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Deforming an ε-Close to Hyperbolic Metric to a Warped Product

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Abstract. We show how to deform a metric of the form $g = g_r + dr^2$ to a warped product $\mathcal{W}g = \sinh^2(r)g' + dr^2$ (g' does not depend on r) for r less than some fixed r_0. Our main result establishes to what extent the *warp forced metric* $\mathcal{W}g$ is *close to being hyperbolic* if we assume g to be close to hyperbolic.

Introduction

We first introduce some notation. The canonical flat metric on $\mathbb{R}^k$ and the round metric on $\mathbb{S}^k$ will be denoted by $\sigma_{\mathbb{R}^k}$ and $\sigma_{\mathbb{S}^k}$, respectively. Let (M^n, g) be a complete Riemannian manifold *with center* $o \in M$, that is, the exponential map $\exp_o : T_oM \to M$ is a diffeomorphism. Using the exponential map $\exp_o$, we shall sometimes identify M with $\mathbb{R}^n$, and thus we can write the metric g on $M - \{o\} = \mathbb{S}^{n-1} \times \mathbb{R}^+$ as $g = g_r + dr^2$, where r is the distance to o. The open ball of radius r in M, centered at o, will be denoted by $B_r = B_r(M)$, and the closed ball by $\bar{B}_r$. We fix a function $\rho : \mathbb{R} \to [0, 1]$ with $\rho(t) = 0$ for $t \le 0$, $\rho(t) = 1$ for $t \ge 1$, and ρ constant near 0 and 1.

Let M have center o and metric $g = g_r + dr^2$. Fix $r_0 > 0$. We define the metric $\bar{g}_{r_0}$ on $M - \{o\}$ by

$$\bar{g}_{r_0} = \sinh^2(r)\left(\frac{1}{\sinh^2(r_0)}\right)g_{r_0} + dr^2.$$

Note that this metric is a warped product (warped by sinh). Note also that to define $\bar{g}_{r_0}$ we are using the identification $M - \{o\} = \mathbb{S}^{n-1} \times \mathbb{R}^+$ given by the original metric g. We now force the metric g to be equal to $\bar{g}_{r_0}$ on $\bar{B}_{r_0} = \bar{B}_{r_0}(M)$ and stay equal to g outside $B_{r_0+1/2}$. For this, we define the *warp forced (on B_{r_0}) metric* as

$$\mathcal{W}_{r_0}g = \rho_{r_0}\bar{g}_{r_0} + (1 - \rho_{r_0})g,$$

where $\rho_{r_0}(t) = \rho(2t - 2r_0)$. Hence, we have

$$\mathcal{W}_{r_0}g = \begin{cases} \bar{g}_{r_0} & \text{on } \bar{B}_{r_0}, \\ g & \text{outside } B_{r_0+\frac{1}{2}}. \end{cases} \tag{0.1}$$

We call the process $g \mapsto \mathcal{W}_{r_0}g$ *warp forcing*. Note that if we choose g to be the warped-by-sinh hyperbolic metric $g = \sinh^2(t)\sigma_{\mathbb{S}^{n-1}} + dt^2$, then $\mathcal{W}_{r_0}g = g$. This suggests that if g is in some sense close to being hyperbolic, then $\mathcal{W}_{r_0}g$

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should also be close to hyperbolic. The purpose of this paper is to quantify this last statement, that is, to answer the following question: if g is *ε-close to a hyperbolic metric,* then to what extent is the warp forced metric $\mathcal{W}_{r_0}g$ close to hyperbolic? The answer is that $\mathcal{W}_{r_0}g$ is η-close to hyperbolic where η depends on ε and r_0. The term "ε-close to a hyperbolic metric" refers to a chart-by-chart concept and is introduced in the next paragraph.

