Michigan Math. J. 66 (2017), 569–580

Infinite Groups Acting Faithfully on the Outer Automorphism Group of a Right-Angled Artin Group

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Abstract. We construct the first known examples of infinite subgroups of the outer automorphism group of $\mathrm{Out}(A_\Gamma)$, for certain right-angled Artin groups A_Γ. This is achieved by introducing a new class of graphs, called *focused graphs,* whose properties allow us to exhibit (infinite) projective linear groups as subgroups of $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$. This demonstrates a marked departure from the known behavior of $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ when A_Γ is free or free abelian since in these cases $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ has order at most 4. We also disprove a previous conjecture of the second author, producing new examples of finite-order members of certain $\mathrm{Out}(\mathrm{Aut}(A_\Gamma))$.

1. Introduction

Right-angled Artin groups, or *RAAGs,* comprise a class of groups that generalize free groups and free Abelian groups. Every finite simplicial graph Γ with vertex set V defines a RAAG A_Γ in the following way. The generating set of A_Γ is in bijection with the vertices of Γ, and the only relations are that two generators commute if their corresponding vertices share an edge in Γ. Thus, if Γ has no edges, then A_Γ is just the free group F_V, whereas if Γ is a complete graph, then A_Γ is the free Abelian group $\mathbb{Z}\langle V\rangle$.

In this paper, we consider the automorphism and outer automorphism groups of general RAAGs in comparison with those of free groups and free Abelian groups. More specifically, we investigate $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ and $\mathrm{Out}(\mathrm{Aut}(A_\Gamma))$. These groups provide a measure of the algebraic rigidity of $\mathrm{Out}(A_\Gamma)$ and $\mathrm{Aut}(A_\Gamma)$, respectively, and their study fits into a more general program of investigating rigidity of groups throughout geometric group theory.

The main goal of this paper is to show that there exist infinitely many graphs Γ for which $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ is infinite. We achieve this by introducing a new class of graphs, which we call *focused graphs.* A graph Γ is said to be *focused* if it has a distinguished vertex c with the following two properties: (i) c is the unique vertex of Γ that may dominate a vertex other than itself, and (ii) c is the only vertex whose star disconnects Γ. Focused graphs are the key construction that allows us to prove our following main theorem.

Theorem A. *For each $n \geq 2$, there exist infinitely many focused graphs Γ such that* $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ *contains* $\mathrm{PGL}_n(\mathbb{Z})$.

Received March 21, 2016. Revision received April 20, 2016.

Our construction of $\mathrm{PGL}_n(\mathbb{Z})$ subgroups of $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ may also be utilized successfully for a subset of the graphs Γ that have no *separating intersection of links,* a property of graphs introduced by Gutierrez, Piggott and Ruane [8]. We discuss this generalized construction after we complete the proof of Theorem A.

Previous work has calculated the groups $\mathrm{Out}(\mathrm{Aut}(A_\Gamma))$ and $\mathrm{Out}(\mathrm{Aut}(A_\Gamma))$ when A_Γ is a free or free Abelian group. A classical result of Hua and Reiner [10] computes $\mathrm{Out}(\mathrm{Aut}(\mathbb{Z}^n)) = \mathrm{Out}(\mathrm{Out}(\mathbb{Z}^n)) = \mathrm{Out}(\mathrm{GL}_n(\mathbb{Z}))$ to be

$$\mathrm{Out}(\mathrm{GL}_n(\mathbb{Z})) = \begin{cases} \mathbb{Z}/2 \times \mathbb{Z}/2, & \text{even } n, \\ \mathbb{Z}/2, & \text{odd } n > 1, \\ 1, & n = 1. \end{cases}$$

For the case of free groups, Dyer and Formanek [5] give an algebraic proof that $\mathrm{Out}(\mathrm{Aut}(F_n))$ is trivial for all n. Khramtsov [11] gave another proof of this fact and also showed that $\mathrm{Out}(\mathrm{Out}(F_n))$ is trivial for $n \geq 3$. Using outer space and auter space for free groups, Bridson and Vogtmann [1] gave a geometric proof that, for $n \geq 3$, both $\mathrm{Out}(\mathrm{Out}(F_n))$ and $\mathrm{Out}(\mathrm{Aut}(F_n))$ are trivial. Note that the cases $n = 1$ and $n = 2$ for $\mathrm{Out}(\mathrm{Out}(F_n))$ are covered by the Hua–Reiner theorem since $F_1 \cong \mathbb{Z}$, and a theorem of Nielsen states that $\mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbb{Z})$ (see [13]).

The results mentioned indicate that both $\mathrm{Out}(\mathrm{Aut}(A_\Gamma))$ and $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ are either small or trivial for $A_\Gamma = \mathbb{Z}^n$ and $A_\Gamma = F_n$, independent of n. For more general RAAGs, the second author has shown in [7] that this behavior does not hold. More precisely, he proves that, for any $n > 0$, there exist graphs Γ_1, Γ_2 such that $|\mathrm{Out}(\mathrm{Aut}(A_{\Gamma_1}))| > n$ and $|\mathrm{Out}(\mathrm{Out}(A_{\Gamma_2}))| > n$. Theorem A of this paper substantially strengthens the second author's result in the case of $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$.

