Michigan Math. J. 65 (2016), 199–224

Seidel Elements and Mirror Transformations for Toric Stacks

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Abstract. We give a precise relation between the mirror transformation and the Seidel elements for weak Fano toric Deligne–Mumford stacks. Our result generalizes the corresponding result for toric varieties proved by González and Iritani [5]. The correction coefficients that we computed match with the instanton corrections from genus 0 open Gromov–Witten invariants for toric Calabi–Yau orbifolds in [3].

1. Introduction

In [5], González and Iritani gave a precise relation between the mirror map and the Seidel elements for a smooth projective weak Fano toric variety X. The goal of this paper is to generalize the main theorem of [5] to a smooth projective weak Fano toric Deligne–Mumford stack $\mathcal{X}$.

Let $\mathcal{X}$ be a smooth projective weak Fano toric Deligne–Mumford stack. The mirror theorem can be stated as an equality between the I-function and the J-function via a change of coordinates, called mirror map (or mirror transformation). We refer to [4] and Section 4.1 of [6] for further discussions.

Let Y be a monotone symplectic manifold. For a loop λ in the group of Hamiltonian symplectomorphisms on Y, Seidel [10] constructed an invertible element $S(\lambda)$ in (small) quantum cohomology counting sections of the associated Hamiltonian Y-bundle $E_\lambda \to \mathbb{P}^1$. The Seidel element $S(\lambda)$ defines an element in $\mathrm{Aut}(QH(Y))$ via quantum multiplication, and the map $\lambda \mapsto S(\lambda)$ gives a representation of $\pi_1(\mathrm{Ham}(Y))$ on $QH(Y)$. McDuff and Tolman [9] extended this construction to all symplectic manifolds. The definition of Seidel representation and Seidel element were extended to symplectic orbifolds by Tseng and Wang [11].

Let $D_1, \ldots, D_m$ be the classes in $H^2(X)$ Poincaré dual to the toric divisors. When the loop λ is a circle action, McDuff and Tolman [9] considered the Seidel element $\tilde{S}_j$ associated to an action λ_j that fixes the toric divisor D_j. Given a circle action on X (resp. $\mathcal{X}$), the Seidel element in [5] (resp. [11]) is defined using the small quantum cohomology ring. In this paper, we need to define it, for smooth projective Deligne–Mumford stack, with deformed quantum cohomology to include the bulk deformations. For weak Fano toric Deligne–Mumford stack, the mirror theorem in [6] shows that the mirror map $\tau(y) \in H^{\leq 2}_{\mathrm{orb}}(\mathcal{X})$; therefore, we will only need bulk deformations with $\tau \in H^{\leq 2}_{\mathrm{orb}}(\mathcal{X})$.

Received February 1, 2015. Revision received May 13, 2015.

We consider the Seidel element $\tilde{S}_j$ associated to the toric divisor D_j and the Seidel element $\tilde{S}_{m+j}$ corresponding to the box element s_j. The Seidel element in Definition 2.3 shows that $S = q_0\tilde{S}$ is a pull-back of a coefficient of the J-function $J_{\mathcal{E}_j}$ of the associated orbifiber bundle $\mathcal{E}_j$; hence, we can use the mirror theorem for $\mathcal{E}_j$ to calculate $\tilde{S}_j$ when $\mathcal{E}_j$ is weak Fano.

We extend the definition of the Batyrev element $\tilde{D}_j$ to weak Fano toric Deligne–Mumford stacks via partial derivatives of the mirror map $\tau(y)$. As analogues of the Seidel elements in B-model, the Batyrev elements can be explicitly computed from the I-function of $\mathcal{X}$. The following theorem states that the Seidel and Batyrev elements only differ by a multiplication of a correction function.

THEOREM 1.1. *Let X be a smooth projective toric Deligne–Mumford stack with $\rho^S \in \mathrm{cl}(C^S(\mathcal{X}))$.*

(i) *The Seidel element $\tilde{S}_j$ associated to the toric divisor D_j is given by*

$$\tilde{S}_j(\tau(y)) = \exp(-g_0^{(j)}(y))\tilde{D}_j(y),$$

where $\tau(y)$ is the mirror map of $\mathcal{X}$, and the function $g_0^{(j)}$ is given explicitly in (40);

(ii) *The Seidel element $\tilde{S}_{m+j}$ corresponding to the box element s_j is given by*

$$\tilde{S}_{m+j}(\tau(y)) = \exp(-g_0^{(m+j)})y^{-D_{m+j}^{S\vee}}\tilde{D}_{m+j}(y),$$

where $\tau(y)$ is the mirror map of $\mathcal{X}$, and the function $g_0^{(m+j)}$ is given explicitly in (51).

It appears that the correction coefficients in the theorem coincide with the instanton corrections in Theorem 1.4 in [3]. This phenomenon also indicates that the deformed quantum cohomology ring of the toric Deligne–Mumford stack $\mathcal{X}$ is isomorphic to the Batyrev ring given in [6].

2. Seidel Elements and J-Functions

2.1. Seidel Elements

In this section, we fix our notation and construct the Seidel elements of smooth projective Deligne–Mumford stacks using τ-deformed quantum cohomology.

Let $\mathcal{X}$ be a smooth projective Deligne–Mumford stack equipped with a $\mathbb{C}^\times$ action.

DEFINITION 2.1. The associated orbifiber bundle of the $\mathbb{C}^\times$-action is the $\mathcal{X}$-bundle over $\mathbb{P}^1$

$$\mathcal{E} := \mathcal{X} \times (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times \to \mathbb{P}^1,$$

where $\mathbb{C}^\times$ acts on $\mathbb{C}^2 \setminus \{0\}$ via the standard diagonal action.

