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On the Moduli of Isotropic and Helical Minimal Immersions between Spheres

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Abstract. DoCarmo–Wallach theory and its subsequent refinements assert the rich abundance of spherical minimal immersions, minimal immersions of round spheres into round spheres. A spherical minimal immersion is a conformal minimal immersion $f : S^m \to S^n$; its components are spherical harmonics of a common order p on S^m, and the conformality constant is λ_p/m, where λ_p is the pth eigenvalue of the Laplace operator on S^m. In this paper, we impose the additional constraint of "isotropy" expressed in terms of the higher fundamental forms of such immersions and determine the dimension of the respective moduli space. By the work of Tsukada, isotropy can be characterized geometrically by "helicality", constancy of initial sequences of curvatures of the image curves of geodesics under the respective spherical minimal immersions.

We first give a simple criterion for (the lowest order) isotropy of a spherical minimal immersion in terms of orthogonality relations in the third (ordinary) derivative of the image curves (Theorem A). This is then applied in the main result of this paper (Theorem B), which gives a full characterization of isotropic $SU(2)$-equivariant spherical minimal immersions of S^3 into the unit sphere of real and complex $SU(2)$-modules. Specific examples include the polyhedral minimal immersions of which the icosahedral minimal immersion (into S^{12}) is isotropic whereas its tetrahedral and octahedral cousins are not.

1. Introduction

Minimal isometric immersions of round spheres into round spheres form a rich and subtle class of objects in differential geometry studied by many authors; see [2; 4; 5; 6; 7; 8; 10; 12; 15; 16; 18; 17; 19; 20; 25; 24; 26; 27] and, for a more complete list, the bibliography at the end of the second author's monograph [23]. Such immersions can be written as $f : S^m_\kappa \to S_V$ of the round m-sphere S^m_κ of (constant) curvature $\kappa > 0$ into the unit sphere S_V of a Euclidean vector space V or, scaling the domain sphere S^m_κ to radius one, as minimal immersions $f : S^m \to S_V$ with *homothety constant* $1/\kappa$. By minimality, the components $\alpha \circ f$, $\alpha \in V^*$ (the

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dual of V), of f are necessarily eigenfunctions of the Laplacian $\triangle$ of S^m corresponding to the *eigenvalue* $\lambda = m/\kappa$. Setting $\lambda = \lambda_p = p(p+m-1)$, $p \geq 1$, the pth *eigenvalue,* and $\mathcal{H}_m^p \subset C^\infty(S^m)$, the corresponding *eigenspace* of spherical harmonics of order p on S^m, a homothetic minimal immersion $f : S^m \to S_V$ with homothety constant λ_p/m is called a *spherical minimal immersion of degree* p. (For the standard results recalled here and further, see [23, Appendix 2], [26], and the summary in [24].)

Beyond the classical *Veronese immersions* $Ver_p : S^2 \to S^{2p}$, $p \geq 2$, and various generalizations, it is well known that spherical minimal immersions abound. (For many specific examples, see [5; 6; 24].)

According to the DoCarmo–Wallach theory, for $m \geq 3$ and $p \geq 4$, the set of spherical minimal immersions $f : S^m \to S_V$ of degree p can be parameterized by a (nontrivial) compact convex body $\mathcal{M}_m^p$ in a linear subspace $\mathcal{F}_m^p$ of the symmetric square $S^2(\mathcal{H}_m^p)$. More precisely, this is a parameterization of the congruence classes of full spherical minimal immersions, where a spherical minimal immersion $f : S^m \to S_V$ is *full* if the image of f spans V, and two full spherical minimal immersions $f : S^m \to S_V$ and $f' : S^m \to S_{V'}$ are *congruent* if $f' = U \circ f$ for some *linear isometry* $U : V \to V'$.

The convex body $\mathcal{M}_m^p$ is called the *moduli space* for spherical minimal immersions $f : S^m \to S_V$ of degree p. (The moduli space $\mathcal{M}_m^p$ is trivial (zero-dimensional singleton) if and only if $m = 2$ (and $p \geq 1$) or $p \leq 3$ (and $m \geq 2$). For the original work of DoCarmo and Wallach, see [7] and [26].)

The group $SO(m+1)$ acts on the set of all spherical minimal immersions by precomposition, and this action naturally carries over to the moduli space $\mathcal{M}_m^p$. This latter action, in turn, is the restriction of the $SO(m+1)$-module structure on $S^2(\mathcal{H}_m^p)$ (extended from that of $\mathcal{H}_m^p$) with $\mathcal{F}_m^p$ being an $SO(m+1)$-submodule. The complexification of $\mathcal{F}_m^p$ decomposes as

$$\mathcal{F}_m^p \otimes_{\mathbb{R}} \mathbb{C} \cong \sum_{(u,v)\in\triangle_2^p;\, u,v \text{ even}} V_{m+1}^{(u,v,0,\ldots,0)}, \tag{1}$$

where $\triangle_2^p \subset \mathbb{R}^2$ is the closed convex triangle with vertices $(4,4)$, (p,p), and $(2p-4,4)$. Here $V_{m+1}^{(u_1,\ldots,u_d)}$, $d = [(m+1)/2]$, denotes the complex irreducible $SO(m+1)$-module with highest weight vector $(u_1,\ldots,u_d)$ relative to the standard maximal torus in $SO(m+1)$. Since the dimension of the irreducible components in (1) can be explicitly calculated by the Weyl dimension formula, we obtain the exact dimension $\dim \mathcal{M}_m^p = \dim \mathcal{F}_m^p$ of the moduli space. (The fact that the right-hand side in (1) is a lower bound for $\mathcal{F}_m^p \otimes_{\mathbb{R}} \mathbb{C}$ is the main result of the DoCarmo–Wallach theory. The equality, the so-called *exact dimension conjecture* of DoCarmo and Wallach, was proved by the second author in [21]; see also [23, Chapter 3] and also a subsequent different proof in [27].)

The dimension and subtlety of the moduli space $\mathcal{M}_m^p$ increase rapidly with $m \geq 3$ and $p \geq 4$. It is therefore natural to impose further geometric restrictions on the spherical minimal immersions. These, on the one hand, reduce the dimension and complexity of the moduli space and, on the other hand, give new examples of spherical minimal immersions with additional properties. As we will see further, two competing natural geometric properties of spherical minimal immersions are "isotropy" and "helicality".

Let $f : S^m \to S_V$ be a spherical minimal immersion of degree p. For $k = 1, \ldots, p$, let $\beta_k(f)$ be the *kth fundamental form of f*, and $\mathcal{O}_f^k$ the *kth osculating bundle of f*, both defined on a (maximal) open and dense subset $D_f \subset S^m$. (For a summary on higher fundamental forms, see [26] or [10], and also Section 2.1.)

Definition of Isotropy. Let $k \geq 2$. A spherical minimal immersion $f : S^m \to S_V$ is said to be *isotropic of order k* if $\|\beta_l(f)(X, X, \ldots, X)\|$ are *universal constants* Λ_l, $2 \leq l \leq k$ (depending only on m and p), for all unit vectors $X \in T_x(S^m)$, $x \in D_f$. The constants Λ_l, $l \geq 2$, are called the *constants of isotropy.* Since the first fundamental form of f is the differential f_*, it is convenient to set $\Lambda_1 = \sqrt{\lambda_p/m}$ with $\Lambda_1^2 = \lambda_p/m$ being the *homothety* constant. (For an extensive study of isotropy, see Tsukada's work [25].)