Our approach is to compute explicitly a large subgroup of $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ for each focused graph Γ. In computing this, we first exhibit $\mathrm{GL}_n(\mathbb{Z})$ subgroups inside $\mathrm{Aut}(\mathrm{Out}(A_\Gamma))$, hence proving that any $\mathbb{Z}$-linear group can be made to act faithfully on $\mathrm{Out}(A_\Gamma)$ via automorphisms for some RAAG A_Γ. This provides a stark constrast to the summarized work of Hua–Reiner, Bridson–Vogtmann, and Dyer–Formanek.

The second author [7] also introduced the notion of an *austere graph.* If Γ is austere, then $\mathrm{Out}(A_\Gamma)$ is, in some sense, as simple as possible. The second author previously conjectured that, for austere graphs Γ, the group $\mathrm{Aut}(A_\Gamma)$ is complete (see the remarks after Proposition 5.1 in [7]). However the following theorem establishes that the order of $\mathrm{Out}(\mathrm{Aut}(A_\Gamma))$ in the austere case is at least exponential in $|V|$.

THEOREM B. *If Γ is austere, then $|\mathrm{Out}(\mathrm{Aut}(A_\Gamma))| \geq 2^K$, where $K = \sum_{v \in V} K_v$, and K_v is the number of vertices of Γ not adjacent to v.*

In particular, we are able to achieve the two main results of [7] simultaneously:

COROLLARY C. *For each $n \geq 1$, there exist infinitely many graphs Γ such that $|\mathrm{Out}(\mathrm{Aut}(A_\Gamma))| > n$ and $|\mathrm{Out}(\mathrm{Out}(A_\Gamma))| > n$.*

A Caveat

One might naïvely expect that to construct automorphisms of Out(A_Γ), say, it would suffice to find a finite index subgroup $K \leq \text{Out}(A_\Gamma)$ that has a rich collection of automorphisms as an abstract group. It could then be hoped that these extend to give many automorphisms of Out(A_Γ) since it is often the case that group-theoretic properties pass easily between a group and its finite index subgroups. Indeed, this is our approach; however, the interplay between the finitely many cosets of K in Out(A_Γ) frequently prohibits any obvious attempts at extending such automorphisms to all of Out(A_Γ).

Considering the abstract commensurator Comm(Out(A_Γ)) instead of Out(Out(A_Γ)) circumvents some of these difficulties since Comm(Out(A_Γ)) is precisely concerned with isomorphisms between finite index subgroups of Out(A_Γ). For details, see the remarks after Corollary 3.5.

Outline of the Paper

In Section 2, we recall some necessary background regarding automorphisms of right-angled Artin groups. In Section 3, we prove Theorem A, whereas in Section 4, we prove Theorem B.

2. Preliminaries

In this section we review basic properties of RAAGs and their automorphism groups. Let $\Gamma = (V, E)$ be a simplicial graph. As stated in the introduction, Γ defines a group A_Γ with generating set $V = \{v_1, \ldots, v_n\}$ and relations $v_i v_j = v_j v_i$ if and only if v_i is adjacent to v_j in Γ. For $v \in V$, denote by lk(v) the *link* of v, by which we mean the set of vertices adjacent to v. The *star* of v, by which we mean the set $\text{lk}(v) \cup \{v\}$, will be denoted st(v). If $u, v \in V$ and $\text{lk}(v) \subseteq \text{st}(u)$, then we say that u *dominates* v and write $v \leq u$.

Elements of A_Γ enjoy nice normal forms in terms of the generators V. Two words w_1 and w_2 in the generators V (and their inverses) are said to be *shuffle-equivalent* if w_2 can be obtained from w_1 by repeatedly exchanging pairs of adjacent commuting generators. Hermiller and Meier [9] show that if w_1 and w_2 are minimal length words, then $w_1 = w_2$ in A_Γ iff w_1 is shuffle-equivalent to w_2 and moreover that any word can be transformed into a minimal-length word by swapping adjacent commuting generators and cancelling pairs of inverses whenever possible. This allows us to define the *support* of $w \in A_\Gamma$, denoted supp(w), to consist of all $v \in V$ such that v (or v^{-1}) appears in a minimal-length word representing w. For a survey of RAAGs and their properties, see [2].

The automorphism group Aut(A_Γ) of a RAAG A_Γ is generated by the following four types of automorphisms, known as the Laurence–Servatius generators:

1. *Inversions:* Given $v \in V$, the automorphism ι_v sends $v \mapsto v^{-1}$ and fixes all other generators.
2. *Graph automorphisms:* Any graph automorphism of Γ induces a permutation of V that extends to an automorphism of A_Γ.

3. *Transvections:* If $v \leq u$, the automorphism τ_{uv} sends $v \mapsto uv$ and fixes all other generators.
4. *Partial conjugations:* If P is a connected component of $\Gamma \setminus \mathrm{st}(v)$ for some $v \in V$, then the automorphism $\chi_{v,P}$ maps $u \mapsto vuv^{-1}$ for every $u \in P$ and acts as the identity elsewhere.

