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The Cotangent Bundle of a Cominuscule Grassmannian

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Abstract. A theorem of the first author states that the cotangent bundle of the type A Grassmannian variety can be embedded as an open subset of a smooth Schubert variety in a two-step affine partial flag variety. We extend this result to cotangent bundles of cominuscule generalized Grassmannians of arbitrary Lie type.

1. Introduction

An earlier work of Lusztig and Strickland suggests possible connections between the conormal varieties to partial flag varieties on the one hand and affine Schubert varieties on the other. In particular, Lusztig [8] relates certain orbit closures arising from the type A cyclic quiver $\hat{A}_h$ to affine Schubert varieties. In the case $h = 2$, Strickland [11] relates such orbit closures to conormal varieties of determinantal varieties; furthermore, any determinantal variety can be canonically realized as an open subset of a Schubert variety in the Grassmannian [6].

Inspired by these results, the first author was interested in finding a relationship between affine Schubert varieties and conormal varieties to the Grassmannian. As a first step, she showed that the compactification of the cotangent bundle to the Grassmannian is canonically isomorphic to a Schubert variety in a two-step affine partial flag variety [5]. In this paper, we extend her result to cominuscule generalized Grassmannians of arbitrary finite type (such Grassmannians occur in types $A - E$).

1.1. Preliminaries

Let G_0 be a simple algebraic group over $\mathbb{C}$ with associated Lie algebra $\mathfrak{g}_0$ and simple roots $\{\alpha_1, \dots, \alpha_n\}$. A simple root α_i is *cominuscule* if the coefficient of α_i in any positive root of $\mathfrak{g}_0$ (written in the simple root basis) is less than or equal to 1.

The Weyl group of G_0 is generated by simple reflections $S_0 := \{s_1, \dots, s_n\}$ corresponding to the simple roots $\{\alpha_1, \dots, \alpha_n\}$. For any subset $K \subset S_0$, we let $P_K \subset G_0$ denote the parabolic subgroup whose Weyl group is generated by the elements of K. For $1 \le i \le n$, set $S_{0,i} := S_0 \setminus \{s_i\}$, so that $P_{S_{0,i}}$ is a maximal parabolic subgroup of G_0. The manifold $G_0/P_{S_{0,i}}$ is called a *generalized Grassmannian of type* G_0 and is said to be *cominuscule* if α_i is cominuscule. For additional background on cominuscule Grassmannians, see [1] or [7].

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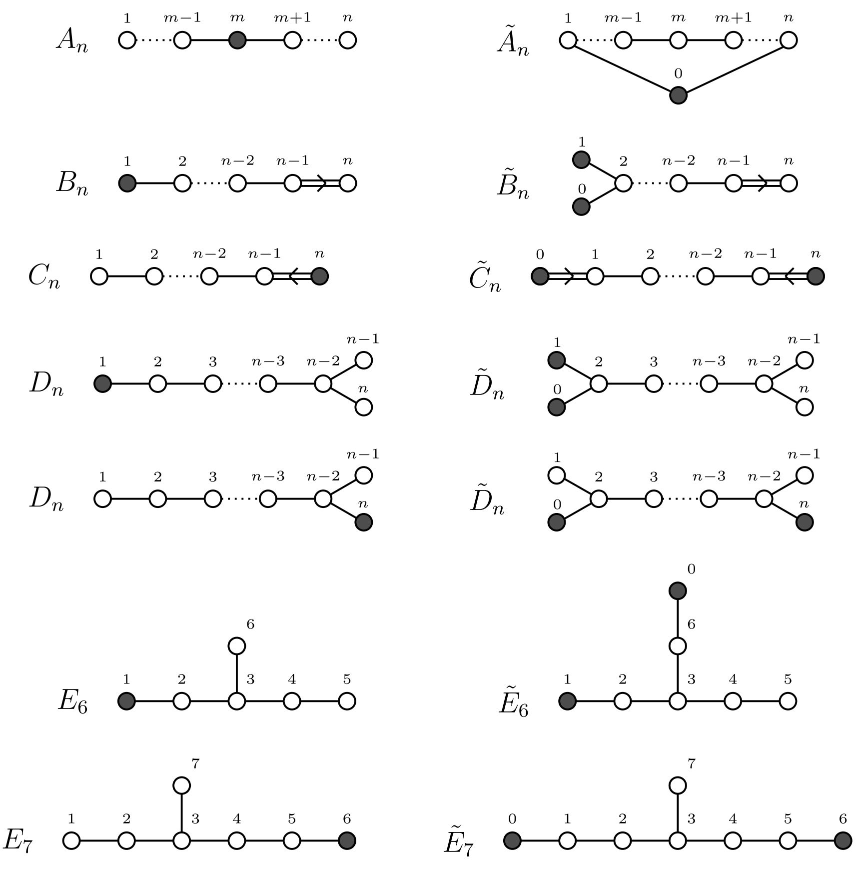
Table 1 Finite type Dynkin diagrams with cominuscule simple root marked in black (left column), and the corresponding affine Dynkin diagrams with both the cominuscule and the additional affine root marked in black (right column)



