Simplicial Embeddings Between Multicurve Graphs

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Abstract. We study some graphs associated to a surface, called k-multicurve graphs, which interpolate between the curve complex and the pants graph. Our main result is that, under certain conditions, simplicial embeddings between multicurve graphs are induced by π_1-injective embeddings of the corresponding surfaces. We also prove the rigidity of the multicurve graphs.

1. Introduction

Since the introduction of graphs associated to surfaces, there have been different questions raised about them. One line of questions concerns maps between such graphs. In particular, as the mapping class group usually acts on these complexes by automorphisms, it is interesting to understand whether these are all automorphisms or there are some not induced by a self-homeomorphism of the underlying surface. This is the *rigidity* problem: a graph associated with a surface is *rigid* if its automorphism group is the mapping class group of its surface. The first proof of the rigidity of the curve graph is due to Ivanov [Iva97] in the case of surfaces of genus at least two; subsequently, Korkmaz [Kor99] proved it for low-genus surfaces and Luo [Luo00] proved it in the general case. Margalit [Mar04] proved the rigidity of the pants graph by reducing it to the rigidity of the curve complex. Multiple other graphs have been proven to be rigid (like the Hatcher–Thurston graph [Irm06] and the graph of nonseparating curves [IK07]), and it is conjectured that graphs associated with surfaces having no obvious obstructions (such as being disconnected) should be rigid.

A more subtle problem is studying *simplicial embeddings* (i.e. injective maps preserving the graph structure) between such objects. There are embeddings that correspond to maps between the underlying surfaces: for instance, in the case of the curve graph, a π_1-injective embedding of a surface into another (which corresponds to seeing the first surface as an essential subsurface of the second) induces a simplicial embedding between the curve graphs. A similar construction holds for the pants graph: suppose S_1 is a subsurface of S_2 and choose a k-multicurve ν in the complement of the subsurface, where k is the difference of the complexities of the surfaces. Then we get a simplicial embedding of the pants graph of S_1 into the pants graph of S_2 by sending a pants decomposition μ of S_1 to $\mu \cup \nu$. Aramayona [Ara10] showed that, except in some low-complexity cases, any simplicial embedding between pants graphs arises this way. On the other hand, for

curve graphs, there are examples of simplicial embeddings not coming from π_1-injective embeddings between the underlying surfaces. For instance, puncturing a surface gives a simplicial embedding between the curve graph of the surface and the curve graph of the punctured one (see [AS] for a precise description). Note also that Aramayona [Ara15] described conditions for graphs built from arcs and/or curves to satisfy a similar superrigidity property. It is natural to ask which graphs, besides the pants graph, satisfy these conditions.

Among the reasons to study maps between graphs associated with surfaces, there is the possibility of using such results to say more about mapping class groups. Examples of these results are the results of Ivanov [Iva88] and McCarthy [McC86], saying that for surfaces of genus at least three, the group of automorphisms of the mapping class group is the extended mapping class group. A tool in their proof is Irmak's rigidity of complex of nonseparating curves. In a similar way, the rigidity of the curve complex and of other complexes built from multicurves is used by Ivanov [Iva97] to show that automorphisms between finite index subgroups of the mapping class groups are restrictions of automorphisms of the whole mapping class group.

In this paper, we are interested in a new set of graphs, called *k-multicurve graphs* (for k between one and the complexity of the surface). These graphs can be thought of as graphs interpolating between the curve graph and the pants graph. Indeed, vertices of the k-multicurve graph $\mathcal{C}^{[k]}(S)$ are k-multicurves, and edges correspond to minimal intersection. In particular for $k = 1$ we obtain the curve graph and for k equal to the complexity the pants graph.

Note that other graphs interpolating between the curve and the pants graph have been constructed by Mj [Mj09], where he computes their geometric rank.

Just as in the case of pants graphs, if S_1 can be seen as a subsurface of S_2, we can see any multicurve graph of S_1 as a subgraph of a multicurve graph of S_2. Indeed, fixing a d-multicurve ν on S_2 disjoint from S_1 and mapping a k-multicurve μ of S_1 to $\mu \cup \nu$ gives a simplicial embedding of $\mathcal{C}^{[k]}(S_1)$ into $\mathcal{C}^{[k+d]}(S_2)$. In general though, not every simplicial embedding between multicurve graphs is induced by a map between the underlying surfaces, as examples in Section 2.3 show. On the other hand, if we give some condition on the topology, we can control exactly how simplicial embeddings arise. This is the content of our main theorem:

THEOREM A. *Let S_1 and S_2 be connected finite-type surfaces such that the complexity $\xi(S_1)$ is at least $4 + k_1$. Let $\varphi : \mathcal{C}^{[k_1]}(S_1) \hookrightarrow \mathcal{C}^{[k_2]}(S_2)$ be a simplicial embedding with $k_2 \geq k_1$, and assume that $\xi(S_2) - \xi(S_1) \leq k_2 - k_1$. Then*

$$\xi(S_2) - \xi(S_1) = k_2 - k_1$$

and:

- *if $k_2 = k_1$, then φ is an isomorphism induced by a homeomorphism $f : S_1 \to S_2$;*
- *if $k_2 > k_1$, then there exist a π_1-injective embedding $f : S_1 \to S_2$ and a $(k_2 - k_1)$-multicurve ν on S_2 such that, for any $\mu \in \mathcal{C}^{[k_1]}(S_1)$, we have*

$$\varphi(\mu) = f(\mu) \cup \nu.$$

We say that φ *is induced by* f *and* ν whenever there exist a π_1-injective embedding $f : S_1 \to S_2$ and a multicurve $\nu \subset S_2$ such that the simplicial embedding φ satisfies $\varphi(\mu) = f(\mu) \cup \nu$ for all μ as above.

An analogous control on the topology is necessary when trying to describe injective homomorphisms between mapping class groups. Although in general not much can be said, restricting the topology allows us to say that injective maps between (extended) mapping class groups are induced by maps between the surfaces (see the results of Aramayona and Souto [AS12], Ivanov and McCarthy [IM99], and Bell and Margalit [BM06]).

While proving Theorem A, we also show that the multicurve graphs are rigid.

THEOREM B. *Let* S *be a finite-type surface of complexity* $\xi(S)$ *at least four. For any* k *between* 1 *and* $\xi(S)$, *the natural map*

$$\text{Mod}(S) \to \text{Aut}(\mathcal{C}^{[k]}(S))$$

is a group isomorphism.

This paper is organized as follows. In Section 2, we introduce the objects of study and prove some basic results. Theorem A is proved by induction on k_1, and Section 3 contains the proof of the base case. Using results of Section 3, in Section 4, we deduce the rigidity of the multicurve graphs (Theorem B). In Section 5, we complete the proof of our main theorem by showing the induction step.

2. Preliminaries

Throughout the paper, we assume all surfaces to be of finite type, connected, and of negative Euler characteristic.

2.1. Definitions and Notation

(Almost) all graphs we consider are defined using curves on surfaces. Here by *curve* we mean the free homotopy class of a simple closed curve which is not homotopic to a point or a puncture. We say that two curves α and β are *disjoint* if there are representatives a of α and b of β that are disjoint.

A k-*multicurve* is a collection of k disjoint curves. The maximal size of a multicurve is the *complexity* $\xi = \xi(S) = 3g(S) - 3 + n(S)$ of S; here $g(S)$ and $n(S)$ are the genus and the number of punctures of S, respectively. A maximal size multicurve is a *pants decomposition*. Given a curve α or a multicurve μ, $S \setminus \alpha$ or $S \setminus \mu$ denotes the surface obtained by removing, respectively, a representative of α or disjoint representatives of the curves of μ. Note that, for us, cutting along a curve gives rise to two punctures (and not two boundary components, as for other authors). From the point of view of the multicurve graphs, there is no difference between surfaces with boundary or punctures, and considering open surfaces makes some of our proofs cleaner.

