Michigan Math. J. 66 (2017), 533–547

Hypercommutative Algebras and Cyclic Cohomology

Benjamin C. Ward

Abstract. We introduce a chain model for the Deligne–Mumford operad formed by homotopically trivializing the circle in a chain model for the framed little disks. We then show that under degeneration of the Hochschild to cyclic cohomology spectral sequence, a known action of the framed little disks on Hochschild cochains lifts to an action of this new chain model. We thus establish homotopy hypercommutative algebra structures on both Hochschild and cyclic cochain complexes, and we interpret the gravity brackets on cyclic cohomology as obstructions to degeneration of this spectral sequence. Our results are given in the language of deformation complexes of cyclic operads.

Introduction

A differential graded Batalin–Vilkovisky (BV) algebra enhanced with a homotopy trivialization of the Δ-operator is equivalent to a hypercommutative (HyCom) algebra [DCV13; KMS13; DC14]. This relationship may be described in the language of operads, where BV and HyCom algebras are represented respectively by the homology operads of genus 0 moduli spaces of surfaces with boundary [Get94a] and by the Deligne–Mumford compactification of the moduli space of surfaces with punctures [Get95]. In practice, BV algebras often arise as the homology or cohomology of a geometric, topological, or algebraic object, and the chain level structure can only be expected to be BV up to homotopy. For example, this is the case when studying Hochschild cochain operations via the cyclic Deligne conjecture [Kau08]. More generally, this is the case when considering the deformation complex of a cyclic operad $\mathcal{O}$ with A_∞ multiplication μ. We denote such a deformation complex $CH^*(\mathcal{O},\mu)$.

On the other hand, the results of [War16] show that the complex of cyclic invariants associated with such data carries a compatible structure of an algebra over a model of the *open* moduli space of punctured Riemann spheres. This complex of invariants is a generalization of Connes' C^*_λ-complex and will be denoted $C^*_\lambda(\mathcal{O},\mu)$. Its cohomology $HC^*(\mathcal{O},\mu)$ generalizes the notion of the cyclic cohomology of a cyclic k-module. It is natural to ask for conditions under which this action of the open moduli space lifts to an action of an operad of chains on the Deligne–Mumford compactification.

In a BV algebra, the Δ-operator corresponds to an action of the circle at a boundary component. In the homotopy theory of S^1-spaces, trivialization of the circle action corresponds to degeneration of the Hochschild to cyclic

Received February 18, 2016. Revision received September 29, 2016.

(co)homology spectral sequence arising from the associated cyclic object. Thus it is reasonable to expect that an analog of this degeneration will permit such a lifting. We prove the following result (see Theorem 4.3).

THEOREM. *Let* $\mu\colon \mathcal{A}_\infty \to \mathcal{O}$ *be a map of cyclic operads. Let* $CH^*(\mathcal{O},\mu)$ *and* $C^*_\lambda(\mathcal{O},\mu)$ *be the associated deformation and cyclic deformation complexes of* μ. *If the morphism* μ *is cyclically degenerate* (*Definition 3.4*), *then the homotopy BV algebra structure on* $CH^*(\mathcal{O},\mu)$ *lifts to a compatible homotopy hypercommutative algebra. Moreover,* $C^*_\lambda(\mathcal{O},\mu)$ *carries the structure of a* $\mathcal{H}y\mathcal{C}om_\infty$*-algebra for which the inclusion* $C^*_\lambda(\mathcal{O},\mu)\to CH^*(\mathcal{O},\mu)$ *extends to an* ∞*-morphism of* $\mathcal{H}y\mathcal{C}om_\infty$*-algebras.*

Examples of the complexes $CH^*(\mathcal{O},\mu)$ and $C^*_\lambda(\mathcal{O},\mu)$ include Hochschild and cyclic cochain complexes of Frobenius or cyclic A_∞-algebras/categories, singular and equivariant cochains of S^1-spaces, complexes computing string topology, and homology and S^1-equivariant homology of the loop space of a closed oriented manifold; see [War16] for these and other examples.

Formulating precise conditions under which the Hochschild to cyclic spectral sequence degenerates is a question of active study [KS09]. This question is of particular interest when studying homological mirror symmetry and categorical models of quantum cohomology [BK98; Man99], which provided an expectation that the known BV/gravity structure should lift under degeneration. Our result further says that, under degeneration and if the chain level structure can be seen as cyclic, the lifting can be performed at the chain level. Pursuing this chain level structure in such geometric examples, with an eye toward a chain level lift of Gromov–Witten invariants, should be an interesting avenue for future study.

In proving this result, we start with a chain model for the framed little disks given in [War12], which we denote fM. Alternatively, we could use cellular chains on Cacti [Vor05; Kau05]. These chain models have $\Delta^2=0$ and thus are susceptible to the language of mixed complexes. Combining the constructions of [KMS13] and formality of the Deligne–Mumford (cyclic) operad [GSNPR05] yields the following:

THEOREM. *Homotopically trivializing the* Δ*-operator in* fM *gives a chain model for the hypercommutative operad.*

This chain model is denoted fM_{hS^1} and is, to an extent, combinatorially tractable. For example, the fundamental class of $\overline{\mathcal{M}}_4$ is represented by a union of fourteen 2-cells. The combinatorics of fM_{hS^1} are based on planar trees with three types of vertices; see Section 5. Table 1 summarizes the chain models for moduli spaces that act on such deformation complexes. (Note that the assumption that $\mathcal{O}$ is cyclic is not needed in the first row.)

In this paper, we assume familiarity with operads and cyclic operads, references for which include [GK95; MSS02; LV12]. We also assume familiarity with moduli space operads in genus 0 and their homology, references for which include [Get94a; Get95; KSV95].

