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A Surface with $q = 2$ and Canonical Map of Degree 16

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Abstract. We construct a surface with irregularity $q = 2$, geometric genus $p_g = 3$, self-intersection of the canonical divisor $K^2 = 16$, and canonical map of degree 16.

1. Introduction

Let S be a smooth minimal surface of general type. Denote by $\phi : S \dashrightarrow \mathbb{P}^{p_g-1}$ the canonical map, and let $d := \deg(\phi)$. The following Beauville's result is well known.

Theorem 1 [Be]. *If the canonical image* $\Sigma := \phi(S)$ *is a surface, then either*:

(i) $p_g(\Sigma) = 0$, *or*

(ii) Σ *is a canonical surface* (*in particular*, $p_g(\Sigma) = p_g(S)$).

Moreover, in case (i) $d \leq 36$, *and in case* (ii) $d \leq 9$.

Beauville has also constructed families of examples with $\chi(\mathcal{O}_S)$ arbitrarily large for $d = 2, 4, 6, 8$ and $p_g(\Sigma) = 0$. Despite being a classical problem, for $d > 8$ the number of known examples drops drastically: only Tan's example [Ta, §5] with $d = 9$, the author's [Ri] example with $d = 12$, and Persson's example [Pe] with $d = 16$ are known. There is a recent preprint of Sai-Kee Yeung [Ye] claiming that the case $d = 36$ does occur. Du and Gao [DuGa] show that if the canonical map is an Abelian cover of $\mathbb{P}^2$, then the examples mentioned with $d = 9$ and $d = 16$ are the only possibilities for $d > 8$. These surfaces are regular, so for irregular surfaces, all known examples satisfy $d \leq 8$. We get from Beauville's proof that lower bounds hold for irregular surfaces. In particular,

$$q = 2 \quad \Longrightarrow \quad d \leq 18.$$

In this note, we construct an example with $q = 2$ and $d = 16$. The idea of the construction is the following. We start with a double plane with geometric genus $p_g = 3$, irregularity $q = 0$, self-intersection of the canonical divisor $K^2 = 2$, and singular set the union of 10 points of type A_1 (nodes) and 8 points of type A_3 (standard notation, the resolution of a singularity of type A_n is a chain of (-2)-curves $C_1, \dots, C_n$ such that $C_iC_{i+1} = 1$ and $C_iC_j = 0$ for $j \neq i \pm 1$). Then we take a double covering ramified over the points of type A_3 and obtain a surface with $p_g = 3$, $q = 0$ and $K^2 = 4$ with 28 nodes. A double covering ramified over 16 of these 28 nodes gives a surface with $p_g = 3$, $q = 0$ and $K^2 = 8$ with 24 nodes (which is a $\mathbb{Z}_2^3$-covering of $\mathbb{P}^2$). Finally, there is a double covering ramified

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over these 24 nodes that gives a surface with $p_g = 3$, $q = 2$ and $K^2 = 16$, and the canonical map factors through these coverings, and thus it is of degree 16.

Notation

We work over the complex numbers. All varieties are assumed to be projective algebraic. A $(-n)$-curve on a surface is a curve isomorphic to $\mathbb{P}^1$ with self-intersection $-n$. Linear equivalence of divisors is denoted by $\equiv$. The rest of the notation is standard in algebraic geometry.

2. $\mathbb{Z}_2^n$-Coverings

The following is taken from [Ca]; the standard reference is [Pa].

PROPOSITION 2. *A normal finite $G \cong \mathbb{Z}_2^r$-covering $Y \to X$ of a smooth variety X is completely determined by the datum of*

1. *reduced effective divisors D_σ, $\forall \sigma \in G$, with no common components*;
2. *divisor classes $L_1, \ldots, L_r$, for $\chi_1, \ldots, \chi_r$ a basis of the dual group of characters $G^\vee$, such that*

$$2L_i \equiv \sum_{\chi_i(\sigma)=-1} D_\sigma .$$

Conversely, given 1 and 2, we obtain a normal scheme Y with finite $G \cong \mathbb{Z}_2^r$-covering $Y \to X$.

The covering $Y \to X$ is embedded in the total space of the direct sum of the line bundles whose sheaves of sections are the $\mathcal{O}_X(-L_i)$ and is there defined by the equations

$$u_{\chi_i} u_{\chi_j} = u_{\chi_i \chi_j} \prod_{\chi_i(\sigma)=\chi_j(\sigma)=-1} x_\sigma ,$$

where x_σ is a section such that $\operatorname{div}(x_\sigma) = D_\sigma$. The scheme Y can be seen as the normalization of the Galois covering given by the equations

$$u_{\chi_i}^2 = \prod_{\chi_i(\sigma)=-1} x_\sigma .$$

The scheme Y is irreducible if $\{\sigma \,|\, D_\sigma > 0\}$ generates G.