We say that a curve α is *nonseparating* if $S \setminus \alpha$ is connected. The curve is an *outer* curve if $S \setminus \alpha$ has a component of complexity zero (i.e., a *pair of pants*).
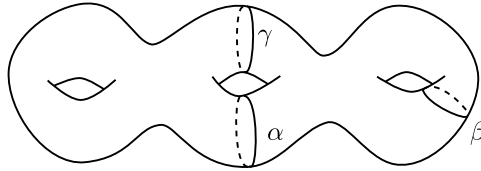
Figure 1 α and β form a nice pair, whereas α and γ do not

Given two disjoint outer or nonseparating curves α and β, we say that they form a *nice pair* if $S \setminus (\alpha \cup \beta)$ has exactly one positive complexity component (see Figure 1). Note that as a consequence of the definition, if (α, β) is a nice pair, then the unique component of $S \setminus (\alpha \cup \beta)$ that is not a pair of pants has complexity $\xi(S) - 2$.

We study the *k-multicurve graphs* for $k \in \{1, \ldots, \xi(S)\}$. For a given k, the vertices of $\mathcal{C}^{[k]}(S)$ are k-multicurves. Two multicurves are joined by an edge if they have a common $(k-1)$-multicurve ν and the remaining two curves intersect minimally on $S \setminus \nu$. Note that the minimum number of intersections of two curves on $S \setminus \nu$ is

- zero if $\xi(S \setminus \nu) \geq 2$,
- one if $S \setminus \nu$ is a once-punctured torus,
- two if $S \setminus \nu$ is a four-punctured sphere.

With this definition, $\mathcal{C}^{[1]}(S)$ is the curve graph $\mathcal{C}(S)$, and $\mathcal{C}^{[\xi(S)]}(S)$ is the pants graph $\mathcal{P}(S)$, so the multicurve graphs can be seen as graphs interpolating between $\mathcal{C}(S)$ and $\mathcal{P}(S)$.

We also need some definitions from graph theory. Given a graph G, we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. A subgraph H of G is an *induced* subgraph if any two vertices of H that are adjacent in G are adjacent in H as well. If W is a set of vertices of G, then the subgraph *spanned* by W is the induced subgraph H with $V(H) = W$. If v is a vertex, then the *star* of v, denoted $\mathrm{St}(v)$, is the subgraph spanned by v and all its adjacent vertices.

An important induced subgraph of $\mathcal{C}^{[k]}(S)$ is the graph spanned by all multicurves containing a fixed multicurve ν (possibly reduces to a single curve). We denote this graph by C_ν. If $d = |\nu|$, then there is a natural isomorphism $C_\nu \simeq \mathcal{C}^{[k-d]}(S \setminus \nu)$ given by $\mu \mapsto \mu \setminus \nu$.

REMARK 2.1. If ν_1 and ν_2 are two distinct $(k-1)$-multicurves such that $C_{\nu_1} \cap C_{\nu_2} \neq \emptyset$, then $C_{\nu_1} \cap C_{\nu_2}$ is exactly one vertex, which is the multicurve $\nu_1 \cup \nu_2$. In particular, $|\nu_1 \cap \nu_2| = k - 2$ (that is, the intersection has maximal possible size).

2.2. Connectedness Results

The connectivity of graphs associated to surfaces has been studied by multiple authors. Note that such graphs are not obviously connected in general: for instance,

the pants graph for infinite-type surfaces is not connected (see [FP15] for the construction of an alternative graph that is connected). On the other hand, Putman [Put08] proved a general criterion that can be used to prove that the multicurve graphs are connected. For completeness, we add here an alternative short proof of this fact. The proof follows same ideas used in [MM00, Section 1.4] to show that the pants graph of the five-holed sphere is connected.

LEMMA 2.2. *The multicurve graphs are connected.*

Proof. We proceed by induction on k. The base case $k = 1$ is the connectivity of the curve graph, which is well known.

Suppose now $\mathcal{C}^{[k-1]}(\Sigma)$ is connected for every surface satisfying $\xi(\Sigma) \geq k - 1$. Consider a surface S of complexity at least k and two k-multicurves $\nu = \{\alpha_1, \ldots, \alpha_k\}$ and $\nu' = \{\beta_1, \ldots, \beta_k\}$. Since the curve graph is connected, there is a path in $\mathcal{C}(S)$ between α_1 and β_2, say $\alpha_1 = \gamma_0, \ldots, \gamma_m = \beta_1$. Choose a k-multicurve ν_1 that contains α_1 and γ_1. Then by induction there is a path in $\mathcal{C}^{[k-1]}(S \setminus \alpha_1)$ between $\nu \setminus \{\alpha_1\}$ and $\nu_1 \setminus \{\alpha_1\}$ and hence a path in $\mathcal{C}^{[k]}(S)$ between ν and ν_1. In the same way, we construct multicurves $\nu_2, \ldots, \nu_m$ such that ν_i contains γ_{i-1} and γ_i for $i \leq m$ and paths between ν_i and ν_{i+1} for $i \leq m$ and between ν_m and ν'. The concatenation of these gives a path in $\mathcal{C}^{[k]}(S)$ between ν and ν'. □

For the proof of Theorem A, we need to consider the subgraph $\mathcal{B}^{[k]}(S)$ of $\mathcal{C}^{[k]}(S)$ given by:

- $V(\mathcal{B}^{[k]}(S)) = \{\mu \mid \mu \text{ contains a nonseparating or outer curve}\}$,
- $E(\mathcal{B}^{[k]}(S)) = \{\mu\nu \mid |\mu \cap \nu| = k - 1, \text{ and the remaining curves form a nice pair}\}$.

For $k = 1$, we simply denote the graph by $\mathcal{B}(S)$. For $k = \xi(S)$, there is no edge in $\mathcal{B}^{[k]}(S)$, but its set of vertices is the same as the set of vertices of $\mathcal{C}^{[k]}(S) = \mathcal{P}(S)$. In general, the graph is not an induced subgraph of the multicurve graph since there are pairs of disjoint curves that do not form nice pairs.

To prove that $\mathcal{B}^{[k]}(S)$ is connected, for $\xi(S)$ big enough with respect to k, we consider two auxiliary graphs. For a surface with genus, we consider the subgraph of $\mathcal{C}(S)$ spanned by nonseparating curves, denoted by $\mathcal{NC}(S)$.

If S has punctures, then by *arc* on S we mean the homotopy class of an embedded arc, starting and ending at punctures, which is not nullhomotopic (relative to the punctures). The *arc graph* $\mathcal{A}(S)$ is the graph whose vertices are arcs and whose edges correspond to a pair of arcs that can be realized disjointly.

LEMMA 2.3. *If $\xi(S) \geq 3$, then the graph $\mathcal{B}(S)$ is connected.*

Proof. Let g be the genus of S.

If $g = 0$, then we have a bijection between outer curves and the subset of $V(\mathcal{A}(S))$ given by arcs with two different endpoints: given an arc between two different punctures, the associated outer curve is the boundary of a regular neighborhood of the arc.
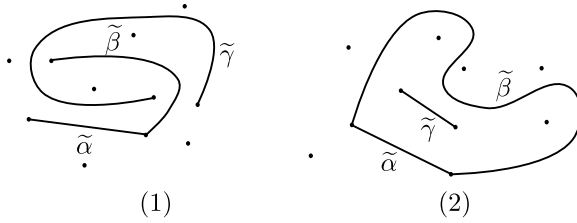
Figure 2 Arcs sharing one or two endpoints and the corresponding arc disjoint from both. In case (2), there are enough cusps either inside or outside the curve formed by $\widetilde{\alpha}$ and $\widetilde{\beta}$
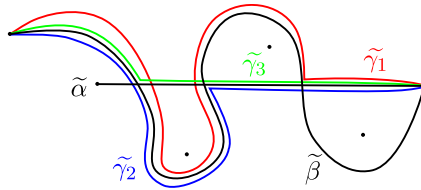

Figure 3 A unicorn path between arcs $\widetilde{\alpha}$ and $\widetilde{\beta}$

Consider two outer curves α and β on S. Because of the condition on the complexity of S, they form a nice pair if and only if they are disjoint. So, in this case, $\mathcal{B}(S)$ is an induced subgraph of $\mathcal{C}(S)$ (spanned by outer curves).