Table 1

Moduli Space	Algebras' name	Chain Model	Acts on / theorem	Combin-atorics
points in the plane: $\mathcal{D}_2$	Gersten-haber	M of [KS00] (minimal operad)	$CH^*(\mathcal{O},\mu)$. A_∞-Deligne conjecture	black and white (or b/w) planar rooted trees
surfaces with boundary: $f\mathcal{D}_2 \sim \widehat{\mathcal{M}}_*$	BV	fM of [War12]	$CH^*(\mathcal{O},\mu)$. cyclic A_∞-Deligne conj.	b/w planar rooted trees with spines
surfaces with punctures: $\mathcal{M}_*$	gravity	$\mathsf{M}_{\circlearrowleft}$ of [War16]	$C^*_\lambda(\mathcal{O},\mu)$ S^1-equivariant Deligne Conj.	b/w planar non-rooted trees
Deligne–Mumford compactification: $\overline{\mathcal{M}}_*$	HyCom or formal Frob. mfd	fM_{hS^1} Theorem 4.1	$CH^*(\mathcal{O},\mu)$. under degen. Theorem 4.3	b/w/gray planar rooted trees w/spines

1. The Model Categorical Framework

Let $\mathcal{O}ps$ be the category of reduced operads valued in the category of differential graded (DG) vector spaces over a field k of characteristic 0. Reduced means that we restrict attention to arities $n \geq 1$. The category $\mathcal{O}ps$ is a model category such that forgetting to symmetric sequences creates weak equivalences and fibrations [BM03]. The reduced assumption will be needed to ensure that this model category is left proper: a pushout of a weak equivalence along a cofibration is a weak equivalence (see [BB13, Theorem 0.1]).

We consider an associative algebra to be an operad concentrated in arity 1. Here we may consider either unital associative algebras or operads without units. For an associative algebra A, we define A-$\mathcal{O}ps$ to be the undercategory $A \searrow \mathcal{O}ps$. Note we often abuse notation by writing, for example, $\mathcal{Q} \in A$-$\mathcal{O}ps$ and not $A \stackrel{\alpha}{\to} \mathcal{Q} \in A$-$\mathcal{O}ps$. Note that the undercategory of a left proper model category is a left proper model category where forgetting to the original category creates all three classes of distinguished morphisms.

Let $f: A \to B$ be a morphism between DG associative algebras A and B. We define $\Gamma_f : B$-$\mathcal{O}ps \to A$-$\mathcal{O}ps$ to be the functor induced by composition with f. The subscript notation will be suppressed when appropriate. Note that Γ preserves weak equivalences and fibrations, and since Γ is a right adjoint, it is a right Quillen functor. The left adjoint L can be realized as a left Kan extension or as a pushout of operads $\mathcal{Q} \leftarrow A \to B$.

Lemma 1.1. *If f is a quasi-isomorphism, the Quillen adjunction (L,Γ) is a Quillen equivalence.*

Proof. If $A \to \mathcal{O}$ is a cofibrant object in A-$\mathcal{O}ps$, then the map $A \to \mathcal{O}$ is a cofibration in $\mathcal{O}ps$. Using left properness of this category, we know that the pushout of f along this cofibration is a weak equivalence $\mathcal{O} \xrightarrow{\sim} L\mathcal{O}$ in $\mathcal{O}ps$. In particular, $\mathcal{O} \to \Gamma L\mathcal{O}$ is a weak equivalence in A-$\mathcal{O}ps$. This, along with the fact that Γ reflects weak equivalences, proves the claim via the standard theory (see [Hov99, Cor. 1.3.16]). □

We now consider the special case of an inclusion $\iota\colon A \hookrightarrow A \oplus B$ between DG associative algebras A and $A \oplus B$. We define $T_\iota\colon A\text{-}\mathcal{O}ps \to (A \oplus B)\text{-}\mathcal{O}ps$ to be the trivial extension by 0. It is again immediately clear from the model structure on the undercategories that T preserves weak equivalences and fibrations, and since T is a right adjoint, it is a right Quillen functor. We define its left adjoint Φ on a morphism $\varepsilon\colon A \oplus B \to \mathcal{P}$ by taking the quotient $\mathcal{P}/(\varepsilon(B))$.

In particular, for such an inclusion, we have a pair of adjunctions:

$$\begin{array}{ccc} & \overset{\Phi}{\longleftarrow} & \\ & \overset{T}{\longrightarrow} & \\ A\text{-}\mathcal{O}ps & & (A \oplus B)\text{-}\mathcal{O}ps \\ & \underset{L}{\longrightarrow} & \\ & \underset{\Gamma}{\longleftarrow} & \end{array}$$

We will typically be concerned with the inclusions $k \hookrightarrow H^*(S^1)$ and $k \hookrightarrow \Omega H_*(BS^1)$.

REMARK 1.2. We may view the categories A-$\mathcal{O}ps$ and so on as categories of monoidal functors [KW17], in which case these adjunctions suggest a formal analogy with Verdier duality, which could be further explored.

The functors T and Γ preserve weak equivalences between fibrant objects (and hence all objects). The functors L and Φ preserve weak equivalences between cofibrant objects, and so cofibrant replacement will yield a well-defined functor on the homotopy category. Let $\tilde{\Phi}(\mathcal{Q}) := \Phi(q\mathcal{Q})$ and similarly for L, where q means a choice of cofibrant replacement *in the undercategory*. Then we have adjunctions $(\tilde{L}_f, \Gamma_f)$ and $(\tilde{\Phi}_\iota, T_\iota)$ between the respective homotopy categories corresponding to any morphism f and any inclusion ι. We may now record the following technical lemma for future use.

LEMMA 1.3. *Let $A \overset{\iota}{\hookrightarrow} A \oplus B \xrightarrow{f} A \oplus C$ be a sequence of morphisms of associative DG algebras where f is of the form $f = id_A \oplus \bar{f}$ and where ι is the inclusion. If f is a weak equivalence, then for every $\mathcal{Q} \in (A \oplus C)$-$\mathcal{O}ps$, there is a zig-zag of weak equivalences of A-$\mathcal{O}ps$ connecting*

$$\tilde{\Phi}_{f\circ\iota}(\mathcal{Q}) \sim \tilde{\Phi}_\iota(\Gamma\mathcal{Q}).$$

Proof. We continue to write q for cofibrant replacement in any of these undercategories.

Since (L_f, Γ_f) is a Quillen equivalence, we know that the composite $Lq\Gamma q\mathcal{Q} \to L\Gamma q\mathcal{Q} \to q\mathcal{Q}$ is a weak equivalence. Moreover, L preserves cofibrant objects (since cofibrations are closed under pushout), and so this composite is a weak equivalence between cofibrant objects. Hence the induced map

$$\Phi_{f\circ\iota}(Lq\Gamma q\mathcal{Q}) \xrightarrow{\sim} \Phi_{f\circ\iota}(q\mathcal{Q})$$

is a weak equivalence.