For the reader's convenience, we leave here the character table for the group $\mathbb{Z}_2^3$ with generators x, y, z:

```
[    -1    -1    -1    -1     1     1     1     1 x*y*z]
[    -1    -1     1     1    -1    -1     1     1     z]
[    -1     1    -1     1    -1     1    -1     1     y]
[    -1     1     1    -1     1    -1    -1     1     x]
[     1    -1    -1     1     1    -1    -1     1   y*z]
[     1    -1     1    -1    -1     1    -1     1   x*z]
[     1     1    -1    -1    -1    -1     1     1   x*y]
[     1     1     1     1     1     1     1     1    Id]
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3. The Construction

Step 1

Let $T_1, \dots, T_4 \subset \mathbb{P}^2$ be distinct lines tangent to a smooth conic H_1, and

$$\pi : X \longrightarrow \mathbb{P}^2$$

be the double cover of the projective plane ramified over $T_1 + \cdots + T_4$. The curve $\pi^*(H_1)$ is of arithmetic genus 3 by the Hurwitz formula and has four nodes, corresponding to the tangencies to $T_1 + \cdots + T_4$. Hence $\pi^*(H_1)$ is reducible:

$$\pi^*(H_1) = A + B$$

with A, B smooth rational curves. From $AB = 4$ and $(A + B)^2 = 8$ we get $A^2 = B^2 = 0$. Now the adjunction formula

$$2g(A) - 2 = AK_X + A^2$$

gives $AK_X = -2$, and then the Riemann–Roch theorem implies

$$h^0(X, \mathcal{O}_X(A)) \geq 1 + \frac{1}{2}A(A - K_X) = 2.$$

Therefore, there exists a smooth rational curve C such that $C \neq A$, $C \equiv A$, and $AC = 0$. The curve

$$H_2 := \pi(C)$$

is smooth rational. The fact $\pi^*(H_2)^2 > C^2$ implies that $\pi^*(H_2)$ is reducible, and thus H_2 is tangent to the lines $T_1, \dots, T_4$. As before, there is a smooth rational curve D such that

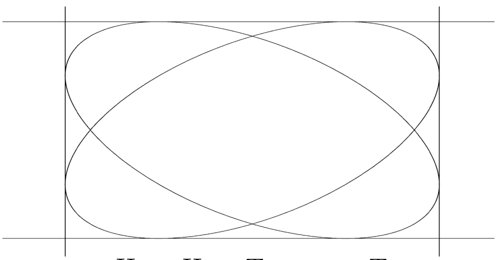
$$\pi^*(H_2) = C + D$$

and $C^2 = D^2 = 0$. Since $A \equiv C$ and $A + B \equiv C + D$, we have $B \equiv D$.

Step 2

Let x, y, z be generators of the group $\mathbb{Z}_2^3$, and

$$\psi : Y \longrightarrow \mathbb{P}^2$$



$H_1 + H_2 + T_1 + \cdots + T_4$

Figure 1

be the $\mathbb{Z}_2^3$-covering defined by

$$D_1 := D_{xyz} := H_1, \qquad D_2 := D_z := H_2, \qquad D_3 := D_y := T_1 + T_2,$$
$$D_4 := D_x := T_3 + T_4, \qquad D_{yz} := D_{xz} := D_{xy} := 0.$$

Let d_i be the defining equation of D_i. According to Section 2, the surface Y is obtained as the normalization of the covering given by the equations

$$u_1^2 = d_1 d_2 d_3 d_4, \qquad u_2^2 = d_1 d_2, \qquad \dots, \qquad u_7^2 = d_3 d_4.$$

Since the branch curve $D_1 + \cdots + D_4$ has only simple singularities, the invariants of Y can be computed directly. Consider divisors $L_{i \dots h}$ such that $2L_{i \dots h} \equiv D_i + \cdots + D_h$ and let T be a general line in $\mathbb{P}^2$. We have

$$L_{1234}(K_{\mathbb{P}^2} + L_{1234}) = 4T \cdot T = 4,$$
$$L_{ij}(K_{\mathbb{P}^2} + L_{ij}) = 2T(-T) = -2,$$

and thus

$$\chi(Y) = 8\chi(\mathbb{P}^2) + \frac{1}{2}(4 + 6 \times (-2)) = 4,$$
$$p_g(Y) = p_g(\mathbb{P}^2) + h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(T)) + 6h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-T)) = 3.$$

So a canonical curve in Y is the pullback of a line in $\mathbb{P}^2$, and then

$$K_Y^2 = 8.$$

Step 3

Notice that the points where two curves D_i meet transversely give rise to smooth points of Y, and hence the singularities of Y are:

- 16 points $p_1, \dots, p_{16}$ corresponding to the tacnodes of $D_1 + \cdots + D_4$;
- 8 nodes $p_{17}, \dots, p_{24}$ corresponding to the nodes of D_3 and D_4.

We want to show that $p_1, \dots, p_{24}$ are nodes with even sum.

REMARK 3. The surface Y attains Myiaoka's bound [Mi, Prop. 2.1.1] for the number of rational double points on a surface of general type.