Suppose α and β intersect and consider the associated arcs $\widetilde{\alpha}$ and $\widetilde{\beta}$. If $\widetilde{\alpha}$ and $\widetilde{\beta}$ are adjacent in $\mathcal{A}(S)$, then they must share at least one endpoint, and it is easy to find an arc disjoint from both sharing no endpoints with either of them (see Figure 2). So we get a path between α and β.

Suppose then that $\widetilde{\alpha}$ and $\widetilde{\beta}$ are not adjacent in $\mathcal{A}(S)$. Using the construction of *unicorn paths* described in [HPW15], we obtain a path $\widetilde{\gamma}_0 = \widetilde{\alpha}, \ldots, \widetilde{\gamma}_m = \widetilde{\beta}$ in $\mathcal{A}(S)$ such that no $\widetilde{\gamma}_j$ starts and ends at the same puncture: indeed, a unicorn path from $\widetilde{\alpha}$ to $\widetilde{\beta}$ contains arcs starting at an endpoint of $\widetilde{\alpha}$ and ending at an endpoint of $\widetilde{\beta}$, and these can be chosen to be different (see Figure 3).

Consider the corresponding outer curves $\gamma_0 = \alpha, \ldots, \gamma_m = \beta$. For any j, the arcs $\widetilde{\gamma}_j$ and $\widetilde{\gamma}_{j+1}$ share two endpoints. As described before, there is an outer curve δ_j that is disjoint from both γ_j and γ_{j+1}. Then

$$\alpha = \gamma_0, \delta_0, \gamma_1, \delta_1, \ldots, \gamma_m = \beta$$

is a path in $\mathcal{B}(S)$.

If $g \geq 1$, then for any outer curve, there is a disjoint nonseparating one (and vice versa). Moreover, if an outer curve and a nonseparating one are disjoint, then they form a nice pair. So to show that $\mathcal{B}(S)$ is connected, it is enough to prove that there is a path in $\mathcal{B}(S)$ between any two nonseparating curves. Let α and β be two such curves.
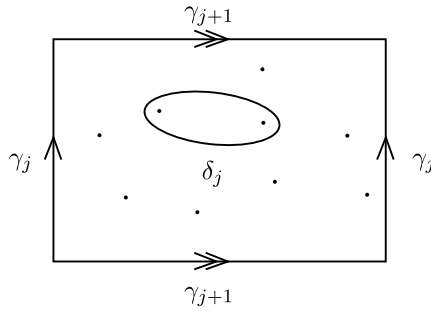
Figure 4   Cutting along γ_j and γ_{j+1}

Suppose $g = 1$. As proven by Schmutz Schaller [SS00], there is a sequence $\gamma_0 = \alpha, \ldots, \gamma_m = \beta$ of nonseparating curves such that γ_j and γ_{j+1} intersect exactly once for every j. Cutting S along γ_j and γ_{j+1} gives us a punctured disk (with at least three punctures), and it is clear that there is an outer curve δ_j disjoint from γ_j and γ_{j+1} (see Figure 4). So the sequence

$$\alpha = \gamma_0, \delta_0, \gamma_1, \delta_1, \gamma_2, \ldots, \delta_{m-1}, \gamma_m = \beta$$

is a path in $\mathcal{B}(S)$.

Suppose now $g \geq 2$. By [FM12, Theorem 4.4] the graph $\mathcal{NC}(S)$ is connected. Let $\gamma_0 = \alpha, \ldots, \gamma_m = \beta$ be a path in $\mathcal{NC}(S)$. If for some j, the curves γ_j and γ_{j+1} do not form a nice pair, then $S \setminus (\gamma_j \cup \gamma_{j+1})$ must have two connected components, Σ_1 and Σ_2. As the genus of S is at least two, one component has genus, and we can pick a nonseparating curve δ_j on it. Then (γ_j, δ_j) and (δ_j, γ_{j+1}) are nice pairs. By adding the curves δ_j when needed we get a path in $\mathcal{B}(S)$ between α and β.   □

COROLLARY 2.4.   *If $\xi(S) \geq k + 2$, then the graph $\mathcal{B}^{[k]}(S)$ is connected.*

Proof. The proof is by induction on k. For $k = 1$, it is the result of Lemma 2.3. Suppose $k \geq 2$ and let $\mu, \nu \in \mathcal{B}^{[k]}(S)$. Fix $\alpha \in \mu$ and $\beta \in \nu$ such that both $\mu \setminus \{\alpha\}$ and $\nu \setminus \{\beta\}$ still contain a nonseparating or outer curve. By the connectivity of $\mathcal{B}(S)$ there is a path $\gamma_0 = \alpha, \gamma_1, \ldots, \gamma_m = \beta$ in $\mathcal{B}(S)$. Choose a k-multicurve η_1 containing α and γ_1. Because $\mathcal{B}^{[k-1]}(S \setminus \alpha)$ is connected by the induction hypothesis, there is a path in it between $\mu \setminus \alpha$ and $\eta_1 \setminus \alpha$, which gives a path between μ and η_1 in $\mathcal{B}^{[k]}(S)$. By repeating the same argument we obtain a path in $\mathcal{B}^{[k]}(S)$ between μ and ν.   □

REMARK 2.5.   Any multicurve μ in $\mathcal{C}^{[k]}(S) \setminus \mathcal{B}^{[k]}(S)$ is adjacent to infinitely many multicurves ν in $\mathcal{B}^{[k]}(S)$: this is vacuously true if $k = \xi(S)$ because $V(\mathcal{C}^{[k]}(S)) = V(\mathcal{B}^{[k]}(S))$. Otherwise:

- if S has genus, then there is a nonseparating curve α disjoint from μ, and we set $\nu := \mu \cup \{\alpha\} \setminus \{\beta\}$ (for some $\beta \in \mu$);

- if S is a sphere, then since μ does not contain any outer curve, there is a component of $S \setminus \mu$ which is a four-holed sphere with only one boundary component belonging to μ. Choose any outer curve α in the four-holed sphere and set $\nu := \mu \cup \{\alpha\} \setminus \{\beta\}$ (for some $\beta \in \mu$).

Since in both cases there are infinitely many choices for the curve α, we get the claim.

2.3. (Counter)Examples

As mentioned in the Introduction, Aramayona [Ara15] stated some conditions ensuring that graphs built from arcs and/or curves satisfy the following superrigidity property: any *alternating* simplicial embedding between such graphs is induced by a map at the surface level. We refer to Aramayona's paper for a precise statement and necessary definitions. Here we show that the multicurve graphs do not satisfy these conditions. Suppose $k \geq 2$ and consider a $(k-2)$-multicurve ν on a surface S of complexity at least $k+1$. Let α, β, and γ be curves on S, mutually disjoint and disjoint from ν. Then $\mu_1 = \{\alpha, \beta\} \cup \nu$, $\mu_2 = \{\beta, \gamma\} \cup \nu$, and $\mu_3 = \{\alpha, \gamma\} \cup \nu$ form a triangle, and their intersection is ν, which is an extendable set of deficiency 2. Hence, for the conditions to be satisfied, there should be an alternating circuit containing them. However, since $\mu_i \cap \mu_j$ contains $k-1$ curves for all pairs $i \neq j$, they cannot be contained in any alternating circuit.

