Michigan Math. J. 66 (2017), 155–174

Jet Schemes and Generating Sequences of Divisorial Valuations in Dimension Two

Hussein Mourtada

Abstract. Using the theory of jet schemes, we give a new approach to the description of a minimal generating sequence of a divisorial valuations on $\mathbf{A}^2$. For this purpose, we show how to recover the approximate roots of an analytically irreducible plane curve from the equations of its jet schemes. As an application, for a given divisorial valuation v centered at the origin of $\mathbf{A}^2$, we construct an algebraic embedding $\mathbf{A}^2 \hookrightarrow \mathbf{A}^N$, $N \geq 2$, such that v is the trace of a monomial valuation on $\mathbf{A}^N$. We explain how results in this direction give a constructive approach to a conjecture of Teissier on resolution of singularities by one toric morphism.

1. Introduction

Let $X = \mathbf{A}^d = \operatorname{Spec} R$, where $R = \mathbf{K}[x_1, \dots, x_d]$ is a polynomial ring over an algebraically closed field $\mathbf{K}$. The arc space of X, which we denote by X_∞, is the scheme whose $\mathbf{K}$-rational points are

$$X_\infty(\mathbf{K}) = \operatorname{Hom}_{\mathbf{K}}(\operatorname{Spec} \mathbf{K}[[t]], X).$$

We have a natural truncation morphism $X_\infty \longrightarrow X$, which we denote by Ψ_0. For $p \in \mathbf{N}$ and the subvariety $Y = V(I) \subset X$ defined by an ideal I, we consider the subset of arcs in X_∞ that have an order of contact p with Y, that is,

$$\operatorname{Cont}^p(Y) = \{\gamma \in X_\infty \mid \operatorname{ord}_t \gamma^*(I) = p\},$$

where $\gamma^* : R \longrightarrow \mathbf{K}[[t]]$ is the $\mathbf{K}$-algebra homomorphism associated with γ, and

$$\operatorname{ord}_t \gamma^*(I) = \min_{h \in I}\{\operatorname{ord}_t \gamma^*(h)\}.$$

With an irreducible component $\mathbb{W}$ of $\operatorname{Cont}^p(Y)$, which is contained in the fiber $\Psi_0^{-1}(0)$ above the origin, we associate a valuation $v_{\mathbb{W}} : R \longrightarrow \mathbf{N}$ as follows:

$$v_{\mathbb{W}}(h) = \min_{\gamma \in \mathbb{W}}\{\operatorname{ord}_t \gamma^*(h)\} \quad \text{for } h \in R.$$

It follows from [ELM] (see also [dFEI; Re], Prop. 3.7(vii)) that $v_{\mathbb{W}}$ is a divisorial valuation centered at the origin $0 \in X$ and that all divisorial valuations centered at $0 \in X$ can be obtained in this way.

Received September 15, 2015. Revision received May 19, 2016.
This research was partially supported by the ANR-12-JS01-0002-01 SUSI.

We are interested in determining a generating sequence of such a valuation that is a sequence of elements of R determining the valuation completely. It is defined as follows. For $\alpha \in \mathbf{N}$, let

$$\mathcal{P}_\alpha = \{h \in R \mid v_{\mathbb{W}}(h) \geq \alpha\}.$$

Following [T3], we define the $\mathbf{K}$-graded algebra

$$\mathrm{gr}_{v_{\mathbb{W}}} R = \bigoplus_{\alpha \in \mathbf{N}} \frac{\mathcal{P}_\alpha}{\mathcal{P}_{\alpha+1}}.$$

We denote by $\mathrm{in}_{v_{\mathbb{W}}}$ the natural map

$$\mathrm{in}_{v_{\mathbb{W}}} : R \longrightarrow \mathrm{gr}_{v_{\mathbb{W}}} R, \qquad h \mapsto h \mod \mathcal{P}_{v_{\mathbb{W}}(h)+1}.$$

DEFINITION 1.1 ([S]). A generating sequence of $v_{\mathbb{W}}$ is a set of elements of R such that their image by $\mathrm{in}_{v_{\mathbb{W}}}$ generates $\mathrm{gr}_{v_{\mathbb{W}}} R$ as a $\mathbf{K}$-algebra.

In this article, we give a new way to determine a generating sequence of $v_{\mathbb{W}}$ in dimension 2, that is, when $d = 2$. Traditionally, there are three approaches to determine such a generating sequence:

(1) By studying the relations in the semigroup $v_{\mathbb{W}}(R)$ [T3]. The new developments of this theory in higher dimensions treat only valuations with maximal rational rank [T1; T2], which do not include divisorial valuations.
(2) By considering curvettes [S]. Let π be the composition of the minimal sequence of blow ups that produces the divisor defining $v_{\mathbb{W}}$. Let G be its dual graph. Then a curvette is a curve that is an image of a transversal arc to a rupture divisor of G. If we choose the equation of a curvette for every rupture divisor, plus the variables of R, then we obtain a generating sequence of $v_{\mathbb{W}}$. This approach has not been generalized to higher dimensions, and this seems to be a difficult mission.
(3) Maclane's method [Mc] (see also [AM; FJ]). A generating sequence is obtained by induction using Euclidean division. The generalizations of this method to higher dimensions [V1; HOS; Ma] do not produce elements in R, which is essential for our applications. See also [CV] for a comparable approach.

Our approach is based on the definition of a divisorial valuation that we gave before in terms of arcs (and jet schemes). It will enable us to build a generating sequence from the equations of the subset $\mathbb{W}$ of the arc space that defines the divisorial valuation. The construction of a generating sequence passes through the extraction of the approximate roots of a plane branch from its jet schemes.

One motivating application that we present and that remains true for a particular type of divisorial valuations in higher dimensions [Mo4] is the following. Given a divisorial valuation v centered at $0 \in \mathbf{A}^2$, we will determine an embedding $e : \mathbf{A}^2 \hookrightarrow \mathbf{A}^n$ (where n depends on v) and a toric proper birational morphism

$\mu : X_\Sigma \longrightarrow \mathbf{A}^n$ such that:

$$\begin{tikzcd} \tilde{\mathbf{A}}^2 \arrow[r] \arrow[d] & X_\Sigma \arrow[d, "\mu"] \\ \mathbf{A}^2 \arrow[r, hook, "e"] & \mathbf{A}^n \end{tikzcd}$$

- X_Σ is a smooth toric variety (i.e., Σ is a fan obtained by a regular subdivision of the positive quadrant $\mathbb{R}^n_+$, which is the cone that defines $\mathbf{A}^n$ as a toric variety),
- the strict transform $\tilde{\mathbf{A}}^2$ of $\mathbf{A}^2$ by μ is smooth,
- a toric divisor E' (associated with one of the edges of Σ and determined by the values of the elements in a generating sequence) intersects $\tilde{\mathbf{A}}^2$ transversally along a divisor E; note that the valuation associated with E' is monomial and is given by the weight vector corresponding to E',
- the valuation defined by the divisor E is v.

Our goal is to use such a construction to answer constructively the following conjecture of Teissier [T2]:

For a subvariety $Y \subset \mathbf{A}^n$, there exists an embedding $\mathbf{A}^n \hookrightarrow \mathbf{A}^N$, $N \geq n$, such that the singularities of Y can be resolved by a birational proper toric map $Z \longrightarrow \mathbf{A}^N$.

A solution of this problem in the case of quasi-ordinary singularities is given in [GP]. A related result was proved in [Te2], but the author starts with a given resolution of singularities.

For a given singular subvariety $Y \subset \mathbf{A}^n$, our idea is to extract a finite number of significant divisorial valuations $v_1, \ldots, v_r$ on $\mathbf{A}^n$ from the jet schemes of Y (this is to compare with the Nash map [I; ELM]), then to embed as before $\mathbf{A}^n$ in a larger affine space $\mathbf{A}^N$ in such a way that all the valuations $v_1, \ldots, v_r$ can be seen as the traces of monomial valuations on $\mathbf{A}^N$. If $v_1, \ldots, v_r$ are well chosen, then this should guarantee Newton nondegeneracy [AGS; Te1] of $Y \subset \mathbf{A}^N$ and hence would give the desired embedding. There remains the subtle matter of detecting the valuations $v_1, \ldots, v_r$ (see [Mo3; LMR] for simple examples) and finding the embedding described before for general divisorial valuations. In [Mo4], we present a progress in this last problem.

This idea corresponds to an approach of resolution of singularities by one toric morphism, which is different from that suggested in [GT], where this resolution of an irreducible plane curve $\mathcal{C}$ is constructed by considering the curve valuation $\nu_{\mathcal{C}}$, whereas the approach suggested by this article is to study the divisorial valuations associated with special components of the jet schemes. The two approaches lead to the same result for plane branches but bifurcate in higher dimensions.

One application of the result of this article would be a resolution of singularities of a reducible plane curve with one toric morphism. This will be treated elsewhere.