The moduli space $\mathcal{M}_m^{p;k}$ parameterizing the spherical minimal immersions $f : S^m \to S_V$ of degree p that are *isotropic of order k* is a linear slice of the moduli space $\mathcal{M}_m^p$ by an $SO(m+1)$-submodule $\mathcal{F}_m^{p;k} \subset \mathcal{F}_m^p$. We have the decomposition

$$\mathcal{F}_m^{p;k} \otimes_{\mathbb{R}} \mathbb{C} \cong \sum_{(u,v) \in \triangle_{k+1}^p;\, u,v \text{ even}} V_{m+1}^{(u,v,0,\ldots,0)}, \tag{2}$$

where the closed convex triangle $\triangle_k^p \subset \mathbb{R}^2$, $k = 2, 3, \ldots, [p/2]$, has vertices $(2k, 2k)$, (p, p), and $(2(p-k), 2k)$. As before, (2) gives the exact dimension of the moduli space: $\dim \mathcal{M}_m^{p;k} = \dim \mathcal{F}_m^{p;k}$. (These results have been proved by Gauchman and the second author, for $m \geq 4$, in [10]; and the case $m = 3$ has been completed in [23].)

We thus have the filtration

$$\mathcal{F}_m^p = \mathcal{F}_{m+1}^{p,1} \supset \mathcal{F}_{m+1}^{p,2} \supset \cdots \supset \mathcal{F}_{m+1}^{p;[p/2]-1},$$

where each term is obtained from decomposition (2) by restriction to the respective triangle in the sequence

$$\triangle_2^p \supset \triangle_3^p \supset \cdots \supset \triangle_{[p/2]}^p.$$

As a byproduct, we obtain that, for $p \leq 2k+1$, the moduli space $\mathcal{M}_m^{p,k}$ is trivial. (For the original proof of this, see again [25].)

A geometric characterization of isotropy lies in the concept of "helicality" introduced and studied by Sakamoto in a series of papers [18; 17; 19].

Definition of Helicality. A spherical minimal immersion $f : S^m \to S_V$ of degree p is called *helical up to order k* if, for any arc-length parameterized geodesic $\gamma : \mathbb{R} \to S^m$, the first $k-1$ curvatures of the image curve $\sigma = f \circ \gamma : \mathbb{R} \to S_V$ are nonzero *constants,* and these constants are universal in that they do not depend on the choice of γ but only on m and p.

(Recall that the curvatures are obtained by taking higher-order *covariant* derivatives of σ'. Note also that the universal constants have been determined in [9].)

Tsukada's characterization of isotropy (with appropriate modifications of his proof of Proposition 5.1 in [25]) is the following:

THEOREM. *A spherical minimal immersion $f : S^m \to S_V$ of degree p is isotropic of order k if and only if it is helical up to order k.*

The applications of this result are severalfold. First, a geometrically transparent interpretation of the moduli space $\mathcal{M}_m^{p;k}$ emerges: it parameterizes the spherical minimal immersions $f : S^m \to S_V$ of degree p that are helical up to order k. Second, as noted before, $\dim \mathcal{M}_m^{p;k}$ can be calculated explicitly. In the past, helical minimal immersions have only been studied individually, and here we have a precise formula for the dimension of the moduli space of such maps. Third, helicality is a much simpler condition than isotropy; therefore, in several instances, this condition can be checked by explicit calculation. (See the examples in Section 2.4 and the computations in Section 3.2.)

The complexity of the condition of isotropy/helicality increases rapidly with the order. The lowest order of isotropy, isotropy of order two, has special significance because of the relative simplicity of the formula expressing the first curvature of the image curve of a geodesic under the immersion. Our first result is the following:

THEOREM A. *Let $f : S^m \to S_V$ be a spherical minimal immersion of degree p. For a unit vector $X \in T_x(S^m)$, let $\gamma_X : \mathbb{R} \to S^m$ be the (arc-length parameterized) geodesic such that $\gamma_X(0) = x$ and $\gamma_X'(0) = X$, and set $\sigma_X = f \circ \gamma_X : \mathbb{R} \to S_V$. Then $f : S^m \to S_V$ is isotropic of order two if and only if, for any $x \in S^m$ and $X, Y \in T_x(S^m)$ with $\langle X, Y\rangle = 0$, we have*

$$\langle \sigma_X^{(3)}(0), \sigma_Y'(0)\rangle = 0. \tag{3}$$

Here $\sigma_X^{(k)}$, $k \geq 1$, is the kth derivative of σ_X as a vector-valued function (with values in V) and viewed as a vector field along the curve σ_X.

If $f : S^m \to S_V$ is an isotropic spherical minimal immersion of degree p, then, for the isotropy constant Λ_2, we have

$$\langle \sigma_X^{(3)}(0), \sigma_X'(0)\rangle = -\Lambda_1^2 - \Lambda_2^2, \quad \|X\| = 1, X \in T_x(S^m), x \in S^m, \tag{4}$$

where $\Lambda_1^2 = \lambda_p/m$.

As shown by the works of DeTurck and Ziller [5; 6], a rich subclass of spherical minimal immersions is comprised by minimal $SU(2)$-orbits in spheres (of $SU(2)$-modules).

Let W_p, $p \geq 0$, be the space of complex homogeneous polynomials of degree p in two variables $z, w \in \mathbb{C}$. Then W_p is a complex irreducible $SU(2)$-module.

Given a (nonzero) polynomial

$$\xi = \sum_{q=0}^{p} c_q z^{p-q} w^q \in W_p, \tag{5}$$

we consider the orbit map $f_\xi : S^3 \to W_p$, $f_\xi(g) = g \cdot \xi = \xi \circ g^{-1}$, $g \in SU(2)$, through ξ. (This so-called *equivariant construction* has been initiated by Mashimo [12].) Now f_ξ maps into a unit sphere S_{W_p} if and only if

$$\|\xi\|^2 = \sum_{q=0}^{p} (p-q)!q!|c_q|^2 = 1. \tag{6}$$

Assuming this, we obtain an $SU(2)$-equivariant map $f_\xi : S^3 \to S_{W_p}$.

DeTurck and Ziller showed that f_ξ is a spherical minimal immersion of degree p, that is, f_ξ is homothetic with homothety constant $\Lambda_1^2 = \lambda_p/3 = p(p+2)/3$, if and only if

$$\sum_{q=0}^{p-2} (p-q)!(q+2)! c_q \bar{c}_{q+2} = 0, \tag{7}$$

$$\sum_{q=0}^{p-1} (p-q)!(q+1)!(p-2q-1) c_q \bar{c}_{q+1} = 0, \tag{8}$$

$$\sum_{q=0}^{p} (p-q)!q!(p-2q)^2 |c_q|^2 = \Lambda_1^2. \tag{9}$$

(For more details, see [5; 6] or [24; 23].)

Our main result gives a full characterization of order two isotropic $SU(2)$-equivariant spherical minimal immersions $f : S^3 \to S_{W_p}$ of degree p as follows:

THEOREM B. *Let $f : S^3 \to S_{W_p}$ be an $SU(2)$-equivariant spherical minimal immersion of degree p. Setting $f = f_\xi$ with $\xi \in W_p$ satisfying* (5)–(9), *f_ξ is isotropic of order two if and only if the following system of equations holds*:

$$\sum_{q=0}^{p-4} (p-q)!(q+4)! c_q \bar{c}_{q+4} = 0, \tag{10}$$

$$\sum_{q=0}^{p-3} (p-q)!(q+3)!(p-2q-3) c_q \bar{c}_{q+3} = 0, \tag{11}$$

$$\sum_{q=0}^{p-2} (p-q)!(q+2)!(p-2q-2)^2 c_q \bar{c}_{q+2} = 0, \tag{12}$$

$$\sum_{q=0}^{p-1} (p-q)!(q+1)!(p-2q-1)^3 c_q \bar{c}_{q+1} = 0, \tag{13}$$

$$\sum_{q=0}^{p}(p-q)!q!(p-2q)^4|c_q|^2 = \Lambda_1^2+\Lambda_2^2, \tag{14}$$

where, for the second constant of isotropy Λ_2, *we have*

$$\Lambda_2^2 = \frac{p(p+2)(p(p+2)-3)}{5}. \tag{15}$$

To exhibit specific examples of isotropic $SU(2)$-equivariant spherical minimal immersions $f_\xi : S^3 \to S_{W_p}$ thus amounts to solve the system of equations (6)–(15). We will do this in Section 2.4.