We now give two examples showing that the hypothesis $\xi(S_2) - \xi(S_1) \leq k_2 - k_1$ in Theorem A is necessary. Both examples are simple generalizations of similar constructions for the curve complex (see [Ara10] and [AS]).

First Example, $k_1 = k_2$. Consider a closed surface S of genus $g \geq 2$ and choose $k \leq 3g - 4$. Fix a hyperbolic structure on S. Pick n points $p_1, \ldots, p_n$ in the complement of all simple closed curves (which can be done by noting that the set of simple closed geodesics is a countable union of measure zero sets; in fact, Birman and Series [BS85] showed that the union of all simple closed geodesics is nowhere dense). We now have a natural way to associate with a curve on S a curve on the surface $\widetilde{S} := S \setminus \{p_1, \ldots, p_n\}$: for α on S, its geodesic representative lies in $\widetilde{S}$, so we can consider the homotopy class of the geodesic representative in $\widetilde{S}$. Because this map respects disjointness, it gives a simplicial embedding $\mathcal{C}^{[k]}(S) \hookrightarrow \mathcal{C}^{[k]}(\widetilde{S})$. Since S is closed, the embedding cannot be induced by an embedding of the surfaces.

Second Example, $k_2 > k_1$. Consider a closed surface S of complexity at least $k+1$ (for some positive k) and fix a hyperbolic structure on it. Choose d points $p_1, \ldots, p_d$ in the complement of the union of all simple closed geodesics on S to obtain an embedding $\mathcal{C}^{[k]}(S) \hookrightarrow \mathcal{C}^{[k]}(S \setminus \{p_1, \ldots, p_d\})$ as before. Clearly,

$$\mathcal{C}^{[k]}(S \setminus \{p_1, \ldots, p_d\}) \simeq \mathcal{C}^{[k]}(\widetilde{S}),$$

where $\widetilde{S}$ has the same genus as S and d boundary components, so we have a simplicial embedding $\Phi : \mathcal{C}^{[k]}(S) \to \mathcal{C}^{[k]}(\widetilde{S})$. By gluing pairs of pants to each boundary component of $\widetilde{S}$ we get a new surface Σ of the same genus as S with

$2d$ boundary components and a distinguished d-multicurve ν (given by the curves corresponding to the boundary components of $\widetilde{S}$). Then

$$\varphi : \mathcal{C}^{[k]}(S) \to \mathcal{C}^{[k+d]}(\Sigma),$$
$$\mu \mapsto \Phi(\mu) \cup \nu$$

is a simplicial embedding that cannot be induced by an embedding at the surface level.

3. Embeddings of the Curve Graph Into Multicurve Graphs

In this section, we prove Theorem A for $k_1 = 1$. In fact, we need a slightly weaker assumption on the complexity of S_1. We prove the following:

PROPOSITION 3.1. *Let S_1 and S_2 be surfaces such that $\xi(S_1) \geq 4$. Consider a simplicial embedding*

$$\varphi : \mathcal{C}(S_1) \hookrightarrow \mathcal{C}^{[k]}(S_2),$$

where $\xi(S_2) - \xi(S_1) \leq k - 1$ and $k \geq 1$. Then $\xi(S_2) - \xi(S_1) = k - 1$ and:
- *if $k = 1$, then φ is an isomorphism induced by a homeomorphism $f : S_1 \to S_2$;*
- *if $k > 1$, then there exist a π_1-injective embedding $f : S_1 \to S_2$ and a $(k-1)$-multicurve ν on S_2 such that, for any $\mu \in \mathcal{C}^{[k_1]}(S_1)$, we have*

$$\varphi(\mu) = f(\mu) \cup \nu.$$

Proof. If $k = 1$, then $\xi(S_2) \leq \xi(S_1)$, so by Shackleton's result in [Sha07], $\xi(S_2) = \xi(S_1)$, and φ is an isomorphism induced by a homeomorphism $f : S_1 \to S_2$.

Suppose $k \geq 2$. Let $\alpha \in \mathcal{C}(S_1)$ and consider its image $a = \varphi(\alpha) = \{\alpha_1, \ldots, \alpha_k\}$. Consider β adjacent to α; then $b = \varphi(\beta)$ is adjacent to a, so we can assume, without loss of generality, that $b = \varphi(\beta) = \{\alpha_1, \ldots, \alpha_{k-1}, \beta_k\}$.

Note that $\text{St}(a) \subseteq C_{\alpha_1} \cup C_{\nu_1}$ for $\nu_1 = \{\alpha_2, \ldots, \alpha_k\}$.

REMARK 3.2. If $k < \xi(S_2)$, then given $x \in \text{St}(a) \cap C_{\alpha_1}$ different from a, there exists a unique $x' \in \text{St}(a) \cap C_{\nu_1}$ different from a that is adjacent to x. Indeed, if $x = \{\alpha_1, \ldots, \beta_i, \ldots, \alpha_k\}$, then $x' = \nu_1 \cup \{\beta_i\}$. Conversely, given $x' \in \text{St}(a) \cap C_{\nu_1}$ different from a, there exists a unique $x \in \text{St}(a) \cap C_{\alpha_1}$ that is adjacent to x' and is different from a. Note that β_i must be disjoint from α_i for all $i = 1, \ldots, k$. Hence, if $k = \xi(S_2)$, such β_i does not exist, and in particular, given $x \in \text{St}(a) \cap C_{\alpha_1}$ different from a, there exists no $x' \in \text{St}(a) \cap C_{\nu_1}$ different from a that is adjacent to x.

We want to show that $\varphi(\text{St}(\alpha)) \subseteq C_{\alpha_1}$. Let $\gamma \in \text{St}(\alpha)$ be different from β. Since (for $\xi(S_1) \geq 2$)

$$\text{St}(\alpha) \setminus \{\alpha\} \simeq \mathcal{C}(S_1 \setminus \alpha),$$

which is connected, there exists a path

$$\gamma_0 = \gamma, \gamma_1, \ldots, \gamma_m = \beta$$

in $\text{St}(\alpha) \setminus \{\alpha\}$. Consider γ_{m-1} and its image $c_{m-1} = \varphi(\gamma_{m-1})$. Since c_{m-1} is adjacent to b, by Remark 3.2 it must belong to C_{α_1} if $\xi(S_2) = k$. If $\xi(S_2) > k$ and we assume by contradiction that $c_{m-1} \in C_{\nu_1}$, again by Remark 3.2 c_{m-1} is completely determined by being adjacent to b. Since $\xi(S_1) \geq 4$, there exists a curve

δ on S_1 that is disjoint from α, β, and γ_{m-1}. So $d = \varphi(\delta) \in \varphi(\mathrm{St}(\alpha))$, and it is adjacent both to b and c_{m-1}. This implies that either $d = b$ or $d = c_{m-1}$, which contradicts the injectivity of φ. Thus again $c_{m-1} \in C_{\alpha_1}$. By repeating the argument we deduce that $\gamma \in C_{\alpha_1}$, so $\varphi(\mathrm{St}(\alpha)) \subseteq C_{\alpha_1}$.

Note that the proof works for every curve in the intersection $a \cap b$, so what we actually obtain is that $\varphi(\mathrm{St}(\alpha)) \subseteq C_\nu$, where $\nu = a \cap b$.