The fact that these four types of automorphisms generate $\mathrm{Aut}(A_\Gamma)$ was conjectured by Servatius [15], and later proven by Laurence [12]. If $v \leq u$ and v is adjacent to u, then τ_{uv} is an *adjacent* transvection. Otherwise, τ_{uv} is a *nonadjacent* transvection. In the sequel, the subgroup generated by the inversions is denoted I_Γ, whereas the subgroup generated by partial conjugations and tranvections is denoted $\mathrm{PCT}(A_\Gamma)$. The images of these four types of generators under the quotient map $\mathrm{Aut}(A_\Gamma) \to \mathrm{Out}(A_\Gamma)$ generate $\mathrm{Out}(A_\Gamma)$. We will use an overline to indicate when we refer to elements or subgroups of $\mathrm{Out}(A_\Gamma)$: for example, $\bar{\tau}_{uv}$, $\bar{\chi}_{v,P}$, and $\overline{\mathrm{PCT}}(A_\Gamma)$.

3. Proof of Theorem A

Let $\Gamma = (V, E)$ be a graph with a distinguished vertex c such that if $v \leq u$ for distinct $v, u \in V$, then $u = c$, and for any $v \in V \setminus \{c\}$, the graph $\Gamma \setminus \mathrm{st}(v)$ is connected. We will call such a graph *focused* (*at* c). Let $L = \{x_1, \ldots, x_l\} \subset V \setminus \{c\}$ denote the set of vertices that are dominated by, but not adjacent to, the vertex c, and let $S = \{x_{l+1}, \ldots, x_m\} \subset V \setminus \{c\}$ denote the set of vertices that are both dominated by and adjacent to c. Finally, let $Q = \{P_1, \ldots, P_k\}$ denote the connected components of the graph $\Gamma \setminus \mathrm{st}(c)$, where $k \geq l$, and we set $P_i = \{x_i\}$ for $1 \leq i \leq l$. See Figure 1 a typical example of a focused graph.

Note that a focused graph Γ may have nontrivial graph automorphism group $\mathrm{Aut}(\Gamma)$. From now on, we assume that $\mathrm{Aut}(\Gamma)$ is trivial since this simplifies the following exposition. This is not a too restrictive condition; our construction still yields infinite subgroups of $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ if $\mathrm{Aut}(\Gamma) \neq 1$; however, the obvious action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{Out}(A_\Gamma)$ would force us to pass to proper subgroups of those we find further. Precisely, the isomorphism given by Proposition 3.1 would become

$$\mathrm{Out}(A_\Gamma) \cong \mathbb{Z}^{k+m-1} \rtimes (I_\Gamma \rtimes \mathrm{Aut}(\Gamma)),$$

and when extending automorphisms of $\mathbb{Z}^{k+m-1}$ to $\mathrm{Out}(A_\Gamma)$ (which will be our strategy), we would need to ensure that they satisfy additional relations arising from the action of $\mathrm{Aut}(\Gamma)$.

Examining the Laurence–Servatius generators of $\mathrm{Aut}(A_\Gamma)$ for a focused graph Γ, we find that

$$\mathrm{Aut}(A_\Gamma) \cong \mathrm{PCT}(A_\Gamma) \rtimes I_\Gamma,$$

with the splitting following from the observation that I_Γ injects into $\mathrm{GL}_n(\mathbb{Z})$ under the canonical map $\Phi : \mathrm{Aut}(A_\Gamma) \to \mathrm{GL}_n(\mathbb{Z})$, and $\Phi(\mathrm{PCT}(A_\Gamma)) \cap \Phi(I_\Gamma) = 1$. This splitting of $\mathrm{Aut}(A_\Gamma)$ descends to one of $\mathrm{Out}(A_\Gamma)$ since $\mathrm{Inn}(A_\Gamma) \leq \mathrm{PCT}(A_\Gamma)$. The image $\overline{\mathrm{PCT}}(A_\Gamma)$ of $\mathrm{PCT}(A_\Gamma)$ in $\mathrm{Out}(A_\Gamma)$ is easy to describe, which we do now explicitly.

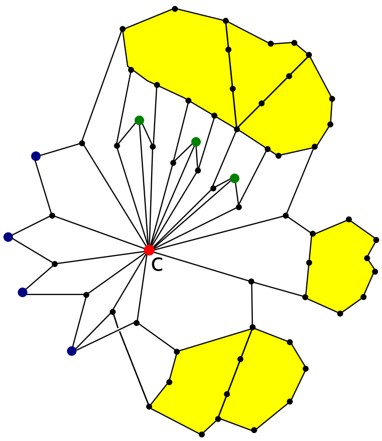

Figure 1 An example of a focused graph with $l = 4$, $m = 7$, and $k = 7$. The distinguished vertex c is shown in red. Vertices dominated by but not adjacent to c are shown in blue, whereas those dominated by and adjacent to c are green. Connected components of $\Gamma \setminus \mathrm{st}(c)$ that are not vertices are yellow

PROPOSITION 3.1. *Let* Γ *be focused at* c. *Then*

$$\mathrm{Out}(A_\Gamma) \cong \mathbb{Z}^{k+m-1} \rtimes I_\Gamma.$$

In particular, $\overline{\mathrm{PCT}}(A_\Gamma) \cong \mathbb{Z}^{k+m-1}$.