I have found inspiration for this article in [T2], and I am thankful to Bernard Teissier for all explanations he gave me about it and for several corrections and suggestions for an earlier version of this article. I would also like to thank Pedro González Pérez, Monique Lejeune-Jalabert, Mohammad Moghaddam, Patrick Popescu-Pampu, and Matteo Ruggiero for several discussions on this subject. Finally, I would like to thank the referee for his numerous corrections and remarks.

The article assumes some knowledge of valuations and toric geometry. This can be found respectively in [V2] and [AGS].

2. Jet Schemes

Let $\mathbf{K}$ be an algebraically closed field of arbitrary characteristic. Let X be a $\mathbf{K}$-algebraic variety, and let $m \in \mathbb{N}$. The functor $F_m : \mathbf{K} - \text{Schemes} \longrightarrow \text{Sets}$ associating with an affine scheme defined by a $\mathbf{K}$-algebra A

$$F_m(\mathrm{Spec}(A)) = \mathrm{Hom}_{\mathbf{K}}(\mathrm{Spec}\, A[t]/(t^{m+1}), X)$$

is representable by a $\mathbf{K}$-scheme X_m [EM; I]. X_m is the mth jet scheme of X, and F_m is isomorphic to its functor of points. So we have the bijection

$$\mathrm{Hom}_{\mathbf{K}}(\mathrm{Spec}\, A, X_m) \simeq \mathrm{Hom}_{\mathbf{K}}(\mathrm{Spec}\, A[t]/(t^{m+1}), X). \tag{1}$$

If $X = \mathrm{Spec}\, R$ is affine, then $X_m = \mathrm{Spec}\, R_m$ is also affine, and by taking $A = R_m$ in bijection (1) we obtain a universal morphism $\Lambda^* : R \longrightarrow R_m[t]/(t^{m+1})$, which is the morphism associated with the image of the identity $\mathrm{id} \in \mathrm{Hom}_{\mathbf{K}}(X_m, X_m)$ by bijection (1). For example, if $X = \mathrm{Spec}\, \mathbf{K}[x_0, x_1]$ and $f \in \mathbf{K}[x_0, x_1]$, then

$$X_m = \mathrm{Spec}\, \mathbf{K}[x_0^{(0)}, x_1^{(0)}, \ldots, x_0^{(m)}, x_1^{(m)}] = \mathrm{Spec}\, R_m$$

and

$$\Lambda^*(f) = F^{(0)} + F^{(1)}t + \cdots + F^{(m)}t^m, \tag{2}$$

where $F^{(i)}$ is the coefficient of t^i in the expansion of

$$f(x_0^{(0)} + x_0^{(1)}t + \cdots + x_0^{(m)}t^m, x_1^{(0)} + x_1^{(1)}t + \cdots + x_1^{(m)}t^m). \tag{3}$$

Note that since we are interested in the ideal generated by the $F^{(i)}$, in characteristic 0, we can reconstruct them in such a way that they are obtained by a derivation process; see Proposition 2.3 in [Mo1].

For $m, p \in \mathbb{N}$, $m > p$, the truncation homomorphism $A[t]/(t^{m+1}) \longrightarrow A[t]/(t^{p+1})$ induces a canonical projection $\pi_{m,p} : X_m \longrightarrow X_p$. These morphisms clearly satisfy $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$ for $p < m < q$, and they are affine morphisms, so that they define a projective system whose limit is a scheme that we denote X_∞; it is the arc space of X.

Note that $X_0 = X$. We denote by π_m the canonical projection $\pi_{m,0} : X_m \longrightarrow X_0$ and by Ψ_m the canonical morphisms $X_\infty \longrightarrow X_m$.

3. Minimal Generating Sequences of a Curve Valuation from the Equations of Jet Schemes

In [Mo1] and [LMR], we have used the approximate roots to study the geometry of the jet schemes of plane branches and to obtain toric resolutions of singularities of these curves. In this section, we show how to obtain a minimal generating sequence of the valuation defined by a plane branch, thats is, a curve valuation, from the jet schemes of the branch. Note that the graph that we have introduced in [Mo1] is not sufficient to determine this generating sequence. The invariants of the jet schemes that we consider further are finer and are not determined by the topological type of the curve singularity.

Let $\mathcal{C}$ be a plane branch defined by an irreducible power series $f \in \mathbf{K}[[x_0, x_1]]$, where $\mathbf{K}$ is an algebraically closed field. We assume that $x_0 = 0$ (resp. $x_1 = 0$) is transversal (resp. tangent) to $\mathcal{C}$, which can always be achieved by a linear change of variables. Let $\bar{\beta}_0, \ldots, \bar{\beta}_g$ be the minimal system of generators of the semigroup $\Gamma(\mathcal{C})$ of $\mathcal{C}$. Let $e_0 = \bar{\beta}_0$ (this is also the multiplicity of $\mathcal{C}$ at the origin) and $e_i = \gcd(e_{i-1}, \bar{\beta}_i)$, $i \geq 1$ (where gcd is the greatest common divisor). Since the sequence of positive integers

$$e_0 > e_1 > \cdots > e_i > \cdots$$

is strictly decreasing, there exists $g \in \mathbb{N}$ such that $e_g = 1$. We set

$$n_i := \frac{e_{i-1}}{e_i}, \qquad m_i := \frac{\beta_i}{e_i}, \quad i = 1, \ldots, g,$$

and by convention we set $\beta_{g+1} = +\infty$ and $n_{g+1} = 1$. We have:

1. $e_i = \gcd(\bar{\beta}_0, \ldots, \bar{\beta}_i)$, $0 \leq i \leq g$.
2. For $1 \leq i \leq g$, there exists a unique system of nonnegative integers b_{ij}, $0 \leq j < i$, such that $b_{ij} < n_j$ for $1 \leq j < i$ and $n_i \bar{\beta}_i = \sum_{0 \leq j < i} b_{ij} \bar{\beta}_j$.

With such a plane branch $\mathcal{C} = \{f = 0\}$, we associate a (curve) valuation

$$\nu_{\mathcal{C}} : \mathbf{K}[[x_0, x_1]] \longrightarrow \mathbb{N} \cup \infty,$$

which is positive on the maximal ideal (x_0, x_1), by using local intersection multiplicity:

$$\nu_{\mathcal{C}}(h) = \dim \frac{\mathbf{K}[[x_0, x_1]]}{(f, h)}$$

for every $h \in \mathbf{K}[[x_0, x_1]]$. Note that $\operatorname{tr.deg}(\nu_{\mathcal{C}}) = 0$ and $\operatorname{rank}(\nu_{\mathcal{C}}) = 2$ (see [FJ], p. 17).

For an irreducible $h \in \mathbf{K}[[x_0, x_1]]$, we have that h is, up to multiplication by a constant, of the form

$$h = (x_1^{n_h} - \alpha_h x_0^{m_h})^{\delta_h} + \sum_{(a,b)} c_{ab} x_0^a x_1^b, \tag{4}$$

where m_h and n_h are coprime, $\alpha_h \in \mathbf{K}^*$, $c_{ab} \in \mathbf{K}$, and the points (a, b) are strictly above the Newton polygon of h [CA].

LEMMA 3.1. *Given f, h in the form* (4), *we have $x_1^{n_f} - \alpha_f x_0^{m_f} \neq x_1^{n_h} - \alpha_h x_0^{m_h}$ if and only if*

$$\nu_{\mathcal{C}}(h) = \min(\bar{\beta}_0 m_h \delta_h, \bar{\beta}_1 n_h \delta_h).$$

Moreover, we have that

$$\begin{cases} \operatorname{in}_{\nu_{\mathcal{C}}} h = x_0^{m_h \delta_h} \text{ or } x_1^{n_h \delta_h} & \text{if } (m_f, n_f) \neq (m_h, n_h), \\ \operatorname{in}_{\nu_{\mathcal{C}}} h = (x_1^{n_h} - \alpha_h x_0^{m_h})^{\delta_h} & \text{if } (m_f, n_f) = (m_h, n_h) \text{ and } \alpha_f \neq \alpha_h. \end{cases}$$

Proof. This follows from the the classical formula of the local intersection multiplicity:

$$\nu_{\mathcal{C}}(h) = \mathrm{ord}_t h(x(t), y(t)),$$

where $(x(t), y(t))$ is a special parameterization of $\mathcal{C}$ obtained by the Newton–Puiseux theorem [CA]. □

Following [Mo1], we describe the irreducible components of the schemes of jets centered at 0, that is, $\mathcal{C}_m^0 := \pi_m^{-1}(0)$, where $\pi_m : \mathcal{C}_m \longrightarrow \mathcal{C}$ is the canonical morphism. We set

$$\mathrm{Cont}^e(x_0)_m (\text{resp. } \mathrm{Cont}^{>e}(x_0)_m) := \{\gamma \in \mathcal{C}_m \mid \mathrm{ord}_t x_0 \circ \gamma = e (\text{resp.} > e)\}.$$

Then we can state the following:

THEOREM 3.2 (Thm. 4.9, [Mo1]). *Let $\mathcal{C}$ be a plane branch with g Puiseux exponents. Let $m \in \mathbb{N}$. For $1 \leq m < n_1\bar{\beta}_1 + e_1$, $\mathcal{C}_m^0 = \mathrm{Cont}^{>0}(x_0)_m$ is irreducible. For $q = [\frac{m-e_1}{n_1\bar{\beta}_1}] \geq 1$, the irreducible components of $\mathcal{C}_m^0$ are*

$$C_{m\kappa I} = \overline{\mathrm{Cont}^{\kappa\bar{\beta}_0}(x_0)_m}$$

for $1 \leq \kappa$ and $\kappa\bar{\beta}_0\bar{\beta}_1 + e_1 \leq m$,

$$C^j_{m\kappa v} = \overline{\mathrm{Cont}^{\frac{\kappa\bar{\beta}_0}{n_j\cdots n_g}}(x_0)_m}$$

for $j = 2, \ldots, g$ and $1 \leq \kappa$, and $\kappa \not\equiv 0 \mod n_j$ such that $\kappa n_1 \cdots n_{j-1}\bar{\beta}_1 + e_1 \leq m < \kappa\bar{\beta}_j$, and

$$B_m = \mathrm{Cont}^{>n_1 q}(x_0)_m.$$

We are interested in the following inverse system of irreducible components:

$$\begin{aligned} \cdots \longrightarrow C_{(\bar{\beta}_0\bar{\beta}_1+e_1+2)1I} \longrightarrow C_{(\bar{\beta}_0\bar{\beta}_1+e_1+1)1I} \longrightarrow C_{(\bar{\beta}_0\bar{\beta}_1+e_1)1I} \\ \longrightarrow B_{\bar{\beta}_0\bar{\beta}_1+e_1-1} \longrightarrow B_{\bar{\beta}_0\bar{\beta}_1}. \end{aligned} \qquad (\star)$$

Let $C_m := \overline{\mathrm{Cont}^{\bar{\beta}_0}(x_0)_m}$ (the notation C_m will be used all over the paper). Let γ_m be the generic point of C_m. From Corollary 4.2 in [Mo1] we can see that, for m large enough,

$$\mathrm{ord}_t x_1 \circ \gamma_m(t) = \bar{\beta}_1.$$

Note that the only data we need to detect the inverse system ($\star$) is the multiplicity $\bar{\beta}_0$ of the curve. Indeed, the components in system ($\star$) are given by the closure of $\mathrm{Cont}^{\bar{\beta}_0}(x_0)_m$, for $m \geq \bar{\beta}_0\bar{\beta}_1 - 1$.

In the following lemma, we compute the intersection multiplicity of two curves in terms of ideals of jet schemes. Our first goal is to give a new way to determine the initial part of an element $h \in \mathbf{K}[[x, y]]$ with respect to the valuation $\nu_{\mathcal{C}}$. This is achieved in Corollary 3.6.

Let $D^m(x_0^{(\bar{\beta}_0)})$ be the open subscheme of $\mathbf{A}_m^2$ defined by $x_0^{(\bar{\beta}_0)} \neq 0$. Let I_m be the ideal defining $\mathrm{Cont}^{\bar{\beta}_0}(x_0)_m$ in $D^m(x_0^{(\bar{\beta}_0)})$, and let I_m^r be its radical. Let $h \in \mathbf{K}[[x, y]]$ be irreducible, and $H^{(i)}$ be the coefficient of t^i in $\Lambda^*(h)$ (see equation (2)).

REMARK 3.3. In what follows, unless stated otherwise, when we use the symbol $\equiv$, we just want to replace elements that are congruent to zero by zero.

LEMMA 3.4. *$\nu_{\mathcal{C}}(h) = l$ if and only if for $m \gg 0$, we have $H^{(i)} \equiv 0 \mod I_m^r$, if $i < l$ and $H^{(l)} \not\equiv 0 \mod I_m^r$.*

Proof. If $\nu_{\mathcal{C}}(h) = l$, then we have $\nu_{\mathcal{C}}(h) = \mathrm{ord}_t h(x_0(t), x_1(t))$ for any good parameterization (i.e., a general point of the curve corresponds to just one value of the parameter) $(x_0(t), x_1(t))$ of $\mathcal{C}$. Let $x_0^{(0)}, \ldots, x_0^{(i_m)}$, $x_1^{(0)}, \ldots, x_1^{(j_m)}$ be the variables that intervene in the generators of I_m^r. Note that $i_m, j_m < m$. By the definition of I_m^r, for any closed point $(a_0^{(0)}, \ldots, a_0^{(i_m)}, a_1^{(0)}, \ldots, a_1^{(j_m)}) \in V(I_m^r) \subset \mathrm{Spec}\, \mathbf{K}[x_0^{(0)}, \ldots, x_0^{(i_m)}, x_1^{(0)}, \ldots, x_1^{(j_m)}]$, there is a good parameterization of $\mathcal{C}$ of the form

$$(a_0^{(0)} + a_0^{(1)} t + \cdots + a_0^{(i_m)} t^{i_m} + \cdots, a_1^{(0)} + a_1^{(1)} t + \cdots + a_1^{(j_m)} t^{j_m} + \cdots).$$

It follows that

$$\mathrm{ord}_t h(a_0^{(0)} + a_0^{(1)} t + \cdots + a_0^{(i_m)} t^{i_m} + \cdots, a_1^{(0)} + a_1^{(1)} t + \cdots + a_1^{(j_m)} t^{j_m} + \cdots) = l,$$

and so $H^{(i)}(a_0^{(0)}, \ldots, a_0^{(i_m)}, a_1^{(0)}, \ldots, a_1^{(j_m)}) = 0$ for every $i < l$, and

$$H^{(l)}(a_0^{(0)}, \ldots, a_0^{(i_m)}, a_1^{(0)}, \ldots, a_1^{(j_m)}) \neq 0.$$

Hence, $H^{(i)} \equiv 0 \mod I_m^r$ for every $i < l$, and $H^{(l)} \not\equiv 0 \mod I_m^r$.

The converse is straightforward. □

REMARK 3.5. 1. In the proof of Lemma 3.4, the fact that, for a closed point of $V(I_m^r) \subset \mathrm{Spec}\, \mathbf{K}[x_0^{(0)}, \ldots, x_0^{(i_m)}, x_1^{(0)}, \ldots, x_1^{(j_m)}]$, we find an arc that "lifts" this point is not equivalent to saying that any m-jet in the irreducible component defined by I_m^r is liftable (which is not true). The reason is that we need more coordinates to define an m-jet, namely there remains to specify $x_0^{(i_m+1)}, \ldots, x_0^{(m)}$, $x_1^{(j_m+1)}, \ldots, x_1^{(m)}$, which can be chosen freely, but for such a jet to be liftable, these coordinates should satisfy more equations.

2. We can estimate the minimum m satisfying the conclusion of Lemma 3.4 by determining the variables that appear in the equations of jet schemes. We find

$$m = \kappa_h := \left[l \frac{\mathrm{mult}(f)}{\mathrm{mult}(h)} \right],$$

where mult denotes the multiplicity, $l = \nu_{\mathcal{C}}(h)$, and the brackets [ ] denote the integral part.

We continue with the settings of Lemma 3.4. Since $H^{(l)} \not\equiv 0 \; mod I^r_{\kappa_h}$, let $P \in \mathbf{K}[[x_0, x_1]]$ be the minimal part of h such that

$$(h-P)^{(i)} \equiv 0 \mod I^r_{\kappa_h} \quad \text{for } i \leq l.$$

This means that the terms (a term is a constant times a monomial in x_0 and x_1) of P are terms of h, and P has the least number of terms with the previous property. We thus obtain the following important corollary of Lemma 3.4.

COROLLARY 3.6. *We have*

$$\text{in}_{\nu_{\mathcal{C}}} P = \text{in}_{\nu_{\mathcal{C}}} h.$$

Moreover, P is the minimal part of h achieving this equality.

Proof. It follows from the definition of P and from Lemma 3.4 that $\nu_{\mathcal{C}}(h-P) > \nu_{\mathcal{C}}(h)$, and the assertion follows. □

REMARK 3.7.

EXAMPLE 1. We assume that the characteristic of $\mathbf{K}$ is zero, which makes the computation easier.