Now, consider any other curve η on S_1. We can construct a path in $\mathcal{C}(S_1)$ between α and η and by what we just proved the image of any edge in the path is in C_ν. So $\mathrm{Im}(\varphi) \subseteq C_\nu$. We have

$$
\begin{array}{ccc}
\mathcal{C}(S_1) & \xrightarrow{\ \varphi\ } & C_\nu \\
 & \searrow{\scriptstyle \Phi} & \ \ \simeq \downarrow \theta \\
 & & \mathcal{C}(S_2 \setminus \nu)
\end{array}
$$

where θ is the isomorphism given by $\theta(\nu \cup \{\varepsilon\}) = \varepsilon$, and Φ is defined to be the composition $\theta \circ \varphi$. Since φ is a simplicial embedding, Φ is a simplicial embedding too. Moreover, $\xi(S_2 \setminus \nu) \le \xi(S_1)$, so by Shackleton's result in [Sha07], $\xi(S_2 \setminus \nu) = \xi(S_1)$, that is, $\xi(S_2) - \xi(S_1) = k - 1$, and Φ is induced by a homeomorphism $f : S_1 \to S_2 \setminus \nu$. This gives us a π_1-injective map $F : S_1 \to S_2$ (given by composing f with the natural injection $S_2 \setminus \nu \hookrightarrow S_2$), and φ is induced by F and ν. $\qquad\square$

4. Rigidity of Multicurves Graphs

This section is dedicated to the proof of Theorem B, which is done by induction on k. Recall that the mapping class group $\mathrm{Mod}(S)$ of a surface S is the group of homeomorphisms of S up to isotopy. The action

$$
\mathrm{Mod}(S) \times \mathcal{C}^{[k]}(S) \to \mathcal{C}^{[k]}(S),
$$
$$
(f, \alpha) \mapsto f(\alpha) := \text{homotopy class of } f_0(\alpha_0),
$$
$$
\text{where } f_0 \in f \text{ and } \alpha_0 \in \alpha
$$

is well defined and induces a map $\mathrm{Mod}(S) \to \mathrm{Aut}(\mathcal{C}^{[k]}(S))$. It is easy to see that this map is a group homomorphism, and the main content of Theorem B is that, for $\xi(S) \ge 4$, it is a bijection.

For $k = 1$, $\mathcal{C}^{[1]}(S)$ is the 1-skeleton of the curve complex, which is rigid if $\xi(S) \ge 4$ (see [Iva97; Kor99; Luo00]).

For the induction step, assume that S has complexity at least four and that

$$
\mathrm{Aut}(\mathcal{C}^{[k-1]}(S)) \simeq \mathrm{Mod}(S).
$$

We will show that

$$
\mathrm{Aut}(\mathcal{C}^{[k]}(S)) \simeq \mathrm{Aut}(\mathcal{C}^{[k-1]}(S)).
$$

The idea for the proof of this fact is the same as in Margalit's proof of the rigidity of the pants graph [Mar04]. Define the map

$$
\theta : E(\mathcal{C}^{[k]}(S)) \longrightarrow V(\mathcal{C}^{[k-1]}(S)),
$$
$$
e = \alpha\beta \longmapsto \mu = \alpha \cap \beta.
$$

It is clearly a surjection.

LEMMA 4.1. *Let $A \in \text{Aut}(\mathcal{C}^{[k]}(S))$ and suppose $e, f \in E(\mathcal{C}^{[k]}(S))$ satisfy $\theta(e) = \theta(f)$. Then $\theta(A(e)) = \theta(A(f))$.*

Proof. Note that $\theta(e) = \theta(f) = \mu$ if and only if $e, f \in C_\mu$. This implies that $A(e), A(f) \in A(C_\mu)$. The automorphism A induces the map

$$\psi : \mathcal{C}(S \setminus \mu) \to \mathcal{C}^{[k]}(S),$$
$$a \mapsto A(\{a\} \cup \mu),$$

which is a simplicial embedding. So, by the results of Section 3 there exists a $(k-1)$-multicurve ν such that the image of ψ is contained in C_ν. This implies that $\theta(A(e)) = \nu = \theta(A(f))$. $\qquad \square$

REMARK 4.2. Since we need only the existence of ν such that $\text{Im}(\psi) \subseteq C_\nu$ and not the full base case, we do not need any condition on the complexities of the components of $S \setminus \mu$.

By Lemma 4.1 any $A \in \text{Aut}(\mathcal{C}^{[k]}(S))$ induces a well-defined map

$$\varphi(A) : V(\mathcal{C}^{[k-1]}(S)) \to V(\mathcal{C}^{[k-1]}(S))$$

given by

$$\varphi(A)(\mu) = \theta(A(e)),$$

where e is any edge of $\mathcal{C}^{[k]}(S)$ such that $\theta(e) = \mu$. Note that $\varphi(A)$ is actually a bijection since it is easy to show that its inverse exists and is equal to $\varphi(A^{-1})$.

We want to prove that $\varphi(A)$ is an automorphism of $\mathcal{C}^{[k-1]}(S)$. We first show that it sends edges to edges.

LEMMA 4.3. *For any $A \in \text{Aut}(\mathcal{C}^{[k]}(S))$, $\varphi(A)$ sends edges to edges.*

Proof. Suppose $\mu\nu$ is an edge of $\mathcal{C}^{[k-1]}(S)$. This means that the intersection $C_\mu \cap C_\nu$ is not empty, so it contains exactly one vertex $\mu \cup \nu$. As a consequence, $A(C_\mu)$ and $A(C_\nu)$ intersect in exactly one vertex too, and thus $C_{\varphi(A)(\mu)} \cap C_{\varphi(A)(\nu)}$ is not empty. By Remark 2.1 this means that either $C_{\varphi(A)(\mu)} = C_{\varphi(A)(\nu)}$ or $C_{\varphi(A)(\mu)} \cap C_{\varphi(A)(\nu)}$ is one vertex. But if $C_{\varphi(A)(\mu)} = C_{\varphi(A)(\nu)}$, then $\varphi(A)(\mu) = \varphi(A)(\nu)$, which contradicts the injectivity of $\varphi(A)$. Hence $C_{\varphi(A)(\mu)} \cap C_{\varphi(A)(\nu)}$ is one vertex, so $\varphi(A)(\mu) \cup \varphi(A)(\nu)$ is a k-multicurve, that is, $\varphi(A)(\mu)\varphi(A)(\nu)$ is an edge of $\mathcal{C}^{[k-1]}(S)$. $\qquad \square$

We have seen that $\varphi(A)$ is a simplicial map of $\mathcal{C}^{[k-1]}(S)$ to itself and is a bijection. Since the same holds for $\varphi(A)^{-1} = \varphi(A^{-1})$, $\varphi(A)$ is an automorphism of $\mathcal{C}^{[k-1]}(S)$. Therefore we have a map

$$\varphi : \text{Aut}(\mathcal{C}^{[k]}(S)) \to \text{Aut}(\mathcal{C}^{[k-1]}(S)).$$

We claim that φ is a group isomorphism.