REMARK. In Question 9 of [4], the authors ask when Out(A_Γ) is abstractly commensurable with a right-angled Artin group. This proposition implies that focused graphs provide an infinite family of graphs such that Out(A_Γ) is virtually free abelian and hence abstractly commensurable with a RAAG.

Proof of Proposition 3.1. The group Out(A_Γ) certainly splits as $\overline{\mathrm{PCT}}(A_\Gamma) \rtimes I_\Gamma$ by the preceding discussion. We must show that $\overline{\mathrm{PCT}}(A_\Gamma)$ is free abelian of rank $k + m - 1$.

The group $\mathrm{PCT}(A_\Gamma) \le \mathrm{Aut}(A_\Gamma)$ is generated by $\mathrm{Inn}(A_\Gamma) \cong A_\Gamma$ and the $m + k$ mutually commuting Laurence–Servatius generators of the form τ_{cx_i} or χ_{c,P_j} for $1 \le i \le m$ and $1 \le j \le k$. For conciseness, we shall write $\tau_i = \tau_{cx_i}$ and $\chi_j = \chi_{c,P_j}$.

The image $\overline{\mathrm{PCT}}(A_\Gamma)$ is, therefore, an Abelian group generated by the images $\bar{\tau}_i$ and $\bar{\chi}_j$ of τ_i and χ_j (respectively), and we use an additive notation to reflect this. Since

$$\gamma_c = \prod_{j=1}^{k} \chi_j$$

in Aut(A_Γ), where $\gamma_c \in \mathrm{Inn}(A_\Gamma)$ denotes conjugation by c, we observe that

$$\bar{\chi}_1 + \cdots + \bar{\chi}_{k-1} = -\bar{\chi}_k$$

in $\mathrm{Out}(A_\Gamma)$. We thus remove $\bar{\chi}_k$ from our generating set for $\mathrm{Out}(A_\Gamma)$.

We now show that the product $\omega := {\tau_1}^{r_1} \dots {\tau_m}^{r_m} \chi_1^{s_1} \dots {\chi_{k-1}}^{s_{k-1}} \in \mathrm{Aut}(A_\Gamma)$ (r_i, $s_j \in \mathbb{Z}$) is inner if and only if it is trivial. First, observe that each of the r_i must be zero. Denoting by $[\omega]$ the induced action of ω on the abelianization of A_Γ, we have $[\omega]\colon [x_i] \mapsto [x_i] + r_i[c]$, where $[x_i]$ and $[c]$ denote the equivalence classes of x_i and c in the abelianization of A_Γ.

Hence we may assume $\omega := \chi_1^{s_1} \cdots {\chi_{k-1}}^{s_{k-1}}$ and suppose that ω is equal to conjugation by $p \in A_\Gamma$. Since ω acts trivially on $\mathrm{st}(c)$, we must have $pvp^{-1} = v$ for each $v \in \mathrm{st}(c)$. This implies that every $u \in \mathrm{supp}(p)$ is adjacent to each such v since pvp^{-1} and v must be shuffle-equivalent. Each vertex in $\mathrm{supp}(p)$ hence dominates the vertex c, and so $\mathrm{supp}(p) = \{c\}$ or $\emptyset$ since Γ is focused at c.

However, ω also acts trivially on P_k, so by the same argument we must have that p is the identity since c is not adjacent to any vertex in P_k. Thus, if ω is nontrivial in $\mathrm{Aut}(A_\Gamma)$, its image is nontrivial in $\mathrm{Out}(A_\Gamma)$, and so the set $\{\bar{\tau}_i, \bar{\chi}_j \mid 1 \le i \le m, 1 \le j \le k-1\}$ is a free abelian basis for $\overline{\mathrm{PCT}}(A_\Gamma)$. □

Since the image of $\mathrm{PCT}(A_\Gamma)$ in $\mathrm{Out}(A_\Gamma)$ is torsion-free, and so is $\mathrm{Inn}(A_\Gamma)$, we obtain the following corollary.

COROLLARY 3.2. *For a focused graph* Γ, *the group* $\mathrm{PCT}(A_\Gamma)$ *is torsion-free.*

Our goal is now to understand the action of I_Γ on $\overline{\mathrm{PCT}}(A_\Gamma) \cong \mathbb{Z}^{k+m-1}$ sufficiently to identify automorphisms of $\mathbb{Z}^{k+m-1}$ that may extend to well-defined automorphisms of $\mathrm{Out}(A_\Gamma)$ by declaring that they act trivially on I_Γ.