The surface X defined in Step 1 is the double plane with equation $u_7^2 = d_3 d_4$, and thus the covering ψ factors trough a $\mathbb{Z}_2^2$-covering

$$\varphi : Y \longrightarrow X.$$

The branch locus of φ is $A + B + C + D$ plus the four nodes given by the points in $D_3 \cap D_4$. The points $p_1, \dots, p_{16}$ are nodes because they are the pullbacks of nodes of $A + B + C + D$.

The divisor $\varphi^*(A + C)$ is even ($A + C \equiv 2A$), double ($A + C$ in the branch locus of φ), with smooth support ($A + C$ smooth), and $p_1, \dots, p_{16} \in \varphi^*(A + C)$, $p_{17}, \dots, p_{24} \notin \varphi^*(A + C)$. Consider the minimal resolution of the singularities of Y

$$\rho : Y' \longrightarrow Y$$

and let $A_1, \dots, A_{24} \subset Y'$ be the (-2)-curves corresponding to the nodes $p_1, \dots, p_{24}$. The divisor $(\varphi \circ \rho)^*(A + C)$ is even, and there exists a divisor E such that

$$(\varphi \circ \rho)^*(A + C) = 2E + \sum_1^{16} A_i.$$

Thus, there exists a divisor L_1 such that $\sum_1^{16} A_i \equiv 2L_1$.

Analogously, we show that the nodes $p_{17}, \dots, p_{24}$ have even sum, that is, there exists a divisor L_2 such that $\sum_{17}^{24} A_i \equiv 2L_2$. This follows from $\psi^*(T_1 + T_3)$ even, double, and with support of multiplicity 1 at $p_{17}, \dots, p_{24}$ and of multiplicity 2 at 8 of the nodes $p_1, \dots, p_{16}$.

Step 4

So there is a divisor $L := L_1 + L_2$ such that

$$\sum_1^{24} A_i \equiv 2L.$$

Consider the double covering $S \longrightarrow Y$ ramified over $p_1, \dots, p_{24}$ and determined by L. More precisely, given the double covering

$$\eta : S' \longrightarrow Y'$$

with branch locus $\sum_1^{24} A_i$, determined by L, S is the minimal model of S'. We have

$$\chi(S') = 2\chi(Y') + \frac{1}{2} L(K_{Y'} + L) = 8 - 6 = 2.$$

Since the canonical system of Y is given by the pullback of the system of lines in $\mathbb{P}^2$, the canonical map of Y is of degree 8 onto $\mathbb{P}^2$. We want to show that the canonical map of S' factors through η.

We have

$$p_g(S') = p_g(Y') + h^0(Y', \mathcal{O}_{Y'}(K_{Y'} + L)),$$

so the canonical map factors if

$$h^0(Y', \mathcal{O}_{Y'}(K_{Y'} + L)) = 0.$$

Let us suppose the opposite. Hence, the linear system $|K_{Y'} + L|$ is not empty, and then $A_i(K_{Y'} + L) = -1$, $i = 1, \dots, 24$, implies that $\sum_1^{24} A_i \equiv 2L$ is a fixed component of $|K_{Y'} + L|$. Therefore,

$$h^0(Y', \mathcal{O}_{Y'}(K_{Y'} + L - 2L)) = h^0(Y', \mathcal{O}_{Y'}(K_{Y'} - L)) > 0,$$

and then

$$h^0(Y', \mathcal{O}_{Y'}(2K_{Y'} - 2L)) = h^0\left(Y', \mathcal{O}_{Y'}\left(2K_{Y'} - \sum_1^{24} A_i\right)\right) > 0.$$

This means that there is a bicanonical curve B through the 24 nodes of Y. We claim that there is exactly one such curve. In fact, the strict transform in Y' of the line T_1 is the union of two double curves $2T_a, 2T_b$ such that

$$T_a \sum_1^{24} A_i = T_b \sum_1^{24} A_i = 6$$

and $T_a\rho^*(B) = T_b\rho^*(B) = 4$. This implies that $\rho^*(B)$ contains T_a and T_b. Analogously, $\rho^*(B)$ contains the reduced strict transform of T_2, T_3, and T_4. There is only one bicanonical curve with this property, with equation $u_7 = 0$ (the bicanonical system of Y is induced by $\mathcal{O}_{\mathbb{P}^2}(2)$ and $u_2, \ldots, u_7$).

Since

$$h^0(Y', \mathcal{O}_{Y'}(2K_{Y'} - 2L)) = 1 \quad \Longrightarrow \quad h^0(Y', \mathcal{O}_{Y'}(K_{Y'} - L)) = 1,$$

such a bicanonical curve is double. This is a contradiction because the curve given by $u_7 = 0$ is not double.

So $h^0(Y', \mathcal{O}_{Y'}(K_{Y'} + L)) = 0$, and we conclude that the surface S has invariants $p_g = 3$, $q = 2$, and $K^2 = 16$ and that the canonical map of S is of degree 16 onto $\mathbb{P}^2$.

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