For the remainder of the paper, we fix $m \in [1, n]$ and consider the generalized Grassmannian $X := G_0/P_{S_{0,m}}$ associated with α_m. Note that α_m is not yet assumed to be cominuscule.

Let $\mathfrak{g}$ denote the affine untwisted Kac–Moody algebra associated to $\mathfrak{g}_0$, and let $\mathcal{G}$ be the corresponding affine Kac–Moody group (see [4, §6]).[1] The Dynkin diagram for $\mathfrak{g}$ depends on the Dynkin diagram for $\mathfrak{g}_0$ and is shown in Table 1 (see [2, §18.1] or [3, §4.8]). We use the convention that the affine node (sometimes called the special node) is labeled by zero, and similarly let α_0 and s_0 be the affine

[1] We use the calligraphic font (e.g., $\mathcal{G}$ and $\mathcal{P}_J$) for infinite-dimensional Kac–Moody groups and a non-calligraphic font for finite-dimensional Lie groups (e.g., G_0, P_J).

simple root and reflection, respectively. The Weyl group W of $\mathfrak{g}$ is generated by $S := \{s_0, \dots, s_n\}$, and there is a parahoric subgroup $\mathcal{P}_K \subset \mathcal{G}$ associated with any subset $K \subset S$. We let $\mathcal{X}_K := \mathcal{G}/\mathcal{P}_K$ denote the associated affine flag variety, and $W_K \subset W$ denote the Weyl group of $\mathcal{P}_K$ or, in other words, the subgroup of W generated by K. For any subsets $I \subset K \subset S$, let $W_K^I \subset W_K$ denote the set of minimal length coset representatives of W_K/W_I. In particular, $W^K := W_S^K$ is the set of minimal length coset representatives of W/W_K, and elements $w \in W_K$ index Schubert varieties $\mathcal{X}_K(w)$ of $\mathcal{X}_K$.

Observe that $S_0 = S \setminus \{s_0\}$. Let $S_m := S \setminus \{s_m\}$ and $J := S_{0,m} = S \setminus \{s_0, s_m\}$. Let w_i be the maximal element of $W_{S_i}^J$, where $i \in \{0, m\}$. It is a standard fact that $\mathcal{X}_{S_m}(w_0) \cong \mathcal{X}_J(w_0) \cong X$ (see Lemma 2.3). The basis of this note is the following elementary but crucial observation.

LEMMA 1.1. *If α_m is cominuscule in $\mathfrak{g}_0$, then $\mathcal{X}_J(w_0)$ and $\mathcal{X}_J(w_m)$ are isomorphic.*

Proof. The list of cominuscule simple roots in each type is well known. We indicate the cominuscule simple roots for each Dynkin diagram (up to diagram automorphism) in the left column of Table 1 and the corresponding untwisted affine Dynkin diagram in the right column. In each case, the Dynkin diagram of W_{S_0} is isomorphic to the Dynkin diagram of W_{S_m}, and this isomorphism identifies α_m with the affine root α_0. Consequently, $\mathcal{X}_J(w_0)$ and $\mathcal{X}_J(w_m)$ are isomorphic. □

1.2. Main Results

Consider the Schubert variety $Y := \mathcal{X}_J(w_0 w_m)$ in $\mathcal{X}_J$. The Kac–Moody group $\mathcal{G}$ acts on $\mathcal{X}_J$ by left multiplication, and since G_0 is the Levi subgroup of $\mathcal{P}_{S_0} \subset \mathcal{G}$, we can regard Y as a G_0-variety.