Let $\mathbb{B}$ be the unit open $(n-1)$-ball with the flat metric $\sigma_{\mathbb{R}^{n-1}}$. Write $I_\xi = (-1-\xi, 1+\xi)$, $\xi \geq 0$. Our *basic models* are $\mathbb{T}_\xi = \mathbb{B} \times I_\xi$ with hyperbolic metric $\sigma = e^{2t}\sigma_{\mathbb{R}^{n-1}} + dt^2$. The number ξ is called the *excess* of $\mathbb{T}_\xi$. (The reason for introducing ξ will become clear in the main theorem below; see also the remark after the theorem.) Let (M, g) be a Riemannian manifold, and let $S \subset M$. We say that g is *ε-close to hyperbolic on S* if there is $\xi \geq 0$ such that for every $p \in S$, there is an *ε-close to hyperbolic chart with center p*, that is, there is a chart $\phi : \mathbb{T}_\xi \to M$, $\phi(0,0) = p$, such that $|\phi^* g - \sigma|_{C^2} < \varepsilon$. The number ξ is called the *excess* of the charts. We stress that ξ is independent of p. Here $|\cdot|_{C^2}$ is the C^2-norm (see Section 1).

Let (M, g) have center o, and let $S \subset M$. We say that g is *radially ε-close to hyperbolic on S (with respect to o)* if, for every $p \in S$, there is an ε-close to hyperbolic chart ϕ with center p and, in addition, the chart ϕ respects the product structure of $\mathbb{T}_\xi$ and $M - o = \mathbb{S}^{n-1} \times \mathbb{R}^+$, that is, $\phi(\cdot, t) = (\phi_1(\cdot), t + a)$, where the constant a depends on ϕ, and ϕ_1 is some function independent of t (equivalently, ϕ_1 is a chart on M). Here the "radial" directions are $(-1-\xi, 1+\xi)$ and $\mathbb{R}^+$ in $\mathbb{T}_\xi$ and $M - o$, respectively.

As mentioned before, our main result below shows that if g is radially ε-close to hyperbolic, then the warp forced metric $\mathcal{W}_{r_0}g$ is radially η-close to hyperbolic, where η depends on ε and r_0. In the next theorem, we assume that $\xi > 1$ and $r_0 \geq 3 + 2\xi$.

THEOREM. *Let (M, g) have center o, and let $S \subset M$. If g is radially ε-close to hyperbolic on S, with charts of excess ξ, then $\mathcal{W}_{r_0}g$ is radially η-close to hyperbolic on $S - \bar{B}_{r_0-1-\xi}$ with charts of excess $\xi - 1$, provided that $\eta \geq e^{27+12\xi}(e^{-2r_0} + \varepsilon)$.*

REMARK. Note that warp forcing reduces the excess of the charts by 1. This was one of the motivations to introduce the excess ξ.

The results in this paper are used to construct negatively curved Riemannian smoothings of Charney–Davis strict hyperbolizations of manifolds [1; 2]. In the next paragraph, we give an idea how the theorem in this paper is used in [2].