The assumptions here imply that $\Gamma_f \circ T_{f\circ\iota} = T_\iota$, and so by adjointness we know that $\Phi_{f\circ\iota} \circ L_f \cong \Phi_\iota$. Therefore $\Phi_{f\circ\iota}(Lq\Gamma q\mathcal{Q}) \cong \Phi_\iota(q\Gamma q\mathcal{Q})$, and so we have

$$\begin{aligned}\tilde{\Phi}_\iota(\Gamma\mathcal{Q}) = \Phi(q\Gamma\mathcal{Q}) \xleftarrow{\sim} \Phi_\iota(q\Gamma q\mathcal{Q}) &\cong \Phi_{f\circ\iota}(Lq\Gamma q\mathcal{Q}) \\ &\xrightarrow{\sim} \Phi_{f\circ\iota}(q\mathcal{Q}) = \tilde{\Phi}_{f\circ\iota}(\mathcal{Q}).\end{aligned}$$ □

2. Trivializing Δ

In this section we revisit the literature on mixed complexes, BV algebras, and trivializing Δ to extract what will be needed. We primarily follow [KMS13], with influence from [DCV13] and [DSV15].

A mixed complex (A, d, Δ) is a chain complex (A, Δ) and a cochain complex (A, d) such that $d\Delta + \Delta d = 0$. Here the degrees have been chosen to be consistent with the example of Hochschild cohomology, but this is merely our convention. Given a mixed complex, let $(\mathrm{End}_A, \partial)$ be the cochain complex of endomorphisms of (A, d). The cycle Δ is homologically trivial if there is β_1 of degree -2 such that $\partial(\beta_1) = \Delta$. Given such β_1, we then encounter the cycle $\beta_1 \circ \Delta$. This cycle will be homologically trivial if there is β_2 of degree -4 such that $\partial(\beta_2) = \beta_1 \circ \Delta$. Given such β_2, we encounter the cycle $\beta_2 \circ \Delta$ and so on. We are thus led to the following definition.

Definition 2.1. Let (A, d, Δ) be a mixed complex. A trivialization of Δ is a sequence $\{\beta_i\}_{i\geq 0} \in \mathrm{End}_{(A,d)}$ with $\beta_0 = id_A$ such that $\partial(\beta_i) = \beta_{i-1} \circ \Delta$. In particular, $|\beta_i| = -2i$.

We write $(k_\Delta\langle\beta_*\rangle, \partial)$ for the DG associative algebra that encodes the unary operations on a trivialized mixed complex. Explicitly, $k_\Delta\langle\beta_*\rangle := k[\Delta]\langle\beta_1, \beta_2, \dots\rangle$, the free graded associative algebra on $\{\beta_i\}$ over the graded commutative algebra $k[\Delta]$. The differential takes $\partial(\beta_i) = \beta_{i-1}\Delta$ and $\partial(\Delta) = 0$, and the degrees of elements are as before. In particular, Δ being of odd degree implies $\Delta^2 = 0$. This complex may be interpreted as a noncommutative analog of ES^1; see Section 5. The operations $\partial(\beta_i)$ correspond to differentials in the Hochschild to cyclic cohomology spectral sequence; see Section 3.

We write $A[[z]] := A \otimes k[[z]]$, meaning the completed tensor product.

Lemma 2.2. *Let z be a variable of degree 2, and let (A, d, Δ) be a mixed complex. A trivialization of Δ is equivalent to a z-linear isomorphism of complexes $(A[[z]], d + z\Delta) \cong (A[[z]], d)$.*

Proof. If $\{\beta_i\}$ are such a trivialization of Δ, then define a z-linear map $F\colon (A \otimes k[[z]], d + z\Delta) \to (A \otimes k[[z]], d)$ by $F(a) = \sum_i \beta_i(a) z^i$ for $a \in A$. This is a DG map, and its leading term is invertible. Conversely, given such an isomorphism, we may extract the sequence $\{\beta_i(a)\}$ as the coefficients of $F(a)$. □

REMARK 2.3. The papers [KMS13; DCV13; DSV15] consider various equivalent forms of the data in Definition 2.1. For example, [KMS13] consider exponential coordinates for a trivialization, that is, elements ϕ_i in the algebra $k_\Delta\langle\beta_*\rangle$ related to β_i via $\beta_1 = \phi_1$, $\beta_2 = \phi_2 + \phi_1^2/2$, $\beta_3 = \phi_3 + (\phi_2\phi_1 + \phi_1\phi_2)/2 + \phi_1^3/6$, etc. Note that in [KMS13], the notation Φ_i is used in place of our β_i. These data are equivalent to what [DCV13] call "Hodge-to-de Rham degeneration data", see Theorem 2.1 in [DSV15]. Note that these data are weaker than the classical $d\bar{d}$-condition of [DGMS75], which will not in general be satisfied in the examples we consider.

2.1. *The KMS Model for* $\tilde{\Phi}$

In this section, we consider the adjunction (Φ, T) with respect to the inclusion $k \to H^*(S^1)$. We further use the shorthand notation $S^1\text{-}\mathcal{O}ps := H^*(S^1)\text{-}\mathcal{O}ps$, so we have $\Phi\colon S^1\text{-}\mathcal{O}ps \leftrightarrows \mathcal{O}ps\colon T$.

If $\mathcal{Q}$ is an S^1-operad with operator $\Delta_\mathcal{Q}$, then we define the S^1-operad $W(\mathcal{Q})$ by

$$W(\mathcal{Q}) := (\mathcal{Q} \star k_\Delta\langle\beta_*\rangle, d_{W(\mathcal{Q})} := d_\mathcal{Q} + \partial_\Delta - \partial_{\Delta_\mathcal{Q}}). \tag{2.1}$$

Here $\star$ means the free product, $\partial_{\Delta_\mathcal{Q}}(\beta_i) = \beta_{i-1}\Delta_\mathcal{Q}$ and is zero on other generators, and similarly for ∂_Δ. This is viewed as an S^1-operad via Δ (not $\Delta_\mathcal{Q}$). We define $\Phi_{\mathrm{KMS}}(\mathcal{Q}) := \Phi(W(\mathcal{Q}))$. Viewing $k_\Delta\langle\beta_*\rangle$ as a noncommutative analog of ES^1, we expect that quotienting by the Δ action in $W(\mathcal{Q})$ will be homotopy invariant. Indeed, [KMS13] shows the following:

THEOREM 2.4 [KMS13]. *As derived functors*, $\Phi_{\mathrm{KMS}} \cong \tilde{\Phi}$.