1. Let $\mathcal{C} = \{f = (x_1^2 - x_0^3)^2 - x_0^6 x_1 = 0\}$, and let $h = (x_1^2 - x_0^3)^2 - 4x_0^5 x_1 - x_0^7$. We have that $\nu_{\mathcal{C}}(h) = 26$. We can see this by applying Lemma 3.4; indeed:

$$I^r_{26} = (x_0^{(0)}, \dots, x_0^{(3)}, x_1^{(0)}, \dots, x_1^{(5)}, {x_1^{(6)}}^2 - {x_0^{(4)}}^3, 2x_1^{(6)} x_1^{(7)} - 3{x_0^{(4)}}^2 x_0^{(5)}) \subset (R_m)_{x_0^{(4)}},$$

   where $(R_m)_{x_0^{(4)}}$ is the ring R_m localized by $x_0^{(4)}$. Note that since in this example f and h have the same multiplicity, we have $\kappa_h = \nu_{\mathcal{C}}(h)$. We observe that, for every $i < 26$, $H_i \equiv 0$ modulo I^r_{26}, and

$$H^{(26)} \equiv -4{x_0^{(4)}}^5 x_1^{(6)} \not\equiv 0 \mod I^r_{26}.$$

   From Corollary 3.6 we deduce that $\text{in}_{\nu_{\mathcal{C}}} h = -4x_0^5 x_1$.
2. Let $\mathcal{C} = \{f = (x_1^2 - x_0^3)^2 - x_0^6 x_1\}$, and let $h = x_1^2 - x_0^3$. We have $\nu_{\mathcal{C}}(h) = 15$, and therefore $\kappa_h = 30$. We have that

$$I^r_{30} = (x_0^{(0)}, \dots, x_0^{(3)}, x_1^{(0)}, \dots, x_1^{(5)}, H^{(12)}, H^{(13)}, H^{(14)}, {H^{(15)}}^2 - {x_0^{(4)}}^6 x_1^{(6)}) \subset (R_m)_{x_0^{(4)}},$$

   where $H^{(12)} = {x_1^{(6)}}^2 - {x_0^{(4)}}^3$ and $H^{(13)} = 2x_1^{(6)} x_1^{(7)} - 3{x_0^{(4)}}^2 x_0^{(5)}$. We observe that, for every $i < 15$, $H^{(i)} \equiv 0$ modulo I^r_{30}, and

$$H^{(15)} \not\equiv 0 \mod I^r_{30}.$$

   From Corollary 3.6 we deduce that $\text{in}_{\nu_{\mathcal{C}}} h = h$.

Let us have a look at the equations of jet schemes. It follows from Corollary 4.2 in [Mo1] that

$$I_{\bar{\beta}_0\bar{\beta}_1-1} = (x_0^{(0)}, \ldots, x_0^{(\bar{\beta}_0-1)}, x_1^{(0)}, \ldots, x_1^{(\bar{\beta}_1-1)}). \tag{5}$$

We get from the same corollary that

$$F^{(\bar{\beta}_0\bar{\beta}_1)} \equiv (x_1^{(\bar{\beta}_1)^{n_1}} - c x_0^{(\bar{\beta}_0)^{m_1}})^{e_1} \mod I_{\bar{\beta}_0\bar{\beta}_1-1} \tag{6}$$

for some $c \in \mathbf{K}$, $c \neq 0$.

REMARK 3.8. Note that equations (5) and (6) are conditional on the hypothesis we have made on the variables x_0 and x_1. These variables permit the best approximation of the valuation $\nu_{\mathcal{C}}$ by a monomial valuation, namely the monomial valuation ν_1 determined by $\nu_1(x) = \nu_{\mathcal{C}}(x)$ and $\nu_1(y) = \nu_{\mathcal{C}}(y)$. Note that if we begin with any choice of variables, then we can use jet schemes to detect variables having this property.

We now give the steps of an algorithm that determine the minimal generating sequence. This will be guided by the fact that we can detect the initial part of a function with respect to $\nu_{\mathcal{C}}$ from the equations of the jet schemes of $\mathcal{C}$; this follows from Lemma 3.4 and Corollary 3.6. So we will determine algorithmically elements in $\mathbf{K}[x, y]$ whose images by the universal morphism Λ^* (see equation (2)) generate the equations of the families of jets that define the valuations $\nu_{\mathcal{C}}$. The idea is to observe the defining equations of the irreducible set $\mathrm{Cont}^{\bar{\beta}_0}(x_0)_m$, $m \geq \bar{\beta}_0\bar{\beta}_1$, and how these equations behave when m varies. For $m = \bar{\beta}_0\bar{\beta}_1$, the scheme $S^{2,0} = \{(x_1^{n_1} - x_0^{m_1})^{e_1} =: x_{2,0}^{e_1} = 0\}$ has the property that the equations defining $\mathrm{Cont}^{\bar{\beta}_0}(x_0, S^{2,0})_m = \{\gamma \in S_m^{2,0}, \mathrm{ord}_t \gamma^*(x_0) = \bar{\beta}_0\}$ in $\mathbf{A}_m^2$ are those defining $\mathrm{Cont}^{\bar{\beta}_0}(x_0)_m \subset \mathbf{A}_m^2$. The algorithm will detect the first $m \geq \bar{\beta}_0\bar{\beta}_1$ (it will be called $\mu_{2,0}+1$) for which this property is not anymore true for $S^{2,0}$ and produces the equation $x_{2,1}^{e_1} = 0$ (resp. $x_{3,0}^{l} = 0$, $l < e_1$) of a new scheme $S^{1,1}$ (resp. $S_{2,0}$) satisfying this property for $m = \mu_{2,0}+1$. These schemes approximate our curve $\mathcal{C}$ from the viewpoint of jet schemes. Moreover, the shape of the equations $x_{2,1}$ or $x_{3,0}$ is governed by the structure of the equations of jet schemes that can be obtained by derivation. We then continue this construction of these "approximating schemes" following the same idea. The fact that the multiplicity e_1 of $S^{2,1}$ does not drop (resp. the multiplicity l of $S^{3,0}$ drops) has an effect on the behavior of the function $m \mapsto \mathrm{codim}(C_m)$, whose behavior is known (see Prop. 4.7 of [Mo1]) and which implies that the multiplicity sequence $(e_1, l_2, \ldots)$ of the approximating schemes will drop until it attains 1, and the algorithm stops. We start now the algorithm.

If $e_1 = 1$, then a minimal generating sequence of $\nu_{\mathcal{C}}$ is given by x_0, x_1, and f itself. We assume that $e_1 > 1$.

We set $x_{2,0} = x_1^{n_1} - x_0^{m_1}$, and to every $C_m := \overline{\mathrm{Cont}^{\bar{\beta}_0}(x_0)_m}$ in ($\star$), we assign a vector $v_m^{3,0} = v^{3,0}(C_m) \in \mathbb{N}^3$ as follows:

$$v_m^{3,0} = (\mathrm{ord}_t x_0 \circ \gamma_m(t), \mathrm{ord}_t x_1 \circ \gamma_m(t), \mathrm{ord}_t x_{2,0} \circ \gamma_m(t)),$$

where γ_m is the generic point of C_m. Let

$$\mu_{2,0}=\min\{m\geq\bar{\beta}_0\bar{\beta}_1 \mid \operatorname{codim}(C_{m+1})>\operatorname{codim}(C_m) \text{ and } v_m^{3,0}=v_{m+1}^{3,0}\}.$$

Let

$$F^{(\mu_{2,0}+1)}\equiv H \mod I^r_{\mu_{2,0}}.$$

Since $C_{\mu_{2,0}+1}$ is irreducible, $H={x_0^{(\bar{\beta}_0)}}^{a_1}{x_1^{(\bar{\beta}_1)}}^{a_2}Q^l$, where Q is an irreducible polynomial (recall that $x_0^{(\bar{\beta}_0)}\neq 0$ and hence $x_1^{(\bar{\beta}_1)}\neq 0$ because of equation (6)). But if $a_1\neq 0$ or $a_2\neq 0$, then this contradicts the form of equation (6), and the geometry of the jet schemes of the irreducible curve $\mathcal{C}$ described in Theorem 3.2 (in particular, this would mean that $B_{(\bar{\beta}_0)(\bar{\beta}_1)}$ is irreducible). Hence, we can write

$$F^{(\mu_{2,0}+1)}\equiv Q^l \mod I^r_{\mu_{2,0}},$$

for some irreducible polynomial Q and positive integer l; moreover, Q is a nonzero polynomial because the equation $F^{(\mu_{2,0}+1)}=0$ forces the inequality $\operatorname{codim}(C_{\mu_{2,0}+1})>\operatorname{codim}(C_{\mu_{2,0}})$.

We then have two cases:

Case 1. If $l=e_1$, then we have the following:

Claim 1.1: If $l=e_1$, then we have

$$Q-x_{2,0}^{(\frac{\mu_{2,0}+1}{e_1})}\equiv Q' \mod I^r_{\mu_{2,0}},$$

where $Q'(x_0^{(\bar{\beta}_0)},x_1^{(\bar{\beta}_1)})$ is a polynomial in the variables $x_0^{(\bar{\beta}_0)}$ and $x_1^{(\bar{\beta}_1)}$.

We then define

$$x_{2,1}=x_{2,0}+Q'(x_0,x_1)$$

and

$$v_m^{3,1}=(\operatorname{ord}_t x_0\circ\gamma_m(t),\operatorname{ord}_t x_1\circ\gamma_m(t),\operatorname{ord}_t x_{2,1}\circ\gamma_m(t)).$$

Case 2. If $l=l_2<e_1$ and $l_2=1$, then we stop.