Systems (7)–(9) and (10)–(14) are special cases of a general pattern, and it is reasonable to pose the following:

MAIN CONJECTURE. *Let* $f_\xi : S^3 \to S_{W_p}$, $\xi \in W_p$, *be an* $SU(2)$*-equivariant spherical minimal immersion of degree* p *and order of isotropy* $k-1$. *Then* f_ξ *is isotropic of order* k *if and only if we have*

$$\sum_{q=0}^{p-l}(p-q)!(q+l)!(p-2q-l)^{2k-l}c_q\bar{c}_{q+l} = \delta_{0l}(\Lambda_1^2+\cdots+\Lambda_k^2),$$
$$l = 0, 1, \ldots, 2k, \tag{16}$$

where $\Lambda_1, \ldots, \Lambda_k$ *are the first* k *constants of isotropy, and* δ *is the Kronecker delta.*

As noted before, for $k=1$ and $k=2$, (16) specializes to (7)–(9) and (10)–(14), respectively. (For easier reference, in these special cases, we preferred to give those expanded systems.)

2. Preliminaries

2.1. Higher Fundamental Forms and Isotropy

Let $f : S^m \to S_V$ be a spherical minimal immersion of degree p. For $k = 1, \ldots, p$, we define $\beta_k(f)$, the *kth fundamental form of* f, and $\mathcal{O}_f^k$, the *kth osculating bundle of* f. For $k=1$, we set $\beta_1(f) = f_*$, the differential of f, and $\mathcal{O}_f^1 = T(S^m)$ regarded as a subbundle of the pull-back $f^*T(S_V)$. For $k \geq 2$, the kth osculating bundle $\mathcal{O}_f^k$ is a subbundle of the normal bundle $\mathcal{N}_f$ of f. The higher fundamental forms and osculating bundles are defined on a (maximal) open dense set $D_f \subset S^m$. On D_f, the kth fundamental form is a bundle map $\beta_k(f) : S^k(T(S^m)) \to \mathcal{O}_f^k$, which is fiberwise *onto*. The higher fundamental forms are defined inductively as

$$\beta_k(f)(X_1, \ldots, X_k) = (\nabla^\perp_{X_k}\beta_{k-1}(f))(X_1, \ldots, X_{k-1})^{\perp_{k-1}},$$
$$X_1, \ldots, X_k \in T_x(S^m), x \in D_f^{k-1}, \tag{17}$$

where $\nabla^\perp$ is the natural connection on the normal bundle $\mathcal{N}_f$, $\perp_{k-1}$ is the orthogonal projection with kernel $\mathcal{O}^0_{f;x} \oplus \mathcal{O}^1_{f;x} \oplus \cdots \oplus \mathcal{O}^{k-1}_{f;x}$ ($\mathcal{O}^0_{f;x} = \mathbb{R}\cdot f(x)$), and

D_f^k is the set of points $x \in D_f^{k-1}$ at which the image $\mathcal{O}_{f;x}^k$ of $\beta_k(f)$ has maximal dimension. We set $D_f = \bigcap_{k=0}^p D_f^k$.

In the definition of isotropy in the previous section, the higher fundamental forms have *identical* (vectorial) arguments. It is desirable and more revealing to have an equivalent formulation of isotropy with independent vectorial arguments. (This has been used by Tsukada [25] and by Gauchman and the second author [10].)

First, we define the *Dirac delta* $\delta_{m,p} : S^m \to S_{(\mathcal{H}_m^p)^*}$ by evaluating spherical harmonics in $\mathcal{H}_m^p$ on points of S^m [27]. (The Dirac delta is also known as the *standard minimal immersion;* see [7; 26].) Then $\delta_{m,p}$ is $SO(m+1)$-equivariant with respect to the $SO(m+1)$-module structure of $(\mathcal{H}_m^p)^* \cong \mathcal{H}_m^p$. We write $S^m = SO(m+1)/SO(m)$ with isotropy subgroup $SO(m+1)_o = SO(m) \oplus [1] \cong SO(m)$ at the base point $o = (0, \dots, 0, 1)$. Since $SO(m)$ acts irreducibly on $T_o(S^m)$, the Dirac delta $\delta_{m,p}$ is homothetic and therefore a spherical minimal immersion.

Moreover, *branching* (from $SO(m+1)$ to $SO(m)$) gives

$$\mathcal{H}_m^p|_{SO(m)} = \mathcal{H}_{m-1}^0 \oplus \mathcal{H}_{m-1}^1 \oplus \cdots \oplus \mathcal{H}_{m-1}^p,$$

and this corresponds to the decomposition of the osculating spaces

$$\mathcal{O}_{\delta_{m,p};o}^0 \oplus \mathcal{O}_{\delta_{m,p};o}^1 \oplus \cdots \oplus \mathcal{O}_{\delta_{m,p};o}^p.$$

(See again [7; 26].)

In a technical argument [25, Proposition 3.1], Tsukada gave the following equivalent formulation of isotropy: A spherical minimal immersion $f : S^m \to S_V$ is *isotropic of order k,* $2 \le k \le p$, if and only if, for $2 \le l \le k$, we have

$$\begin{aligned} &\langle \beta_l(f)(X_1, \dots, X_l), \beta_l(f)(X_{l+1}, \dots, X_{2l}) \rangle \\ &\quad = \langle \beta_l(\delta_{m,p})(X_1, \dots, X_l), \beta_l(\delta_{m,p})(X_{l+1}, \dots, X_{2l}) \rangle, \\ &\quad X_1, \dots, X_{2l} \in T_x(S^m), x \in D_f. \end{aligned} \tag{18}$$

(Note that, for $X = X_1 = \cdots = X_{2l}$ (and of unit length) this specializes to the definition of isotropy we gave in Section 1.)

The condition of isotropy (18) implies that, for $2 \le l \le k$, the osculating bundles $\mathcal{O}_f^l$ of f are isomorphic with those of the Dirac delta $\delta_{m,p}$. In view of the decomposition of the osculating spaces for $\delta_{m,p}$, for a spherical minimal immersion $f : S^m \to S_V$ that is isotropic of order k, we have the *lower bound*

$$\dim V \ge \dim(\mathcal{H}_{m-1}^0 \oplus \mathcal{H}_{m-1}^1 \oplus \cdots \oplus \mathcal{H}_{m-1}^k) = \dim \mathcal{H}_m^k. \tag{19}$$

2.2. *The Lowest Order Isotropy*

In this short section, we obtain a simple condition for isotropy of order two of a spherical minimal immersion. We will use this to prove Theorem A in Section 3.1.

For brevity, we will suppress the order and refer to a spherical minimal immersion of degree p and order of isotropy two simply as an *isotropic spherical minimal immersion* (of degree p).

REMARK. The moduli space parameterizing the (congruence classes of full) isotropic spherical minimal immersions is $\mathcal{M}_m^{p;2}$, which by (2) is nontrivial if and only if $p \geq 6$.

By definition, a spherical minimal immersion $f : S^m \to S_V$ is isotropic (of order two) if $\|\beta(f)(X, X)\|$ is a *universal* constant Λ_2 for all unit vectors $X \in T_x(S^m)$, $x \in S^m$.