COROLLARY 2.5. *Suppose $\mathcal{Q}$ is an S^1-operad, and suppose (A, d) is a $\mathcal{Q}$-algebra via a unary square-zero operator $\Delta_\mathcal{Q}$. A trivialization of $\Delta_\mathcal{Q}$ in $(A, d, \Delta_\mathcal{Q})$ permits the lifting of (A, d) from a $\mathcal{Q}$-algebra to a $\Phi_{\mathrm{KMS}}(\mathcal{Q})$-algebra.*

Proof. Define the map $Hom(\Gamma(\mathcal{Q}), \mathrm{End}_A) \to Hom(W(\mathcal{Q}), T\mathrm{End}_A)$ as follows. Given $\Gamma(\mathcal{Q}) \to \mathrm{End}_A$, we define a map of S^1-operads $W(\mathcal{Q}) \to T\mathrm{End}_A$ by mapping $\mathcal{Q}$ via said morphism, sending $\Delta \mapsto 0$ and sending β_i to the coefficients of the degeneration, as in Lemma 2.2. Then $Hom(\Phi_{\mathrm{KMS}}(\mathcal{Q}), \mathrm{End}_A) \cong Hom(W(\mathcal{Q}), T\mathrm{End}_A)$ by adjointness, hence the claim. □

Note that although there is a zig-zag of morphisms of DG operads

$$\mathcal{Q} \xleftarrow{\sim} W(\mathcal{Q}) \to \Phi_{\mathrm{KMS}}(\mathcal{Q}),$$

when we speak of lifting a $\mathcal{Q}$-algebra to a $\Phi_{\mathrm{KMS}}(\mathcal{Q})$-algebra, we use the corollary (i.e., inclusion), which is not the same as composition in the diagram.

3. Cyclic Cohomology Operations

In this section, we give an overview of the mixed complex associated to a multiplicative cyclic operad, its associated spectral sequence, and the accompanying homotopy BV and gravity structures. The material on cyclic operads and associated complexes comes primarily from [War16, Sections 2–4]. The material on spectral sequences associated to mixed complexes can be found in [Lod98].

3.1. Deformation Complexes and Mixed Complexes from Cyclic Operads

Let $\mathcal{O}$ be a unital cyclic operad valued in DG vector spaces graded cohomologically. We define $\mathcal{O}^* := \prod_n \Sigma\mathfrak{s}\mathcal{O}(n)$, where Σ is the degree shift operator, and $\mathfrak{s}$ is the operadic suspension. This vector space has a natural Lie bracket $\{-,-\}$, for which a Maurer–Cartan element is equivalent to a morphism $\mu\colon \mathcal{A}_\infty \to \mathcal{O}$. Given this data, we form a differential in the usual way: $d_\mu := d_{\mathcal{O}} + \{\mu, -\}$, where $d_{\mathcal{O}}$ is the original (aritywise) differential in $\mathcal{O}$.

Definition 3.1. Let $\mu\colon \mathcal{A}_\infty \to \mathcal{O}$ be a map of cyclic operads. Define $CH^*(\mathcal{O},\mu)$ to be the complex $(\mathcal{O}^*, d_\mu)$. This complex is called the deformation complex of μ.

The complex $CH^*(\mathcal{O},\mu)$ can be endowed with a square zero operator Δ of degree -1 analogous to the construction of Connes' B operator; explicitly, $\Delta := Ns_0(1-t)$, where s_0 is the extra degeneracy (via the unit), t is the aritywise cyclic operator, and N is the aritywise sum $\Sigma_i t^i$. This operator is square zero and commutes with d_μ, and so $(\mathcal{O}^*, d_\mu, \Delta)$ is a mixed complex. See [War16, Section 3.2], for the details of this construction.

With this mixed complex we will associate a bicomplex with vertical differential d_μ (pointing up) and horizontal differential Δ (pointing right) and filter this bicomplex by columns. We would like this filtration to be exhaustive, so for simplicity, we impose the following assumption: the complex $\mathcal{O}^*$ is supported in nonnegative degrees. This will be the case if $\mathcal{O}(n)$ is contained in degrees $\geq -n$. Of course, there are weaker boundedness conditions under which this filtration is exhaustive. In this case, we may form the bicomplex in the first quadrant. We filter this bicomplex by columns and construct the associated spectral sequence. The 0-page has $E_0^{pq} = \Sigma^{2p}\mathcal{O}^{q-p} = \prod_{q=0}^{\infty} \Sigma^{q+2p}\mathcal{O}(n)_{q-p-n}$, the 1-page has $E_1^{pq} = \Sigma^{2p}H^{q-p}(\mathcal{O}^*, d) = \Sigma^{2p}HH^{q-p}(\mathcal{O},\mu)$, and the 2-page has $E_2^{pq} = H^p(\Sigma^{2*}HH^{q-*}(\mathcal{O},\mu),[\Delta])$. In particular, the 1-page differential is $d^1 := [\Delta]$.

If $[\Delta] = 0$ on HH, then there exists a chain operation of degree -2 whose boundary is Δ, from which we can build the 2-page differential. Explicitly, in the notation of the previous section, $d^2 = \Delta\beta_1$. Likewise, d^2 is zero if there exists a chain operation of degree -4 whose boundary is $\Delta\beta_1$, and we may explicitly calculate $d^3 = \Delta(\beta_2 - \beta_1^2)$. Continuing in this vein, we see that the existence of the β_i is necessary and sufficient for degeneration:

LEMMA 3.2 [DSV15, Proposition 1.5]. *Degeneration in this spectral sequence at the* 1*-page is equivalent to a trivialization of* Δ *in the sense of Definition 2.1.*

This spectral sequence converges to the product total complex ToT of the original bicomplex. The homology of the total complex is the cyclic cohomology of the pair $(\mathcal{O}, \mu)$ and is denoted $HC^*(\mathcal{O}, \mu)$. The lemma is a general statement about mixed complexes, but here we have additional structure allowing us to compute $HC^*(\mathcal{O}, \mu)$ in two ways, via the total complex $(\mathcal{O}^*[[z]], d + z\Delta)$, where $\deg(z) = 2$, or by the product complex of invariants $C^*_\lambda(\mathcal{O}, \mu) := (\prod_n \mathcal{O}(n)^{\mathbb{Z}_{n+1}}, d_\mu)$. The complex $C^*_\lambda(\mathcal{O}, \mu)$ may also be called the cyclic deformation complex of μ, and it computes $HC^*(\mathcal{O}, \mu)$ by the following lemma.