In the same way a cubical complex is made of basic pieces (the cubes $\square^k$), the hyperbolization $h(K)$ of a cubical complex K is also made of basic pieces, prefixed hyperbolization pieces X^k. Indeed, we begin with a cubical complex K and replace each cube of dimension k by the hyperbolization piece of the same dimension. Cube complexes have a piecewise flat metric induced from the flat geometry of the cubes. Likewise, the Charney–Davis hyperbolizations have a piecewise hyperbolic structure because the Charney–Davis hyperbolization pieces are hyperbolic manifolds (compact, with boundary and corners). To see how singularities appear, we can first think about the manifold two-dimensional cube case. If K^2 is a two-dimensional manifold cube complex, then its piecewise flat metric is Riemannian outside the vertices. A vertex is a singularity if and only if the vertex does not meet exactly four cubes. The picture is exactly the same for $h(K^2)$. These point singularities in $h(K^2)$ can be smoothed out easily using warping methods. In higher dimensions, the singularities of K^n and $h(K)$ appear in (possibly the whole of) the codimension 2 skeletons $K^{(n-2)}$ and $h(K^{(n-2)})$, respectively. In [2] the idea of smoothing the piecewise hyperbolic metric on $h(K)$ is to do it inductively down the dimension of the skeleta. We begin with the $(n-2)$-dimensional pieces X^{n-2}. Transversally to each X^{n-2} (that is, on the union of geodesic segments emanating perpendicularly to X^{n-2}, from a fixed point in X^{n-2}), we have essentially the two-dimensional picture mentioned before. Once we solve this transversal problem, we extend this transversal smoothing by taking a warp product with X^{n-2}; we called this product method *hyperbolic extension* [3]. This gives a smoothing on a (tubular) neighborhood of the piece X^{n-2}. Caveat: we do not want to actually have a smoothing on a neighborhood of the whole of X^{n-2} since we will certainly have matching problems for different X^{n-2} meeting on a common X^{n-3}; so we only want a smoothing on a neighborhood of the Z^{n-2}, where $Z^{n-2} \subset X^{n-2}$ is just a bit "smaller" than X^{n-2}, so that the neighborhoods of the Z^{n-2} are all disjoint. Next step is to smooth around the X^{n-3} (or, specifically, the Z^{n-3}). The metric is already smooth outside a neighborhood of the $(n-3)$-skeleton. Transversally to each X^{n-3}, we have a three-dimensional problem. (It helps to have a three-dimensional picture in mind, like in dimension 2.) It happens that if we did things with care in the first step (around the Z^{n-2}), the metric in the three-dimensional transversal problem would be radially ε-close to hyperbolic outside some large ball B. If this metric was a warped product, we could use the *two-variable warping deformation* given in [3] to extend the metric to a Riemannian metric on the ball B, getting rid, in this way, of the transverse singularity. But the metric in the three-dimensional transversal problem is not warped; hence, the need for the theorem in this paper: we take a radially ε-close to hyperbolic metric and deform it to a warped metric inside a ball, and the resulting metric is still radially η-close to hyperbolic, with an η that can be controlled. Once the transversal three-dimensional problem is solved, we extend this smoothing to neighborhoods of the Z^{n-3} using hyperbolic extension. Next, we do the same for the Z^{n-4}, and so on. About the excess: since warp forcing reduces the excess by 1, we begin with a large excess at codimension 2, so that when we arrive at codimension n, we still have a positive excess; therefore, in the theorem, we should think of the ξ as fixed, whereas of the r_0 as being as large as wanted, ε as small as desired, and the set S as the complement of the ball of radius $r_0 - 1 - \xi$.

In Section 1, we give some definitions and a useful lemma. In Section 2, we give some estimates on changing warping functions. In Section 3, we do warp forcing locally. In Section 4, we prove the theorem.

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1. Preliminaries

Let $A \subset \mathbb{R}^n$ be an open set. Let $|\cdot|_{C^2(A)}$ denote the uniform C^2-norm of $\mathbb{R}^l$-valued functions on A, that is, if $f = (f_1, \dots, f_l) : A \to \mathbb{R}^l$, then $|f|_{C^2(A)} = \sup_{z \in A, 1 \leq i \leq l, 1 \leq j,k \leq n} \{|f_i(z)|, |\partial_j f_i(z)|, |\partial_{j,k} f_i(z)|\}$. Sometimes, we will write $|\cdot|_{C^2} = |\cdot|_{C^2(A)}$ when the context is clear. Given a Riemannian metric g on A, the number $|g|_{C^2(A)}$ is computed considering g as the $\mathbb{R}^{n^2}$-valued function $z \mapsto (g_{ij}(z))$ where, as usual, $g_{ij} = g(e_i, e_j)$, and e_i are the canonical vectors in $\mathbb{R}^n$.

The C^2-norm $|\cdot|_{C^2}$ mentioned in the definition of an ε-close to hyperbolic Riemannian manifold in the Introduction is $|\cdot|_{C^2} = |\cdot|_{C^2(\mathbb{T}_\xi)}$. If (M, g) is ε-close to hyperbolic (or radially ε-close to hyperbolic), then we will also say that the metric g is ε-close to hyperbolic (or radially ε-close to hyperbolic).

Note that for the metric $\sigma = e^{2t}\sigma_{\mathbb{R}^{n-1}} + dt^2$ on our model $\mathbb{T}_\xi$, we have $|\sigma|_{C^2(\mathbb{T}_\xi)} = 4e^{2+2\xi}$.