Let $\phi_1, \dots, \phi_N$ be a basis for the orbifold cohomology ring $H^*_{\mathrm{orb}}(\mathcal{X}) := H^*(\mathcal{I}\mathcal{X}; \mathbb{Q})$ of $\mathcal{X}$, where $\mathcal{I}\mathcal{X}$ is the inertia stack of $\mathcal{X}$. Let $\phi^1, \dots, \phi^N$ be the dual basis of $\phi_1, \dots, \phi_N$ with respect to the orbifold Poincaré pairing. Furthermore, let $\hat{\phi}_1, \dots, \hat{\phi}_M$ denote a basis for the orbifold cohomology $H^*_{\mathrm{orb}}(\mathcal{E}) := H^*(\mathcal{IE}; \mathbb{Q})$ of $\mathcal{E}$. Let $\hat{\phi}^1, \dots, \hat{\phi}^M$ be the dual basis of $\hat{\phi}_1, \dots, \hat{\phi}_M$ with respect to the orbifold Poincaré pairing.

We will use X to denote the coarse moduli space of $\mathcal{X}$ and use E to denote the coarse moduli space of $\mathcal{E}$. Then the $\mathbb{C}^\times$ action on $\mathcal{X}$ descends to the $\mathbb{C}^\times$ action on X with E being the associated bundle. Following [8] and [5], there is a (noncanonical) splitting

$$H^*(\mathcal{E}; \mathbb{Q}) \cong H^*(E; \mathbb{Q}) \cong H^*(X; \mathbb{Q}) \otimes H^*(\mathbb{P}^1; \mathbb{Q}) \cong H^*(\mathcal{X}; \mathbb{Q}) \otimes H^*(\mathbb{P}^1; \mathbb{Q}).$$

According to [5], there is a unique $\mathbb{C}^\times$-fixed component $F_{\max} \subset X^{\mathbb{C}^\times}$ such that the normal bundle of $F_{\max}$ has only negative $\mathbb{C}^\times$-weights. Let σ_0 be the section associated to a fixed point in $F_{\max}$. Following [5], there is a splitting defined by this maximal section,

$$\begin{aligned} H_2(\mathcal{E}; \mathbb{Z})/tors &\cong H_2(E; \mathbb{Z})/tors \cong \mathbb{Z}[\sigma_0] \oplus (H_2(X, \mathbb{Z})/tors) \\ &\cong \mathbb{Z}[\sigma_0] \oplus (H_2(\mathcal{X}, \mathbb{Z})/tors). \end{aligned} \tag{1}$$

Let $\mathrm{NE}(X) \subset H_2(X; \mathbb{R})$ denote the Mori cone, that is, the cone generated by effective curves, and set

$$\mathrm{NE}(X)_{\mathbb{Z}} := \mathrm{NE}(X) \cap (H_2(X, \mathbb{Z})/tors).$$

Then, by Lemma 2.2 of [5] we have

$$\mathrm{NE}(E)_{\mathbb{Z}} = \mathbb{Z}_{\geq 0}[\sigma_0] + \mathrm{NE}(X)_{\mathbb{Z}}. \tag{2}$$

Let $H_2^{\sec}(E; \mathbb{Z})$ be the affine subspace of $H_2(E, \mathbb{Z})/tors$ that consists of the classes that project to the positive generator of $H_2(\mathbb{P}^1; \mathbb{Z})$. Setting

$$\mathrm{NE}(E)_{\mathbb{Z}}^{\sec} := \mathrm{NE}(E)_{\mathbb{Z}} \cap H_2^{\sec}(E; \mathbb{Z}),$$

we obtain

$$\mathrm{NE}(E)_{\mathbb{Z}}^{\sec} = [\sigma_0] + \mathrm{NE}(X)_{\mathbb{Z}}. \tag{3}$$

We choose a nef integral basis $\{p_1, \dots, p_r\}$ of $H^2(\mathcal{X}; \mathbb{Q})$; then there are unique lifts of $p_1, \dots, p_r$ in $H^2(\mathcal{E}; \mathbb{Q})$ that vanish on $[\sigma_0]$. By abuse of notation, we also denote these lifts as $p_1, \dots, p_r$; these lifts are also nef. Let p_0 be the pullback of the positive generator of $H^2(\mathbb{P}^1; \mathbb{Z})$ in $H^2(\mathcal{E}; \mathbb{Q})$. Therefore, $\{p_0, p_1, \dots, p_r\}$ is an integral basis of $H^2(\mathcal{E}; \mathbb{Q})$.

Let $q_0, q_1, \dots, q_r$ be the Novikov variables of $\mathcal{E}$ dual to $p_0, p_1, \dots, p_r$, and $q_1, \dots, q_r$ be the Novikov variables of $\mathcal{X}$ dual to $p_1, \dots, p_r$. We denote the Novikov ring of $\mathcal{X}$ and the Novikov ring of $\mathcal{E}$ by

$$\Lambda_{\mathcal{X}} := \mathbb{Q}[[q_1, \dots, q_r]] \quad \text{and} \quad \Lambda_{\mathcal{E}} := \mathbb{Q}[[q_0, q_1, \dots, q_r]],$$

respectively. For each $d \in \mathrm{NE}(X)_{\mathbb{Z}}$, we write

$$q^d := q_1^{\langle p_1, d\rangle} \cdots q_r^{\langle p_r, d\rangle} \in \Lambda_{\mathcal{X}},$$

and for each $\beta \in \mathrm{NE}(E)_{\mathbb{Z}}$, we write

$$q^\beta := q_0^{\langle p_0, \beta\rangle} q_1^{\langle p_1, \beta\rangle} \cdots q_r^{\langle p_r, \beta\rangle} \in \Lambda_{\mathcal{E}}.$$

The τ-deformed orbifold quantum product is defined as follows:

$$\alpha \bullet_\tau \beta = \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}}} \sum_{l \geq 0} \sum_{k=1}^{N} \frac{1}{l!} \langle \alpha, \beta, \tau, \ldots, \tau, \phi_k \rangle^{\mathcal{X}}_{0,l+3,d} q^d \phi^k; \tag{4}$$

the associated quantum cohomology ring is denoted by

$$QH_\tau(\mathcal{X}) := (H(\mathcal{X}) \otimes_{\mathbb{Q}} \Lambda_{\mathcal{X}}, \bullet_\tau).$$

