal{B}_{\varepsilon_p + 2\varepsilon_q})}$,
- if $(p - r, q - s) \equiv (1, 1)$ (mod 2), then $\Phi_{K_{r,s}}(t_{B_2}) \notin \mathcal{G}_{\Gamma(\mathcal{B}_{\varepsilon_p + 2\varepsilon_q})}$.

Therefore, whenever $(r, s) \not\equiv (p, q)$ (mod 2),

$$G_F(\Phi_{K_{r,s}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2) \not\leq \mathcal{G}_{\Gamma(\mathcal{B}_{\varepsilon_p + 2\varepsilon_q})},$$

and $\Phi_{K_{p,q}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2$ is not isomorphic to $\Phi_{K_{r,s}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2$ as a genus 5 Lefschetz fibration by Corollary 2.14.

Finally, because $(n, 0) \not\equiv (n - 1, 1)$ (mod 2) for each integer n, it is clear that $E(2)_{K_n}$ admits (at least) two nonisomorphic genus 5 Lefschetz fibration structures. $\qquad\square$

Now, we extend this result on $E(2)_K$ to the case of a family of connected sums of n copies of Stallings twist knots as follows.

THEOREM 3.3. *For each integer $n > 0$ and $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, the knot surgery 4-manifold*

$$E(2)_{K_{m_1} \sharp K_{m_2} \sharp \cdots \sharp K_{m_n}}$$

admits (at least) 2^n nonisomorphic genus $(4n + 1)$ Lefschetz fibrations over S^2. Here, K_{m_i} $(1 \leq i \leq n)$ denotes a knot obtained by performing $|m_i|$ left/right-handed full twist on the double knot K_0, as in Figure 1.

Proof. Let us decompose each m_i as a sum of two integers $p_i, q_i \in \mathbb{Z}$ such that $m_i = p_i + q_i$. Let $\varepsilon_{p_i} = \varepsilon_{2i-1} \equiv p_i$ (mod 2) and $\varepsilon_{q_i} = \varepsilon_{2i} \equiv q_i$ (mod 2),
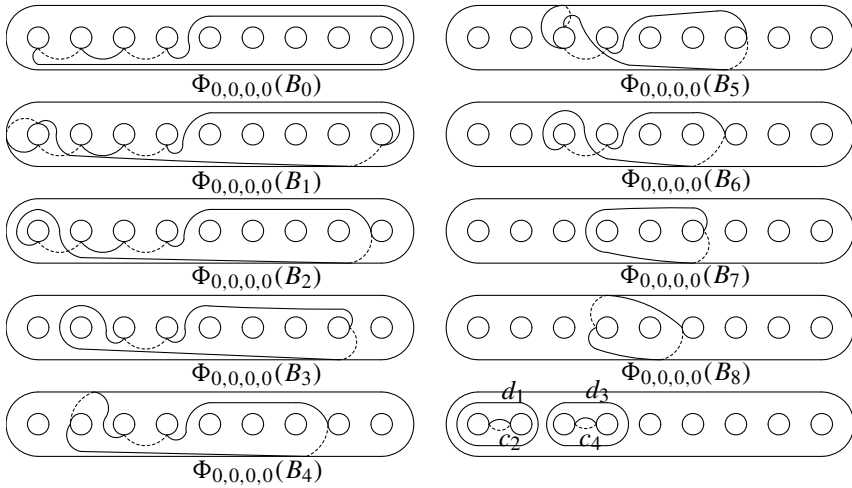
Figure 5 Simple closed curves $\Phi_{K_0 \sharp K_0}(B_i)$

where $\varepsilon_{p_i}, \varepsilon_{q_i} \in \{0, 1\}$ for $1 \leq i \leq n$ and $\varepsilon_i \in \{0, 1\}$ for $1 \leq i \leq 2n$. Because $K_{m_1} \sharp K_{m_2} \sharp \cdots \sharp K_{m_n}$ is a fibered knot of genus $2n$ and

$$K_{m_1} \sharp K_{m_2} \sharp \cdots \sharp K_{m_n} \sim K_{p_1, q_1} \sharp K_{p_2, q_2} \sharp \cdots \sharp K_{p_n, q_n}$$

whenever $m_i = p_i + q_i$ for $1 \leq i \leq n$, we obtain a monodromy map

$$\phi_{p_1, q_1, \ldots, p_n, q_n} := \prod_{i=1}^{n} (t_{d_{2i-1}}^{q_i} \circ t_{c_{2i}}^{p_i} \circ t_{a_{2i}}^{-1} \circ t_{b_{2i}}^{-1} \circ t_{a_{2i-1}} \circ t_{b_{2i-1}}).$$