LEMMA 3.3. *Inclusion* $C^*_\lambda(\mathcal{O}, \mu) \hookrightarrow (\mathcal{O}^*[[z]], d + z\Delta)$ *is a quasi-isomorphism.*

Proof. This follows from Proposition 2.16 of [War16] and a standard argument of "killing contractible complexes", as in [Lod98, Section 2.4.3]. □

In the case that the spectral sequence associated with the mixed complex $(\mathcal{O}^*, d_\mu, \Delta)$ degenerates at the 1-page, we have $HC^*(\mathcal{O}, \mu) \cong HH^*(\mathcal{O}, \mu)[[z]]$. This happens precisely when there is a trivialization of Δ, so we make the following definition.

DEFINITION 3.4. We say that a morphism of cyclic operads $\mu\colon \mathcal{A}_\infty \to \mathcal{O}$ is *cyclically degenerate* if there exists a trivialization of Δ in the mixed complex $(\mathcal{O}^*, d_\mu, \Delta)$.

3.2. *Operations on CH and C_λ*

In the case that $\mu\colon \mathcal{A}_\infty \to \mathcal{O}$ is simply a map of operads (not cyclic), the classical Deligne conjecture says that $CH^*(\mathcal{O}, \mu)$ is an algebra over a chain model for the little disks operad. In the A_∞ case, this chain operad was constructed by Kontsevich and Soibelman [KS00] and was denoted M for minimal operad. It is an insertion operad of planar 2-colored trees.

If we now recall the cyclic structure of $\mu\colon \mathcal{A}_\infty \to \mathcal{O}$, then we may upgrade the complex $CH^*(\mathcal{O}, \mu)$ to an algebra over a chain model for the framed little disks. In the cyclic A_∞ setting, this operad was constructed in [War12], where it was denoted $\mathcal{TS}_\infty$. In hindsight, this ∞ notation might be confusing, so we will instead use the notation fM for "framed minimal operad". For details, see [War12], but the important properties of the chain operad fM are:

(1) There exists a zig-zag of quasi-isomorphisms of operads between fM and singular chains on the framed little disks.
(2) In particular, there exists a zig-zag of DG operads: $\mathsf{fM} \stackrel{\sim}{\leftarrow} \mathcal{BV}_\infty \stackrel{\sim}{\rightarrow} \mathcal{BV}$.
(3) The DG operad fM acts on $CH^*(\mathcal{O}, \mu)$ for any such $\mu\colon \mathcal{A}_\infty \to \mathcal{O}$ recovering the known BV structure on cohomology.

(4) The DG operad fM is constructed as an insertion operad of planar 2-colored trees along with marked points on the boundaries of vertices, which are called spines.

We could instead consider the chain model provided by cellular chains on normalized cacti [Vor05; Kau05], which has all these properties and requires that the multiplication be associative.

Since the invariants $C^*_\lambda(\mathcal{O}, \mu)$ form a d_μ-closed subspace of $CH^*(\mathcal{O}, \mu)$, we can ask which of the operations in fM restrict to this complex. In other words, which operations preserve the property of t-invariance? This question was answered in [War16], and the answer forms a suboperad $\mathsf{M}_\circlearrowleft \subset \mathsf{fM}$ whose homology is the homology operad of the punctured Riemann spheres, called the gravity operad in [Get94b]. In particular, this endows the cyclic cohomology $HC^*(\mathcal{O}, \mu)$ with the structure of a gravity algebra, and the natural map $HC^*(\mathcal{O}, \mu) \to HH^*(\mathcal{O}, \mu)$ is a map of gravity algebras. It is natural to ask when this structure lifts to the compactification, and this question is answered in the next section.

4. Collecting Results

We now collect our main results. We continue to write Φ (with suppressed subscript) in place of $\Phi_{k\hookrightarrow H^*(S^1)}$. We define $\mathsf{fM}_{hS^1} := \Phi_{\mathrm{KMS}}(\mathsf{fM})$.

THEOREM 4.1. *The DG operad* fM_{hS^1} *is a chain model for the hypercommutative operad. Explicitly, there exists a zig-zag of weak equivalences of DG operads* $\mathsf{fM}_{hS^1} \xleftarrow{\sim} \cdots \xrightarrow{\sim} S_*(\overline{\mathcal{M}}_{*+1})$.

Proof. Here S_* denotes singular chains with coefficients in the field k. Using formality of the Deligne–Mumford operad $\overline{\mathcal{M}}_{*+1}$ (see [GSNPR05, Corollary 7.2.1]), we know that there exists a zig-zag of weak equivalences connecting $\mathcal{H}y\mathcal{C}om \sim S_*(\overline{\mathcal{M}}_{*+1})$. From [KMS13] we know that there is a weak equivalence $\mathcal{H}y\mathcal{C}om \xrightarrow{\sim} \Phi_{\mathrm{KMS}}(BV)$. So it remains to show that there is a zig-zag of weak equivalences $\tilde{\Phi}(BV) \sim \tilde{\Phi}(\mathsf{fM})$.

The algebra $H^*(S^1)$ has a Koszul resolution given by taking the cobar construction Ω of the coalgebra $H_*(BS^1)$. Let $\iota\colon k \hookrightarrow \Omega H_*(BS^1)$ be an inclusion, and let $f\colon \Omega H_*(BS^1) \to H^*(S^1)$ be the standard weak equivalence. Then, since $f \circ \iota$ is the standard injection $k \hookrightarrow H^*(S^1)$, we endeavor to show that $\tilde{\Phi}_{f\circ\iota}(\mathcal{BV}) \sim \tilde{\Phi}_{f\circ\iota}(\mathsf{fM})$. Note that fM is equivalent to $\mathcal{BV}$ in the category under $\Omega H_*(BS^1)$. In particular, there is a zig-zag of weak equivalences $\mathsf{fM} \xleftarrow{\sim} \mathcal{BV}_\infty \xrightarrow{\sim} \mathcal{BV}$ under $\Omega H_*(BS^1)$. Therefore $\tilde{\Phi}_\iota(\Gamma_f \mathsf{fM}) \sim \tilde{\Phi}_\iota(\Gamma_f \mathcal{BV})$. Applying Lemma 1.3 proves the claim. □

We let $\mathcal{H}y\mathcal{C}om_\infty$ denote the cobar resolution via the Koszul dual operad $\mathcal{G}rav$, that is, $\mathcal{H}y\mathcal{C}om_\infty := \Omega\mathcal{G}rav^*$. Cofibrancy of this operad gives us the following:

COROLLARY 4.2. *There exists a zig-zag of weak equivalences*: $\mathsf{fM}_{hS^1} \xleftarrow{\sim} \mathcal{H}y\mathcal{C}om_\infty \xrightarrow{\sim} \mathcal{H}y\mathcal{C}om$.