REMARKS. 1. The definition of radially ε-close to hyperbolic metrics is well suited to studying metrics of the form $g_t + dt^2$ for t large, but for small t, this definition has some drawbacks: (1) we need some space to fit the charts and (2) the form of our specific fixed model $\mathbb{T}_\xi$. An undesired consequence is that the punctured hyperbolic space $\mathbb{H}^n - \{o\} = \mathbb{S}^{n-1} \times \mathbb{R}^+$ (with warped product $\sinh^2(t)\sigma_{\mathbb{S}^{n-1}} + dt^2$) is not radially ε-close to hyperbolic for t small.

2. In [2], we actually need warped metrics with warping functions that are multiples of hyperbolic functions. All these functions are close to the exponential e^t (for t large), so instead of introducing one model for each hyperbolic function, we introduced only the exponential model. In the next section, we show the effect of changing warping functions.

We will need the following lemma.

LEMMA 1.1. *Let g_i be metrics on $\mathbb{T}_\xi$ such that $|g_i - \sigma|_{C^2(\mathbb{T}_\xi)} < \varepsilon_i$ for $i = 1, 2$. Let $\lambda : \mathbb{T}_\xi \to [0, 1]$ be smooth with $|\lambda|_{C^2(\mathbb{T}_\xi)}$ finite. Then*

$$|\lambda g_1 + (1 - \lambda)g_2 - \sigma|_{C^2(\mathbb{T}_\xi)} < 4(1 + |\lambda|_{C^2(\mathbb{T}_\xi)})(\varepsilon_1 + \varepsilon_2).$$

Proof. The proof follows from the triangle inequality, Leibniz rule, and the equality $(\lambda g_1 + (1 - \lambda)g_2) - \sigma = \lambda(g_1 - \sigma) + (1 - \lambda)(g_2 - \sigma)$. This proves the lemma. □

2. Warping with $\sinh t$

The metric of our basic hyperbolic model $\mathbb{T}_\xi$ is an exponentially warped metric. Here we show that we can change the exponential by multiples of $\sinh(t)$ for t large.

In what follows, we will often consider metrics h on $\mathbb{T}_\xi$ of the form $h = h_t + dt^2$. Recall that $I_\xi = (-1 - \xi, 1 + \xi)$.

LEMMA 2.1. *For* $\mathsf{r} \geq 2 + \xi$, *we have* $|e^{-2t}(\frac{\sinh(t+\mathsf{r})}{\sinh(\mathsf{r})})^2 - 1|_{C^2(I_\xi)} < 43e^{2+2\xi}e^{-2\mathsf{r}}$.

Proof. Write $e^{-t}\frac{\sinh(t+\mathsf{r})}{\sinh(\mathsf{r})} - 1 = ((1-e^{-2t})/(1-e^{-2\mathsf{r}}))e^{-2\mathsf{r}}$. Since $\mathsf{r} \geq 2$, we have $1/(1-e^{-2\mathsf{r}}) \leq 1/(1-e^{-4}) < 1.02$. Differentiating $(1/(1-e^{-2\mathsf{r}}))(1-e^{-2t})e^{-2\mathsf{r}}$ twice, together with the previous two facts, gives the following estimate:

$$\begin{aligned}\left|e^{-t}\left(\frac{\sinh(t+\mathsf{r})}{\sinh(\mathsf{r})}\right) - 1\right|_{C^2(I_\xi)} &< (1.02)(4e^{2+2\xi})e^{-2\mathsf{r}} \\ &= 4.08e^{2+2\xi}e^{-2\mathsf{r}}.\end{aligned}$$

This estimate, together with the triangle inequality and the hypothesis $\mathsf{r} \geq 2 + \xi$, gives the following estimate:

$$\begin{aligned}\left|e^{-t}\left(\frac{\sinh(t+\mathsf{r})}{\sinh(\mathsf{r})}\right) + 1\right|_{C^2(I_\xi)} &\leq 2 + 4.08e^{2+2\xi}e^{-2\mathsf{r}} \\ &= 2 + 4.08e^{2+2\xi-2\mathsf{r}} \\ &\leq 2 + 4.08e^{-2} < 2.6.\end{aligned}$$