Note that we have the following relations in $H_1(\Sigma_{4n+1}; \mathbb{Z}_2)$:

$$B_0 = \sum_{i=1}^{4n} B_i + a_{2n+1}, \qquad B_{4n+1} = a_{2n+1},$$

and from Figure 5, the following hold for $1 \leq j \leq n$:

$$\Phi_{p_1, q_1, \ldots, p_n, q_n}(B_0) = B_0 + \sum_{i=1}^{n}(a_{2i-1} + a_{2i} + b_{2i-1} + b_{2i}), \qquad (3.1)$$

$$\Phi_{p_1, q_1, \ldots, p_n, q_n}(B_{4(j-1)+1}) = B_{4(j-1)+1} + b_{2j-1} + b_{2j} + a_{2j} + \varepsilon_{2j-1}c_{2j}$$
$$+ \varepsilon_{2j}d_{2j-1}$$
$$+ \sum_{i=j+1}^{n}(a_{2i-1} + a_{2i} + b_{2i-1} + b_{2i}), \qquad (3.2)$$

$$\Phi_{p_1, q_1, \ldots, p_n, q_n}(B_{4(j-1)+2}) = B_{4(j-1)+2} + b_{2j} + a_{2j-1} + a_{2j} + \varepsilon_{2j-1}c_{2j}$$

$$+ \sum_{i=j+1}^{n} (a_{2i-1} + a_{2i} + b_{2i-1} + b_{2i}), \qquad (3.3)$$

$$\Phi_{p_1,q_1,\ldots,p_n,q_n}(B_{4(j-1)+3}) = B_{4(j-1)+3} + b_{2j} + \varepsilon_{2j-1}c_{2j}$$

$$+ \sum_{i=j+1}^{n} (a_{2i-1} + a_{2i} + b_{2i-1} + b_{2i}), \qquad (3.4)$$

$$\Phi_{p_1,q_1,\ldots,p_n,q_n}(B_{4(j-1)+4}) = B_{4(j-1)+4} + a_{2j} + \varepsilon_{2j-1}c_{2j} + \epsilon_{2j}d_{2j-1}$$

$$+ \sum_{i=j+1}^{n} (a_{2i-1} + a_{2i} + b_{2i-1} + b_{2i}). \qquad (3.5)$$

Now, for each $\sum_{i=1}^{2n} \varepsilon_i 2^{i-1} \in \{0, 1, \ldots, 2^{2n} - 1\}$, we want to construct a basis $\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}}$ of $H_1(\Sigma_{4n+1}; \mathbb{Z}_2)$ that satisfies

- $\{B_1, B_2, \ldots, B_{4n}, a_{2n+1}, b_{2n+1}\} \subset \mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}}$ and
- $G_F(\Phi_{p_1,q_1,\ldots,p_n,q_n}(\eta_{1,2n}^2) \cdot \eta_{1,2n}^2) \leq \mathcal{G}_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})}$.

Note that equations (3.1)–(3.5) and the second condition for $\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}}$ imply that (a)–(e) hold for $1 \leq j \leq n$, and we add one further condition (f) as follows:

(a) an even number of $\bigcup_{i=1}^{n}\{a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}\}$ have $\chi_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})} = 0$,

(b) an even number of

$$\{b_{2j-1}, b_{2j}, a_{2j}, \varepsilon_{2j-1}c_{2j}, \varepsilon_{2j}d_{2j-1}\} \cup \bigcup_{i=j+1}^{n} \{a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}\}$$

have $\chi_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})} = 0$,

(c) an even number of

$$\{b_{2j}, a_{2j-1}, a_{2j}, \varepsilon_{2j-1}c_{2j}\} \cup \bigcup_{i=j+1}^{n} \{a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}\}$$

have $\chi_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})} = 0$,

(d) an even number of

$$\{b_{2j}, \varepsilon_{2j-1}c_{2j}\} \cup \bigcup_{i=j+1}^{n} \{a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}\}$$

have $\chi_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})} = 0$,

(e) an even number of

$$\{a_{2j}, \varepsilon_{2j-1}c_{2j}, \varepsilon_{2j}d_{2j-1}\} \cup \bigcup_{i=j+1}^{n} \{a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}\}$$

have $\chi_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})} = 0$,

(f) $\chi_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})}(c_{2j}) = 0 = \chi_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})}(d_{2j-1})$ for $1 \leq j \leq n$.