In fact, Y can naturally be considered as a G_0-homogeneous fibre bundle over X. More precisely:

THEOREM 1.2. *The affine Schubert variety $Y = \mathcal{X}_J(w_0 w_m)$ is stable under the left action of $G_0 \subset \mathcal{G}$, and the natural projection $Y \to \mathcal{X}_{S_m}(w_0) \cong X$ is a G_0-homogeneous fiber bundle map with fiber $\mathcal{X}_J(w_m)$. In particular, Y is smooth.*

Our main result is that if X is cominuscule, then Y is a natural compactification of the cotangent bundle T^*X:

THEOREM 1.3. *If X is cominuscule, then the fiber $\mathcal{X}_J(w_m)$ is isomorphic to X, and there is a G_0-equivariant map $\tilde{\mu} : T^*X \to Y$ of fiber bundles over X under which T^*X is isomorphic to a dense open subset of Y.*

We prove Theorem 1.2 in Section 2 and Theorem 1.3 in Section 3. In order to prove Theorem 1.3, we explicitly construct the G_0-equivariant embedding $\tilde{\mu} : T^*X \to Y$ that maps the base X isomorphically onto the Schubert variety $\mathcal{X}_J(w_0)$ and maps the fiber over the identity to a dense open subset of the Schubert variety $\mathcal{X}_J(w_m)$.

When X is minuscule rather than cominuscule, it is natural to replace $\mathcal{G}$ with a twisted affine Kac–Moody group. Theorem 1.2 still holds in this case, but as we show in Section 4, Theorem 1.3 does not hold. In this case, the variety Y is not the compactification of the cotangent bundle T^*X, but of a different bundle over X.

We note that Manivel and Michalek [9] have recently studied the local geometry of tangential varieties (which are compactifications of the tangent bundle) to cominuscule Grassmannians. We do not know if there is a version of our compactification for the tangent bundle.

2. The Fiber Bundle Structure on Y

We note that α_m is not assumed to be cominuscule in this section. Given $I \subset K \subset S$, we can write any $w \in W^I$ uniquely as $w = vu$, where $v \in W^K$ and $u \in W_K^I$. In this case, the projection $\mathcal{G}/\mathcal{P}_I \to \mathcal{G}/\mathcal{P}_K$ induces a projection $\mathcal{X}_I(w) \to \mathcal{X}_K(v)$, and the generic fiber of this projection is $\mathcal{X}_I(u)$. We say that $w = vu$ is a *parabolic decomposition* with respect to K.

For any $v \in W$, we define $\operatorname{Supp}(v) := \{s \in S \mid s \le v\}$ to be the set of simple reflections contained in a reduced expression for v. For any $u \in W$, let $D^I(u) := \{s \in S \mid su \le_I u\}$, where $\le_I$ is the Bruhat order on W/W_I. We have the following proposition from [10, Thm. 2.3 and Prop. 3.2].

Proposition 2.1. *The projection $\mathcal{X}_I(w) \to \mathcal{X}_K(v)$ is a fiber bundle with fiber $\mathcal{X}_I(u)$ if and only if* $\operatorname{Supp}(v) \cap K \subset D^I(u)$.

When the condition $\operatorname{Supp}(v) \cap K \subset D^I(u)$ is satisfied, we say that $w = vu$ is a *Billey–Postnikov decomposition* with respect to K.

Recall that for any $s \in S$, we have $sw \le_I w$ if and only if $\mathcal{X}_I(w)$ is stable under left multiplication by the rank 1 parahoric subgroup $\mathcal{P}_{\{s\}}$. It follows that if $L = D^I(w)$, then $\mathcal{X}_I(w)$ is stable under the action of the parahoric subgroup $\mathcal{P}_L$ ([1], see also [10, Lemma 3.9]).

Lemma 2.2. *Let $y = w_0 w_m$, so $Y = \mathcal{X}_J(y)$.*

(a) *$y = w_0 w_m$ is a Billey–Postnikov decomposition with respect to S_m.*
(b) *$D^J(y) = S_0$.*

Proof. Since w_i is maximal in $W_{S_i}^J$, we know that $D^J(w_i) = S_i$ for $i \in \{0, m\}$. It is clear that $w_0 w_m$ is a parabolic decomposition with respect to S_m and $\operatorname{Supp}(w_0) \cap S_m = S_0 \cap S_m = J \subset D^J(w_m)$, proving part (a).

For part (b), if $z \in W_{S_0}$, then $zw_0 \le_J w_0$, and hence $zw_0 = v_0 z'$, where $v_0 \in W_{S_0}^J$ and $z' \in W_J$. Similarly, $z' w_m = v_m z''$, where $v_m \in W_{S_m}^J$ and $z'' \in W_J$. So $zy = v_0 v_m z'' \le_J y$, and hence $S_0 \subset D^J(y)$. But $D^J(y)$ must be a proper subset of S, so $D^J(y) = S_0$. □

Given $K \subseteq S$, the Levi subgroup G_K of $\mathcal{P}_K$ is a Kac–Moody group with Weyl group W_K. Since $\mathcal{G}$ is affine, if K is a strict subset of S, then G_K is finite-dimensional, and similarly W_K is finite. In order to prove Theorem 1.2, we need the following standard lemma.