To prove the lemma, write

$$e^{-2t}\left(\frac{\sinh(t+\mathsf{r})}{\sinh(\mathsf{r})}\right)^2 - 1 = \left(e^{-t}\left(\frac{\sinh(t+\mathsf{r})}{\sinh(\mathsf{r})}\right) - 1\right)\left(e^{-t}\left(\frac{\sinh(t+\mathsf{r})}{\sinh(\mathsf{r})}\right) + 1\right).$$

This, together with the previous two estimates and the Leibniz rule, gives

$$\left|e^{-2t}\left(\frac{\sinh(t+\mathsf{r})}{\sinh(\mathsf{r})}\right)^2 - 1\right|_{C^2(I_\xi)} \leq 4(4.08e^{2+2\xi}e^{-2\mathsf{r}})2.6 < 43e^{2+2\xi}e^{-2\mathsf{r}}.$$

This proves the lemma. □

Let $\nu : I_\xi \to \mathbb{R}^+$ be smooth. For a metric $f = f_t + dt^2$ on $\mathbb{T}_\xi$, we write $f^\nu = \nu f_t + dt^2$.

LEMMA 2.2. *We have* $|f^\nu - f|_{C^2(\mathbb{T}_\xi)} \leq 4|\nu - 1|_{C^2(I_\xi)}|f|_{C^2(\mathbb{T}_\xi)}$.

Proof. Just note that $f^\nu - f = (\nu - 1)f_t$ and differentiate twice. This proves the lemma. □

Recall that the metric on our model $\mathbb{T}_\xi$ is $\sigma = e^{2t}\sigma_{\mathbb{R}^{n-1}} + dt^2$.

LEMMA 2.3. *Let* $f = f_t + dt^2$ *be a metric on* $\mathbb{T}_\xi$ *such that* $|f - \sigma|_{C^2(\mathbb{T}_\xi)} < \varepsilon$. *Let* $\nu = e^{-2t}(\frac{\sinh(t+\mathsf{r})}{\sinh(\mathsf{r})})^2$. *Assume that* $\mathsf{r} \geq 2 + \xi$. *Then*

(1) $|f^\nu - f|_{C^2(\mathbb{T}_\xi)} < 172e^{2+2\xi}(\varepsilon + 4e^{2+2\xi})e^{-2\mathsf{r}}$;
(2) $|f^\nu - \sigma|_{C^2(\mathbb{T}_\xi)} < 688e^{4+4\xi}(\varepsilon + e^{-2\mathsf{r}})$.

Proof. Item 1 follows from Lemmas 2.1, 2.2, and the fact that $|f|_{C^2(\mathbb{T}_\xi)} \leq |f - \sigma|_{C^2(\mathbb{T}_\xi)} + |\sigma|_{C^2(\mathbb{T}_\xi)} < \varepsilon + 4e^{2+2\xi}$. To prove item 2, note that from item 1 and the hypothesis $|f-\sigma|_{C^2(\mathbb{T}_\xi)}<\varepsilon$ it follows that

$$\begin{aligned}|f^\nu-\sigma|_{C^2(\mathbb{T}_\xi)}&\le|f-\sigma|_{C^2(\mathbb{T}_\xi)}+|f^\nu-f|_{C^2(\mathbb{T}_\xi)}\\&<\varepsilon+172e^{2+2\xi}(\varepsilon+4e^{2+2\xi})e^{-2\mathsf{r}}\\&=(1+172e^{2+2\xi-2\mathsf{r}})\varepsilon+172e^{2+2\xi}4e^{2+2\xi}e^{-2\mathsf{r}}\\&<172e^{2+2\xi}4e^{2+2\xi}(\varepsilon+e^{-2\mathsf{r}})\\&=688e^{4+4\xi}(\varepsilon+e^{-2\mathsf{r}}).\end{aligned}$$

This proves the lemma. □

As in Lemma 2.3, let $\nu=e^{-2t}(\frac{\sinh(t+\mathsf{r})}{\sinh(\mathsf{r})})^2$. Lemma 2.1 says that $|\nu-1|_{C^2(I_\xi)}<43e^{2+2\xi}e^{-2\mathsf{r}}$.

Let $s\in I_\xi$. Write $\nu_s(t)=\nu(t-s)$ with ν as before.