Table 1 The system of conditions

	b_1	b_2	a_1	a_2			b_3	b_4	a_3	a_4		
$\Phi_{p_1,q_1,p_2,q_2}(B_0)$	b_1	b_2	a_1	a_2			b_3	b_4	a_3	a_4		
$\Phi_{p_1,q_1,p_2,q_2}(B_1)$	b_1	b_2		a_2	$\varepsilon_{p_1}c_2$	$\varepsilon_{q_1}d_1$	b_3	b_4	a_3	a_4		
$\Phi_{p_1,q_1,p_2,q_2}(B_2)$		b_2	a_1	a_2	$\varepsilon_{p_1}c_2$		b_3	b_4	a_3	a_4		
$\Phi_{p_1,q_1,p_2,q_2}(B_3)$		b_2			$\varepsilon_{p_1}c_2$		b_3	b_4	a_3	a_4		
$\Phi_{p_1,q_1,p_2,q_2}(B_4)$				a_2	$\varepsilon_{p_1}c_2$	$\varepsilon_{q_1}d_1$	b_3	b_4	a_3	a_4		
$\Phi_{p_1,q_1,p_2,q_2}(B_5)$							b_3	b_4		a_4	$\varepsilon_{p_2}c_4$	$\varepsilon_{q_2}d_3$
$\Phi_{p_1,q_1,p_2,q_2}(B_6)$								b_4	a_3	a_4	$\varepsilon_{p_2}c_4$	
$\Phi_{p_1,q_1,p_2,q_2}(B_7)$								b_4			$\varepsilon_{p_2}c_4$	
$\Phi_{p_1,q_1,p_2,q_2}(B_8)$										a_4	$\varepsilon_{p_2}c_4$	$\varepsilon_{q_2}d_3$

Then, these systems of conditions specify a unique solution, which we can prove by using induction on n, the number of connected summed Stallings knots. For example, the case $n = 1$ was already proved in Proposition 3.2. Now, we will demonstrate how the inductive step works. Let us consider the following Table 1.

Then, the last four rows, concerning $\Phi_{p_1,q_1,p_2,q_2}(B_j)$ ($5 \le j \le 8$), have a unique solution for each given $(\varepsilon_{p_2}, \varepsilon_{q_2}) \in \{0, 1\} \times \{0, 1\}$. In each case, an even number of elements from $\{b_3, b_4, a_3, a_4\}$ have a modulo 2 Euler number 0. Therefore, this has no effect on the solution of the first four rows, concerning $\Phi_{p_1,q_1,p_2,q_2}(B_j)$ ($1 \le j \le 4$), and the solution has exactly the same pattern as the solution for

$$\{\Phi_{p_1,q_1}(B_1), \Phi_{p_1,q_1}(B_2), \Phi_{p_1,q_1}(B_3), \Phi_{p_1,q_1}(B_4)\}.$$

Note that the condition for $\Phi_{p_1,q_1,p_2,q_2}(B_0)$ is automatically satisfied.

Therefore, $\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}}$ satisfies the following conditions for $1 \le i \le n$:

- If $(\varepsilon_{2i-1}, \varepsilon_{2i}) \equiv (0, 0) \pmod 2$, then

$$\{t_{a_{2i-1}}, t_{a_{2i}}, t_{b_{2i-1}}, t_{b_{2i}}\} \subset \mathcal{G}_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})};$$

- if $(\varepsilon_{2i-1}, \varepsilon_{2i}) \equiv (1, 0) \pmod 2$, then

$$t_{a_{2i-1}}, t_{a_{2i}}, t_{b_{2i-1}}, t_{b_{2i}} \notin \mathcal{G}_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})};$$

- if $(\varepsilon_{2i-1}, \varepsilon_{2i}) \equiv (0, 1) \pmod 2$, then

$$\{t_{b_{2i-1}}, t_{b_{2i}}\} \subset \mathcal{G}_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})} \text{ and } t_{a_{2i-1}}, t_{a_{2i}} \notin \mathcal{G}_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})};$$

- if $(\varepsilon_{2i-1}, \varepsilon_{2i}) \equiv (1, 1) \pmod 2$, then

$$\{t_{a_{2i-1}}, t_{a_{2i}}\} \subset \mathcal{G}_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})} \text{ and } t_{b_{2i-1}}, t_{b_{2i}} \notin \mathcal{G}_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})}.$$

These observations suggest that we can define $\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}}$ as follows:

- Start with $\{B_1, B_2, \ldots, B_{4n}, a_{2n+1}, b_{2n+1}\}$;
- For $1 \le i \le n$, proceed as follows:
  - If $(\varepsilon_{2i-1}, \varepsilon_{2i}) \equiv (0, 0) \pmod 2$, then add $\{a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}\}$ to the set.