It is well known that this holds if (and only if) the second fundamental form $\beta(f)$ is *pointwise isotropic,* that is, *for any* $x \in S^m$, $\beta(f)$ is isotropic on the tangent space $T_x(S^m)$ as a symmetric bilinear form in the classical sense (with $\|\beta(f)(X, X)\|$ being independent of the unit vector $X \in T_x(S^m)$). (See, e.g., [25, Proposition 3.1].)

Isotropy (at a point) can be conveniently reformulated in terms of the *shape operator* $\mathcal{A}(f)$ of $f : S^m \to S_V$ as

$$\mathcal{A}(f)_{\beta(f)(X,X)} X \wedge X = 0, \quad X \in T_x(S^m), x \in S^m. \tag{20}$$

Indeed, for $x \in S^m$, polarizing $\|\beta(f)(X, X)\|^2$, $X \in T_x(S^m)$, we see that $\beta(f)$ is isotropic on $T_x(S^m)$ if and only if

$$\langle \beta(f)(X, X), \beta(f)(X, Y)\rangle = \langle \mathcal{A}_{\beta(f)(X,X)} X, Y\rangle = 0$$

for all $X, Y \in T_x(S^m)$ with $\langle X, Y\rangle = 0$. (See also [17, (2.2)] or [4, Section 2].)

As expected, higher-order isotropy is more complex. For completeness, we briefly indicate the condition analogous to (20). Let $X \in T_x(S^m)$, $x \in D_f$, and denote by ζ_k a (locally defined) section of the osculating bundle $\mathcal{O}_f^k$. (We use the notation in the previous section and tacitly assume that we work over $D_f \subset S^m$ so that all osculating bundles are well defined.) We define T^k by

$$T_X^k(\zeta_{k-1}) = (\nabla_X^{\perp} \zeta_{k-1})^{\mathcal{O}_f^k},$$

where $\nabla^{\perp}$ is the connection of the normal bundle $\mathcal{N}_f$, and the osculating bundle in the superscript indicates orthogonal projection. By (17) we have

$$\beta_k(f)(X_1, \ldots, X_k) = T_{X_1}^k(\beta_{k-1}(f)(X_2, \ldots, X_k))$$

for (locally defined) vector fields $X_1, \ldots, X_k$ on D_f.

Let S_X^{k-1} be the adjoint of T_X^k (with respect to the bundle metrics on the respective osculating bundles induced by the Riemannian metric on S_V). Clearly, we have

$$S_X^{k-1}(\zeta_k) = -(\nabla_X^{\perp} \zeta_k)^{\mathcal{O}_f^{k-1}}.$$

Now, polarizing $\|\beta_k(f)(X, \ldots, X)\|^2$ as before, we obtain that, for $x \in S^m$, $\beta_k(f)$ is isotropic on $T_x(S^m)$ if and only if

$$\langle \beta_k(f)(X, \ldots, X), \beta_k(f)(X, \ldots, X, Y)\rangle = 0$$

whenever $X, Y \in T_x(S^m)$ with $\langle X, Y\rangle = 0$. We now calculate

$$\begin{aligned}&\langle \beta_k(f)(X, \ldots, X), \beta_k(f)(X, \ldots, X, Y)\rangle \\ &\quad = \langle \beta_k(f)(X, \ldots, X), T_X^k \beta_{k-1}(f)(X, \ldots, X, Y)\rangle\end{aligned}$$

$$= \langle S_X^{k-1}\beta_k(f)(X,\dots,X), \beta_{k-1}(f)(X,\dots,X,Y)\rangle$$
$$= \langle S_X^2 S_X^3 \cdots S_X^{k-1}\beta_k(f)(X,\dots,X), \beta(f)(X,Y)\rangle$$
$$= \langle \mathcal{A}(f)_{S_X^2 S_X^3 \cdots S_X^{k-1}\beta_k(f)(X,\dots,X)}X, Y\rangle.$$

Summarizing, we obtain that, for $x \in S^m$, $\beta_k(f)$, $k \geq 3$, is isotropic on $T_x(S^m)$ if and only if

$$\mathcal{A}(f)_{S_X^2 S_X^3 \cdots S_X^{k-1}\beta_k(f)(X,\dots,X)}X \wedge X = 0, \quad X \in T_x(S^m).$$

Remark. Another approach for order k isotropy in general is derived by Hong and Houh [11, Theorem 2.3]. The first $k-1$ curvatures are constant if and only if, for $2 \leq l \leq 2k-1$, we have

$$\mathcal{A}(f)_{(D^{l-2}\beta(f))(X,\dots,X)}X \wedge X = 0, \quad X \in T_x(S^m), x \in S^m,$$

where D is the covariant differentiation on $T(M) \oplus \mathcal{N}_f$ with $\mathcal{N}_f$ being the normal bundle of f. (Note that, in this case, $\mathcal{A}_{(D^{l-2}\beta(f))(X\dots,X)}X = 0$ for l odd.)

These conditions are formulated in terms of the notion of *contact number* of Euclidean submanifolds. See [3; 1] for details for pseudo-Euclidean submanifolds. The first author generalized this notion for the case of affine immersions in projectively flat space; see [14].

2.3. $SU(2)$-*Equivariant Minimal Immersions*

As noted in Section 1, the moduli space $\mathcal{M}_m^p$ parameterizing the congruence classes of full spherical minimal immersions $f : S^m \to S_V$ of degree p is nontrivial if and only if $m \geq 3$ and $p \geq 4$. The lowest dimension of the domain S^m for nontrivial moduli is $m = 3$. This case is of special interest since the product $SU(2) \times SU(2)$ double covers the acting isometry group $SO(4)$. The (projection of the) first factor $SU(2)$ in this product is an isomorphic copy of $SU(2)$, and it can be realized as a subgroup of $SO(4)$ by identifying $\mathbb{R}^4$ and $\mathbb{C}^2$ in the usual way: $\mathbb{R}^4 \ni (x, y, u, v) \mapsto (z, w) = (x + \imath y, u + \imath v) \in \mathbb{C}^2$. With this identification,

$$SU(2) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \,\middle|\, |a|^2 + |b|^2 = 1, a, b \in \mathbb{C} \right\} \tag{21}$$

becomes a subgroup of $SO(4)$. (Note that this also shows that $SU(2) = S^3$, where the latter is the unit sphere in $\mathbb{C}^2$.)

The orthogonal matrix $\gamma = \mathrm{diag}(1, 1, 1, -1) \in O(4)$ (or, in complex coordinates, $\gamma : z \mapsto z$, $w \mapsto \bar{w}$, $(z, w) \in \mathbb{C}^2$) conjugates $SU(2)$ to the subgroup

$$SU(2)' = \gamma SU(2)\gamma \subset SO(4), \quad \gamma^{-1} = \gamma.$$

This is the projection of the second factor in the product $SU(2) \times SU(2)$ to $SO(4)$ via the double cover. Both subgroups $SU(2)$ and $SU(2)'$ are normal in $SO(4)$, and we have $SU(2) \cap SU(2)' = \{\pm I\}$.

In view of this, it is natural to consider (full) spherical minimal immersions $f : S^3 \to S_V$ of degree p that are $SU(2)$*-equivariant,* that is, there exists a homomorphism $\rho_f : SU(2) \to SO(V)$ such that

$$f \circ g = \rho_f(g) \circ f, \quad g \in S^3. \tag{22}$$

The homomorphism ρ_f (associated to $SU(2)$-equivariance) defines an $SU(2)$-module structure on the Euclidean vector space V. Moreover, the natural isomorphism between V and the space of components $V_f = \{\alpha \circ f \mid \alpha \in V^*\} \subset \mathcal{H}_3^p$ (through the dual V^*) is $SU(2)$-equivariant, and we obtain that V is an $SU(2)$-submodule of the restriction $\mathcal{H}_3^p|_{SU(2)}$.