LEMMA 2.4. *For $\mathsf{r}\ge 2+\xi$ and $s\in I_\xi$, we have $|\nu_s-1|_{C^2(I_\xi)}<43e^{4+4\xi}e^{-2\mathsf{r}}$.*

Proof. For $t\in I_\xi$, we have $t-s\in I_{1+2\xi}$. This, together with Lemma 2.1, implies $|\nu_s-1|_{C^2(I_\xi)}<43e^{2+2(2\xi+1)}e^{-2\mathsf{r}}$. This proves the lemma. □

The next lemma is similar to Lemma 2.3, with ν_s replacing ν in the conclusion.

LEMMA 2.5. *Let $f=f_t+dt^2$ be a metric on $\mathbb{T}_\xi$ such that $|f-\sigma|_{C^2(\mathbb{T}_\xi)}<\varepsilon$. Let $\nu=e^{-2t}(\frac{\sinh(t+\mathsf{r})}{\sinh(\mathsf{r})})^2$. Assume that $\mathsf{r}\ge 2+\xi$. Then*

(1) $|f^{\nu_s}-f|_{C^2(\mathbb{T}_\xi)}<172e^{4+4\xi}(\varepsilon+4e^{2+2\xi})e^{-2\mathsf{r}}$;

(2) $|f^{\nu_s}-\sigma|_{C^2(\mathbb{T}_\xi)}<688e^{6+6\xi}(\varepsilon+e^{-2\mathsf{r}})$.

The proof is the same as that of Lemma 2.3 but uses Lemma 2.4 instead of Lemma 2.1.

3. Local Warp Forcing

Here we give a kind of a local version to warp forcing.

Let a be a metric on $\mathbb{B}^{n-1}$. For a fixed $s\in I_\xi$, we denote by $\underline{a}_s$ the warped metric $e^{2(t-s)}a+dt^2$ on $\mathbb{T}_\xi=\mathbb{B}^{n-1}\times I_\xi$.

LEMMA 3.1. *Let $s\in I_\xi$, and let a,b be metrics on $\mathbb{B}^{n-1}$ with $|a-b|_{C^2(\mathbb{B}^{n-1})}<\varepsilon$. Then $|\underline{a}_s-\underline{b}_s|_{C^2(\mathbb{T}_\xi)}<16e^{4+4\xi}\varepsilon$.*

Proof. Just compute the derivatives of $\underline{a}_s-\underline{b}_s=e^{2(t-s)}(a-b)$. This proves the lemma. □

The next lemma gives local estimates (that is, on the model $\mathbb{T}_\xi$) needed for global warp forcing estimates.

LEMMA 3.2. *Let $h = h_t + dt^2$ be a metric on $\mathbb{T}_\xi$ with $|h - \sigma|_{C^2(\mathbb{T}_\xi)} < \varepsilon$. Fix $s \in I_\xi$ and consider the warped-by-exponential metric $\underline{h_s}_s = e^{2(t-s)} h_s + dt^2$ on $\mathbb{T}_\xi$. Then $|\underline{h_s}_s - \sigma|_{C^2(\mathbb{T}_\xi)} < 16e^{4+4\xi}\varepsilon$.*

Proof. By hypothesis we have $|(h_t + dt^2) - (e^{2t}\sigma_{\mathbb{R}^{n-1}} + dt^2)|_{C^2(\mathbb{T}_\xi)} < \varepsilon$. Therefore, taking $t = s$, we get $|h_s - e^{2s}\sigma_{\mathbb{R}^{n-1}}|_{C^2(\mathbb{B}^{n-1})} < \varepsilon$. Note that $\underline{e^{2s}\sigma_{\mathbb{R}^{n-1}}}_s = e^{2t}\sigma_{\mathbb{R}^{n-1}} + dt^2 = \sigma$. This, together with Lemma 3.1, implies that $|\underline{h_s}_s - \sigma|_{C^2(\mathbb{T}_\xi)} < 16e^{4+4\xi}\varepsilon$. This completes the proof of Lemma 3.2. □

4. Proof of the Theorem

Let (M^n, g) be a complete Riemannian manifold with center $o \in M$. Recall that we can write the metric on $M - \{o\} = \mathbb{S}^{n-1} \times \mathbb{R}^+$ as $g = g_r + dr^2$. Also, B_r is the closed ball on M of radius r centered at the center o. Let $r_0 \geq 3 + 2\xi$. We assume that g is radially ε-close to hyperbolic on some $S \subset M$, with charts of excess ξ. We have to prove that $\mathcal{W}_{r_0} g$ is radially η-close to hyperbolic on $S - B_{r_0-1-\xi}$, with charts of excess $\xi - 1$, where $\eta = e^{27+12\xi}(e^{-2r_0} + \varepsilon)$.