LEMMA 4.4. *The map φ is a group homomorphism.*

Proof. Consider $A, B \in \mathrm{Aut}(\mathcal{C}^{[k]}(S))$ and let μ be a $(k-1)$-multicurve. Suppose $\mu = \theta(e)$. Then

$$\varphi(AB)(\mu) = \theta(AB(e)) = \theta(A(B(e))) = \varphi(A)(\theta(B(e)))$$
$$= \varphi(A)(\varphi(B)(\theta(e))) = (\varphi(A) \circ \varphi(B))(\mu),$$

which shows that $\varphi(AB) = \varphi(A) \circ \varphi(B)$, as claimed. $\hspace{1em}\square$

LEMMA 4.5. *The map φ is injective.*

Proof. Since φ is a group homomorphism, it suffices to prove that its kernel is reduced to the identity. Suppose that $\varphi(A)$ it the identity on $\mathcal{C}^{[k-1]}(S)$ and let $\mu = \{\alpha_1, \ldots, \alpha_k\}$ be a k-multicurve. Define μ_i to be $\mu \setminus \{\alpha_i\}$ for any i between 1 and k. Then $\mu_i = \theta(e_i)$, where we can choose e_i incident to α for every i (say $e_i = \mu v_i$). Since, by hypothesis, $\varphi(A)$ is the identity, we obtain

$$\mu_i = \varphi(A(\mu_i)) = \theta(A(e_i)) = A(\mu) \cap A(v_i),$$

which means that $\mu_i \subseteq A(\mu)$ for every i. So $A(\mu) = \mu$, that is, A is the identity in $\mathcal{C}^{[k]}(S)$. $\hspace{1em}\square$

LEMMA 4.6. *The map φ is surjective.*

Proof. Let $F : \mathrm{Mod}(S) \to \mathrm{Aut}(\mathcal{C}^{[k]}(S))$ and $G : \mathrm{Mod}(S) \to \mathrm{Aut}(\mathcal{C}^{[k-1]}(S))$ be the natural maps from the mapping class group to the multicurve graphs. By the induction hypothesis we know that G is surjective. To show that φ is surjective, it suffices to prove that $G = \varphi \circ F$. Let f be any mapping class, and $\mu = \{a_1, \ldots, a_{k-1}\}$ any $(k-1)$-multicurve. Suppose $e = \alpha\beta$ is an edge of $\mathcal{C}^{[k]}(S)$ such that $\theta(e) = \mu$, say, $\alpha = \mu \cup \{a\}$ and $\beta = \mu \cup \{b\}$. Then

$$\varphi(F(f))(\mu) = \theta(F(f)(\alpha)F(f)(\beta)) = F(f)(\alpha) \cap F(f)(\beta)$$
$$= \{f(a_1), \ldots, f(a_{k-1}), f(a)\} \cap \{f(a_1), \ldots, f(a_{k-1}), f(b)\}$$
$$= \{f(a_1), \ldots, f(a_{k-1})\} = G(f)(\mu). \hspace{1em}\square$$

So $\mathrm{Aut}(\mathcal{C}^{[k]}(S)) \simeq \mathrm{Aut}(\mathcal{C}^{[k-1]}(S)) \simeq \mathrm{Mod}(S)$.

5. Proof of Theorem A

We prove Theorem A by induction on k_1. The base case $k_1 = 1$ is proven in Section 3.

For the induction step, assume that the theorem holds until $k_1 - 1$ and consider a simplicial embedding $\varphi : \mathcal{C}^{[k_1]}(S_1) \hookrightarrow \mathcal{C}^{[k_2]}(S_2)$ satisfying the hypothesis of Theorem A. The proof will proceed as follows: we show that there exist a π_1-injective embedding $f : S_1 \to S_2$ and a multicurve $\nu \subset S_2$ such that φ is induced by f and ν on the *subgraph* $\mathcal{B}^{[k_1]}(S_1)$ defined in Section 2.2. Next, we show that this implies that the image of φ must lie in $\mathcal{C}_\nu$, giving us a map Φ defined by $\Phi(\mu) = \varphi(\mu) \setminus \nu$. This means that f is induced by Φ and ν everywhere. Finally, we show that Φ is actually (induced by) f, completing the proof.

Let α be a nonseparating curve on S_1. Then φ induces a simplicial embedding

$$\varphi_\alpha : \mathcal{C}^{[k_1-1]}(S_1 \setminus \alpha) \simeq C_\alpha \hookrightarrow \mathcal{C}^{[k_2]}(S_2)$$

that satisfies the hypothesis of Theorem A. So by induction we know that

$$\xi(S_2) - \xi(S_1 \setminus \alpha) = k_2 - (k_1 - 1),$$

and thus

$$\xi(S_2) - \xi(S_1) = k_2 - k_1.$$

Define $d := k_2 - k_1$. The induction hypothesis also tells us that φ_α is induced by a π_1-injective embedding f_α of $S_1 \setminus \alpha$ into S_2 and a $(d+1)$-multicurve ν_α of S_2.

If α is an outer curve on S_1 that cuts off a pair of pants P_α, then the same holds if we replace $S_1 \setminus \alpha$ by $S_1 \setminus \alpha \setminus P_\alpha$.

For ease of notation, in what follows, we interpret a d-multicurve when $d = 0$ as the empty set. Moreover, given a possibly disconnected surface S, we denote by $(S)^{\mathrm{pos}}$ the union of the components of S that are not pairs of pants. For instance, if α is an outer curve on S_1 cutting off a pair of pants P_α, then $(S_1 \setminus \alpha)^{\mathrm{pos}} = S_1 \setminus \alpha \setminus P_\alpha$.

LEMMA 5.1. *If α and β form a nice pair, then there exist a π_1-injective embedding $f : S_1 \to S_2$ and a d-multicurve ν on S_2 such that φ is induced by f and ν on $C_\alpha \cup C_\beta$. Moreover, if $d = 0$, then f can be taken to be a homeomorphism.*

Note that if α, β form a nice pair, then $(S_1 \setminus (\alpha \cup \beta))^{\mathrm{pos}}$ has complexity $\xi(S_1) - 2 \geq k_1 + 2 \geq 4$ and hence satisfies Theorem B. In particular, the curve graph $\mathcal{C}((S_1 \setminus (\alpha \cup \beta))^{\mathrm{pos}})$ and, for $k_1 \geq 3$, the multicurve graph $\mathcal{C}^{[k_1-2]}((S_1 \setminus (\alpha \cup \beta))^{\mathrm{pos}})$ are rigid.

Assuming Lemma 5.1, fix a nice pair (α, β) and consider f and ν obtained via the lemma. We will prove the following:

LEMMA 5.2. *The embedding φ is induced by f and ν on $\mathcal{B}^{[k_1]}(S_1)$.*

We end the proof showing that if φ is induced by f and ν on $\mathcal{B}^{[k_1]}(S_1)$, then it is induced by f and ν everywhere.

Proof of Lemma 5.1. As seen at the beginning of this section, by induction we get two π_1-injective embeddings $f_\alpha : (S_1 \setminus \alpha)^{\mathrm{pos}} \to S_2$, $f_\beta : (S_1 \setminus \beta)^{\mathrm{pos}} \to S_2$ and two $(d+1)$-multicurves on S_2 such that φ is induced by f_α, ν_α on C_α and by f_β, ν_β on C_β.

Our objective is to show that $\nu_\alpha \cap \nu_\beta$ is the d-multicurve we are looking for (in particular, it is empty when $d = 0$) and that f_α and f_β define a π_1-injective embedding of S_1 into S_2 that is a homeomorphism when $d = 0$.

Case $k_1 = 2$: since $\varphi(\{\alpha, \beta\}) = \nu_\alpha \cup \{f_\alpha(\beta)\} = \nu_\beta \cup \{f_\beta(\alpha)\}$, $\nu_\alpha \cap \nu_\beta$ is a d-multicurve if (and only if) $\nu_\alpha \neq \nu_\beta$. So suppose by contradiction that $\nu_\alpha = \nu_\beta$. We know that

$$\varphi(C_\alpha \simeq \mathcal{C}((S_1 \setminus \alpha))^{\mathrm{pos}}) \subseteq C_{\nu_\alpha} \simeq \mathcal{C}((S_2 \setminus \nu_\alpha)^{\mathrm{pos}})$$

and
$$\varphi(C_\beta \simeq \mathcal{C}((S_1 \setminus \beta))^{\text{pos}}) \subseteq C_{v_\alpha} \simeq \mathcal{C}((S_2 \setminus v_\alpha)^{\text{pos}}).$$
But if we compute the complexities, then we have
$$\xi((S_2 \setminus v_\alpha)^{\text{pos}}) = \xi(S_2) - (d+1) = (\xi(S_2) - d) - 1 = \xi(S_1) - 1$$
$$= \xi((S_1 \setminus \alpha)^{\text{pos}}) = \xi((S_1 \setminus \beta)^{\text{pos}}).$$
By a result in [Sha07] the inclusions are actually equalities, that is, $\varphi(C_\alpha) = \varphi(C_\beta) = C_{v_\alpha}$.