Due to its distinguished role, we denote by ι_c the automorphism of A_Γ that inverts $c \in V$ and fixes every $v \in V \setminus \{c\}$. The action of I_Γ is fully encoded by the following six types of relation:

$$\iota_c \bar{\chi}_j \iota_c = -\bar{\chi}_j \quad (1 \le j \le k-1), \tag{1}$$

$$\iota_c \bar{\tau}_i \iota_c = -\bar{\tau}_i \quad (1 \le i \le m), \tag{2}$$

$$\iota_r \bar{\chi}_j \iota_r = \bar{\chi}_j \quad (1 \le j \le k-1, \iota_r \ne \iota_c), \tag{3}$$

$$\iota_r \bar{\tau}_i \iota_r = \bar{\tau}_i \quad (1 \le i \le m, \iota_r \ne \iota_c \text{ or } \iota_i), \tag{4}$$

$$\iota_i \bar{\tau}_i \iota_i = \bar{\chi}_i - \bar{\tau}_i \quad (1 \le i \le l), \tag{5}$$

$$\iota_i \bar{\tau}_i \iota_i = -\bar{\tau}_i \quad (l+1 \le i \le m). \tag{6}$$

Note that relations (5) and (6) distinguish $\iota_i \bar{\tau}_i \iota_i$ depending upon whether $\bar{\tau}_i$ is a nonadjacent or adjacent transvection, respectively. The action of I_Γ on $\overline{\mathrm{PCT}}(A_\Gamma)$ in the semidirect product decomposition of $\mathrm{Out}(A_\Gamma)$ is given by a homomorphism

$$\alpha : I_\Gamma \to \mathrm{Aut}(\mathbb{Z}^{k+m-1}) \cong \mathrm{GL}_{k+m-1}(\mathbb{Z}).$$

Let $\mathcal{C}$ denote the centralizer of $\alpha(I_\Gamma)$ in $\mathrm{GL}_{k+m-1}(\mathbb{Z})$. We may view $\mathcal{C}$ as a subgroup of $\mathrm{Aut}(\mathrm{Out}(A_\Gamma))$ by extending each $M \in \mathcal{C}$ to an automorphism $\tilde{M} \in \mathrm{Aut}(\mathrm{Out}(A_\Gamma))$ by declaring that $\tilde{M}$ restricts to the identity on I_Γ (see [7, Section 3.1] for a more detailed discussion).

To give a tractable description of $\alpha(I_\Gamma)$ and $\mathcal{C}$, we order the free basis for $\overline{\mathrm{PCT}}(A_\Gamma)$ found in Proposition 3.1 as follows:

$$(\bar{\chi}_1, \bar{\tau}_1, \ldots, \bar{\chi}_l, \bar{\tau}_l, \bar{\tau}_{l+1}, \ldots, \bar{\tau}_m, \bar{\chi}_{l+1}, \ldots, \bar{\chi}_{k-1}).$$

As is usual, we denote the $q \times q$ identity matrix by I_q. Looking at relations (1)–(6), we see that the subgroup $\alpha(I_\Gamma)$ consists of $-I_{k+m-1}$ together with block-diagonal matrices of the form $\mathrm{Diag}(D_1, D_2, D_3)$, where D_3 is $\pm I_{k-l-1}$, and D_2 is any diagonal matrix in $\mathrm{GL}_{m-l}(\mathbb{Z})$. The matrix D_1 is any matrix in $\mathrm{GL}_{2l}(\mathbb{Z})$ with block decomposition

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_l \end{pmatrix},$$

where each A_i $(1 \le i \le l)$ is either I_2 or $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$. We denote the subgroup of $\mathrm{GL}_{2l}(\mathbb{Z})$ consisting of such matrices by $\mathcal{L}$.

With this description of $\alpha(I_\Gamma)$, we now identify the centralizer $\mathcal{C}$. We denote by $\Lambda_l[2]$ the *principal level 2 congruence subgroup of* $\mathrm{GL}_l(\mathbb{Z})$ (that is, the kernel of the epimorphism $\mathrm{GL}_l(\mathbb{Z}) \to \mathrm{GL}_l(\mathbb{Z}/2)$ that reduces matrix entries mod 2).

Proposition 3.3. *The centralizer $\mathcal{C}$ of $\alpha(I_\Gamma)$ in $\mathrm{GL}_{k+m-1}(\mathbb{Z})$ is isomorphic to*

$$\Lambda_l[2] \times (\mathbb{Z}/2)^m \times \mathrm{GL}_{k-1}(\mathbb{Z}).$$

Proof. Let $M \in \mathrm{GL}_{k+m-1}(\mathbb{Z})$, and suppose that M centralizes $\alpha(I_\Gamma)$. We specify a 3×3 block decomposition on M by declaring that the $(1,1)$ block is $2l \times 2l$, the $(2,2)$ block is $(m-l) \times (m-l)$, and the $(3,3)$ block is $(k-1) \times (k-1)$. First, we show that M is block-diagonal with respect to this block decomposition.