We now consider the effect that degeneration has upon the algebraic operations on $CH^*(\mathcal{O},\mu)$ and $C^*_\lambda(\mathcal{O},\mu)$ and their cohomologies.

THEOREM 4.3. *Let $\mu\colon \mathcal{A}_\infty \to \mathcal{O}$ be a morphism of cyclic DG operads that is cyclically degenerate (Definition 3.4). Then*:

(1) *The fM-algebra structure on $CH^*(\mathcal{O},\mu)$ lifts along the inclusion $\mathsf{fM} \to \mathsf{fM}_{hS^1}$ to an fM_{hS^1}-algebra, and hence a $\mathcal{H}y\mathcal{C}om_\infty$-algebra.*
(2) *There is a $\mathcal{H}y\mathcal{C}om_\infty$-algebra structure on $C^*_\lambda(\mathcal{O},\mu)$ for which the inclusion $C^*_\lambda(\mathcal{O},\mu) \to CH^*(\mathcal{O},\mu)$ extends to an ∞-morphism between these $\mathcal{H}y\mathcal{C}om_\infty$-algebra structures.*
(3) *The gravity structure on $HC^*(\mathcal{O},\mu)$ vanishes.*

Proof. The first statement follows from Corollaries 2.5 and 4.2.

For the second statement, we first use the obvious inclusion and projection maps $in\colon (\mathcal{O}^*, d_\mu) \leftrightarrows (\mathcal{O}^*[[z]], d_\mu)\colon \pi$, with $\pi \circ in = id$, to furnish a morphism of operads $\mathrm{End}_{(\mathcal{O}^*,d_\mu)} \to \mathrm{End}_{(\mathcal{O}^*[[z]],d_\mu)}$. Observe that π is a morphism of fM_{hS^1}-algebras and hence of $\mathcal{H}y\mathcal{C}om_\infty$-algebras with respect to the induced z-linear extension.

Under the degeneration hypothesis, we have the commutative diagram

$$\begin{array}{ccc} C^*_\lambda(\mathcal{O},\mu) & \xrightarrow{\quad in \quad} & (\mathcal{O}^*, d_\mu) \\ \downarrow \sim & \overset{\pi}{\nearrow} & \\ (\mathcal{O}^*[[z]], d_\mu + \Delta z) \cong (\mathcal{O}^*[[z]], d_\mu) & & \end{array}$$

where the isomorphism is given by applying Lemma 2.2 to the mixed complex $(\mathcal{O}^*, d_\mu, \Delta)$. Since π is a morphism of $\mathcal{H}y\mathcal{C}om_\infty$-algebras, we can, via a standard transfer argument (see [BM03, Theorem 3.5], or [LV12, Chapter 10]), endow $C^*_\lambda(\mathcal{O},\mu)$ with the structure of a $\mathcal{H}y\mathcal{C}om_\infty$-algebra such that the weak equivalence $\downarrow$ extends to an ∞-quasi-isomorphism of such. Composition in the diagram proves statement 2.

For statement 3, we see from [War16] that the gravity bracket g_n on $HC^*(\mathcal{O},\mu)$, induced via the chain level action of $\mathsf{M}_{\circlearrowleft}$, satisfies the Chas–Sullivan formula (see [CS99, p. 21])

$$g_n([a_1],\dots,[a_n]) = B(I([a_1]) \bullet \cdots \bullet I([a_n])), \tag{4.1}$$

where

$$\cdots \to HC^{m-1}(\mathcal{O},\mu) \xrightarrow{S} HC^{m+1}(\mathcal{O},\mu) \xrightarrow{I} HH^{m+1}(\mathcal{O},\mu) \xrightarrow{B} HC^{m}(\mathcal{O},\mu) \to \cdots$$

is the cyclic cohomology long exact sequence, and $\bullet$ is the commutative product. Since μ is assumed to be cyclically degenerate, we know that on cohomology $B = 0$, from which the claim follows. □

From statement 3 of the theorem we derive the following consequence, which is a piece with the deformation theoretic interpretation of hypercommutativity (via the Quantum cup product).

COROLLARY 4.4. *The gravity brackets in* $\mathrm{End}_{HC^*(\mathcal{O},\mu)}$ *are obstructions to* μ *being cyclically degenerate.*

REMARK 4.5. If we consider the usual sequence of operads $\mathcal{G}rav \to \mathcal{BV} \to \mathcal{H}y\mathcal{C}om$ and note that composition in this sequence is the zero map, then we may choose to interpret statement 3 of Theorem 4.3 as a lifting statement. Indeed, it says that under degeneration, the gravity structure on $HC^*(\mathcal{O}, \mu)$ lifts via this sequence to a hypercommutative algebra.

It would be interesting to prove a chain level refinement of this lifting statement asserting that under degeneration, the $\mathcal{G}rav_\infty$ structure on $C^*_\lambda(\mathcal{O}, \mu)$ induced by $\mathsf{M}_\circlearrowleft$ lifts (up to an ∞-quasi-isomorphism) to the $\mathcal{H}y\mathcal{C}om_\infty$-algebra structure given in statement 2 of the theorem. A first step in this direction is to show the existence of a weak equivalence of DG operads $\mathcal{G}rav_\infty \xrightarrow{\sim} \mathsf{M}_\circlearrowleft$, a result which will appear as part of upcoming joint work with R. Campos.

5. Topology and Combinatorics of the Chain Model fM_{hS^1}

The chain models $\mathsf{M}_\circlearrowleft$, M, and fM may be described using the combinatorics of trees. We now briefly describe such an interpretation of the chain model fM_{hS^1}. To begin, we recall that fM is described via rooted, planar, black and white trees with spines; see Figure 1. In particular, the figure depicts the set of cells that ensure that the BV equation holds up to homotopy. Here we have suppressed associahedra labels of black vertices for simplicity; see [War12] for details.