NOTATION 2.2. For a smooth projective Deligne–Mumford stack $\mathcal{X}$, we denote by $H^{\leq 2}_{tw}(\mathcal{X})$ the complementary subspace of $H^2(\mathcal{X})$ in $H^{\leq 2}_{\mathrm{orb}}(\mathcal{X})$ supported on the twisted sectors, that is, we have the decomposition

$$H^{\leq 2}_{\mathrm{orb}}(\mathcal{X}) = H^2(\mathcal{X}) \oplus H^{\leq 2}_{tw}(\mathcal{X}).$$

DEFINITION 2.3. The Seidel element of $\mathcal{X}$ is the class

$$S(\hat{\tau}) := \sum_{\alpha} \sum_{\beta \in \mathrm{NE}(E)^{\mathrm{sec}}_{\mathbb{Z}}} \sum_{l \geq 0} \frac{1}{l!} \langle \mathbf{1}, \hat{\tau}_{tw}, \ldots, \hat{\tau}_{tw}, \iota_* \phi_\alpha \psi \rangle^{\mathcal{E}}_{0,l+2,\beta} \phi^\alpha e^{\langle \hat{\tau}_{0,2}, \beta \rangle} \tag{5}$$

in $QH_\tau(\mathcal{X}) \otimes_{\Lambda_{\mathcal{X}}} \Lambda_{\mathcal{E}}$. Here $\iota : \mathcal{X} \to \mathcal{E}$ is the inclusion of a fiber, and

$$\iota_* : H^*(\mathcal{I}\mathcal{X}; \mathbb{Q}) \to H^{*+2}(\mathcal{I}\mathcal{E}; \mathbb{Q})$$

is the Gysin map. Moreover,

$$e^{\langle \hat{\tau}_{0,2}, \beta \rangle} = q^\beta = q_0^{\langle p_0, \beta \rangle} \cdots q_r^{\langle p_r, \beta \rangle},$$

where

$$\hat{\tau}_{0,2} = \sum_{a=0}^{r} p_a \log q_a \in H^2(\mathcal{E})$$

and

$$\hat{\tau} = \hat{\tau}_{0,2} + \hat{\tau}_{tw} \in H^2(\mathcal{E}) \oplus H^{\leq 2}_{tw}(\mathcal{E}) = H^{\leq 2}_{\mathrm{orb}}(\mathcal{E}).$$

The Seidel element can be factorized as

$$S(\hat{\tau}) = q_0 \tilde{S}(\hat{\tau}) \quad \text{with } \tilde{S}(\hat{\tau}) \in QH_\tau(\mathcal{X}). \tag{6}$$

REMARK 2.4. The descendant class ψ in equation (5) can be eliminated by the string equation; hence, our definition of Seidel elements matches with the definition in [5]

2.2. *J-Functions*

We will explain the relation between the Seidel element and the J-function of the associated bundle $\mathcal{E}$.

Definition 2.5. The J-function of $\mathcal{E}$ is the cohomology-valued function

$$J_{\mathcal{E}}(\hat{\tau}, z) = e^{\hat{\tau}_{0,2}/z}\Bigg(1 + \sum_{\alpha} \sum_{(\beta,l)\neq(0,0),\beta\in\mathrm{NE}(E)_{\mathbb{Z}}} \frac{e^{\langle \hat{\tau}_{0,2},\beta\rangle}}{l!} \times \left\langle \mathbf{1}, \hat{\tau}_{tw}, \ldots, \hat{\tau}_{tw}, \frac{\hat{\phi}_\alpha}{z-\psi} \right\rangle^{\mathcal{E}}_{0,l+2,\beta} \hat{\phi}^\alpha \Bigg), \tag{7}$$

where $\hat{\phi}_\alpha/(z-\psi) = \sum_{n\geq 0} z^{-1-n}\hat{\phi}_\alpha \psi^n$.

Note that when $n = 0$, we have

(i) $\sum_\alpha \langle \mathbf{1}, \hat{\tau}_{tw}, \ldots, \hat{\tau}_{tw}, \hat{\phi}_\alpha\rangle^{\mathcal{E}}_{0,l+2,\beta}\hat{\phi}^\alpha = 0$ for $(l,\beta) \neq (1,0)$;

(ii) $\sum_\alpha \langle \mathbf{1}, \hat{\tau}_{tw}, \ldots, \hat{\tau}_{tw}, \hat{\phi}_\alpha\rangle^{\mathcal{E}}_{0,l+2,\beta}\hat{\phi}^\alpha = \hat{\tau}_{tw}$ for $(l,\beta) = (1,0)$.

The J-function can be expanded in terms of powers of z^{-1} as follows:

$$J_{\mathcal{E}}(\hat{\tau}, z) = e^{\sum_{a=0}^{r} p_a \log q_a/z}\Bigg(1 + z^{-1}\hat{\tau}_{tw} + z^{-2}\sum_{n=0}^{\infty} F_n(q_1, \ldots, q_r; \hat{\tau}) q_0^n + O(z^{-3})\Bigg), \tag{8}$$

where

$$F_n(q_1, \ldots, q_r; \hat{\tau}) = \sum_{\alpha=1}^{M} \sum_{d\in\mathrm{NE}(X)_{\mathbb{Z}}} \sum_{l\geq 0} \frac{1}{l!} \langle \mathbf{1}, \hat{\tau}_{tw}, \ldots, \hat{\tau}_{tw}, \hat{\phi}_\alpha \psi\rangle^{\mathcal{E}}_{0,l+2,d+n\sigma_0} q^d \hat{\phi}^\alpha. \tag{9}$$

Proposition 2.6. *The Seidel element corresponding to the $\mathbb{C}^\times$ action on $\mathcal{X}$ is given by*

$$S(\hat{\tau}) = \iota^*(F_1(q_1, \ldots, q_r; \hat{\tau})q_0). \tag{10}$$

Proof. The proof here is identical to the proof given in Proposition 2.5 of [5] for smooth projective varieties:

Using the duality identity

$$\sum_{\alpha=1}^{M} \hat{\phi}_\alpha \otimes \iota^*\hat{\phi}^\alpha = \sum_{\alpha=1}^{N} \iota_*\phi_\alpha \otimes \phi^\alpha,$$

we can see that

$$\iota^* F_1(q_1, \ldots, q_r; \hat{\tau}) = \sum_{\alpha=1}^{N} \sum_{d\in\mathrm{NE}(X)_{\mathbb{Z}}} \sum_{l\geq 0} \frac{1}{l!} \langle \mathbf{1}, \hat{\tau}_{tw}, \ldots, \hat{\tau}_{tw}, \iota_*\phi_\alpha \psi\rangle^{\mathcal{E}}_{0,l+2,d+\sigma_0} q^d \phi^\alpha.$$

Hence, the conclusion follows, that is,

$$S(\hat{\tau}) = \iota^*(F_1(q_1, \ldots, q_r; \hat{\tau})q_0). \qquad \square$$

3. Seidel Elements Corresponding to Toric Divisors

3.1. A Review of Toric Deligne–Mumford Stacks

In this section, we define toric Deligne–Mumford stacks following the construction of [2] and [6].

A toric Deligne–Mumford stack is defined by a stacky fan $\mathbf{\Sigma} = (\mathbf{N}, \Sigma, \beta)$, where $\mathbf{N}$ is a finitely generated Abelian group, $\Sigma \subset \mathbf{N}_{\mathbb{Q}} = \mathbf{N} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a rational simplicial fan, and $\beta : \mathbb{Z}^m \to \mathbf{N}$ is a homomorphism. We assume that β has finite cokernel and the rank of $\mathbf{N}$ is n. The map $\mathbb{Z}^m \xrightarrow{\beta} \mathbf{N} \to \mathbf{N}_{\mathbb{Q}}$ generates the 1-skeleton of the fan Σ. Let b_i be the image under β of the standard basis of $\mathbb{Z}^m$, and $\bar{b_i}$ be the image of b_i under the canonical map $\mathbf{N} \to \mathbf{N}_{\mathbb{Q}}$. Let $\mathbb{L} \subset \mathbb{Z}^m$ be the kernel of β. Then the fan sequence is the following exact sequence:

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^m \xrightarrow{\beta} \mathbf{N}. \tag{11}$$

Let $\beta^\vee : (\mathbb{Z}^*)^m \to \mathbb{L}^\vee$ be the Gale dual of β in [2], where $\mathbb{L}^\vee := H^1(\mathrm{Cone}(\beta)^*)$ is an extension of $\mathbb{L}^* = \mathrm{Hom}(\mathbb{L}, \mathbb{Z})$ by a torsion subgroup. The divisor sequence is the following exact sequence:

$$0 \longrightarrow \mathbf{N}^* \xrightarrow{\beta^*} (\mathbb{Z}^*)^m \xrightarrow{\beta^\vee} \mathbb{L}^\vee. \tag{12}$$

By applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ to the dual map $\beta^\vee$ we have a homomorphism

$$\alpha : G \to (\mathbb{C}^\times)^m, \quad \text{where } G := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{L}^\vee, \mathbb{C}^\times),$$

and we let G act on $\mathbb{C}^m$ via this homomorphism.

The collection of anticones $\mathcal{A}$ is defined as follows:

$$\mathcal{A} := \left\{ I : \sum_{i \notin I} \mathbb{R}_{\geq 0} \bar{b}_i \in \Sigma \right\}.$$

Let $\mathcal{U}$ denote the open subset of $\mathbb{C}^m$ defined by $\mathcal{A}$:

$$\mathcal{U} := \mathbb{C}^m \setminus \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I,$$

where

$$\mathbb{C}^I = \{(z_1, \dots, z_m) : z_i = 0 \text{ for } i \notin I\}.$$

DEFINITION 3.1. Following [6], the toric Deligne–Mumford stack $\mathcal{X}$ is defined as the quotient stack

$$\mathcal{X} := [\mathcal{U}/G].$$

REMARK 3.2. The toric variety X associated to the fan Σ is the coarse moduli space of $\mathcal{X}$ ([2]).

DEFINITION 3.3 ([6]). Given a stacky fan $\mathbf{\Sigma} = (\mathbf{N}, \Sigma, \beta)$, we define the set of box elements $\mathrm{Box}(\mathbf{\Sigma})$ as follows:

$$\mathrm{Box}(\mathbf{\Sigma}) =: \left\{ v \in \mathbf{N} : \bar{v} = \sum_{k \notin I} c_k \bar{b}_k \text{ for some } 0 \leq c_k < 1, I \in \mathcal{A} \right\}.$$

We assume that Σ is complete; then the connected components of the inertia stack $\mathcal{IX}$ are indexed by the elements of $\mathrm{Box}(\mathbf{\Sigma})$ (see [2]). Moreover, given $v \in \mathrm{Box}(\mathbf{\Sigma})$, the age of the corresponding connected component of $\mathcal{IX}$ is defined by $\mathrm{age}(v) := \sum_{k \notin I} c_k$.

The Picard group $\mathrm{Pic}(\mathcal{X})$ of $\mathcal{X}$ can be identified with the character group $\mathrm{Hom}(G, \mathbb{C}^\times)$. Hence,

$$\mathbb{L}^\vee = \mathrm{Hom}(G, \mathbb{C}^\times) \cong \mathrm{Pic}(\mathcal{X}) \cong H^2(\mathcal{X}; \mathbb{Z}). \tag{13}$$

We can also use the extended stacky fans introduced by Jiang [7] to define the toric Deligne–Mumford stacks. Given a stacky fan $\mathbf{\Sigma} = (\mathbf{N}, \Sigma, \beta)$ and a finite set

$$S = \{s_1, \dots, s_l\} \subset \mathbf{N}_\Sigma := \{c \in \mathbf{N} : \bar{c} \in |\Sigma|\},$$

the S-extended stacky fan is given by $(\mathbf{N}, \Sigma, \beta^S)$, where $\beta^S : \mathbb{Z}^{m+l} \to \mathbf{N}$ is defined by

$$\beta^S(e_i) = \begin{cases} b_i, & 1 \le i \le m; \\ s_{i-m}, & m+1 \le i \le m+l. \end{cases} \tag{14}$$