In general, the irreducible complex $SU(2)$-modules are parameterized by their dimension, and they can be realized as submodules appearing in the (multiplicity one) decomposition of the $SU(2)$-module of complex homogeneous polynomials $\mathbb{C}[z, w]$ in two variables. For $p \geq 0$, the pth submodule W_p, $\dim_{\mathbb{C}} W_p = p + 1$, comprises the homogeneous polynomials of degree p. With respect to the L^2-scalar product (suitably scaled), the standard orthonormal basis for W_p is $\{z^{p-q} w^q / \sqrt{(p-q)!q!}\}_{q=0}^p$. For p odd, W_p is irreducible *as a real* $SU(2)$*-module.* For p even, the fixed point set R_p of the complex antilinear self map $z^q w^{p-q} \mapsto (-1)^q z^{p-q} w^q$, $q = 0, \dots, p$, of W_p is an irreducible real submodule with $W_p = R_p \otimes_{\mathbb{R}} \mathbb{C}$.

For the space of *complex-valued* spherical harmonics $\mathcal{H}_3^p$ of order p, we have

$$\mathcal{H}_3^p = W_p \otimes W_p',$$

as complex $SO(4)$-modules, where W_p' is the $SU(2)'$-module obtained from the $SU(2)$-module W_p via conjugation by γ, and the tensor product is understood by the double cover $SU(2) \times SU(2) \to SO(4)$. Restricting to $SU(2)$, we obtain

$$\mathcal{H}_3^p = (p+1) W_p$$

as complex $SU(2)$-modules.

For *real-valued* spherical harmonics, for p even, this gives

$$\mathcal{H}_3^p = (p+1) R_p.$$

Similarly, for p odd, we have

$$\mathcal{H}_3^p = \frac{p+1}{2} W_p$$

as real $SU(2)$-modules.

Returning to our $SU(2)$-equivariant spherical minimal immersion $f : S^3 \to S_V$, we see that the $SU(2)$-module V is isomorphic with a multiple of R_p for p even and with a multiple of W_p for p odd. As a byproduct, we also obtain that the dimension of V is *divisible* by $p + 1$ if p is even and by $2(p + 1)$ if p is odd.

REMARK. $SU(2)$-equivariant spherical minimal immersions $f : S^3 \to S_V$ of degree p that are isotropic of order k are parameterized by the $SU(2)$*-equivariant moduli space* $(\mathcal{M}_3^{p,k})^{SU(2)}$. It is a compact convex body in the fixed point set

$(\mathcal{F}_3^{p;k})^{SU(2)}$, which, in view of the double cover $SU(2) \times SU(2) \to SO(4)$, is an $SU(2)'$-module. We have

$$(\mathcal{F}_3^{p,k})^{SU(2)} = \sum_{q=k+1}^{[p/2]} R'_{4q}$$

as real $SU(2)'$-modules. In particular, we have the dimension formula

$$\dim(\mathcal{M}_3^{p;k})^{SU(2)} = \dim(\mathcal{F}_3^{p;k})^{SU(2)} = \left(2\left[\frac{p}{2}\right] + 2k + 3\right)\left(\left[\frac{p}{2}\right] - k\right), \quad p \geq 2k+2. \quad (23)$$

To seek explicit examples of $SU(2)$-equivariant spherical minimal immersions $f : S^3 \to S_V$, it is natural to consider the simplest case where $V = W_p$ (regardless the parity of p).

Examples. The quartic ($p = 4$) minimal immersion $\mathcal{I} : S^3 \to S_{W_4} = S^9$, the $SU(2)$-orbit map of the polynomial $\xi = (\sqrt{6}/24)(z^4 - w^4) + (\sqrt{2}/4)z^2w^2 \in W_4$, is archetypal in understanding the structure of the moduli space $(\mathcal{M}_3^4)^{SU(2)}$ and thereby $\mathcal{M}_3^4$; see [24]. Moreover, the sextic ($p = 6$) *tetrahedral minimal immersion Tet* : $S^3 \to S_{R_6} = S^6$, the $SU(2)$-orbit map of the polynomial $\xi = (1/(4\sqrt{15}))zw(z^4 - w^4) \in R_6 \subset W_6$, is a famous example because it realizes the minimum range dimension among all nonstandard spherical minimal immersions of S^3. (For more details, see [12; 13], and for an extensive list of $SU(2)$-equivariant spherical minimal immersions, see [5; 6; 23].)

2.4. Isotropic and Nonisotropic Examples

The archetypal $SU(2)$-equivariant spherical minimal immersions are the *tetrahedral, octahedral,* and *icosahedral* minimal immersions. As recognized by DeTurck and Ziller [5; 6], they are the $SU(2)$-orbits of Felix Klein's minimum-degree absolute invariants of the tetrahedral, T, octahedral, O, and icosahedral, I, groups in $R_{2d} \subset W_{2d}$ for $d = 3, 4, 6$. As such, they realize minimal embeddings of the tetrahedral, S^3/T^*, octahedral, S^3/O^*, and icosahedral, S^3/I^*, manifolds, where the asterisk indicates the respective binary groups. (For more details, see also [23, Section 1.5].)

Example 1. The *tetrahedral minimal immersion Tet* : $S^3 \to S_{R_6} = S^6$ cannot be isotropic for reasons of dimension since, for any isotropic $SU(2)$-equivariant spherical minimal immersion $f : S^3 \to S_V$, by (19) we have $\dim V \geq \dim \mathcal{H}_3^2 = 9$.

Example 2. The dimension restriction in the previous example does not exclude the *octahedral minimal immersion Oct* : $S^3 \to S_{R_8} = S^8$ to be isotropic; however, it is the $SU(2)$-orbit of the octahedral invariant $\xi = c_0(z^8 + 14z^4w^4 + w^8) \in R_8$, $c_0 = 1/(96\sqrt{21})$, which does not satisfy (10) or (14). Hence the octahedral minimal immersion is not isotropic.

EXAMPLE 3. The icosahedral minimal immersion $\mathcal{I} : S^3 \to S_{R_{12}} = S^{12}$ is the $SU(2)$-orbit of Klein's icosahedral invariant $\xi = c_1(z^{11}w + 11z^6w^6 - zw^{11}) \in R_{12}$, $c_1 = 1/(3600\sqrt{11})$. It follows by direct substitution that it is isotropic. Note that this has been proved by Escher and Weingart [8] using basic representation theoretical tools. (See also [23, Remark 2 in Section 4.5].)

CONJECTURE 1. *There are no isotropic spherical minimal immersions* $f : S^3 \to S_{R_8}$ *or* $f : S^3 \to S_{R_{10}}$. (*Over the reals*, (6)–(14) *represent* 15 *quadratic equations*, *for* R_8, *in* 9 *variables*, *and*, *for* R_{10}, *in* 11 *variables*; *both highly overdetermined systems*.)

Note that if Conjecture 1 holds, then the icosahedral minimal immersion is the minimum-(co)dimension *isotropic* spherical minimal immersion.

CONJECTURE 2. *The icosahedral minimal immersion is unique* (*up to isometries of the domain and the range*) *among all isotropic* $SU(2)$*-equivariant spherical minimal immersions with range* R_{12}. (*Note that even for* R_{12}, *system* (6)–(14) *is slightly overdetermined*: 15 *equations in* 13 *variables*.)