Claim 2.1. If $1<l_2<e_1$, then we have

$$Q-{x_{2,0}^{(\frac{\mu_{2,0}+1}{e_1})}}^{\frac{e_1}{l_2}}\equiv Q' \mod I^r_{\mu_{2,0}},$$

where $Q'(x_0^{(\bar{\beta}_0)},x_1^{(\bar{\beta}_1)})$ is a polynomial in the variables $x_0^{(\bar{\beta}_0)}$ and $x_1^{(\bar{\beta}_1)}$.

We then set $x_2:=x_{2,0}$, $\mu_2:=\mu_{2,0}$ and define

$$x_{3,0}=x_2^{\frac{e_1}{l_2}}+Q'(x_0,x_1)$$

and

$$v_m^{4,0}=(\operatorname{ord}_t x_0\circ\gamma_m(t),\operatorname{ord}_t x_1\circ\gamma_m(t),\operatorname{ord}_t x_2\circ\gamma_m(t),\operatorname{ord}_t x_{3,0}\circ\gamma_m(t)).$$

We assume that we have recursively determined $(x_2,\dots,x_{i-1},x_{i,j})$, $(e_1,l_2,\dots,l_{i-1})$, and $(\mu_2,\dots,\mu_{i-1},\mu_{i,j-1})$ (if $j=0$, then we set $\mu_{i,j-1}=\mu_{i-1}$). We define

$$v_m^{i,j}=(\operatorname{ord}_t x_0\circ\gamma_m(t),\operatorname{ord}_t x_1\circ\gamma_m(t),\dots,\operatorname{ord}_t x_{i,j}\circ\gamma_m(t)),$$

$$\mu_{i,j}=\min\{m\geq\mu_{i,j-1}+1 \mid \operatorname{codim}(C_{m+1})>\operatorname{codim}(C_m), \text{ and } v_m^{i,j}=v_{m+1}^{i,j}\}.$$

Let

$$F^{(\mu_{i,j}+1)} \equiv Q^l \mod I^r_{\mu_{i,j}}$$

for some reduced polynomial Q and positive integer l; note as before that Q is an irreducible; it is a nonzero polynomial because the equation $F^{(\mu_{i,j}+1)} = 0$ forces the inequality $\operatorname{codim}(C_{\mu_{i,j}+1}) > \operatorname{codim}(C_{\mu_{i,j}})$.

We then have two cases.

Case 1. If $l = l_{i-1}$, then we have the following:

Claim 1.2: We have

$$Q - x_{i,j}^{(\frac{\mu_{i,j}+1}{l_{i-1}})} \equiv Q' \mod I^r_{\mu_{i,j}},$$

where Q' is a polynomial in $x_0^{(\bar{\beta}_0)}, x_1^{(\bar{\beta}_1)}, x_2^{(\frac{\mu_2+1}{e_1})}, \ldots, x_{i-1}^{(\frac{\mu_{i-1}+1}{l_{i-2}})}$. We then define

$$x_{i,j+1} = x_{i,j} + Q'(x_0, x_1, \ldots, x_{i-1})$$

and

$$v_m^{i,j+1} = (\mathrm{ord}_t x_0 \circ \gamma_m(t), \ldots, \mathrm{ord}_t x_{i,j+1} \circ \gamma_m(t)).$$

Case 2. If $l = l_i < l_{i-1}$ and $l_i = 1$, then we stop. If $1 < l_i < l_{i-1}$, then we have the following:

Claim 2.2: We have that

$$Q - x_{i,j}^{(\frac{\mu_{i,j}+1}{e_1})^{\frac{l_{i-1}}{l_i}}} \equiv Q' \mod I^r_{\mu_{i,j}},$$

where Q' is a polynomial in $x_0^{(\bar{\beta}_0)}, x_1^{(\bar{\beta}_1)}, x_2^{(\frac{\mu_2+1}{e_1})}, \ldots, x_{i-1}^{(\frac{\mu_{i-1}+1}{l_{i-2}})}$.

We then set $x_i := x_{i,j}$, $\mu_i := \mu_{i,j}$ and define

$$x_{i+1,0} = x_i^{\frac{l_{i-1}}{l_i}} + Q'(x_0, x_1, \ldots, x_{i-1}).$$

REMARK 3.9. If we want the elements of a generating sequence to be polynomials (which is more consistent with the terminology key polynomials), then we might need an infinite number of elements to form a generating sequence [FJ]. These polynomials can be found by continuing the same algorithm, without stopping if we reach $l_g = 1$, but only if we reach f, a case that occurs after finitely many steps if and only if f is a polynomial. Here, we permit, as in [T1], elements in the ring $\mathbf{K}[[x_0, x_1]]$. Hence, even if f is a power series and not a polynomial, we will take f as an element of a "minimal" generating sequence. In that case, we can view f as a limit key polynomial.

Proof of Claim 1. By the definition of $F^{(\mu_{2,0}+1)}$ the term Q' comes from a polynomial P such that the terms of P^{e_1} appear in f. More precisely,

$$Q' \equiv P^{(\frac{\mu_{2,0}+1}{e_1})} \mod I^r_{\mu_{2,0}}.$$

By construction we have

$$x_{2,0}^{(\frac{\mu_{2,0}+1}{e_1})} \equiv P^{(\frac{\mu_{2,0}+1}{e_1})} \mod I^r_{\mu_{2,0}},$$

and both members are not congruent to 0 modulo $I^r_{\mu_{2,0}+1}$ (because the codimension of the irreducible component of the $(\mu_{2,0}+1)$-jet scheme we consider increases). We deduce from Lemma 3.4 that

$$\nu_{\mathcal{C}}(x_{2,0}-P) > \nu_{\mathcal{C}}(x_{2,0}) = \nu_{\mathcal{C}}(P) = \frac{\mu_{2,0}+1}{e_1},$$

which implies that $\mathrm{in}_{\nu_{\mathcal{C}}} x_{2,0} = x_{2,0} = \mathrm{in}_{\nu_{\mathcal{C}}} P$. Indeed, any polynomial whose terms are also terms of $x_{2,0}$, namely $x_1^{n_1}$ and $x_0^{m_1}$, has values less than $\nu_{\mathcal{C}}(x_{2,0})$.

We have that P is of the form

$$P = P_1^{a_1} \cdots P_s^{a_s},$$

where $P_i, i = 1, \ldots, s$, are irreducible. This follows from the fact that the residue field of $\nu_{\mathcal{C}}$ is $\mathbf{K}$ since $\mathrm{tr.deg}(\nu_{\mathcal{C}}) = 0$ and $\mathbf{K}$ is algebraically closed. We want to prove that the $\mathrm{in}_{\nu_{\mathcal{C}}} P_j$ are sums of monomials in x_0 and x_1 for every j. If not, then by Lemma 3.1 we have that

$$P_j = (x_1^{n_1} - \alpha_f x_0^{m_1})^{\delta_{P_j}} + \sum c_{ab} x_0^a x_1^b,$$

where (a, b) is above the Newton polygon of P_j. If $(x_1^{n_1} - \alpha_f x_0^{m_1})^{\delta_{P_j}}$ is a part of $\mathrm{in}_{\nu_{\mathcal{C}}} P_j$, then this implies that $\nu_{\mathcal{C}}(P_j) \geq \nu_{\mathcal{C}}(x_{2,0})$, and the equality follows from $\mathrm{in}_{\nu_{\mathcal{C}}} x_{2,0} = \mathrm{in}_{\nu_{\mathcal{C}}} P$. We deduce that $\delta_{P_j} = 1$. Then $\mathrm{in}_{\nu_{\mathcal{C}}} P = \mathrm{in}_{\nu_{\mathcal{C}}} P_j$ contains $x_{2,0}$, which contradicts the form of equation (4) for f. It follows that $(x_1^{n_1} - c x_0^{m_1})^{\delta_{P_j}}$ is not a part of $\mathrm{in}_{\nu_{\mathcal{C}}} P_j$, and we deduce by Lemma 3.1 that $\mathrm{in}_{\nu_{\mathcal{C}}} P_j$ is a sum of monomials in x_0 and x_1.

Let us prove the remaining part of Claim 1. The proof is by induction on i. Assume that the claim is true till $i-1$. Again, the term Q' (in "Claim 1 continues") comes from a polynomial P such that the terms of P^l appear in f. We have that P is of the form

$$P = P_1^{a_1} \cdots P_s^{a_s},$$

where P_r are irreducible for $r = 1, \ldots, s$. This again follows from the fact that $\mathrm{tr.deg}(\nu_{\mathcal{C}}) = 0$. Note that, as before,

$$\nu_{\mathcal{C}}(P) = \frac{\mu_{i,j}+1}{l},$$

and we have $\nu_{\mathcal{C}}(P_r) \leq \frac{\mu_{i-1}}{l_{i-1}}$. It follows from Corollary 3.6 and from the hypothesis of induction that $\mathrm{in}_{\nu_{\mathcal{C}}} P_r$ is a polynomial in $x_0^{(\bar{\beta}_0)}, x_1^{(\bar{\beta}_1)}, x_2^{(\frac{\mu_2+1}{e_1})}, \ldots, x_{i-1}^{(\frac{\mu_{i-1}+1}{l_{i-2}})}$. The proof of Claim 2 is similar to the proof of Claim 1. □

THEOREM 3.10. *We have that*:

1. *For* $i = 2, \ldots, g$, $\mu_i = e_{i-1}\bar{\beta}_i - 1$ *and* $l_i = e_i$. *Therefore*, $l_g = 1$, *and the algorithm stops at* $\mu_g = e_{g-1}\bar{\beta}_g$.
2. $x_0, x_1, \ldots, x_g, f$ *is a minimal generating sequence of* $\nu_{\mathcal{C}}$.