Let $\mathbb{L}^S$ be the kernel of $\beta^S : \mathbb{Z}^{m+l} \to \mathbf{N}$. Then we have the following S-extended fan sequence:

$$0 \longrightarrow \mathbb{L}^S \longrightarrow \mathbb{Z}^{m+l} \xrightarrow{\beta^S} \mathbf{N}. \tag{15}$$

By the Gale duality we have the S-extended divisor sequence

$$0 \longrightarrow \mathbf{N}^* \xrightarrow{\beta^*} (\mathbb{Z}^*)^{m+l} \xrightarrow{\beta^{S\vee}} \mathbb{L}^{S\vee}, \tag{16}$$

where $\mathbb{L}^{S\vee} := H^1(\mathrm{Cone}(\beta^S)^*)$.

Assumption 3.4. *In the rest of the paper, we assume that the set*

$$\{v \in \mathrm{Box}(\mathbf{\Sigma}); \mathrm{age}(v) \le 1\} \cup \{b_1, \dots, b_m\}$$

generates $\mathbf{N}$ *over* $\mathbb{Z}$, *and we choose the set*

$$S = \{s_1, \dots, s_l\} \subset \mathrm{Box}(\mathbf{\Sigma})$$

such that the set $\{b_1, \dots, b_m, s_1, \dots, s_l\}$ *generates* $\mathbf{N}$ *over* $\mathbb{Z}$ *and* $\mathrm{age}(s_j) \le 1$ *for* $1 \le j \le l$.

Let D_i^S be the image of the standard basis of $(\mathbb{Z}^*)^{m+l}$ under the map $\beta^{S\vee}$. Then there is a canonical isomorphism

$$\mathbb{L}^{S\vee} \otimes \mathbb{Q} \cong (\mathbb{L}^\vee \otimes \mathbb{Q}) \oplus \bigoplus_{i=m+1}^{m+l} \mathbb{Q} D_i^S, \tag{17}$$

which can be constructed as follows [6].

Since Σ is complete, for $m < j \le m+l$, the box element s_{j-m} is contained in some cone in Σ. Namely,

$$s_{j-m} = \sum_{i \notin I_j^S} c_{ji} b_i \quad \text{in } \mathbf{N} \otimes \mathbb{Q}, c_{ji} \ge 0, \exists I_j^S \in \mathcal{A}^S,$$

where I_j^S is the "anticone" of the cone containing s_{j-m}.

By the S-extended fan sequence (15) tensored with $\mathbb{Q}$, we have the following short exact sequence:

$$0 \longrightarrow \mathbb{L}^S \otimes \mathbb{Q} \longrightarrow \mathbb{Q}^{m+l} \xrightarrow{\beta^S} \mathbf{N} \otimes \mathbb{Q} \longrightarrow 0.$$

Hence, there exists a unique $D_j^{S\vee} \in \mathbb{L}^S \otimes \mathbb{Q}$ such that

$$\langle D_i^S, D_j^{S\vee} \rangle = \begin{cases} 1, & i = j; \\ -c_{ji}, & i \notin I_j^S; \\ 0, & i \in I_j^S \setminus \{j\}. \end{cases} \tag{18}$$

These vectors $D_j^{S\vee}$ define the decomposition

$$\mathbb{L}^{S\vee} \otimes \mathbb{Q} = \operatorname{Ker}((D_{m+1}^{S\vee}, \dots, D_{m+l}^{S\vee}) : \mathbb{L}^{S\vee} \otimes \mathbb{Q} \to \mathbb{Q}^l) \oplus \bigoplus_{j=m+1}^{m+l} \mathbb{Q} D_j^S.$$

We identify the first factor $\operatorname{Ker}(D_{m+1}^{S\vee}, \dots, D_{m+l}^{S\vee})$ with $\mathbb{L}^\vee \otimes \mathbb{Q}$. Via this decomposition, we can regard $H^2(\mathcal{X}, \mathbb{Q}) \cong \mathbb{L}^\vee \otimes \mathbb{Q}$ as a subspace of $\mathbb{L}^{S\vee} \otimes \mathbb{Q}$.

Let D_i be the image of D_i^S in $\mathbb{L}^\vee \otimes \mathbb{Q}$ under this decomposition. Then

$$D_i = 0 \quad \text{for } m+1 \le i \le m+l.$$

Let $\mathcal{A}^S$ be the collection of S-extended anti-cones, that is,

$$\mathcal{A}^S := \Big\{ I^S : \sum_{i \notin I^S} \mathbb{R}_{\ge 0} \overline{\beta^S(e_i)} \in \Sigma \Big\}.$$

Note that

$$\{s_1, \dots, s_l\} \subset I^S, \quad \forall I^S \in \mathcal{A}^S.$$

By applying $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ to the S-extended dual map $\beta^\vee$ we have a homomorphism

$$\alpha^S : G^S \to (\mathbb{C}^\times)^{m+l}, \quad \text{where } G^S := \operatorname{Hom}_{\mathbb{Z}}(\mathbb{L}^{S\vee}, \mathbb{C}^\times).$$

We define $\mathcal{U}$ to be the open subset of $\mathbb{C}^{m+l}$ defined by $\mathcal{A}^S$:

$$\mathcal{U}^S := \mathbb{C}^{m+l} \setminus \bigcup_{I^S \notin \mathcal{A}^S} \mathbb{C}^{I^S} = \mathcal{U} \times (\mathbb{C}^\times)^l,$$

where

$$\mathbb{C}^{I^S} = \{(z_1, \dots, z_{m+l}) : z_i = 0 \text{ for } i \notin I^S\}.$$

Let G^S act on $\mathcal{U}^S$ via α^S. Then we obtain the quotient stack $[\mathcal{U}^S/G^S]$. Jiang [7] showed that

$$[\mathcal{U}^S/G^S] \cong [\mathcal{U}/G] = \mathcal{X}.$$

3.2. Mirror Theorem for Toric Stacks

Coates et al. [4] defined the S-extended I-function of a smooth toric Deligne–Mumford stack $\mathcal{X}$ with semiprojective coarse moduli space and proved that this I-function is a point of Givental's Lagrangian cone $\mathcal{L}$ for the Gromov–Witten theory of $\mathcal{X}$. In this paper, we only need this theorem for the weak Fano case. In this case, the mirror theorem will take a particularly simple form, which can be stated as an equality of I-function and J-function via a change of variables, called mirror map.