Assume that $p = (x, \mathsf{r}) \in S \subset \mathbb{S}^{n-1} \times \mathbb{R}_+ = M - \{o\}$ and $p \notin \bar{B}_{r_0-(1+\xi)}$ (equivalently, $\mathsf{r} > r_0 - (1 + \xi)$). Since the metric g is radially ε-close to hyperbolic on S, with charts of excess ξ, there is a radially ε-close to hyperbolic chart $\phi : \mathbb{T}_\xi \to M$ centered at p. This means that $\phi(0, 0) = p$, ϕ is *radial,* and $|\phi^* g - \sigma|_{C^2(\mathbb{T}_\xi)} < \varepsilon$. Here by *radial* we mean that ϕ respects product structures (see the definition of a radially ε-close to hyperbolic chart in the Introduction). To prove the theorem, we will prove that the restriction $\phi|_{\mathbb{T}_{\xi-1}} : \mathbb{T}_{\xi-1} \to M$ is a radially η-close to hyperbolic chart for $\mathcal{W}_{r_0}$ centered at p. That is, we will show that $|\phi^*(\mathcal{W}_{r_0} g) - \sigma|_{C^2(\mathbb{T}_{\xi-1})} < \eta$. We have three cases.

First case. $p \notin B_{r_0+1/2+(1+\xi)}$.

Then the image of ϕ lies outside $B_{r_0+1/2}$. By (0.1) we have that $\mathcal{W}_{r_0} = g$ outside $B_{r_0+1/2}$. Hence, the chart ϕ is also a radially ε-close to hyperbolic chart for $\mathcal{W}_{r_0} g$ centered at p with excess ξ. This shows that the metric $\mathcal{W}_{r_0} g$ is radially ε-close to hyperbolic outside $B_{r_0+1/2+(1+\xi)}$, with charts of excess ξ.

Second case. $p \in B_{r_0+1/2+(1+\xi)} - B_{r_0+1/2+\xi}$.

Then the image of the restriction $\phi|_{\mathbb{T}_{\xi-1}}$ of ϕ to $\mathbb{T}_{\xi-1}$ does not intersect $B_{r_0+1/2}$. Hence, as in the first case, by (0.1) the chart $\phi|_{\mathbb{T}_{\xi-1}}$ is an ε-close to hyperbolic chart for $\mathcal{W}_{r_0} g$, centered at p, but with excess $\xi - 1$. Clearly, $\phi|_{\mathbb{T}_{\xi-1}}$ is also radial. This shows that the metric $\mathcal{W}_{r_0} g$ is radially ε-close to hyperbolic on $B_{r_0+1/2+(1+\xi)} - B_{r_0+1/2+\xi}$, with charts of excess $\xi - 1$.

Third case. $p \in B_{r_0+1/2+\xi}$.

The condition $p \in B_{r_0+1/2+\xi}$ is equivalent to $\mathsf{r} < r_0 + \frac{1}{2} + \xi$. Since by hypothesis $p \notin B_{r_0-1-\xi}$, we get $r_0 - (1 + \xi) < \mathsf{r} < r_0 + \frac{1}{2} + \xi$. Recall that $\phi : \mathbb{T}_\xi \to M$ is a radially ε-close to hyperbolic chart of g centered at $p = (x, \mathsf{r})$. Write $h = \phi^* g$. Since ϕ is radial, we have that h has the form $h = h_t + dt^2$ with $h_t = \phi^* g_{t+\mathsf{r}}$.