Proof. The first part follows from the formula for the codimension of C_m in Proposition 4.7 of [Mo1] and the construction of the μ_i. We also recover that $\nu_{\mathcal{C}}(x_i) = \bar{\beta}_i$, $i = 0, \ldots, g$. The second part follows from Corollary 3.6 and the description of the equations defining C_m in terms of the equations of the jet schemes of the curves defined by x_i, $i = 1, \ldots, g$. Note that according to Claim 1, $\mathrm{in}_{\nu_{\mathcal{C}}} x_{i,j}$ is generated by $x_0, \ldots, x_i$. $\square$

EXAMPLE 2. Let $f = ((x_1^2 - x_0^3 - x_0^4)^2 - x_0^8 x_1)^2 - x_0^3 x_1 (x_1^2 - x_0^3 - x_0^4)$, and let $\mathcal{C}$ be the curve defined by f. We have that $e_1 = 4$, $x_{2,0} = x_1^2 - x_0^3$, and $\mu_{2,0} = 127$. Let

$$F^{(\mu_{2,0}+1)} \equiv Q^l \mod I^r_{\mu_{2,0}},$$

then $Q = {x_{2,0}}^{(32)} - {x_0^{(8)}}^4$ and $l = 4 = e_1$, and hence we define

$$x_{2,1} = x_{2,0} - x_0^4 = x_1^2 - x_0^3 - x_0^4.$$

We have that $\mu_{2,1} = 151$. Let

$$F^{(\mu_{2,1}+1)} \equiv Q^l \mod I^r_{\mu_{2,1}}.$$

Then $Q = {x_{2,1}^{(38)}}^2 - {x_0^{(8)}}^8 x_1^{(12)}$ and $l = l_2 = 2 < e_1$. Since $l_2 = 2 < e_1$, we set $\mu_2 := \mu_{2,1}$, $x_{2,1} = x_2$, and we define

$$x_{3,0} = x_2^2 - x_0^8 x_1 = (x_1^2 - x_0^3 - x_0^4)^2 - x_0^8 x_1.$$

We have that $\mu_{3,0} = 153$, and we find that $l_3 = 1 < l_2$; hence, we set $\mu_3 := \mu_{3,0}$, $x_3 = x_{3,0}$, and we stop. A minimal system of generators is then given by x_0, x_1, x_2, x_3, and f.

4. Generating Sequences of Divisorial Valuations

We now apply the results of the previous section to determine from the jet schemes a minimal generating sequence for a divisorial valuation centered at the origin of $\mathbf{A}^2$. The key point is that in dimension 2, a divisorial valuation ν_E determined by a divisor E is defined by an irreducible component of $\mathrm{Cont}^p(\mathcal{C})$, where $p \in \mathbb{N}$, and $\mathcal{C}$ is an analytically irreducible plane curve. More precisely, the valuation is given by an irreducible component of $\mathcal{C}_{p-1}$, which is of type C_{p-1} (see the definition of C_m after Theorem 3.2) for $p \geq n_g \bar{\beta}_g$, where $\bar{\beta}_0, \ldots, \bar{\beta}_g$ give a minimal system of generators of the semigroup $\Gamma(\mathcal{C})$. Note that these numbers $\bar{\beta}_i$ are also extracted from the jet schemes, and this is the first part of Theorem 3.10.

The existence of $\mathcal{C}$ follows, for instance, from Theorem 2.7 in [LMR]: $\mathcal{C}$ is chosen to be a curvette of E. Recall that $\mathcal{C}$ is a curvette of E if there exists $\pi : X \longrightarrow \mathbf{A}^2$, a composition of point blow ups above the origin, where E is an irreducible component of the exceptional divisor of π, and the strict transform of $\mathcal{C}$ by π is smooth and transversal to E at a point that is not an intersection of E with another component of the exceptional divisor, that is, a free point [GB; FJ].

We will obtain a generating sequence of ν_E from the equations of the jet schemes of the curvette $\mathcal{C}$, more precisely, from the irreducible component C_{p-1}. There are two cases.

If $p = n_g\bar{\beta}_g$, then let $x_2, \dots, x_g$ be constructed by the algorithm of the previous section. Then a minimal generating sequence of the valuation ν_E is given by $x_0, \dots, x_g$. This follows from the definition of ν_E in terms of jet schemes. Indeed, C_{p-1} gives rise to an irreducible component $\mathbb{W}$ of $\mathrm{Cont}^p(\mathcal{C})$ (see the discussion after Theorem 3.2 in [Mo2]), and we have

$$v_{\mathbb{W}}(h) = \min_{\gamma \in \mathbb{W}} \{\mathrm{ord}_t \gamma^*(h)\}$$

for $h \in R = \mathbf{K}[x_0, x_1]$.

If $p > n_g\bar{\beta}_g$, then we need to continue the algorithm in the previous section. Assume that we have constructed $x_0, \dots, x_{g-1}$, and hence we have

$$F^{(n_g\bar{\beta}_g)} \equiv Q \mod I^r_{n_g\bar{\beta}_g - 1}$$

for some reduced polynomial Q. Note that we do not have a power of Q because we have reached the step where $l = 1$ in the previous section. This is governed by the behavior of the codimension of C_m, which grows of 1 whenever $m \geq n_g\bar{\beta}_g$ grows of 1 (Prop. 4.7 in [Mo1]). If the power of Q is $l > 1$, then the codimension of of C_m grows only when m is a multiple of l. We have

$$Q - {x_g^{(\bar{\beta}_g)}}^{n_g} \equiv Q' \mod I^r_{n_g\bar{\beta}_g - 1},$$

where Q' is a polynomial in $x_0^{(\bar{\beta}_0)}, x_1^{(\bar{\beta}_1)}, x_2{}^{(\bar{\beta}_2)}, \dots, x_{g-1}^{(\bar{\beta}_{g-1})}$. We then define

$$x_{g+1,0} = x_g^{n_g} + Q',$$

$$v_m^{g+2,0} = (\mathrm{ord}_t x_0 \circ \gamma_m(t), \mathrm{ord}_t x_1 \circ \gamma_m(t), \dots, \mathrm{ord}_t x_{g+1,0} \circ \gamma_m(t)),$$

and

$$\mu_{g+1,0} = \min\{n_g\bar{\beta}_g \leq m < p \mid \text{ and } v_m^{g+2,0} = v_{m+1}^{g+2,0}\}.$$

We have not imposed any conditions on the codimension in the definition of $\mu_{g+1,0}$ because, as we said before, for $m \geq n_g\bar{\beta}_g$, the codimension of C_m grows by 1 when m grows by 1.

If $\mu_{g+1,0} = p - 1$, then a minimal generating sequence of ν_E is given by

$$x_0, \dots, x_{g+1} := x_{g+1,0}.$$

If not, let

$$F^{(\mu_{g+2,0}+1)} \equiv Q \mod I^r_{\mu_{g+2,0}}$$

for some reduced polynomial Q. We have that

$$Q - x_{g+1,0}^{(\mu_{g+1,0}+1)} \equiv Q' \mod I^r_{\mu_{g+1,0}},$$

where Q' is a polynomial in $x_0^{(\bar{\beta}_0)}, x_1^{(\bar{\beta}_1)}, x_2{}^{(\bar{\beta}_2)}, \dots, x_g^{(\bar{\beta}_{g-1})}$. We then define

$$x_{g+1,1} = x_{g+1,0} + Q'.$$

Again, we define as before $v_m^{g+2,1}, \mu_{g+1,1}, x_{g+1,2}, \ldots, v_m^{g+2,j}, \mu_{g+1,j}$, until we have $\mu_{g+1,j} = p-1$ (note that $\mu_{g+1,i+1} > \mu_{g+1,i}, i \geq 0$). Then a minimal generating sequence of ν_E is given by

$$x_0, \ldots, x_{g+1} := x_{g+1,j}.$$

Note that $\nu_E(x_g) = \bar{\beta}_g$, $\nu_E(x_{g+1}) = p$, and all the x_i are polynomials in $\mathbf{K}[x_0, x_1]$. In fact, if we let $\mathcal{D} = \{x_{g+1} = 0\}$, then it follows from the definitions of ν_E and $\mathcal{D}$ that, for an irreducible $h \in \mathbf{K}[[x, y]]$, we have

$$\nu_E(h) = \nu_{\mathcal{D}}(h),$$

and the initial part $\mathrm{in}_{\nu_E}(h) = \mathrm{in}_{\nu_{\mathcal{D}}}(h)$ is a polynomial in $x_0, \ldots, x_g, x_{g+1,j-1}$, unless $\mathrm{in}_{\nu_E}(h) = x_{g+1}^r$ is a power of x_{g+1}, in which case, we have that $\nu_E(h) = rp$. Note that $x_{g+1,j-1}$ is a polynomial in the variables $x_0, \ldots, x_{g+1}$.