To state the mirror theorem for weak Fano toric Deligne–Mumford stack, we need the following definitions.

We define the S-extended Kähler cone $C^S_{\mathcal{X}}$ as

$$C^S_{\mathcal{X}} := \bigcap_{I^S \in \mathcal{A}^S} \Sigma_{i\in I^S} \mathbb{R}_{>0} D^S_i$$

and the Kähler cone $C_{\mathcal{X}}$ as

$$C_{\mathcal{X}} := \bigcap_{I \in \mathcal{A}} \Sigma_{i\in I} \mathbb{R}_{>0} D_i .$$

Let $p^S_1, \ldots, p^S_{r+l}$ be an integral basis of $\mathbb{L}^{S\vee}$, where $r = m - n$, such that p^S_i is in the closure $\mathrm{cl}(C^S_{\mathcal{X}})$ of the S-extended Kähler cone $C^S_{\mathcal{X}}$ for all $1 \le i \le r + l$ and $p^S_{r+1}, \ldots, p^S_{r+l}$ are in $\sum_{i=m+1}^{m+l} \mathbb{R}_{\ge 0} D^S_i$. We denote the image of p^S_i in $\mathbb{L}^\vee \otimes \mathbb{R}$ by p_i; therefore, $p_1, \ldots, p_r$ are nef, and $p_{r+1}, \ldots, p_{r+l}$ are zero. We define the matrix (m_{ia}) by

$$D^S_i = \sum_{a=1}^{r+l} m_{ia} p^S_a, \quad m_{ia} \in \mathbb{Z}.$$

Then the class D_i of toric divisor is given by

$$D_i = \sum_{a=1}^{r} m_{ia} p_a .$$

DEFINITION 3.5 ([6], Sect. 3.1.4). A toric Deligne–Mumford stack $\mathcal{X}$ is called weak Fano if the first Chern class ρ satisfies

$$\rho = c_1(T\mathcal{X}) = \sum_{i=1}^{m} D_i \in \mathrm{cl}(C_{\mathcal{X}}),$$

where $C_{\mathcal{X}}$ is the Kähler cone of $\mathcal{X}$.

We will need a slightly stronger condition:

$$\rho^S := D^S_1 + \cdots + D^S_{m+l} \in \mathrm{cl}(C^S_{\mathcal{X}}),$$

where $C^S_{\mathcal{X}}$ is the S-extended Kähler cone. By Lemma 3.3 of [6] we can see that $\rho^S \in \mathrm{cl}(C^S_{\mathcal{X}})$ implies $\rho \in \mathrm{cl}(C_{\mathcal{X}})$. Moreover, under Assumption 3.4, we have

$$\rho^S \in \mathrm{cl}(C^S_{\mathcal{X}}) \quad \text{if and only if} \quad \rho \in \mathrm{cl}(C_{\mathcal{X}}).$$

For a real number r, let $\lceil r \rceil$, $\lfloor r \rfloor$, and $\{r\}$ be the ceiling, floor, and fractional part of r, respectively.

DEFINITION 3.6. We define two subsets $\mathbb{K}$ and $\mathbb{K}_{\mathrm{eff}}$ of $\mathbb{L}^S \otimes \mathbb{Q}$ as follows:

$$\mathbb{K} := \{d \in \mathbb{L}^S \otimes \mathbb{Q}; \{i \in \{1, \ldots, m+l\}; \langle D_i^S, d\rangle \in \mathbb{Z}\} \in \mathcal{A}^S\},$$
$$\mathbb{K}_{\mathrm{eff}} := \{d \in \mathbb{L}^S \otimes \mathbb{Q}; \{i \in \{1, \ldots, m+l\}; \langle D_i^S, d\rangle \in \mathbb{Z}_{\geq 0}\} \in \mathcal{A}^S\}.$$

REMARK 3.7. We use $\mathbb{K}_{\mathcal{E}_j}$ and $\mathbb{K}_{\mathrm{eff},\mathcal{E}_j}$ to denote the corresponding sets for the associated bundle $\mathcal{E}_j$ and use $\mathbb{K}_{\mathcal{X}}$ and $\mathbb{K}_{\mathrm{eff},\mathcal{X}}$ to denote the corresponding sets for $\mathcal{X}$.

DEFINITION 3.8 ([6], Sect. 3.1.3). The reduction function v is defined as follows:

$$v : \mathbb{K} \longrightarrow \mathrm{Box}(\mathbf{\Sigma}),$$
$$d \longmapsto \sum_{i=1}^{m} \lceil \langle D_i^S, d\rangle \rceil b_i + \sum_{j=1}^{l} \lceil \langle D_{m+j}^S, d\rangle \rceil s_j.$$

By the S-extended fan exact sequence we have

$$\sum_{i=1}^{m} \langle D_i^S, d\rangle b_i + \sum_{j=1}^{l} \langle D_{m+j}^S, d\rangle s_j = 0 \in \mathbf{N} \otimes \mathbb{Q}.$$

Moreover, by the definition of $\mathbb{K}$ we have

$$\langle D_{m+j}^S, d\rangle \in \mathbb{Z} \quad \text{for all } d \in \mathbb{K} \text{ and } 1 \leq j \leq l.$$

Hence,

$$v(d) = \sum_{i=1}^{m} \{-\langle D_i^S, d\rangle\} b_i + \sum_{j=1}^{l} \{-\langle D_{m+j}^S, d\rangle\} s_j = \sum_{i=1}^{m} \{-\langle D_i^S, d\rangle\} b_i.$$

The corresponding inertia component $\mathcal{X}_{v(d)}$ is given by

$$\mathcal{X}_{v(d)} := \{[z_1, \ldots, z_{m+l}] \in \mathcal{X} \mid z_i = 0 \text{ if } \langle D_i^S, d\rangle \notin \mathbb{Z}\},$$

and the unit class of $H^*(\mathcal{X}_{v(d)})$ is denoted by $\mathbf{1}_{v(d)}$.