Let $M = (M_{ij})$ where M_{ij} is the matrix in the (i,j) block of M. Let

$$D = \mathrm{Diag}(D_1, D_2, D_3) \in \alpha(I_\Gamma),$$

as discussed prior to the statement of the proposition. Since $DM = MD$, it must be the case that $M_{32}D_2 = \pm M_{32}$ and $\pm M_{23} = D_2 M_{23}$ for any choice of the diagonal matrix D_2. This forces M_{23} and M_{32} to be the zero matrix. We must also have $M_{21}D_1 = D_2 M_{21}$ and $M_{12}D_2 = D_1 M_{12}$ for any choice of D_2 and D_1. Taking D_1 to be the identity matrix and choosing D_2 appropriately allow us to conclude that M_{21} and M_{12} are both the zero matrix. Finally, a similar argument forces M_{13} and M_{31} to also be the zero matrix. Thus, $M = \mathrm{Diag}(M_{11}, M_{22}, M_{33})$.

Since $M \in \mathcal{C}$ and D are both block-diagonal, to determine $\mathcal{C}$, it is necessary and sufficient to centralize within the three diagonal blocks of D. For the second and third diagonal blocks, these centralizers are the diagonal subgroup of $\mathrm{GL}_{m-l}(\mathbb{Z})$ and all of $\mathrm{GL}_{k-1}(\mathbb{Z})$, respectively. The only task that remains is to determine the centralizer in $\mathrm{GL}_{2l}(\mathbb{Z})$ of the subgroup $\mathcal{L}$ defined previously.

Suppose that $N \in \mathrm{GL}_{2l}(\mathbb{Z})$ lies in $C(\mathcal{L})$, the centralizer of the subgroup $\mathcal{L}$. Endow N with a block decomposition compatible with that used to define $\mathcal{L}$: let N have an $l \times l$ block decomposition, where each block is of size 2×2, writing

$N = (N_{ij})$, where N_{ij} is the matrix in the (i, j) block of N. Carrying out block matrix multiplication, we see that for N to centralize $\mathcal{L}$, it is necessary that N_{ii} $(1 \le i \le l)$ commutes with each member of the order 2 subgroup $\mathcal{P} := \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right) \rangle$ and that $N_{ij} S = T N_{ij}$ $(1 \le i \ne j \le l)$ for all $S, T \in \mathcal{P}$.

Let $K = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ appear in some 2×2 block of N. If K lies in a diagonal block, then by the preceding discussion we necessarily have

$$\begin{aligned} a &= a + c, \\ a - b &= b + d, \\ c &= -c, \\ c - d &= -d. \end{aligned}$$

If K lies off the diagonal of N, then its entries must further satisfy the relations

$$\begin{aligned} b &= b + d, \\ d &= -d. \end{aligned}$$

In summary, we conclude that in the block decomposition of N, the ith diagonal blocks are of the form $\left(\begin{smallmatrix} 2b_i + d_i & b_i \\ 0 & d_i \end{smallmatrix}\right)$ for some $b_i, d_i \in \mathbb{Z}$, and the off-diagonal (i, j) block is of the form $\left(\begin{smallmatrix} 2e_{ij} & e_{ij} \\ 0 & 0 \end{smallmatrix}\right)$ for some $e_{ij} \in \mathbb{Z}$. Notice that the even-numbered rows each have precisely one nonzero entry, and so each d_i must lie in $\{\pm 1\}$. We have necessary conditions on the entries of the centralizing matrix N; we now use these even-numbered rows to obtain sufficient conditions.

To calculate the determinant of N, we may consider the determinants of the minors obtained by expanding along these even-numbered rows. Since $\det N = \pm 1$, this expansion allows us to conclude that the matrix

$$N' := \begin{pmatrix} 2b_1 + d_1 & 2e_{12} & \cdots & 2e_{1l} \\ 2e_{21} & 2b_2 + d_2 & \cdots & 2e_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ 2e_{l1} & 2e_{l2} & \cdots & 2b_l + d_l \end{pmatrix}$$

lies in $\mathrm{GL}_l(\mathbb{Z})$. Indeed, $N' \in \Lambda_l[2]$. This observation allows us to define a function $\theta : \Lambda_l[2] \to C(\mathcal{L})$ by declaring each d_i in $\theta(A)$ to be 1. Moreover, by the placement of zeroes in $N \in C(\mathcal{L})$, θ is a homomorphism, and an injective one at that. The image of θ is clearly not surjective since it cannot contain matrices whose d_i entries are -1; however, the image is finite index in $C(\mathcal{L})$, as we shall see.

Let $\{f_i\}_{i=1}^{l}$ be a basis for $(\mathbb{Z}/2)^l$. Define $\xi : (\mathbb{Z}/2)^l \to C(\mathcal{L})$ with $\xi(f_i)$ being the diagonal matrix in $C(\mathcal{L})$ with $d_i = -1$. We claim that

$$\theta \times \xi : \Lambda_l[2] \times (\mathbb{Z}/2)^l \to C(\mathcal{L})$$

is an isomorphism. Verifying this is a straightforward exercise.

Assembling the centralizers of the three diagonal blocks of $M \in \mathcal{C} \le \mathrm{GL}_{k+m-1}(\mathbb{Z})$, we thus obtain that $\mathcal{C}$ is isomorphic to the direct sum in the statement of the proposition. □

REMARK. Note that in the proof of Proposition 3.3, we realized the principal level 2 congruence subgroup $\Lambda_l[2]$ as a finite index subgroup of a centralizer in $\mathrm{GL}_{2l}(\mathbb{Z})$ of a finite subgroup. We are not aware of this being exhibited elsewhere in the literature, and it may be of independent interest.