We now assume that, for a divisorial valuation ν_E, defined by the irreducible component C_{p-1} of the $(p-1)$th jet scheme of an irreducible curve $\mathcal{C}$, we have determined a minimal generating sequence $x_0, \ldots, x_g$ as before. Then, by construction we have that, for $i = 2, \ldots, g$, there exist polynomials f_i such that

$$x_i = f_i(x_0, \ldots, x_{i-1}).$$

We will use this to prove the following proposition, which is the goal of this article.

Proposition 4.1. *There exist an embedding $e : \mathbf{A}^2 \hookrightarrow \mathbf{A}^{g+1}$ and a toric proper birational morphism $\mu : X_\Sigma \longrightarrow \mathbf{A}^{g+1}$ such that*:

$$\begin{tikzcd} \tilde{\mathbf{A}}^2 \arrow[r] \arrow[d, "\eta"'] & X_\Sigma \arrow[d, "\mu"] \\ \mathbf{A}^2 \arrow[r, hook, "e"] & \mathbf{A}^{g+1} \end{tikzcd}$$

1. *X_Σ is smooth, that is, the fan Σ is a regular subdivision of $\mathbb{R}_+^{g+1}$, and the vector*
$$v_{\nu_E} := (\nu_E(x_0), \ldots, \nu_E(x_g))$$
is an edge of a cone that belongs to Σ;
2. *The strict transform $\tilde{\mathbf{A}}^2$ of $\mathbf{A}^2$ by $\mu : X_\Sigma \longrightarrow \mathbf{A}^{g+1}$ is smooth*;
3. *The divisor $E' \subset X_\Sigma$ that corresponds to the vector v_{ν_E} intersects $\tilde{\mathbf{A}}^2$ transversally along a divisor E*;
4. *The valuation defined by the divisor E is v.*

Proof. The functions f_i provide an embedding $\mathbf{A}^2 \hookrightarrow \mathbf{A}^{g+1}$, which is the geometric counterpart of the morphism

$$\mathbf{K}[x_0, x_1, y_1, \ldots, y_g] \longrightarrow \frac{\mathbf{K}[x_0, x_1, y_2 \ldots, y_g]}{(y_2 - f_2(x_0, x_1), \ldots, y_g - f_g(x_0, x_1, y_2, \ldots, y_{g-1}))} \simeq \mathbf{K}[x_0, x_1].$$

Let Σ' be a regular subdivision of $\mathbb{R}_+^{g+1}$, which is compatible with the Newton dual fan of $y_i - f_i$, $i = 2, \ldots, g$ (see Sect. 5 of [GT] for the construction of Σ'), and let Σ'' be the Stellar subdivision of Σ' associated with the vector v_{ν_E}. Finally, let Σ be a regular subdivision of Σ''. Then the first three properties of the proposition follow from Theorem 5.2 in [GT]. Now by construction of the embedding e we have that if $\mathbb{W}$ is the irreducible component of $\mathrm{Cont}^p(\mathcal{C})$ that defines ν_E, then

$$\begin{aligned} e_\infty(\mathbb{W}) = e_\infty(\mathbf{A}^2_\infty) &\cap \mathrm{Cont}^{\nu_E(x_0)}(x_0) \cap \mathrm{Cont}^{\nu_E(x_1)}(x_1) \\ &\cap \mathrm{Cont}^{\nu_E(x_2)}(y_2) \cap \cdots \cap \mathrm{Cont}^{\nu_E(x_g)}(y_g), \end{aligned}$$

where $e_\infty : \mathbf{A}^2_\infty \hookrightarrow \mathbf{A}^{g+1}_\infty$ is the canonical morphism. But the divisorial valuation associated with

$$\begin{aligned} \mathbb{U} &= \mathrm{Cont}^{\nu_E(x_0)}(x_0) \cap \mathrm{Cont}^{\nu_E(x_1)}(x_1) \cap \mathrm{Cont}^{\nu_E(x_2)}(y_2) \cap \cdots \cap \mathrm{Cont}^{\nu_E(x_g)}(y_g) \\ &\subset \mathbf{A}^{g+1}_\infty \end{aligned}$$

is $\nu_{E'}$, which in terms of arcs means that $\mu_\infty(\mathrm{Cont}^1(E'))$ dominates $\mathbb{U}$, and hence we have that $\eta_\infty(\mathrm{Cont}^1(E))$ dominates $\mathbb{W}$ where η is the restriction of μ to $\tilde{\mathbf{A}}^2$. Property 4 in the proposition follows from the description of the valuation associated with $\mathbb{W}$. $\square$

REMARK 4.2. Note that that we can use the equations f_i to define an overweight deformation in the sense of [T2], and hence ν_E can be obtained from the monomial valuation $\nu_{E'}$ as in Proposition 3.3 in [T2].

EXAMPLE 3. Let $\mathcal{C}$ be the irreducible curve defined by the equation $x_1^2 - x_0^3 = 0$. Let ν be the valuation defined by $C_6 \subset \mathcal{C}_6$ or, equivalently, by the corresponding irreducible component of $\mathrm{Cont}^7(x_1^2 - x_0^3)$. Note that the ideal of C_6 is generated by

$$(x_0^{(0)}, x_0^{(1)}, x_1^{(0)}, \ldots, x_1^{(2)}, {x_1^{(3)}}^2 - {x_0^{(2)}}^3).$$

Then by the discussion at the beginning of this section we have that x_0, x_1, and $x_2 = x_1^2 - x_0^3$ give a minimal generating sequence of ν. We embed $\mathbf{A}^2 = \mathrm{Spec}\,\mathbf{K}[x_0, x_1] \hookrightarrow \mathbf{A}^3 = \mathrm{Spec}\,\mathbf{K}[x_0, x_1, y_2]$ by the equation $y_2 - (x_1^2 - x_0^3) = 0$. A subdivision of $\mathbb{R}^3_+$ as in Proposition 4.1 is given by a fan Σ whose edge vectors are the vectors

$$(1, 1, 1), (1, 2, 3), (2, 3, 5), (2, 3, 6), (2, 3, 7),$$

where the last vector is the $v_\nu = (\nu(x_0), \nu(x_1), \nu(x_2))$. We are interested in a chart of X_Σ, where we can see the divisor E' associated with the vector v_ν. We consider the chart $X_\sigma = \mathbf{A}^3 = \mathrm{Spec}\,\mathbf{K}[u, v, w]$ generated by the vectors $(1, 2, 3)$, $(2, 3, 6)$, $(2, 3, 7)$. The restriction of μ to this chart is given by

$$\begin{aligned} x_0 &= uv^2w^2, \\ x_1 &= u^2v^3w^3, \\ y_2 &= u^3v^6w^7. \end{aligned}$$

The strict transform of $\mathbf{A}^2 = \{y_2 - (x_1^2 - x_0^3) = 0\} \subset \mathbf{A}^3$ is given by

$$\tilde{\mathbf{A}}^2 = \{w - u + 1 = 0\} \simeq \operatorname{Spec} \mathbf{K}[u, v] \subset \mathbf{A}^3 = \operatorname{Spec} \mathbf{K}[u, v, w],$$

and E' is defined by $w = 0$. Thus the divisor E is defined in $\tilde{\mathbf{A}}^2$ by the equation $u - 1 = 0$. The restriction η of μ to $\tilde{\mathbf{A}}^2$ is obtained from the description of μ by substituting w by $u - 1$. Hence, η is given by

$$\begin{aligned} x_0 &= uv^2(u-1)^2, \\ x_1 &= u^2v^3(u-1)^3. \end{aligned}$$

We can directly verify that η is obtained as follows: First, we consider the minimal embedded resolution of the curve $\mathcal{C} = \{x_1^2 - x_0^3 = 0\}$ (which is obtained by three consecutive point blowing ups), then we blow up the intersection of the strict transform of $\mathcal{C}$ with the exceptional divisor. The divisor obtained from this last blowing up satisfies $\nu = \nu_E$. We see that the total transform of $\mathcal{C}$ by η is given by the equation $u^3v^6(u-1)^7$ and hence that $\nu_E(x_1^2 - x_0^3) = 7$.

This result shows a different approach from the one of [GT] to the resolution of singularities of an irreducible plane curve $\mathcal{C}$ by one toric morphism. Indeed, in loc. cit., the embedding e is constructed from the study of the curve valuation $\nu_{\mathcal{C}}$, whereas the approach suggested by this article is to study the divisorial valuation associated with the irreducible component C_{p-1} of $\mathcal{C}_{p-1}$ (where $p = n_g\bar{\beta}_g$ is detected via invariants of jet schemes). The two approaches lead to the same embedding in this case; in higher dimensions, they may differ.