By abuse of notation, we use D_i to denote the divisor $\{z_i = 0\} \subset \mathcal{X}$ and the cohomology class in $H^2(\mathcal{X}; \mathbb{Z}) \cong \mathbb{L}^\vee$ for $1 \leq i \leq m$.

We consider the $\mathbb{C}^\times$-action fixing a toric divisor D_j, $1 \leq j \leq m$; the action of $\mathbb{C}^\times$ on $\mathbb{C}^m$ is given by

$$(z_1, \ldots, z_m) \mapsto (z_1, \ldots, t^{-1} z_j, \ldots, z_m), \quad t \in \mathbb{C}^\times.$$

We can extend this to the diagonal $\mathbb{C}^\times$-action on $\mathcal{U} \times (\mathbb{C}^2 \setminus \{0\})$ by

$$(z_1, \ldots, z_m, u, v) \mapsto (z_1, \ldots, t^{-1} z_j, \ldots, z_m, tu, tv), \quad t \in \mathbb{C}^\times.$$

The associated bundle $\mathcal{E}_j$ of the $\mathbb{C}^\times$-action on $\mathcal{X}$ is given by

$$\mathcal{E}_j = \mathcal{U} \times (\mathbb{C}^2 \setminus \{0\}) / G \times \mathbb{C}^\times.$$

We can also use the S-extended stacky fan of $\mathcal{X}$ to define $\mathcal{E}_j$:

$$\mathcal{E}_j = \mathcal{U}^{\mathcal{S}} \times (\mathbb{C}^2 \setminus \{0\})/G^S \times \mathbb{C}^\times.$$

Therefore, $\mathcal{E}_j$ is also a toric Deligne–Mumford stack. We can identify $H^2(\mathcal{E}_j;\mathbb{Z})$ with the lattice of the characters of $G \times \mathbb{C}^\times$:

$$H^2(\mathcal{E}_j;\mathbb{Z}) \cong \mathbb{L}^\vee \oplus \mathbb{Z} \cong H^2(\mathcal{X};\mathbb{Z}) \oplus \mathbb{Z}. \tag{19}$$

Moreover, we have the divisor sequence

$$0 \to \mathbf{N}^* \oplus \mathbb{Z} \to (\mathbb{Z}^*)^{m+2} \to \mathbb{L}^\vee \oplus \mathbb{Z}$$

and the S-extended divisor sequence

$$0 \to \mathbf{N}^* \oplus \mathbb{Z} \to (\mathbb{Z}^*)^{m+l+2} \to \mathbb{L}^{S\vee} \oplus \mathbb{Z}.$$

Let $\hat{D}_i^S$ be the image of the standard basis of $(\mathbb{Z}^*)^{m+l+2}$ in $\mathbb{L}^{S\vee} \oplus \mathbb{Z}$. Then

$$\begin{aligned} &\hat{D}_i^S = (D_i^S, 0), \quad \text{for } i \neq j; \\ &\hat{D}_j^S = (D_j^S, -1); \qquad \hat{D}_{m+l+1}^S = \hat{D}_{m+l+2}^S = (0,1); \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\hat{D}_i = (D_i, 0), \quad \text{for } i \neq j; \\ &\hat{D}_j = (D_j, -1); \qquad \hat{D}_{m+1} = \hat{D}_{m+2} = (0,1). \end{aligned} \tag{21}$$

The fan Σ_j of $\mathcal{E}_j$ is a rational simplicial fan contained in $\mathbf{N}_\mathbb{Q} \oplus \mathbb{Q}$. The 1-skeleton is given by

$$\begin{aligned} \hat{b}_i &= (b_i, 0), \quad \text{for } 1 \leq i \leq m; \\ \hat{b}_{m+1} &= (0,1); \qquad \hat{b}_{m+2} = (b_j, -1). \end{aligned} \tag{22}$$

We set

$$\begin{aligned} p_0^S &:= \hat{D}_{m+l+1}^S = \hat{D}_{m+l+2}^S \in \mathbb{L}^{S\vee} \oplus \mathbb{Z}; \\ p_0 &:= \hat{D}_{m+1} = \hat{D}_{m+2} \in H^2(\mathcal{E}_j;\mathbb{Q}). \end{aligned}$$

Then a nef integral basis $\{p_1, \ldots, p_r\}$ of $H^2(\mathcal{X};\mathbb{Q})$ can be lifted to a nef integral basis $\{p_0, p_1, \ldots, p_r\}$ of $H^2(\mathcal{E}_j;\mathbb{Q})$ under the splitting (19) (this gives the same splitting as the one in Section 2.1). Recall that $\{p_1^S, \ldots, p_{r+l}^S\}$ is an integral basis of $\mathbb{L}^{S\vee}$ such that p_i is the image of p_i^S in $\mathbb{L}^\vee \otimes \mathbb{R}$. Moreover, $p_0^S, p_1^S, \ldots, p_{r+l}^S$ is an integral basis of $\mathbb{L}^{S\vee} \oplus \mathbb{Z}$, and p_0 is the image of p_0^S in $(\mathbb{L}^\vee \oplus \mathbb{Z}) \otimes \mathbb{R}$. Note that $p_{r+1}, \ldots, p_{r+l}$ are all zero. We have

$$C_{\mathcal{E}_j}^S = C_\mathcal{X}^S + \mathbb{R}_{>0} p_0^S, \qquad \rho_{\mathcal{E}_j}^S = \rho_\mathcal{X}^S + p_0^S.$$

The following result is straightforward.