Finally, we determine the image of $\mathcal{C} \leq \mathrm{Aut}(\mathrm{Out}(A_\Gamma))$ in $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ when we take the quotient by $\mathrm{Inn}(\mathrm{Out}(A_\Gamma))$.

PROPOSITION 3.4. *The image* $\bar{\mathcal{C}}$ *of* $\mathcal{C}$ *in* $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ *is isomorphic to*

$$((\Lambda_l[2] \times (\mathbb{Z}/2)^l)/\mathcal{L}) \times \mathrm{PGL}_{k-1}(\mathbb{Z}).$$

Proof. Consider $w \in \mathbb{Z}^{k+m-1} \leq \mathrm{Out}(A_\Gamma)$. Direct computation gives that for any $\beta := uh \in \mathrm{Out}(A_\Gamma)$ (where $u \in \mathbb{Z}^{k+m-1}$ and $h \in I_\Gamma$), we have $\beta w \beta^{-1} = \alpha(h)(w)$. Since any member of $\mathcal{C}$ preserves $\mathbb{Z}^{k+m-1}$ inside $\mathrm{Out}(A_\Gamma)$, we see that $\phi \in \mathcal{C}$ is inner in $\mathrm{Aut}(\mathrm{Out}(A_\Gamma))$ if and only if there exists $h \in I_\Gamma$ such that $\phi(\bar{x}_i) = \alpha(h)(\bar{x}_i)$ for all $x_i \in V$. Precisely, we have that

$$\mathcal{C} \cap \mathrm{Inn}(\mathrm{Out}(A_\Gamma)) \cong \alpha(I_\Gamma).$$

We immediately see that $\mathcal{C}$ has infinite image in $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$, but we can determine its structure exactly.

Recall that

$$\mathcal{C} \cong (\Lambda_l[2] \times (\mathbb{Z}/2)^l) \times (\mathbb{Z}/2)^{m-l} \times \mathrm{GL}_{k-1}(\mathbb{Z})$$

by Proposition 3.3. Since $\alpha(I_\Gamma) \cong I_\Gamma \cong (\mathbb{Z}/2)^n$, the $(\mathbb{Z}/2)^{m-l}$ factor vanishes in $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$, and we have

$$\bar{\mathcal{C}} \cong ((\Lambda_l[2] \times (\mathbb{Z}/2)^l)/\mathcal{L}) \times \mathrm{PGL}_{k-1}(\mathbb{Z}). \qquad \square$$

We have thus established Theorem A. Over the course of our proofs of Propositions 3.3 and 3.4, we also obtained the following corollary.

COROLLARY 3.5. *Given any* $\mathbb{Z}$*-linear* (*resp. projective* $\mathbb{Z}$*-linear*) *group* G, *there exist infinitely many right-angled Artin groups* A_Γ *for which* G *acts faithfully on* $\mathrm{Out}(A_\Gamma)$ *via automorphisms* (*resp. outer automorphisms*).

Generalizations

Although the statement of Theorem A is concerned only with focused graphs Γ, our construction also produces large subgroups in $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ for other types of graphs.

As was shown in [3], if a graph Γ has no *separating intersection of links,* then the group generated by the set of partial conjugations has free Abelian image in $\mathrm{Out}(A_\Gamma)$. Assuming that there is no domination in Γ, we have that $\mathrm{Out}(A_\Gamma)$ splits as a free Abelian-by-finite group

$$\overline{\mathrm{PCT}}(A_\Gamma) \rtimes (I_\Gamma \rtimes \mathrm{Aut}(\Gamma)).$$

We are then permitted to construct $\mathrm{PGL}_\ell(\mathbb{Z})$ subgroups inside $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ as before, where the value of ℓ will depend upon the rank of $\overline{\mathrm{PCT}}(A_\Gamma)$ and the action of $\mathrm{Aut}(\Gamma)$.

It is also possible to assemble focused graphs together in such a way that the properties required for our construction are preserved. Let $\{\Gamma_i\}$ be a finite set of finite focused graphs with Γ_i focused at the vertex c_i. For each pair of graphs Γ_i and Γ_j, select distinct connected components $P_1^i, \dots, P_q^i$ of $\Gamma_i \setminus \mathrm{st}(c_i)$ and $P_1^j, \dots, P_q^j$ of $\Gamma_j \setminus \mathrm{st}(c_j)$ such that $P_k^i \cong P_k^j$ for each $1 \le k \le q$. Build a graph Δ by gluing the graphs $\{\Gamma_i\}$ along the isomorphic subgraphs P_k^l. While Δ will not be a focused graph, we will call it *locally focused* since it retains enough of the necessary features to run the construction given in this section.

We end this discussion by noting that these three different classes of graphs (focused, locally focused, and no SILs or transvections) have large pairwise intersection, but no single class lies in the union of the two others.

Commensurators

If we consider abstract commensurators $\mathrm{Comm}(\mathrm{Out}(A_\Gamma))$ instead of $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$, the discrepancy between our examples and free or free Abelian groups is not quite so severe. We can make this observation precise as follows.