Operadic composition at the β_i is free, so a general operation in fM_{hS^1} may be depicted as a (nonplanar) rooted tree with tails, each of whose vertices are labeled by an arity appropriate 2-colored planar rooted trees with spines or by β_i. Since the arity of each of the β_i is 1, this is the same thing as a (composition of) planar tree(s) with three types of vertices; the black and white from fM and the β_i; see Figure 2. To distinguish these new vertices, they will be drawn as gray rectangles.

In particular, we identify $\mathsf{fM}_{hS^1}(n)$ with the span of planar black, white, and gray rooted trees having n white vertices, along with the appropriate vertex labels: white vertices carry labels by $\{1, \ldots, n\}$ and spines, black vertices carry associahedra cells of appropriate arity as labels, and gray vertices are each labeled by a natural number. These trees are subject to combinatorial restrictions including that all black vertices have at least two incoming edges, and all gray vertices have exactly one incoming and one outgoing (half) edge (allowed to be the root). Notice these trees do not carry tails (unmatched half-edges) except the root. Also notice that the gray vertices do not contribute to the arity.

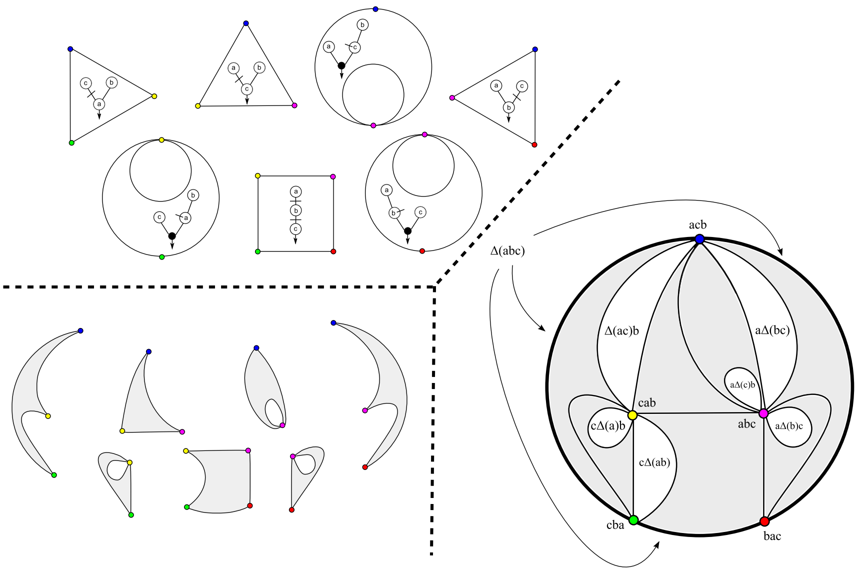


Figure 1 The BV equation on a sphere via fM. Here we see one hemisphere and six holes. The seventh hole is the missing hemisphere. If an fM-algebra lifts to an fM_{hS^1}-algebra, we may fill in the seven holes to form a ternary cycle corresponding to the fundamental class of $\overline{\mathcal{M}}_4$

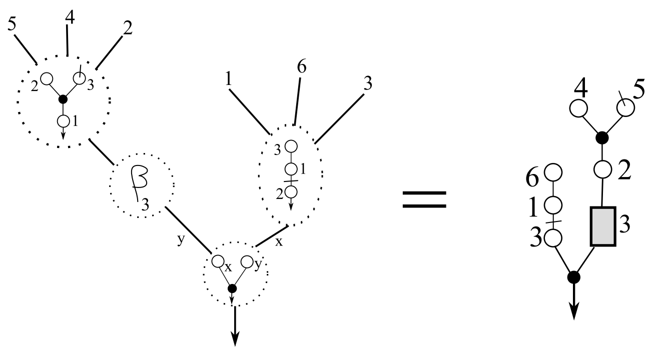
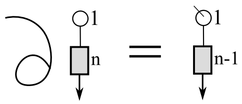
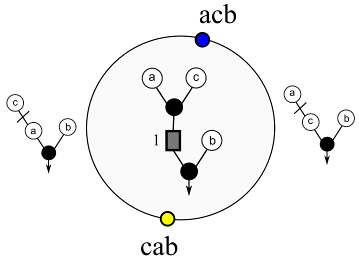


Figure 2 A generator of fM_{hS^1} as a 3-colored tree on the right-hand side. In general, such a correspondence uses the operad composition in fM and can produce a sum of 3-colored trees if composing at white vertices of submaximum height

The differential can be described as a sum over the vertices and works as in fM away from the gray vertices. At gray vertices, the label is reduced by 1, and we take Δ of the input. For example,



Consequently, $\mathsf{fM}_{hS^1}(3)$ includes the 2-cells needed to fill in the holes in the sphere corresponding to the BV equation in Figure 1. For example, we fill in the hole labeled by $\Delta(ac)b$ with



The sum of these fourteen cells represents the fundamental class of $\overline{\mathcal{M}}_4$ in $\mathsf{fM}_{hS^1}(3)$. We could similarly describe the fundamental class of $\overline{\mathcal{M}}_5$ by adding the cells prescribed in [LS07] to the failure of the appropriate 4-ary relation.

To conclude, let us recall that each chain complex $\mathsf{fM}(n)$ may be viewed as the cell complex of a CW complex [War12]. We may also consider such a description for fM_{hS^1}. Cellularly β_n corresponds to an open $2n$ disk bounded by the $2n-1$ sphere $\overline{\beta_{n-1}\Delta}$, and we may use composition in Cacti to form a CW complex whose cellular chain complex is

$$\cdots \to k_{\beta_1\Delta} \xrightarrow{0} k_{\beta_1} \xrightarrow{1} k_{\Delta} \xrightarrow{0} k_{id}.$$

For example, if we consider $\mathrm{Int}(\beta_1\Delta) = \mathrm{Int}(D_2 \times I)$, then the boundary of the disk should be attached to a composition in Cacti; recall that this composition adds the angles [Vor05]. We thus attach the cell $D_2 \times I$ to the closed pointed 2-disk via $(p,0) \mapsto p$ and $(p,1) \mapsto p$ for $p \in \mathrm{Int}(D_2)$ and $(e^{2\pi i\theta_1}, \theta_2) \mapsto e^{2\pi i(\theta_1+\theta_2)}$ for $e^{2\pi i\theta_1} \in \partial D_2$. This 3-skeleton is identified with $S^3 = \{(r_1e^{i\theta_1}, r_2e^{i\theta_2}) : r_1^2 + r_2^2 = 1\}$ by the map $(r_1e^{i\theta_1}, r_2e^{i\theta_2}) \mapsto (r_1e^{2\pi i(\theta_1-\theta_2)}, \theta_2 + r_1\theta_1)$. In particular, the cell Δ is the fiber direction, that is, concatenating this map with projection to the first factor and then identifying the unit disk mod boundary with $\mathbb{C}\cup\infty$ via $p \mapsto p/\sqrt{1-|p|^2}$, we find $(r_1e^{i\theta_1}, r_2e^{i\theta_2}) \mapsto r_1/r_2e^{2\pi i(\theta_1-\theta_2)}$, the Hopf map.