LEMMA 2.3. *If $K, I \subsetneq S$ and $w \in W_K^{K\cap I}$, then $P_{K,I} := G_K \cap \mathcal{P}_I$ is the parabolic subgroup of G_K corresponding to the subgroup $W_{K\cap I} \subseteq W_K$, and $\mathcal{X}_I(w)$ is isomorphic to a Schubert variety in the flag variety $G_K/P_{K,I}$. In particular, if w is the maximal element of $W_K^{K\cap I}$, then $\mathcal{X}_I(w)$ is isomorphic to $G_K/P_{K,I}$.*

Now the proof of Theorem 1.2 follows immediately from Lemma 2.2.

Proof of Theorem 1.2. By part (b) of Lemma 2.2 the variety $Y = \mathcal{X}_J(w_0 w_m)$ is stable under the left action of G_0. The base $\mathcal{X}_{S_m}(w_0)$ is clearly G_0-stable as well, and the natural projection $Y \to \mathcal{X}_{S_m}(w_0)$ is G_0-equivariant. By part (a) of Lemma 2.2 and Proposition 2.1 the projection $Y \to \mathcal{X}_{S_m}(w_0)$ is a G_0-homogeneous fiber bundle with fiber $\mathcal{X}_J(w_m)$.

Now the Levi subgroup G_{S_0} of $\mathcal{P}_{S_0}$ is simply G_0. Since $S_m \cap S_0 = J$ and w_0 is the maximal element of $W_{S_0}^J$, we can alternately set $I = S_m$ in Lemma 2.3 to get $\mathcal{X}_{S_m}(w_0) \cong \mathcal{X}_J(w_0) \cong X = G_0/P_J$. Similarly, $\mathcal{X}_J(w_m)$ is isomorphic to the flag variety G_{S_m}/P_J. Since Y is a fiber bundle with smooth fiber and base, it follows that Y is smooth. □

3. The Cotangent Bundle

If X is cominuscule, then $\mathcal{X}_J(w_m)$ is isomorphic to X by Lemma 1.1. To prove Theorem 1.3, we explicitly construct the map $T^*X \hookrightarrow Y$. Let $\mathcal{B}$ be the Borel subgroup of the Kac–Moody group $\mathcal{G}$ (in the literature, $\mathcal{B}$ is also known as the Iwahori subgroup of $\mathcal{G}$). For convenience, we write G_i for the Levi subgroup G_{S_i} of the parahoric subgroup $\mathcal{P}_{S_i} \subset \mathcal{G}$, where $i \in \{0, m\}$ (in particular, G_0 is the same as before). We let $B_i := G_i \cap \mathcal{B}$ be the induced Borel of G_i, and $P_i := B_i W_J B_i = G_i \cap \mathcal{P}_J$. Finally, let $U_i \subset P_i$ be the unipotent radical of P_i. As in the previous section, $P_0 = P_J$, $X = G_0/P_0$, and moreover $\mathcal{X}_J(w_i) \cong G_i/P_i$ for $i \in \{0, m\}$. Note that P_0 depends on α_m as well as α_0.

We will also need to use the underlying Lie algebras. We assume the standard construction of $\mathfrak{g}$, in which

$$\mathfrak{g} \cong \mathfrak{g}_0 \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

as a vector space (see [2, §18.1] or [3, §7.2]). Let $\mathfrak{h}$ be the Cartan subalgebra of the Kac–Moody algebra $\mathfrak{g}$. Let $\mathfrak{g}_i \subset \mathfrak{g}$ be the (finite-dimensional) Lie algebra of G_i, where $i \in \{0, m\}$. Let $\mathfrak{u}_i \subset \mathfrak{g}_i$ be the (nilpotent) Lie algebra of U_i. Finally, let $\mathfrak{u}_m^-$ be the opposite nilpotent radical to $\mathfrak{u}_m$ inside $\mathfrak{g}_m$. We consider the linear map

$$\phi : \mathfrak{u}_0 \to \mathfrak{g} \text{ defined by } x \mapsto x \otimes z^{-1}.$$

In order to prove Theorem 1.3, we need the following lemma.