Moreover,

$$|h-\sigma|_{C^2(\mathbb{T}_\xi)} < \varepsilon. \tag{1}$$

Write $s = r_0 - \mathsf{r}$. Then $-\frac{1}{2} - \xi < s < 1 + \xi$. In particular, we have $s \in I_\xi$. Also, $h_s = \phi^* g_{r_0}$. Recall that in the Introduction we defined the warped product $\bar{g}_{r_0}$ as $\bar{g}_{r_0} = \sinh^2(r)(1/\sinh^2(r_0))g_{r_0} + dr^2$. Since ϕ is radial, we have $\phi^*(\bar{g}_{r_0}) = \sinh^2(t+\mathsf{r})(1/\sinh^2(r_0))\phi^* g_{r_0} + dt^2$. Therefore,

$$\phi^*(\bar{g}_{r_0}) = \sinh^2((t-s)+r_0)\left(\frac{1}{\sinh^2(r_0)}\right)h_s + dt^2. \tag{2}$$

Note that $e^{2(t-s)}\nu_s(t) = \sinh^2((t-s)+r_0)/\sinh^2(r_0)$, where $\nu(t) = e^{-2t}\sinh^2(t+r_0)/\sinh^2(r_0)$ and $\nu_s(t) = \nu(t-s)$, as in Section 2. Using this and the notation in Sections 2 and 3, equation (2) can be rewritten as

$$\phi^*(\bar{g}_{r_0}) = f^{\nu_s}, \tag{3}$$

where $f = \underline{h_s}_s$. Equation (1) and Lemma 3.2 imply that $|f-\sigma|_{C^2(\mathbb{T}_\xi)} < 16e^{4+4\xi}\varepsilon$. This, together with the second item of Lemma 2.5, implies

$$|f^{\nu_s} - \sigma|_{C^2(\mathbb{T}_\xi)} < 688e^{6+6\xi}(e^{-2\mathsf{r}} + 16e^{4+4\xi}\varepsilon). \tag{4}$$

(To apply Lemma 2.5, we need the condition $\mathsf{r} \geq 2+\xi$. This follows from $\mathsf{r} > r_0 - (1+\xi)$ and the hypothesis $r_0 \geq 3 + 2\xi$.)

From the definition of $\mathcal{W}_{r_0}g$ given in the Introduction and the fact that ϕ is radial we have

$$\phi^*(\mathcal{W}_{r_0}g) = \rho_s\phi^*(\bar{g}_{r_0}) + (1-\rho_s)\phi^* g = \rho_s f^{\nu_s} + (1-\rho_s)h, \tag{5}$$

where $\rho_s(t) = \rho(2t-2s)$ with ρ as in the Introduction. From (5), (4), (1), and Lemma 1.1 we get that $|\phi^*(\mathcal{W}_{r_0}g) - \sigma|_{C^2(\mathbb{T}_\xi)} < \varepsilon'$ with

$$\varepsilon' = 4(1+|\rho_s|_{C^2(I_\xi)})(\varepsilon + 688e^{6+6\xi}(e^{-2\mathsf{r}} + 16e^{4+4\xi}\varepsilon)).$$

Note that

$$\begin{aligned}\varepsilon' &< 4(1+|\rho_s|_{C^2(I_\xi)})[1+688e^{6+6\xi}16e^{4+4\xi}](e^{-2\mathsf{r}}+\varepsilon)\\ &< 44{,}033(1+|\rho_s|_{C^2(I_\xi)})e^{10+10\xi}(e^{-2\mathsf{r}}+\varepsilon).\end{aligned}$$

A calculation shows that we can take $|\rho_s(*)|_{C^2(I_\xi)} = |\rho(2*)|_{C^2(\mathbb{R})} < 48$. This implies that we can take $\varepsilon' < (44{,}033)(49)e^{10+10\xi}(e^{-2\mathsf{r}}+\varepsilon)$. This, together with $\mathsf{r} > r_0 - (1+\xi)$, implies that we can take $\varepsilon' < (44{,}033)(49)e^{12+12\xi}(e^{-2r_0}+\varepsilon) = 2{,}157{,}617e^{12+12\xi}(e^{-2r_0}+\varepsilon) < e^{27+12\xi}(e^{-2r_0}+\varepsilon)$. Note that the excess of the charts in this third case is also ξ. This proves the theorem.

References

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