- If $(\varepsilon_{2i-1}, \varepsilon_{2i}) \equiv (1,0) \pmod 2$, then add $\{e_{2i-1}, e_{2i}, f_{2i-1}, f_{2i}\}$ to the set.
- If $(\varepsilon_{2i-1}, \varepsilon_{2i}) \equiv (0,1) \pmod 2$, then add $\{e_{2i-1}, e_{2i}, b_{2i-1}, b_{2i}\}$ to the set.
- If $(\varepsilon_{2i-1}, \varepsilon_{2i}) \equiv (1,1) \pmod 2$, then add $\{a_{2i-1}, a_{2i}, f_{2i-1}, f_{2i}\}$ to the set.

Here,

- e_i is a simple closed curve representing $a_i + a_{2n+1}$ in $H_1(\Sigma_{4n+1}; \mathbb{Z}_2)$,
- f_i is a simple closed curve representing $b_i + b_{2n+1}$ in $H_1(\Sigma_{4n+1}; \mathbb{Z}_2)$.

Then, by Lemma 3.1 each resulting set $\mathcal{B}_i$ $(0 \le i \le 2^{2n} - 1)$ is a basis of $H_1(\Sigma_{4n+1}; \mathbb{Z}_2)$ satisfying

$$G_F(\Phi_{p_1, q_1, \dots, p_n, q_n}(\eta_{1,2n}^2) \cdot \eta_{1,2n}^2) \le \mathcal{G}_{\Gamma(\mathcal{B}_{\sum_{i=1}^{2n} \varepsilon_i 2^{i-1}})}.$$

Now, we will show that if $(p_1, q_1, \dots, p_n, q_n) \not\equiv (r_1, s_1, \dots, r_n, s_n) \pmod 2$, then $\Phi_{p_1, q_1, \dots, p_n, q_n}(\eta_{1,2n}^2) \cdot \eta_{1,2n}^2$ is not isomorphic to $\Phi_{r_1, s_1, \dots, r_n, s_n}(\eta_{1,2n}^2) \cdot \eta_{1,2n}^2$ as a Lefschetz fibration.

Let us observe that for $1 \le j \le n$,

- $i_2(\Phi_{p_1, q_1, \dots, p_n, q_n}(B_{4(j-1)+1}), c_{2j}) = 1 = i_2(\Phi_{p_1, q_1, \dots, p_n, q_n}(B_{4(j-1)+1}), d_{2j-1})$,
- $i_2(\Phi_{p_1, q_1, \dots, p_n, q_n}(B_{4(j-1)+2}), c_{2j}) = 1$ and $i_2(\Phi_{p_1, q_1, \dots, p_n, q_n}(B_{4(j-1)+2}), d_{2j-1}) = 0$,
- $\chi_{\Gamma(\mathcal{B}_i)}(c_{2j}) = 0 = \chi_{\Gamma(\mathcal{B}_i)}(d_{2j-1})$ for $0 \le i \le 2^{2n} - 1$.

Then, this observation, together with Corollary 2.13, implies that

- if $(p_j, q_j) - (r_j, s_j) \equiv (1,0)$ or $(0,1) \pmod 2$ for some $j \in \{1, 2, \dots, n\}$, then

$$\Phi_{r_1, s_1, \dots, r_n, s_n}(t_{B_{4(j-1)+1}}) \notin \mathcal{G}_{\Gamma(\mathcal{B}_{\sum_{i=1}^{n}(\varepsilon_{p_i} + 2\varepsilon_{q_i}) 2^{2(i-1)}})},$$

- if $(p_j, q_j) - (r_j, s_j) \equiv (1,1) \pmod 2$ for some $j \in \{1, 2, \dots, n\}$, then

$$\Phi_{r_1, s_1, \dots, r_n, s_n}(t_{B_{4(j-1)+2}}) \notin \mathcal{G}_{\Gamma(\mathcal{B}_{\sum_{i=1}^{n}(\varepsilon_{p_i} + 2\varepsilon_{q_i}) 2^{2(i-1)}})}.$$

Therefore, the assertion follows from Corollary 2.14. $\square$

REMARK 3.4. We can obtain similar results on $E(2)_K$ using a family of Kanenobu knots K with a parity of type $(1,0)$ or $(0,1)$. However, one advantage of Stallings twist knots compared to Kanenobu knots in the construction of inequivalent Lefschetz fibration structures on the same smooth 4-manifold is that it is always possible to construct a pair of inequivalent Lefschetz fibration structures on $E(2)_{K_n}$ for the Stallings twist knot K_n.

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