EXAMPLE 4. As a slight modification of Example 3, we let $\xi = c_1(z^{11}w + 11\imath z^6w^6 - zw^{11})$ (with c_1 as there). Then ξ belongs to W_{12} (and not R_{12}), and the corresponding (full) isotropic $SU(2)$-equivariant spherical minimal immersion $f_\xi : S^3 \to S_{W_{12}} = S^{25}$ has the binary dihedral group D_5^* as its invariance group, and it gives a minimal embedding of the dihedral manifold S^3/D_5^* into S^{25}.

The isocahedral minimal immersion and this last example are in the complete list of DeTurck and Ziller of all spherical minimal embeddings of three-dimensional space forms. (See [5; 6] and also [23, Section 1.5].) Using Theorem B, a simple case-by-case check shows that these are the only isotropic spherical minimal immersions in this list.

We have $W_{12} = 2R_{12}$ as real $SU(2)$-modules, so that Example 4 immediately raises the problem of minimal *multiplicity;* that is, for given $p \geq 6$ even, what is the *minimal* $1 \leq k \leq p+1$ such that an isotropic $SU(2)$-equivariant spherical minimal immersion $f : S^3 \to S_{kR_p}$ exists. Using deeper representation theoretical tools, the second author in [22, Corollary to Theorem 3] showed the existence of isotropic $SU(2)$-equivariant spherical minimal immersions $f : S^3 \to S_{4R_6}$ and $f : S^3 \to S_{6R_8}$ (the latter of order of isotropy 3).

Isotropic $SU(2)$-equivariant spherical minimal immersions with range W_p abound for $p \geq 11$ as the following examples show.

EXAMPLE 5. Letting $c_q = 0$ for $q \not\equiv 0 \pmod 5$, $q = 0, \ldots, 11$, (6)–(14) give

$$|c_0|^2 = \frac{1}{2^9 \cdot 3^5 \cdot 5^4 \cdot 11}, \qquad |c_5|^2 = \frac{11}{2^7 \cdot 3^3 \cdot 5^4}, \qquad |c_{10}|^2 = \frac{1}{2^9 \cdot 3^5 \cdot 5^4}.$$

Setting $\xi = c_0z^{11} + c_5z^6w^5 + c_{10}zw^{10} \in W_{11}$, we obtain isotropic $SU(2)$-equivariant spherical minimal immersions $f_\xi : S^3 \to S_{W_{11}} = S^{23}$.

EXAMPLE 6. For a somewhat more symmetric example in W_{12}, once again letting $c_q = 0$ for $q \not\equiv 0 \pmod 5$, $q = 0, \ldots, 12$, by (6)–(14), we have

$$|c_0|^2 = \frac{2^5}{12! \cdot 5^2 \cdot 7}, \qquad |c_5|^2 = \frac{2 \cdot 3 \cdot 11}{5! \cdot 7! \cdot 5^2 \cdot 7}, \qquad |c_{10}|^2 = \frac{11}{2! \cdot 10! \cdot 5^2}.$$

Setting $\xi = c_0 z^{12} + c_5 z^7 w^5 + c_{10} z^2 w^{10} \in W_{12}$, we obtain isotropic $SU(2)$-equivariant spherical minimal immersions $f_\xi : S^3 \to S_{W_{12}} = S^{25}$.

3. Proofs

3.1. Proof of Theorem A

We let ∇ denote the Levi–Civita covariant differentiation on S^m and D the covariant (ordinary) differentiation on the Euclidean vector space V. Letting $\iota : S_V \to V$ denote the inclusion, we have

$$D_X Y = \nabla_X Y + \beta(f)(X, Y) - \langle X, Y \rangle \iota \tag{24}$$

for any locally defined vector fields X, Y on S^m. As usual, we identify locally defined vector fields with their images under any immersions (such as $f : S^m \to S_V$, $\iota \circ f : S^m \to V$, etc.). With this, for any unit tangent vector $X \in T_x(S^m)$, $x \in S^m$, we have

$$D_{\sigma'_X} \sigma_X^{(k)} = \sigma_X^{(k+1)}, \quad k \geq 0, \tag{25}$$

as vector fields along σ_X. Using (24)–(25), we now calculate

$$\sigma''_X = D_{\sigma'_X} \sigma'_X = \beta(f)(\sigma'_X, \sigma'_X) - (\lambda_p/m)\sigma_X,$$

where $\nabla_{\sigma'_X} \sigma'_X = 0$ since γ_X is a geodesic. Using this, we have

$$\begin{aligned} \sigma_X^{(3)} &= D_{\sigma'_X} \sigma''_X = D_{\sigma'_X} (\beta(f)(\sigma'_X, \sigma'_X) - (\lambda_p/m)\sigma'_X) \\ &= \nabla^\perp_{\sigma'_X} \beta(f)(\sigma'_X, \sigma'_X) - \mathcal{A}(f)_{\beta(f)(\sigma'_X, \sigma'_X)} \sigma'_X - (\lambda_p/m)\sigma'_X, \end{aligned}$$

where $\nabla^\perp$ denotes the covariant differentiation of the normal bundle $\mathcal{N}_f$ of $f : S^m \to S_V$, and $\mathcal{A}(f)$ is the shape operator of f. For unit tangent vectors $X, Y \in T_x(S^m)$, $x \in S^m$, this gives

$$\langle \sigma_X^{(3)}(0), \sigma'_Y(0) \rangle = -\langle \mathcal{A}(f)_{\beta(f)(X,X)} X, Y \rangle - (\lambda_p/m)\langle X, Y \rangle.$$

The equivalence of (3) and (20) is now clear.

Setting $X = Y \in T_x(S^m)$, $x \in S^m$, with $\|X\| = 1$, we obtain

$$\langle \sigma_X^{(3)}(0), \sigma'_X(0) \rangle = -\|\beta(f)(X, X)\|^2 - \frac{\lambda_p}{m} = -\Lambda_2^2 - \frac{\lambda_p}{m}.$$

The last statement in (4) and thereby Theorem A follows.

3.2. *Proof of Theorem* B

We first need to develop several computational tools.

In the Lie algebra $su(2)$, we take the standard (orthonormal) basis

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & \imath \\ \imath & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} \imath & 0 \\ 0 & -\imath \end{bmatrix}.$$

The *unit sphere* $S_{su(2)} \subset su(2)$ can then be parameterized by spherical coordinates as

$$\begin{aligned} U = U(\theta, \varphi) &= \cos\theta \cos\varphi \cdot X + \sin\theta \cos\varphi \cdot Y + \sin\varphi \cdot Z \\ &= \begin{bmatrix} \imath \sin\varphi & e^{\imath\theta} \cos\varphi \\ -e^{-\imath\theta} \cos\varphi & -\imath \sin\varphi \end{bmatrix} \in S_{su(2)}, \quad \theta, \varphi \in \mathbb{R}. \end{aligned}$$

(For simplicity, unless needed, we suppress the angular variables.) An important feature of the spherical coordinates to be used in the sequel is that, for *given* $\theta, \varphi \in \mathbb{R}$, the vectors $U(\theta, \varphi)$, $U(\theta + \pi/2, 0)$, and $U(\theta, \varphi + \pi/2)$ form an orthonormal basis of $su(2)$ (which, for $\theta = \varphi = 0$, reduces to the standard basis).