LEMMA 3.1. *The map $\phi : \mathfrak{u}_0 \to \mathfrak{u}_m^-$ is a P_0-equivariant isomorphism of vector spaces.*

Proof. Let R denote the set of roots of $\mathfrak{g}$, with simple roots $\Delta := \{\alpha_0, \dots, \alpha_n\}$. The simple roots of $\mathfrak{g}_0$ and $\mathfrak{g}_m$ are the subsets of Δ obtained by omitting α_0 and α_m, respectively. For any subalgebra $\mathfrak{a} \subset \mathfrak{g}$, we let $R(\mathfrak{a})$ denote the set of $\mathfrak{h}$-weights of $\mathfrak{a}$, and let $R^+(\mathfrak{a})$ and $R^-(\mathfrak{a})$ denote the subsets of positive and negative roots, respectively. Let θ be the highest root of $\mathfrak{g}_0$, and let $\delta = \alpha_0 + \theta$ be the basic imaginary root of $\mathfrak{g}$ ([2, §17.1] or [3, §5.6]).

We can describe the set of roots of $\mathfrak{g}$ by

$$R(\mathfrak{g}) = \{\alpha + k\delta \mid \alpha \in R(\mathfrak{g}_0), k \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z}_{\neq 0}\}.$$

The set of positive roots of $\mathfrak{g}$ is given by

$$R^+(\mathfrak{g}) = R^+(\mathfrak{g}_0) \cup \{\alpha + k\delta \in R \mid \alpha \in R(\mathfrak{g}_0), k \in \mathbb{Z}_{>0}\} \cup \{k\delta \mid k \in \mathbb{Z}_{>0}\}.$$

Note that $R(\mathfrak{u}_0) \subset R^+(\mathfrak{g}_0)$ and $R(\mathfrak{u}_m) \subset R^+(\mathfrak{g}_m)$. Using the simple roots of $\mathfrak{g}_0$ and $\mathfrak{g}_m$, the roots of $\mathfrak{u}_0$ and $\mathfrak{u}_m$ can then be written

$$R(\mathfrak{u}_0) = \left\{ \sum_{i=1}^{n} a_i\alpha_i \in R^+(\mathfrak{g}) \;\middle|\; a_m = 1 \right\} \quad \text{and}$$
$$R(\mathfrak{u}_m) = \left\{ a_0\alpha_0 + \sum_{i \in [1,n]\setminus\{m\}} a_i\alpha_i \in R^+(\mathfrak{g}) \;\middle|\; a_0 = 1 \right\},$$

where the requirement that $a_m = 1$ (resp. $a_0 = 1$) follows from the fact that α_m is cominuscule in $\mathfrak{g}_0$ (resp. α_0 is cominuscule in $\mathfrak{g}_m$).

Every root of $\mathfrak{g}_0$ can be written uniquely as $\theta - \sum_{i=1}^n a_i\alpha_i$, where $a_i \geq 0$ for all $1 \leq i \leq n$. Since α_m is cominuscule, the coefficient of α_m in θ is equal to 1. Using the previous description of $R(\mathfrak{u}_0)$, it follows that $\alpha \in R(\mathfrak{g}_0)$ is an element of $R(\mathfrak{u}_0)$ if and only if

$$\alpha = \theta - \sum_{i \in [1,n]\setminus\{m\}} a_i\alpha_i$$

for some coefficients $a_i \geq 0$ (in particular, note that any α of this form cannot belong to $R^-(\mathfrak{g}_0)$ since it will have a positive α_m-coefficient).

Note that for any $\alpha \in R(\mathfrak{u}_0)$, the homomorphism ϕ maps $\mathfrak{g}_\alpha$ isomorphically onto $\mathfrak{g}_{\alpha-\delta}$. Thus, the $\mathfrak{h}$-weights of $\phi(\mathfrak{u}_0)$ are precisely

$$\left\{ \alpha - \delta \in R \;\middle|\; \alpha = \theta - \sum_{i \in [1,n]\setminus\{m\}} a_i\alpha_i, \text{ where } a_i \geq 0 \text{ for } i \in [1,n] \setminus \{m\} \right\}$$
$$= \left\{ -(-\theta + \delta) - \sum_{i \in [1,n]\setminus\{m\}} a_i\alpha_i \in R^- \;\middle|\; a_i \geq 0 \text{ for all } i \in [1,n] \setminus \{m\} \right\}.$$

This latter set is exactly the negative of the $\mathfrak{h}$-weights of $\mathfrak{u}_m$ since $(-\theta + \delta) = \alpha_0$ and since α_0 is cominuscule in $\mathfrak{g}_m$ as in Lemma 1.1. We conclude that $\phi(\mathfrak{u}_0) = \mathfrak{u}_m^-$. Since ϕ is a clearly bijective, it is a vector space isomorphism.