Choose $\mu \in C_{v_\alpha}$ different from $\varphi(\{\alpha, \beta\})$. Then there exist $a \in C_\alpha$ and $b \in C_\beta$ such that $\varphi(a) = \varphi(b) = \mu$. By the injectivity of φ this means that $a = b \in C_\alpha \cap C_\beta$. But the only multicurve in the intersection is $\{\alpha, \beta\}$, and this contradicts the fact that $\mu \neq \varphi(\{\alpha, \beta\})$. So v_α and v_β must be different, and $v := v_\alpha \cap v_\beta$ is a d-multicurve. In particular, if $d = 0$, then v_α and v_β are distinct single curves, and v is empty.

Consider any curve γ on $S_1 \setminus (\alpha \cup \beta)$. Since $\{\alpha, \gamma\}$ and $\{\beta, \gamma\}$ are multicurves joined by an edge, so are their images via φ. We have
$$\varphi(\{\alpha, \gamma\}) = v_\alpha \cup \{f_\alpha(\gamma)\},$$
$$\varphi(\{\beta, \gamma\}) = v_\beta \cup \{f_\beta(\gamma)\},$$
and since $v_\alpha \neq v_\beta$, we must have $f_\alpha(\gamma) = f_\beta(\gamma)$. Thus f_α and f_β induce the same map
$$\mathcal{C}((S_1 \setminus (\alpha \cup \beta))^{\text{pos}}) \to \mathcal{C}((S_2 \setminus \varphi(\{\alpha, \beta\}))^{\text{pos}}).$$
Since $\xi((S_1 \setminus (\alpha \cup \beta))^{\text{pos}}) = \xi((S_2 \setminus \varphi(\{\alpha, \beta\}))^{\text{pos}})$, we can assume that $g_\alpha = f_\alpha|_{(S_1 \setminus (\alpha \cup \beta))^{\text{pos}}}$ and $g_\beta = f_\beta|_{(S_1 \setminus (\alpha \cup \beta))^{\text{pos}}}$ are homeomorphisms onto $(S_2 \setminus \varphi(\{\alpha, \beta\}))^{\text{pos}}$. Moreover, $g_\beta^{-1} \circ g_\alpha$ induces the identity map on $\mathcal{C}((S_1 \setminus (\alpha \cup \beta))^{\text{pos}})$, which implies (by the rigidity of the curve graph) that the class of $g_\beta^{-1} \circ g_\alpha$ is trivial in $\mathrm{Mod}((S_1 \setminus (\alpha \cup \beta))^{\text{pos}})$. Thus $g_\alpha = g_\beta \circ h$, where h is a diffeomorphism of $(S_1 \setminus (\alpha \cup \beta))^{\text{pos}}$ isotopic to the identity. Up to replacing f_β with an extension of $g_\beta \circ h$, we can assume f_α and f_β agree on $(S_1 \setminus (\alpha \cup \beta))^{\text{pos}}$. Hence they define a map $f : S_1 \to S_2$. By elementary topology arguments, f is a π_1-injective embedding.

When $d = 0$, $f(\alpha) = v_\alpha$ and $f(\beta) = v_\beta$. In particular, we know that v_α and v_β are of the same type (outer or nonseparating) as α and β (by [Sha07, Lemmas 10 and 11]). Now f_α is a map inducing a simplicial embedding between the curve complexes of $(S_1 \setminus \alpha)^{\text{pos}}$ and $(S_2 \setminus v_\alpha)^{\text{pos}}$. The surfaces have the same complexity, so by induction[1] we could choose it to be a homeomorphism between $(S_1 \setminus \alpha)^{\text{pos}}$ and $(S_2 \setminus v_\alpha)^{\text{pos}}$. The same holds for f_β. As a consequence, f can be chosen to be a homeomorphism between S_1 and S_2.

Case $k_1 \geq 3$: consider $\mu \in C_{\alpha \cup \beta}$, that is, $\mu = \{\alpha\} \cup \{\beta\} \cup \widetilde{\mu}$, where $\widetilde{\mu}$ is a $(k_1 - 2)$-multicurve. Then
$$\varphi(\mu) = v_\alpha \cup f_\alpha(\mu \setminus \{\alpha\}) = v_\alpha \cup \{f_\alpha(\beta)\} \cup f_\alpha(\widetilde{\mu})$$
$$= v_\beta \cup f_\beta(\mu \setminus \{\beta\}) = v_\beta \cup \{f_\beta(\alpha)\} \cup f_\beta(\widetilde{\mu}).$$

---

[1] Actually, for $k_1 = 2$, the tool we need is Shackleton's result in [Sha07] used in the base case. For $k_1 \geq 3$, we really need the induction hypothesis.

If we let μ vary in $C_{\alpha \cup \beta}$, that is, we let $\widetilde{\mu}$ vary in $\mathcal{C}^{[k_1-2]}(S_1 \setminus (\alpha \cup \beta))$, then, by the injectivity of φ, $f_\alpha(\widetilde{\mu})$ and $f_\beta(\widetilde{\mu})$ vary (and the rest is fixed since it does not depend on $\widetilde{\mu}$), so

$$\nu_\alpha \cup \{f_\alpha(\beta)\} = \nu_\beta \cup \{f_\beta(\alpha)\} \quad \text{and} \quad f_\alpha(\widetilde{\mu}) = f_\beta(\widetilde{\mu}).$$

This implies that f_α and f_β induce the same map from

$$\mathcal{C}^{[k_1-2]}((S_1 \setminus (\alpha \cup \beta))^{\mathrm{pos}}) \to \mathcal{C}^{[k_1-2]}((S_2 \setminus (\nu_\alpha \cup f_\alpha(\beta)))^{\mathrm{pos}}),$$

and with the same argument as in the case $k_1 = 2$, this time by Theorem B, we obtain that $f_\alpha|_{(S_1 \setminus (\alpha \cup \beta))^{\mathrm{pos}}} = f_\beta|_{(S_1 \setminus (\alpha \cup \beta))^{\mathrm{pos}}} \circ h$, where h is a diffeomorphism of $S_1 \setminus (\alpha \cup \beta)$ isotopic to the identity. Again, up to replacing f_β with an extension of $f_\beta|_{S_1 \setminus (\alpha \cup \beta)} \circ h$, we can assume that f_α and f_β agree on $(S_1 \setminus (\alpha \cup \beta))^{\mathrm{pos}}$, so that they define a map $f : S_1 \to S_2$. As in the case $k_1 = 2$, f is a π_1-injective embedding and a homeomorphism if $d = 0$. Since f is injective, $f(\alpha) \neq f(\beta)$. This implies that $\nu_\alpha \cap \nu_\beta$ is a d-multicurve, which we denote by ν. □

REMARK 5.3. Different choices in this construction give different maps from S_1 to S_2. This is not surprising since if there is a map inducing a simplicial embedding, then it is not unique. For instance, if $\psi : \mathcal{C}^{[k_1]}(S_1) \hookrightarrow \mathcal{C}^{[k_2]}(S_2)$ is a simplicial embedding induced by a map g, it is also induced by $g \circ h$, where h is any homeomorphism of S_1 isotopic to the identity. On the other hand, the multicurve is uniquely determined by the simplicial embedding.