Let $\Gamma = (V, E)$ be a focused graph and suppose that $k + m \ge 4$, where k and m are as in the statement of Proposition 3.1. In particular, we have $n = |V| \ge k + m \ge 4$. At one end, it is a result of Farb and Handel [6] that for $n \ge 4$, the abstract commensurator $\mathrm{Comm}(\mathrm{Out}(F_n))$ is just equal to $\mathrm{Out}(F_n)$. For our focused graph Γ, by Proposition 3.1, $\mathrm{Out}(A_\Gamma)$ is virtually free abelian of rank $k + m - 1$, and hence its abstract commensurator is $\mathrm{GL}_{k+m-1}(\mathbb{Q})$. On the other hand, a theorem of Margulis (see [14; 16]) implies that for $k \ge 3$, the abstract commensurator of $\mathrm{GL}_n(\mathbb{Z})$ is commensurable with $\mathrm{GL}_n(\mathbb{Q})$.

4. Proof of Theorem B

First, we recall the definition of an austere graph as defined in [7]. A finite simplicial graph $\Gamma = (V, E)$ is called *austere* if it has trivial symmetry group, no dominated vertices, and $\Gamma \setminus \mathrm{st}(v)$ is connected for any $v \in V$. In particular, we have that no vertex is adjacent to every other vertex, and hence the associated RAAG A_Γ has trivial center.

Let Γ be an austere graph. By inspecting the Laurence–Servatius generators, this implies that $\mathrm{Out}(A_\Gamma)$ consists only of inversions. In this case, we know that the automorphism group is a semidirect product:

$$\mathrm{Aut}(A_\Gamma) \cong \mathrm{Inn}(A_\Gamma) \rtimes \mathrm{Out}(A_\Gamma) \cong A_\Gamma \rtimes I_\Gamma,$$

where I_Γ is the group of inversions. It is easy to write down a presentation for $\mathrm{Aut}(A_\Gamma)$ in terms of the usual presentation for A_Γ. If $V = \{v_1, \dots, v_n\}$ is the vertex set of Γ, then

$$\mathrm{Aut}(A_\Gamma) = \langle \gamma_i, \iota_j \text{ for } 1 \le i, j \le n \mid \widetilde{\mathcal{R}}_\Gamma \rangle.$$

Here γ_i represents conjugation by v_i, ι_j is inversion of v_j, and $\widetilde{\mathcal{R}}_\Gamma$ is comprised of the following five types of relation:

$$[\gamma_i, \gamma_k] \quad \text{if } v_i \text{ commutes with } v_k \text{ in } A_\Gamma, \tag{7}$$

$$[\iota_j, \iota_l], \quad 1 \le j, l \le n, \tag{8}$$

$$(\iota_j)^2, \quad 1 \le j \le n, \tag{9}$$

$$[\gamma_i, \iota_j], \quad 1 \le i \ne j \le n, \tag{10}$$

$$(\gamma_i \iota_i)^2, \quad 1 \le i \le n. \tag{11}$$

Using this presentation, we are now ready to prove Theorem B.

Proof of Theorem B. Let $\pi_k : I_\Gamma = (\mathbb{Z}/2)^n \to \mathbb{Z}/2$ be the projection onto the kth factor. Consider a function $\phi : \{1, \dots, n\} \to I_\Gamma$ satisfying the following two properties:

(i) $\pi_k(\phi(k)) = 0$ for $1 \le k \le n$.
(ii) $\pi_j(\phi(k)) = 0$ whenever v_k and v_j commute in A_Γ.

Then ϕ induces the map $\Phi : \text{Aut}(A_\Gamma) \to \text{Aut}(A_\Gamma)$ defined by

$$\Phi(\gamma_k) = \gamma_k \cdot \phi(k), \qquad \Phi(\iota_k) = \iota_k.$$

Observe that Φ is an involution and hence a bijection. To see that Φ is moreover an automorphism, we simply check that it preserves relations. Since Φ is the identity on I_Γ, it is clear that the relations of the form (8) and (9) hold. Moreover, since all inversions commute with each other, it is clear that the relations of the form (10) are preserved by Φ. Finally, condition (i) ensures that the relations of type (11) are satisfied, and condition (ii) ensures that the relations of the form (7) still hold.

The automorphism Φ is not inner since it does not preserve the subgroup $\text{Inn}(A_\Gamma)$, and by composing Φ with the projection onto I_Γ we see that each distinct ϕ constructed before gives a distinct $\Phi \in \text{Out}(\text{Aut}(A_\Gamma))$. Thus, given any two nonadjacent vertices, it is possible to define a nonzero function ϕ as before, and hence if K_i denotes the number of vertices not adjacent to v_i, then the number of automorphisms we have constructed is 2^K, where $K = \sum_i K_i$. □

Acknowledgments. The authors are grateful to Dan Margalit and Benson Farb for their helpful remarks on a draft of this paper. The authors would also like to thank Tara Brendle and Andrew Putman for many useful comments and discussions and for their persistent encouragement. They would also like to thank an anonymous referee for several useful comments.

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