Consider the left adjoint action of $\mathfrak{p}_0 := \mathrm{Lie}(P_0)$ on $\mathfrak{g}$. Under this action, each element of the weight space $\mathfrak{g}_\beta \subset \mathfrak{p}_0$ maps $\mathfrak{g}_\alpha$ into $\mathfrak{g}_{\alpha+\beta}$ whenever $\alpha + \beta \in R(\mathfrak{g})$ and annihilates $\mathfrak{g}_\alpha$ otherwise. Recall that $R(\mathfrak{p}_0) = R^+(\mathfrak{g}_0) \cup \{\sum_{i=1}^n a_i\alpha_i \in R^-(\mathfrak{g}_0) \mid a_m = 0\}$ and observe that both $\mathfrak{u}_0$ and $\mathfrak{u}_m^-$ are stable under the left adjoint action of $\mathfrak{p}_0$ and, moreover, that ϕ is $\mathfrak{p}_0$-equivariant. It follows that ϕ is P_0-equivariant. □

Using the map $\phi : \mathfrak{u}_0 \to \mathfrak{u}_m^-$, we construct a map

$$\Phi : \mathfrak{u}_0 \to \mathcal{X}_J = \mathcal{G}/\mathcal{P}_J : x \mapsto [\exp(\phi(x)) \cdot \mathcal{P}_J].$$

LEMMA 3.2. *Φ is a P_0-equivariant algebraic isomorphism from $\mathfrak{u}_0$ to an open dense subset of $\mathcal{X}_J(w_m)$.*

Proof. The exponential map $\mathfrak{u}_m^- \to \exp(\mathfrak{u}_m^-) =: U_m^-$ is an algebraic isomorphism, and $U_m^- \cong U_m^- \cdot [e\mathcal{P}_J]$ is an open dense subset of $\mathcal{X}_J(w_m) = G_m/P_m$, where $e \in \mathcal{G}$ is the identity. Since ϕ is a P_0-equivariant bijection and $P_0 \subset \mathcal{P}_J$, the result follows. □

We can now finish the proof of the main theorem.

Proof of Theorem 1.3. As in Section 2, we let $Y = \mathcal{X}_J(y)$, where $y = w_0 w_m$. The cotangent bundle of X is

$$T^*X = G_0 \times_{P_0} \mathfrak{u}_0,$$

the quotient of $G_0 \times \mathfrak{u}_0$ by the P_0-action $p \cdot (g, x) = (gp, p^{-1}x)$. We can define a map

$$\mu : G_0 \times \mathfrak{u}_0 \to \mathcal{X}_J(y) : (g, x) \mapsto g \cdot \Phi(x),$$

where we use the fact that $\Phi(x) \in \mathcal{X}_J(w_m) \subset \mathcal{X}_J(y)$, which is stable under the left action of G_0 by Theorem 1.2. But μ is P_0-equivariant, so we get an induced map

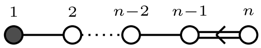
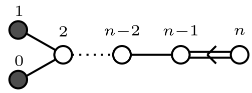
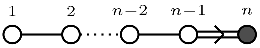
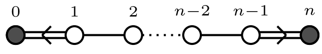
$$\tilde{\mu} : G_0 \times_{P_0} \mathfrak{u}_0 \to \mathcal{X}_J(y).$$

The cotangent bundle map $T^*X \to X$ sends $(g, x) \mapsto [gP_0]$. Since the projection $\mathcal{G} \mapsto \mathcal{G}/\mathcal{P}_{S_m}$ sends $g \cdot \Phi(x) \mapsto [g\mathcal{P}_{S_m}]$, we conclude that the diagram

$$\begin{array}{ccc} G_0 \times_{P_0} \mathfrak{u}_0 & \longrightarrow & \mathcal{X}_J(y) \\ & \searrow \quad \swarrow & \\ & G_0/P_0 \cong \mathcal{X}_{S_m}(w_0) & \end{array}$$

commutes, and thus $\tilde{\mu}$ is a morphism of G_0-homogeneous fibre bundles. Over $[eP_0]$, this map restricts to $\Phi : \mathfrak{u}_0 \to \mathcal{X}_J(w_m)$, which is injective and has open dense image in the fiber $\mathcal{X}_J(w_m)$. We conclude that the total map $G_0 \times_{P_0} \mathfrak{u}_0 \to \mathcal{X}_J(y)$ is injective and has an open image. □