Moreover, since $U^2 = -I$, we have

$$U^{2l} = (-1)^l I \quad \text{and} \quad U^{2l+1} = (-1)^l U, \quad l \geq 1.$$

Hence, for the *exponential map exp* : $su(2) \to SU(2)$, we obtain

$$\begin{aligned} \exp(t \cdot U) &= \sum_{j=0}^{\infty} \frac{1}{j!}(tU)^j = \sum_{l=0}^{\infty} (-1)^l \frac{t^{2l}}{(2l)!} U^{2l} + \sum_{l=0}^{\infty} (-1)^l \frac{t^{2l+1}}{(2l+1)!} U^{2l+1} \\ &= \cos t \cdot I + \sin t \cdot U \\ &= \begin{bmatrix} \cos t + \imath \sin\varphi \sin t & e^{\imath\theta} \cos\varphi \sin t \\ -e^{-\imath\theta} \cos\varphi \sin t & \cos t - \imath \sin\varphi \sin t \end{bmatrix}, \quad t \in \mathbb{R}. \end{aligned} \tag{26}$$

Recall from Section 1 the equivariant construction, which associates with a unit vector $\xi \in W_p$, $p \geq 4$, the orbit map $f_\xi : S^3 \to S_{W_p}$ defined by

$$f_\xi(g) = g \cdot \xi = \xi \circ g^{-1}, \quad g \in SU(2).$$

Here $SU(2) = S^3$, the unit sphere in $\mathbb{C}^2$. For computational purposes, it is convenient to identify $\mathbb{C}^2$ with the space of quaternions $\mathbb{H}$ via $(a, b) \mapsto a + \jmath b$, $(a, b) \in \mathbb{C}^2$. With this, S^3 becomes the unit sphere $S_{\mathbb{H}}$. The unit quaternion $g = a + \jmath b \in S_{\mathbb{H}}$ has the inverse

$$g^{-1} = g^* = (\bar{a}, -b) = (a + \jmath b)^{-1} = \bar{a} - \jmath b.$$

Using the realization W_p as an $SU(2)$-submodule of $\mathbb{C}[z, w]$, we obtain the explicit representation

$$\begin{aligned} f_\xi(g)(z, w) &= \xi(g^{-1}(z, w)) = \xi((\bar{a} - \jmath b)(z + \jmath w)) \\ &= \xi((\bar{a}z + \bar{b}w) + \jmath(-bz + aw)) \\ &= \xi(\bar{a}z + \bar{b}w, -bz + aw), \quad g = (a, b) = a + \jmath b \in S^3. \end{aligned}$$

For the proof of Theorem B, we need to simplify the condition of isotropy of f_ξ in Theorem A. As the first step, we note that, since f_ξ is $SU(2)$-equivariant, the vanishing of the scalar products in (3) need to hold only for unit vectors in the tangent space $T_1(S^3) = su(2)$.

Let $U \in T_1(S^3) = T_I(SU(2)) = su(2)$ be a unit vector, and consider the geodesic $\gamma_U : \mathbb{R} \to S^3$, $\gamma_U(0) = 1$, and $\gamma_U'(0) = U$. Letting $U = U(\theta, \varphi)$, $\theta, \varphi \in \mathbb{R}$, by (26) we have

$$\gamma_U(t) = (\cos t + \imath \sin\varphi \sin t, -e^{-\imath\theta}\cos\varphi \sin t) \in S^3, \quad t \in \mathbb{R}.$$

Following Theorem A, we let $\sigma_U = f_\xi \circ \gamma_U : \mathbb{R} \to S_{W_p}$ be the image curve under f_ξ. By the explicit representation above, we obtain

$$\sigma_U(t) = \xi(a(t), b(t)), \quad t \in \mathbb{R}, \tag{27}$$

where

$$\begin{aligned} a(t) = a(t,\theta,\varphi) &:= (\cos t - \imath \sin\varphi \sin t) z - (e^{\imath\theta}\cos\varphi \sin t) w \\ &= z \cdot \cos t + (-\imath \sin\varphi \cdot z - e^{\imath\theta}\cos\varphi \cdot w)\sin t, \\ b(t) = b(t,\theta,\varphi) &:= (e^{-\imath\theta}\cos\varphi \sin t) z + (\cos t + \imath \sin\varphi \sin t) w \\ &= w \cdot \cos t + (e^{-\imath\theta}\cos\varphi \cdot z + \imath \sin\varphi \cdot w)\sin t. \end{aligned}$$

It is a simple but crucial fact that, for given $\theta, \varphi \in \mathbb{R}$, the pair $(a(t), b(t))$, $t \in \mathbb{R}$, satisfies the system of differential equations

$$\begin{aligned} \frac{da}{dt} &= -\imath \sin\varphi \cdot a(t) - e^{\imath\theta}\cos\varphi \cdot b(t), \\ \frac{db}{dt} &= e^{-\imath\theta}\cos\varphi \cdot a(t) + \imath \sin\varphi \cdot b(t), \end{aligned}$$

with initial conditions $a(0) = z$, $b(0) = w$. (Note that the coefficient matrix is in $SU(2)$.)

We now expand $\xi \in W_p$ as in (5). Evaluating ξ on the pair $(a(t), b(t))$, $t \in \mathbb{R}$, by (27), we obtain

$$\sigma_U(t) = \sum_{q=0}^{p} c_q a(t)^{p-q} b(t)^q, \quad t \in \mathbb{R}.$$

(It will be convenient to define $c_q = 0$ for the out-of-range indices $q < 0$ and $q > p$.) Taking derivatives and using the last system of differential equations, a simple induction gives the following:

Lemma 1. *Given $\theta, \varphi \in \mathbb{R}$, for any $k \in \mathbb{N}$, we have*

$$\sigma_U^{(k)}(t) = \sum_{q=0}^{p} c_q^{(k)} a(t)^{p-q} b(t)^q, \quad t \in \mathbb{R},$$

where the coefficients $c_q^{(k)} = c_q^{(k)}(\theta, \varphi)$ are given by

$$\begin{aligned} c_q^{(k)} &= e^{-\imath\theta}\cos\varphi \cdot (q+1) c_{q+1}^{(k-1)} - \imath \sin\varphi \cdot (p-2q) c_q^{(k-1)} \\ &\quad - e^{\imath\theta}\cos\varphi \cdot (p-q+1) c_{q-1}^{(k-1)}, \quad q = 0, \ldots, p. \end{aligned} \tag{28}$$

Here $c_q^{(0)} = c_q$, $q \in \mathbb{Z}$, and $c_q^{(k)} = 0$ for the out-of-range indices $q < 0$ and $q > p$.

We now assume that $f_\xi : S^3 \to S_{W_p}$ is a spherical minimal immersion, that is, the coefficients of ξ in the expansion (5) satisfy (6)–(9). Our task is to give a necessary and sufficient condition for f_ξ to be isotropic (of order two).

We now let

$$U_1 := U(\theta, \varphi), \qquad U_2 := U(\theta + \pi/2, 0),$$
$$U_3 := U(\theta, \varphi + \pi/2), \quad \theta, \varphi \in \mathbb{R}.$$

We observe that, for given $\theta, \varphi \in \mathbb{R}$, $\{U_1, U_2, U_3\} \subset T_1(S^3)$ is an orthonormal basis. Because of the arbitrary position of U_1 (given by the arbitrary choices of θ and φ), and linearity in the first derivative in (3), Theorem A gives the following:

LEMMA 2. *Let $f_\xi : S^3 \to S_{W_p}$ be an $SU(2)$-equivariant spherical immersion. Then f_ξ is isotropic if and only if, for any $\theta, \varphi \in \mathbb{R}$, we have*

$$\langle \sigma_{U_1}^{(3)}(0), \sigma'_{U_2}(0)\rangle = \langle \sigma_{U_1}^{(3)}(0), \sigma'_{U_3}(0)\rangle = 0. \tag{29}$$

In this case, for the constant of isotropy Λ_2, we have $\langle \sigma_{U_1}^{(3)}(0), \sigma'_{U_1}(0)\rangle = -\Lambda_1^2 - \Lambda_2^2$.