We do not expect that these CW complexes form a topological operad on the nose, but rather a weak topological operad, due to normalization issues, as in [Kau05].

Acknowledgments. This paper was written at the Simons Center for Geometry and Physics, whose support I gratefully acknowledge. Thanks are due to Gabriel C. Drummond-Cole, Yuan Gao, Ralph Kaufmann, Dennis Sullivan, and Bruno Vallette for helpful discussions along the way. Many thanks are also due to the anonymous referee for many helpful improvements.

References

[BB13] M. Batanin and C. Berger, *Homotopy theory for algebras over polynomial monads,* 2013, ⟨http://arxiv.org/abs/1305.0086⟩.

[BK98] S. Barannikov and M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields,* Int. Math. Res. Not. IMRN 4 (1998), 201–215.

[BM03] C. Berger and I. Moerdijk, *Axiomatic homotopy theory for operads,* Comment. Math. Helv. 78 (2003), no. 4, 805–831.

[CS99] M. Chas and D. Sullivan, *String topology,* 1999, arXiv.org/abs/math/9911159.

[DC14] G. C. Drummond-Cole, *Homotopically trivializing the circle in the framed little disks,* J. Topol. 7 (2014), no. 3, 641–676.

[DCV13] G. C. Drummond-Cole and B. Vallette, *The minimal model for the Batalin-Vilkovisky operad,* Selecta Math. (N.S.) 19 (2013), no. 1, 1–47.

[DGMS75] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds,* Invent. Math. 29 (1975), no. 3, 245–274.

[DSV15] V. Dotsenko, S. Shadrin, and B. Vallette, *De Rham cohomology and homotopy Frobenius manifolds,* J. Eur. Math. Soc. (JEMS) 17 (2015), no. 3, 535–547.

[Get94a] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories,* Comm. Math. Phys. 159 (1994), no. 2, 265–285.

[Get94b] ———, *Two-dimensional topological gravity and equivariant cohomology,* Comm. Math. Phys. 163 (1994), no. 3, 473–489.

[Get95] ———, *Operads and moduli spaces of genus* 0 *Riemann surfaces,* The moduli space of curves, Texel Island, 1994, Progr. Math., 129, pp. 199–230, Birkhäuser Boston, Boston, MA, 1995.

[GK95] E. Getzler and M. M. Kapranov, *Cyclic operads and cyclic homology,* Geometry, topology, & physics, Conf. Proc. Lect. Notes Geom. Topol., IV, pp. 167–201, Int. Press, Cambridge, MA, 1995.

[GSNPR05] F. Guillén Santos, V. Navarro, P. Pascual, and A. Roig, *Moduli spaces and formal operads,* Duke Math. J. 129 (2005), no. 2, 291–335.

[Hov99] M. Hovey, *Model categories,* Math. Surveys Monogr., 63, American Mathematical Society, Providence, RI, 1999.

[Kau05] R. M. Kaufmann, *On several varieties of cacti and their relations,* Algebr. Geom. Topol. 5 (2005), 237–300 (electronic).

[Kau08] ———, *A proof of a cyclic version of Deligne's conjecture via cacti,* Math. Res. Lett. 15 (2008), no. 5, 901–921.

[KMS13] A. Khoroshkin, N. Markarian, and S. Shadrin, *Hypercommutative operad as a homotopy quotient of BV,* Comm. Math. Phys. 322 (2013), no. 3, 697–729.

[KS00] M. Kontsevich and Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture,* Conférence Moshé Flato 1999, vol. I (Dijon), Math. Phys. Stud., 21, pp. 255–307, Kluwer Acad. Publ., Dordrecht, 2000.

[KS09] ———, *Notes on A_∞-algebras, A_∞-categories and non-commutative geometry,* Homological mirror symmetry, Lecture Notes in Phys., 757, pp. 153–219, Springer, Berlin, 2009.

[KSV95] T. Kimura, J. Stasheff, and A. A. Voronov, *On operad structures of moduli spaces and string theory,* Comm. Math. Phys. 171 (1995), no. 1, 1–25.

[KW17] R. M. Kaufmann and B. C. Ward, *Feynman categories,* Astérisque 387 (2017).

[Lod98] J.-L. Loday, *Cyclic homology,* second edition, Grundlehren Math. Wiss., 301, Springer-Verlag, Berlin, 1998.

[LS07] A. Losev and S. Shadrin, *From Zwiebach invariants to Getzler relation,* Comm. Math. Phys. 271 (2007), no. 3, 649–679.

[LV12] J.-L. Loday and B. Vallette, *Algebraic operads,* Grundlehren Math. Wiss., 346, Springer, Heidelberg, 2012.

[Man99] Y. I. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces,* American Mathematical Society Colloquium Publications, 47, American Mathematical Society, Providence, RI, 1999.

[MSS02] M. Markl, S. Shnider, and J. Stasheff, *Operads in algebra, topology and physics,* Math. Surveys Monogr., 96, American Mathematical Society, Providence, RI, 2002.

[Vor05] A. A. Voronov, *Notes on universal algebra,* Graphs and patterns in mathematics and theoretical physics, Proc. Sympos. Pure Math., 73, pp. 81–103, Amer. Math. Soc., Providence, RI, 2005.

[War12] B. C. Ward, *Cyclic A_∞-structures and Deligne's conjecture,* Algebr. Geom. Topol. 12 (2012), no. 3, 1487–1551.

[War16] ———, *Maurer-Cartan elements and cyclic operads,* J. Noncommut. Geom. 10 (2016), no. 4, 1403–1464.

Department of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden

bward@math.su.se