Let us explain a little bit more the point of view suggested in this article about the embedding e. Let $\nu = \nu_\alpha$ be the monomial valuation defined on $\mathbf{A}^n = \operatorname{Spec} \mathbf{K}[x_1, \ldots, x_n]$ by a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \mathbb{N}$, $i = 1, \ldots, n$. Let $I \subset \mathbf{K}[x_1, \ldots, x_n]$ be an ideal and assume that the origin O belongs to the variety $V(I) \subset \mathbf{A}^n = \operatorname{Spec} \mathbf{K}[x_1, \ldots, x_n]$ defined by this ideal. We will say that I or $V(I)$ is nondegenerate with respect to ν at O if the singular locus of the variety defined by the initial ideal $\mathrm{in}_\nu(I)$ of I does not intersect the torus $(\mathbf{K}^*)^n$. Note that in this context, the initial ideal of I is defined by

$$\mathrm{in}_\nu(I) = \{\mathrm{in}_\nu(f),\, f \in I\},$$

where for $f = \sum a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbf{K}[x_1, \ldots, x_n]$,

$$\mathrm{in}_\nu(f) = \sum_{a_{i_1,\ldots,i_n} \neq 0,\, i_1\alpha_1 + \cdots + i_n\alpha_n = \nu(f)} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

It follows from [AGS; Te1] (see also [Va] for the hypersurface case) that if for every $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N}$, $i = 1, \ldots, n$, I is nondegenerate with respect to ν_α at O, then we can construct a proper toric birational morphism $Z \longrightarrow \mathbf{A}^n$ that resolves the singularities of $V(I)$ in a neighborhood of O. Note that I can be degenerate with respect to a valuation defined by a vector α if there exists an irreducible family of jets (having a large contact with $V(I)$) or arcs on $V(I)$ such that, for a generic $\gamma = (\gamma_1(t), \ldots, \gamma_n(t))$ in this family, $(\mathrm{ord}_t \gamma_1(t), \ldots, \mathrm{ord}_t \gamma_n(t)) = \alpha$; indeed, by the Newton–Puiseux type theorem, if this were not satisfied, $\mathrm{in}_{\nu_\alpha}(f)$ would contain monomials, and hence by definition I would be nondegenerate with respect to ν_α. By studying irreducible components of jet schemes of a plane branch $\mathcal{C}$, as we have done, we are also looking for the degeneracy with respect to the first Newton polygon. The embedding we have constructed by applying Proposition 4.1 to the divisorial valuation associated with the irreducible component $C_{n_g\bar{\beta}_g-1}$ of $\mathcal{C}_{n_g\bar{\beta}_g-1}$ has the following property. Let I be the defining ideal of the curve $\mathcal{C}$ in $\mathbf{A}^{g+1}$, and let $\alpha = (\bar{\beta}_0, \ldots, \bar{\beta}_g)$; then the initial ideal in_{ν_α}(I) is the defining ideal of the monomial curve defined by $\{(t^{\bar{\beta}_0}, \ldots, t^{\bar{\beta}_g}), t \in \mathbf{K}\}$, which has an isolated singularity at O, and hence I is nondegenerate with respect to ν_α. Moreover, this is the only relevant vector α with respect to which we should check degeneracy, the reason being that the initial ideal with respect to any other vector will contain monomials. One crucial thing is that in the curve case, the initial ideal we found is binomial, and thus it defines a toric variety. In higher dimensions, it will not be the case, and more technology will be needed.

References

[AM] S. Abhyankar and T. T. Moh, *Newton–Puiseux expansion and generalized Tschirnhausen transformation. I, II,* J. Reine Angew. Math. 260 (1973), 47–83, Ibid., 261 (1973) 29–54.

[AGS] F. Aroca, M. Gomez-Morales, and K. Shabbir, *Torical modification of Newton non-degenerate ideals,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM 107 (2013), no. 1, 221–239.

[CA] E. Casas-Alvero, *Singularities of plane curves,* London Math. Soc. Lecture Note Ser., 276, Cambridge University Press, Cambridge, 2000.

[CV] S. D. Cutkosky and P. A. Vinh, *Valuation semigroups of two-dimensional local rings,* Proc. Lond. Math. Soc. (3) 108 (2014), no. 2, 350–384.

[dFEI] T. de Fernex, L. Ein, and S. Ishii, *Divisorial valuations via arcs,* Publ. Res. Inst. Math. Sci. 44 (2008), no. 2, 425–448.

[ELM] L. Ein, R. Lazarsfeld, and M. Mustata, *Contact loci in arc spaces,* Compos. Math. 140 (2004), no. 5, 1229–1244.

[EM] L. Ein and M. Mustata, *Jet schemes and singularities,* Algebraic geometry-seattle, Part 2, Proc. Sympos. Pure Math., 80, pp. 505–546, Amer. Math. Soc., Providence, RI, 2005.

[FJ] C. Favre and M. Jonsson, *The valuative tree,* Lecture Notes in Math., 1853, xiv+234 pp., Springer-Verlag, Berlin, 2004.

[GB] E. Garcia Barroso, *Invariants des singularités de courbes planes et courbure des fibres de milnor,* Servicio de Publicaciones de la ULL, 2004.

[GT] R. Goldin and B. Teissier, *Resolving singularities of plane analytic branches with one toric morphism. resolution of singularities,* Progr. Math., 181, pp. 315–340, Birkhauser, Basel, 2000.

[GP] P. Gonzalez-Perez, *Toric embedded resolutions of quasi-ordinary hypersurface singularities,* Ann. Inst. Fourier (Grenoble) 56 (2003), no. 6, 1819–1881.

[HOS] F. J. Herrera Govantes, M. A. Olalla Acosta, and M. Spivakovsky, *Valuations in algebraic field extensions,* J. Algebra 312 (2007), no. 2, 1033–1074.

[I] S. Ishii, *Jet schemes, arc spaces and the Nash problem,* C. R. Math. Acad. Sci. Soc. R. Can. 29 (2007), 1–21.

[LMR] M. Lejeune-Jalabert, H. Mourtada, and A. Reguera, *Jet schemes and minimal embedded desingularization of plane branches,* Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math., doi:10.1007/s13398-012-0091-5. Special issue dedicated to Professor H. Hironaka.

[Mc] S. MacLane, *A construction for absolute values in polynomial rings,* Trans. Amer. Math. Soc. 40 (1936), no. 3, 363–395.

[Ma] W. Mahboub, *Key polynomials,* J. Pure Appl. Algebra 217 (2013), no. 6, 989–1006.

[Mo1] H. Mourtada, *Jet schemes of complex plane branches and equisingularity,* Ann. Inst. Fourier (Grenoble) 61 (2011), no. 6, 2313–2336.

[Mo3] ———, *Jet schemes of toric surfaces,* C. R. Math. Acad. Sci. Paris 349 (2011), no. 9–10, 563–566.

[Mo2] ———, *Jet schemes of rational double point singularities,* Valuation theory in interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., pp. 373–388, 2014.

[Mo4] ———, *Jet schemes and generating sequences of $\mathbb{H}$-divisorial valuations* (in preparation).

[Re] A. Reguera, *Towards the singular locus of the space of arcs,* Amer. J. Math. 131 (2009), no. 2, 313–350.

[S] M. Spivakovsky, *Valuations in function fields of durfaces,* Amer. J. Math. 112 (1990), no. 1.

[T1] B. Teissier, *Valuations, deformations, and toric geometry. Valuation theory and its applications, vol. II (Saskatoon, SK, 1999),* 33, pp. 361–459, Amer. Math. Soc., Providence, RI, 2003.

[T3] ———, *Appendix to O. Zariski's course: the moduli problem for plane branches,* Univ. Lecture Ser., 39, American Mathematical Society, Providence, RI, 2006.

[T2] ———, *Overweight deformations of affine toric varieties and local uniformization,* Valuation theory in interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., pp. 252–265, 2014.

[Te1] J. Tevelev, *Compactifications of subvarieties of tori,* Amer. J. Math. 129 (2007), no. 4, 1087–1104.

[Te2] ———, *On a question of B,* Teissier. Collect. Math. 65 (2014), no. 1, 61–66.

[V2] M. Vaquié, *Valuations and local uniformization. Singularity theory and its applications,* Adv. Stud. Pure Math., 43, pp. 477–527, Math. Soc. Japan, Tokyo, 2006.

[V1] ———, *Extension d'une valuation,* Trans. Amer. Math. Soc. 359 (2007), no. 7, 3439–3481.

[Va] A. N. Varchenko, *Zeta-function of monodromy and Newton's diagram,* Invent. Math. 37 (1976), no. 3, 253–262.

Equipe Géométrie et Dynamique
Institut Mathématiques de
Jussieu-Paris Rive Gauche
Université Paris 7
Bâtiment Sophie Germain, case 7012
75205 Paris Cedex 13
France

hussein.mourtada@imj-prg.fr