Table 2 Finite type Dynkin diagrams with minuscule simple root marked in black (left column), and the corresponding twisted affine Dynkin diagrams with both the minuscule and the additional affine root marked in black (right column). We use Kac's notation for the twisted affine Dynkin diagrams. Note that in Dynkin's notation, $A_{2n-1}^{(2)}$ is denoted $\tilde{B}_n^t$, and $D_{n+1}^{(2)}$ is denoted $\tilde{C}_n^t$

C_n		$A_{2n-1}^{(2)}$	
B_n		$D_{n+1}^{(2)}$	

4. Minuscule Grassmannians

A Grassmannian $X = G_0/P_{S_{0,m}}$ is *minuscule* if $\alpha_m^\vee$ is cominuscule in the dual root system. The minuscule and cominuscule Grassmannians coincide in types A, D, and E but are disjoint in the other types. There are just two families of Grassmannians that are minuscule but not cominuscule: $\mathrm{SO}(2n+1)/P_{S_{0,n}}$, the Grassmannian corresponding to the root α_n in type B_n, and $\mathrm{Sp}(2n)/P_{S_{0,1}}$, the Grassmannian corresponding to the root α_1 in C_n. The corresponding Dynkin diagrams are listed in Table 2. As algebraic varieties, $\mathrm{Sp}(2n)/P_{S_{0,1}}$ is isomorphic to $\mathbb{P}^{2n-1}$, and $\mathrm{SO}(2n+1)/P_{S_{0,n}}$ is isomorphic to $\mathrm{SO}(2n+2)/P_n \cong \mathrm{SO}(2n+2)/P_{n+1}$, so each minuscule Grassmannian is isomorphic to a cominuscule Grassmannian. However, the minuscule Grassmannians are distinct as homogeneous spaces, and their cotangent bundles are distinct as homogeneous bundles.

Suppose that α_m is minuscule but not cominuscule, and let $\mathfrak{g}$ and $\mathcal{G}$ be the affine *twisted* Kac–Moody algebra and group associated to $\mathfrak{g}_0$ (see [2, §18.4] or [3], and [4, §6]). The proof of Theorem 1.2 still works in this setting, and consequently the affine Schubert variety $Y = \mathcal{X}_J(w_0 w_1) \subseteq \mathcal{X}_J := \mathcal{G}/\mathcal{P}_J$ is a fiber bundle over X with fiber $\mathcal{X}_J(w_m)$. Furthermore, following Lemma 1.1, we have $\mathcal{X}_J(w_m) \cong \mathcal{X}_J(w_0) \cong X$ (see Table 2 for the proof).

With all these pieces in place, we might expect that Y is a compactification of T^*X as in the cominuscule case. However, the argument from the cominuscule setting breaks down at this point. Specifically, the argument from Section 3 shows that Y is a compactification of the homogeneous vector bundle $T := G_0 \times_{P_0} \mathfrak{u}_m^-$ on X. However, T is not the cotangent bundle of X. Indeed, by the following lemma, T splits as the direct sum of two G_0-homogeneous vector bundles on X, whereas T^*X does not.

Lemma 4.1. *As P_0-modules, $\mathfrak{u}_m^-$ splits as the direct sum of two submodules, whereas $\mathfrak{u}_0$ does not.*

Proof. Let $\delta = \alpha_0 + \theta_0$ be the basic imaginary root of $\mathfrak{g}$, where θ_0 is the highest *short* root of $\mathfrak{g}_0$ ([2, §17.1] or [3, §8.3]). For any subalgebra $\mathfrak{a} \subset \mathfrak{g}$, let $R_s(\mathfrak{a})$ (resp. $R_l(\mathfrak{a})$) denote the set of real short (resp. long) $\mathfrak{h}$-weights of $\mathfrak{a}$ (see [2, §17.2] or [3, §5.1]). The set of roots of $\mathfrak{g}$ is given by

$$\{\alpha + k\delta : \alpha \in R_s(\mathfrak{g}_0), k \in \mathbb{Z}\} \cup \{\alpha + 2k\delta : \alpha \in R_l(\mathfrak{g}_0), k \in \mathbb{Z}\} \cup \{k\delta : k \in \mathbb{Z}_{\neq 0}\}.$$