For the proof of Theorem B, we need a convenient scalar product on $W_p \subset \mathbb{C}[z, w]$ or, more generally, on the space of *complex* spherical harmonics $\mathcal{H}_3^p$. As usual, we identify $\mathcal{H}_3^p$ with the space of complex-valued degree p harmonic homogeneous polynomials on $\mathbb{C}^2 = \mathbb{R}^4$. To define this scalar product, we will regard a complex polynomial χ in the complex variables $z, w \in \mathbb{C}$ as a *real* polynomial in the variables $z, w, \bar{z}, \bar{w}$. Then, for $\chi_1, \chi_2 \in \mathcal{H}_3^p$, we define the scalar product on $\mathcal{H}_3^p$ by

$$\langle \chi_1, \chi_2\rangle = \Re\left(\chi_1\left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial w}\right)\bar{\chi}_2\right),$$

where $\Re$ stands for *real part,* and χ_1 acts on the conjugate $\bar{\chi}_2$ as a *polynomial differential operator.* (This form of the scalar product on $\mathcal{H}_3^p$ has been used in [5; 6; 24].) Note that, with respect to this scalar product, $\{z^{p-q}w^q/\sqrt{(p-q)!q!}\}_{q=0}^p$ is an orthonormal basis of W_p as stated in Section 1.

Proof of Theorem B. We need to work out the two scalar products in (29) in terms of the coefficients c_q, $q = 0, \ldots, p$, in (5). In both cases the explicit calculations are very similar. The vanishing of the first scalar product will imply (10)–(13), whereas the vanishing of the second will give (10)–(14). Hence we will treat only the second scalar product in (29).

Using Lemma 1, for fixed $\theta, \varphi \in \mathbb{R}$, we have

$$\begin{aligned}\langle \sigma_{U_1}^{(3)}(0), \sigma'_{U_3}(0)\rangle &= \Re\left(\sum_{q=0}^{p}(p-q)!q! \cdot c_q^{(3)}(\theta, \varphi)\overline{c_q^{(1)}(\theta, \varphi + \pi/2)}\right)\\ &= \sum_{k=-4}^{4} e^{k\imath\theta} B_k,\end{aligned} \tag{30}$$

where the last exponential sum is obtained by repeated application of the recurrence in (28). In this last sum, each B_k, $k = -4, \ldots, 4$, is independent of the variable θ. In particular, the scalar product on the left-hand side of (30) vanishes for all $\theta, \varphi \in \mathbb{R}$ if and only if the (Fourier) coefficients B_k, $k = -4, \ldots, 4$, vanish for all $\varphi \in \mathbb{R}$.

Expanding the factors $c_q^{(3)}(\theta, \varphi)\overline{c_q^{(1)}(\theta, \varphi + \pi/2)}$, $q = 0, \ldots, p$, in (30) in terms of the coefficients c_q, $q = 0, \ldots, p$, requires long but straightforward computations. It turns out that the expressions

$$e^{k\imath\theta} B_k + e^{-k\imath\theta} B_{-k}, \quad k = 0, \ldots, 4, \tag{31}$$

are the least cumbersome to determine. (For $k = 0$, this reduces to $2B_0$, which we included here.)

We begin with the simplest case, namely $k = 4$. As noted before, a straightforward computation gives

$$e^{4\imath\theta} B_4 + e^{-4\imath\theta} B_{-4} = 2\cos^3\varphi \sin\varphi \sum_{q=0}^{p-4} (p-q)!(q+4)! \cdot \Re(e^{4\imath\theta} c_q \bar{c}_{q+4}).$$

Clearly, this vanishes for all $\theta, \varphi \in \mathbb{R}$ if and only if (10) holds.

The cases $k = 1, 2, 3$ are similar but longer. We will discuss only the case $k = 1$. We have

$$\begin{aligned} e^{\imath\theta} B_1 + e^{-\imath\theta} B_{-1} = {} & \frac{\cos^4\varphi}{2} \sum_{q=0}^{p-1} (p-q)!(q+1)! \\ & \cdot [3(p-2q-1)^3 + 2(4-(p+1)^2)(p-2q-1)] \\ & \cdot \Im(e^{\imath\theta} c_q \bar{c}_{q+1}) \\ & - \frac{3\cos^2\varphi \sin^2\varphi}{2} \sum_{q=0}^{p-1} (p-q)!(q+1)! \\ & \cdot [7(p-2q-1)^3 - (3(p+1)^2-20)(p-2q-1)] \\ & \cdot \Im(e^{\imath\theta} c_q \bar{c}_{q+1}) \\ & + 2\sin^4\varphi \sum_{q=0}^{p-1} (p-q)!(q+1)! \\ & \cdot [(p-2q-1)^3 + (p-2q-1)] \cdot \Im(e^{\imath\theta} c_q \bar{c}_{q+1}), \end{aligned}$$

where $\Im$ stands for *imaginary part.* By (8) the second term (with common factor $(p-2q-1)$) in each square bracket cancels. With this, the simplified expression vanishes for all $\theta, \varphi \in \mathbb{R}$ if and only if (13) holds. (Note that we recover (13) three times corresponding to each sum.)

The cases $k = 3$ and $k = 2$ are similar, and they yield (11) and (12), respectively.

Finally, we treat the case $k = 0$. We have

$$
\begin{aligned}
B_0 = {} & \frac{\cos^3\varphi \sin\varphi}{8} \sum_{q=0}^{p} (p-q)!q! \\
& \cdot [15(p-2q)^4 + 18(2-p(p+2))(p-2q)^2 + 3p^2(p+2)^2 \\
& - 8p(p+2)]|c_q|^2 - \frac{\cos\varphi \sin^3\varphi}{2} \sum_{q=0}^{p} (p-q)!q! \\
& \cdot [5(p-2q)^4 - (3p(p+2)-16)(p-2q)^2 - 4p(p+2)]|c_q|^2. \quad (32)
\end{aligned}
$$

(We keep the factor $p(p+2)$ intact as it is the pth eigenvalue of the Laplacian on S^3.) Now, $B_0 = 0$ for all $\theta, \varphi \in \mathbb{R}$ if and only if each of the two sums vanish separately. We split the first as

$$
\begin{aligned}
& 15 \sum_{q=0}^{p} (p-q)!q!(p-2q)^4 |c_q|^2 \\
& \quad + 18(2-p(p+2)) \sum_{q=0}^{p} (p-q)!q!(p-2q)^2 |c_q|^2 \\
& \quad + (3p^2(p+2)^2 - 8p(p+2)) \sum_{q=0}^{p} (p-q)!q! |c_q|^2 = 0.
\end{aligned}
$$

By (9) and (6), the second and third sums are equal to $p(p+2)/3$ and 1, respectively. Rearranging, we obtain (14). The second sum in (32) gives the same result.

Finally, to determine the constant of isotropy Λ_2, by (4) in the last statement of Theorem A, we need to calculate

$$
\langle \sigma_{U_1}^{(3)}(0), \sigma'_{U_1}(0) \rangle = \Re\left(\sum_{q=0}^{p} (p-q)!q! \cdot c_q^{(3)}(\theta,\varphi) \overline{c_q^{(1)}(\theta,\varphi)} \right).
$$

Once again expanding, akin to the previous computations, we obtain

$$
\langle \sigma_{U_1}^{(3)}(0), \sigma'_{U_1}(0) \rangle = -\frac{p(p+2)(3p(p+2)-4)}{15}.
$$

Combining this with (4), the last statement of Theorem B follows. □

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NOTE ADDED IN PROOF. Most recently the second author resolved Conjectures 1 and 2 in Section 2.4. Details will appear elsewhere.

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