Proof of Lemma 5.2. Let f and ν be given by the nice pair (α, β) according to Lemma 5.1. Consider any curve $\gamma \in \mathcal{B}(S_1)$. Since $\mathcal{B}(S_1)$ is connected, there is a path

$$\gamma_0 = \alpha, \gamma_1, \ldots, \gamma_m = \gamma.$$

Consider α and γ_1. Since they form a nice pair, they determine a map f' and a d-multicurve ν' by Lemma 5.1, and we can assume that $f' = f$ on $(S_1 \setminus \alpha)^{\mathrm{pos}}$ (since they both agree with f_α). Note that this implies $\nu' = \nu$ since $f'(\mu) \cup \nu' = \varphi(\mu) = f(\mu) \cup \nu$ for any k-multicurve μ on $(S_1 \setminus \alpha)^{\mathrm{pos}}$. Our goal is to show that $f' = f$ on S_1. This implies, by repeating the argument along the path, that φ is induced by f and ν on C_γ.

If α is nonseparating, then we must have $f = f'$ on S_1 by continuity. So assume that α is an outer curve. Since $f' = f$ on $(S_1 \setminus \alpha)^{\mathrm{pos}}$, the two maps differ a priori by a twist about α. To see that this is not the case, we will show that there exists a curve δ intersecting α such that $f(\delta) = f'(\delta)$.

First consider the case where β and γ_1 are disjoint. Let δ be a curve disjoint from both β and γ_1 but intersecting α. If $k_1 \geq 3$, then there exists a k_1-multicurve μ containing β, γ_1, and δ. Since $\mu \in C_\beta \cup C_{\gamma_1}$,

$$f(\mu) \cup \nu = \varphi(\mu) = f'(\mu) \cup \nu.$$

Moreover, $\mu \setminus \delta \in (S_1 \setminus \alpha)^{\mathrm{pos}}$, so $f(\mu \setminus \delta) = f'(\mu \setminus \delta)$, and we must have $f(\delta) = f'(\delta)$. If $k_1 = 2$, then we instead consider the multicurves $\mu_1 = \{\beta, \delta\}$ and $\mu_2 =$
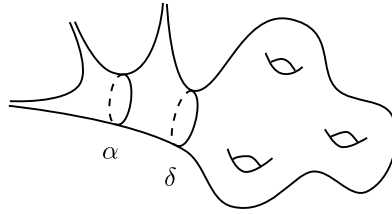
Figure 5 δ is an outer curve in $(S_1 \setminus \alpha)^{\text{pos}}$ but not in S_1

$\{\gamma_1, \delta\}$. Since μ_1 and μ_2 are adjacent in the multicurve graph, so are $\varphi(\mu_1)$ and $\varphi(\mu_2)$. In particular, they differ by exactly one curve. Now,

$$\varphi(\mu_1) = f(\beta) \cup f(\delta) \cup \nu$$

and

$$\varphi(\mu_2) = f'(\gamma_1) \cup f'(\delta) \cup \nu = f(\gamma_1) \cup f'(\delta) \cup \nu.$$

This implies $f(\delta) = f'(\delta)$ because $f(\beta) \neq f(\gamma_1)$.

Now consider the case where β and γ_1 intersect. We can show that there is a path

$$\beta_0 = \beta, \beta_1, \ldots, \beta_m = \gamma_1$$

in $\mathcal{C}((S_1 \setminus \alpha)^{\text{pos}})$ such that (α, β_i) form a nice pair for each i. To prove this, the idea is to construct curves in $\mathcal{B}((S_1 \setminus \alpha)^{\text{pos}})$ with a similar procedure as that for Lemma 2.3. We just need to be careful not to choose in the process any outer curve of $(S_1 \setminus \alpha)^{\text{pos}}$ that bounds a pair of pants with α on S_1. Such a choice would give a curve that is not outer in S_1 (as in Figure 5). This can be done because $\xi((S_1 \setminus \alpha)^{\text{pos}}) = \xi(S_1) - 1 \geq 5$.

By replacing β and γ_1 by β_i and β_{i+1} we can apply this argument along this path to obtain again that $f(\delta) = f'(\delta)$. □

The final step is showing that if φ is induced by f and ν on $\mathcal{B}^{[k_1]}(S_1)$, then it is induced by f and ν everywhere.

LEMMA 5.4. *The image of φ is in C_ν.*

Proof. Suppose not. Then let $\mu \in \mathcal{C}^{[k_1]}(S_1) \setminus \mathcal{B}^{[k_1]}(S_1)$ be such that $\varphi(\mu) \notin C_\nu$, that is, there is some curve $\alpha \in \nu$ such that $\alpha \notin \varphi(\mu)$. Consider any $\bar{\mu} \in \mathcal{B}^{[k_1]}(S_1)$ adjacent to μ ($\bar{\mu}$ exists by Remark 2.5). Its image is

$$\varphi(\bar{\mu}) = \nu \cup f(\bar{\mu}).$$

Then

$$|\varphi(\mu) \cap \varphi(\bar{\mu})| = k_2 - 1,$$

so

$$\varphi(\mu) \cap \varphi(\bar{\mu}) = \nu \setminus \{\alpha\} \cup f(\bar{\mu}).$$

This implies that

$$f(\bar{\mu}) \subseteq \varphi(\mu) \setminus \nu =: \eta.$$

Note that since $\varphi(\mu)$ and $\varphi(\bar{\mu})$ are adjacent, they differ by one curve, which must be α. Thus, $\nu \setminus \alpha \subseteq \varphi(\mu)$, and η has cardinality $k_2 - (d-1) = k_1 + 1$. So $\bar{\mu}$ is contained in $f^{-1}(\eta)$, which has cardinality at most $k_1 + 1$ since f is injective.

This holds for every $\bar{\mu} \in \mathcal{B}^{[k_1]}(S_1)$ that is adjacent to μ, and η does not depend on $\bar{\mu}$. There are at most $k_1 + 1$ multicurves contained in a set of cardinality $k_1 + 1$, but we know from Remark 2.5 that μ has infinitely many adjacent multicurves in $\mathcal{B}^{[k_1]}(S_1)$, a contradiction. $\qquad \square$

As a consequence, we obtain the map $\Phi : \mathcal{C}^{[k_1]}(S_1) \to \mathcal{C}^{[k_1]}(S_2)$ given by

$$\Phi(\mu) := \varphi(\mu) \setminus \nu.$$

Clearly,

$$\varphi(\mu) = \Phi(\mu) \cup \nu$$

for every $\mu \in \mathcal{C}^{[k_1]}(S_1)$. So we just want to show that Φ is (induced by) f.

LEMMA 5.5. *For any β on S_1, $\Phi(C_\beta) \subseteq C_{f(\beta)}$.*

Proof. The idea of the proof is the same as for Lemma 5.4.

If $\mu \in C_\beta \cap \mathcal{B}^{[k_1]}(S_1)$, then $\Phi(\mu) = f(\mu) \in C_{f(\beta)}$.

Suppose now $\mu \in C_\beta$ is not in $\mathcal{B}^{[k_1]}(S_1)$. Any $\bar{\mu} \in \mathcal{B}^{[k_1]}(S_1)$ adjacent to μ satisfies

$$\Phi(\bar{\mu}) \cap \Phi(\mu) = f(\bar{\mu}) \setminus \{f(\beta)\}$$

so $\bar{\mu}$ is a subset $\Phi(\mu) \cup \{f(\beta)\}$. This implies that there are finitely many multicurves in $\mathcal{B}^{[k_1]}(S_1)$ adjacent to μ, contradicting Remark 2.5. $\qquad \square$

Now take any $\mu \in \mathcal{C}^{[k_1]}(S_1)$, say $\mu = \{\beta_1, \dots, \beta_{k_1}\}$. Then

$$\{\Phi(\mu)\} = \bigcap_{j=1}^{k_1} \Phi(C_{\beta_j}) \subseteq \bigcap_{j=1}^{k_1} C_{f(\beta_j)} = f(\mu).$$

So $\Phi(\mu) = f(\mu)$ for all $\mu \in \mathcal{C}^{[k_1]}(S_1)$, which ends the proof of the theorem.

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