Moreover, the $\mathfrak{h}$-weights of $\mathfrak{u}_0$ and $\mathfrak{u}_m^-$ are given by

$$R(\mathfrak{u}_0) = \left\{ \sum_{i\in[1,n]} a_i\alpha_i \in R^+(\mathfrak{g}) : a_m \in \{1,2\} \right\} \quad \text{and}$$

$$R(\mathfrak{u}_m^-) = \left\{ a_0\alpha_0 + \sum_{i\in[1,n]\setminus\{m\}} a_i\alpha_i \in R^-(\mathfrak{g}) : a_0 \in \{-1,-2\} \right\}.$$

Write $\mathfrak{u}_m^- = \mathfrak{u}_{m,s}^- \oplus \mathfrak{u}_{m,l}^-$, where $\mathfrak{u}_{m,s}^- := \bigoplus_{\alpha\in R_s(\mathfrak{u}_m^-)} \mathfrak{g}_\alpha$ and $\mathfrak{u}_{m,l}^- := \bigoplus_{\alpha\in R_l(\mathfrak{u}_m^-)} \mathfrak{g}_\alpha$. The short (resp. long) $\mathfrak{h}$-weights of $\mathfrak{u}_m^-$ are precisely those with α_0 coefficient $a_0 = -1$ (resp. $a_0 = -2$) in the simple root basis. It follows that the left adjoint action of P_0 preserves the long and short roots of u_m^-, and hence $T = G_0 \times_{P_0} u_{m,s}^- \oplus G_0 \times_{P_0} u_{m,l}^-$ is a direct sum of two homogeneous vector bundles. On the other hand, $\mathfrak{u}_0$ does not split as a P_0-module since the Lie algebra $\mathfrak{p}_0$ of P_0 can take short roots of $\mathfrak{u}_0$ (which have α_m coefficient 1) to long roots (which have α_m coefficient 2). □

Let $\mathfrak{h}_0$ and H_0 denote the Cartan subalgebra and subgroup of $\mathfrak{g}_0$ and G_0, respectively. An H_0-module M is *attractive* if there is some ω in $\mathfrak{h}_0$ such that $\alpha(\omega) > 0$ for all $\mathfrak{h}_0$-weights α of M. The fact that Y cannot be the compactification of T^*X follows from the following more general result.

LEMMA 4.2. *Given P_0-modules U and V, suppose that there exists an element $\omega \in \mathfrak{h}_0$ with the property that $\alpha(\omega) > 0$ for any $\mathfrak{h}_0$-weight α of U or V. Furthermore, if both $G_0 \times_{P_0} U$ and $G_0 \times_{P_0} V$ embed as open dense homogeneous G_0-bundles in a homogeneous G_0-fiber bundle Y, then U and V are isomorphic as P_0-modules.*

Proof. We can think of U and V as open dense subsets of the fiber over the identity in Y. As such, the intersection of U and V is nonempty. Let y be a point of the intersection. Since $\alpha(\omega) > 0$ for all $\mathfrak{h}_0$-weights α of U or V, the limit

$$\lim_{n\to\infty} \exp(-n\omega) \cdot y$$

exists and is equal to both 0_U and 0_V, the zero elements of U and V, which in particular must be equal. The sets U and V are both open, and $0 := 0_U = 0_V$ is a P_0-fixed point in both U and V, so $U \cong T_0U = T_0V \cong V$ as P_0-modules. □

COROLLARY 4.3. *There is no open embedding of T^*X into Y as G_0-homogeneous fiber bundles over X.*

Proof. Suppose on the contrary that such an embedding exists. Note that $\alpha(\omega_m) > 0$ for all $\alpha \in R(\mathfrak{u}_0)$, where $\omega_m \in \mathfrak{h}_0$ is the fundamental weight dual to the simple root α_m. Moreover, the roots of $\mathfrak{u}_m^-$ are all of the form $\alpha - \delta$ or $\alpha - 2\delta$ with $\alpha \in R(\mathfrak{u}_0)$. Since $\delta(\omega_m) = 0$, it follows that $\beta(\omega_m) > 0$ for all $\beta \in R(\mathfrak{u}_m^-)$ (indeed, $\mathfrak{u}_0$ and $\mathfrak{u}_m^-$ have the same $\mathfrak{h}_0$-weights since $\delta(\omega) = 0$ for any $\omega \in \mathfrak{h}_0$). By Lemma 4.2 it follows that $\mathfrak{u}_0$ and $\mathfrak{u}_m^-$ are isomorphic as P_0-modules, contradicting Lemma 4.1. □

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