Lemma 3.9. *If $\rho_\mathcal{X}^S \in \mathrm{cl}(C_\mathcal{X}^S)$, then $\rho_{\mathcal{E}_j}^S \in \mathrm{cl}(C_{\mathcal{E}_j}^S)$ for $1 \leq j \leq m$.*

DEFINITION 3.10. The I-function of $\mathcal{X}$ is the $H^*_{\mathrm{orb}}(\mathcal{X})$-valued function

$$I_{\mathcal{X}}(y,z)=e^{\sum_{i=1}^r p_i\log y_i/z}\times\sum_{d\in\mathbb{K}_{\mathrm{eff},\mathcal{X}}}\prod_{i=1}^{m+l}\left(\frac{\prod_{k=\lceil\langle D_i^S,d\rangle\rceil}^{\infty}(D_i+(\langle D_i^S,d\rangle-k)z)}{\prod_{k=0}^{\infty}(D_i+(\langle D_i^S,d\rangle-k)z)}\right)y^d\mathbf{1}_{v(d)},\quad(23)$$

where $y^d=y_1^{\langle p_1^S,d\rangle}\cdots y_{r+l}^{\langle p_{r+l}^S,d\rangle}$. Similarly, the I-function of $\mathcal{E}$ is the $H^*_{\mathrm{orb}}(\mathcal{E})$-valued function

$$I_{\mathcal{E}_j}(y,z)=e^{\sum_{i=0}^r p_i\log y_i/z}\times\sum_{\beta\in\mathbb{K}_{\mathrm{eff},\mathcal{E}_j}}\prod_{i=1}^{m+l+2}\left(\frac{\prod_{k=\lceil\langle \hat{D}_i^S,\beta\rangle\rceil}^{\infty}(\hat{D}_i+(\langle \hat{D}_i^S,\beta\rangle-k)z)}{\prod_{k=0}^{\infty}(\hat{D}_i+(\langle \hat{D}_i^S,\beta\rangle-k)z)}\right)y^\beta\mathbf{1}_{v(\beta)},\quad(24)$$

where $y^\beta=y_0^{\langle p_0^S,\beta,\rangle}y_1^{\langle p_1^S,\beta\rangle}\cdots y_{r+l}^{\langle p_{r+l}^S,\beta\rangle}$.

Following Section 4.1 of [6], the I-functions of $\mathcal{X}$ and $\mathcal{E}_j$ can be rewritten in the forms

$$I_{\mathcal{X}}(y,z)=e^{\sum_{i=1}^r p_i\log y_i/z}\times\sum_{d\in\mathbb{K}_{\mathcal{X}}}\prod_{i=1}^{m+l}\left(\frac{\prod_{k=\lceil\langle D_i^S,d\rangle\rceil}^{\infty}(D_i+(\langle D_i^S,d\rangle-k)z)}{\prod_{k=0}^{\infty}(D_i+(\langle D_i^S,d\rangle-k)z)}\right)y^d\mathbf{1}_{v(d)}\quad(25)$$

and

$$I_{\mathcal{E}_j}(y,z)=e^{\sum_{i=0}^r p_i\log y_i/z}\times\sum_{\beta\in\mathbb{K}_{\mathcal{E}_j}}\prod_{i=1}^{m+l+2}\left(\frac{\prod_{k=\lceil\langle \hat{D}_i^S,\beta\rangle\rceil}^{\infty}(\hat{D}_i+(\langle \hat{D}_i^S,\beta\rangle-k)z)}{\prod_{k=0}^{\infty}(\hat{D}_i+(\langle \hat{D}_i^S,\beta\rangle-k)z)}\right)y^\beta\mathbf{1}_{v(\beta)},\quad(26)$$

respectively, because the summand with $d\in\mathbb{K}\setminus\mathbb{K}_{\mathrm{eff}}$ vanishes. We refer to [6] for more details.

THEOREM 3.11 ([4], Thm. 31). *Assume that $\rho^S\in\mathrm{cl}(C^S_{\mathcal{X}})$. Then the I-function and the J-function satisfy the following relation*:

$$I_{\mathcal{X}}(y,z)=J_{\mathcal{X}}(\tau(y),z),\quad(27)$$

where

$$\tau(y)=\tau_{0,2}(y)+\tau_{tw}(y)=\sum_{i=1}^r(\log y_i)p_i+\sum_{j=m+1}^{m+l}y^{D_j^{S\vee}}\mathfrak{D}_j+h.o.t.\in H^{\le 2}_{\mathrm{orb}}(\mathcal{X}),\quad(28)$$

with

$$\tau_{0,2}(y)\in H^2(\mathcal{X}),\qquad\tau_{tw}(y)\in H^{\le 2}_{tw}(\mathcal{X}),$$

$$\mathfrak{D}_j = \prod_{i \notin I_j^S} D_i^{\lfloor c_{ji} \rfloor} \mathbf{1}_{v(D_j^{S\vee})} \in H^*_{\mathrm{orb}}(\mathcal{X}),$$

where h.o.t. (higher-order term) is a fractional power series in $y_1, \ldots, y_{r+l}$. Note that under Assumption 3.4, we have $\mathfrak{D}_j = 1_{s_j}$ for $m+1 \leq j \leq m+l$. Furthermore, $\tau(y)$ is called the mirror map and takes values in $H^{\leq 2}_{\mathrm{orb}}(\mathcal{X})$.

REMARK 3.12. We only stated a particular case of [4, Thm. 31] when $\mathcal{X}$ is weak Fano. See [6, Sect. 4.1] for more details.

For $\tau_{0,2}(y) = \sum_{a=1}^r p_a \log q_a \in H^2(\mathcal{X})$, we have

$$\log q_i = \log y_i + g_i(y_1, \ldots, y_{r+l}), \quad \text{for } i = 1, \ldots, r,$$

where g_i is a (fractional) power series in $y_1, \ldots, y_{r+l}$, which is homogeneous of degree zero with respect to the degree $\deg y^d = 2\langle \rho^S_{\mathcal{X}}, d\rangle$.

By Lemma 3.9, under the assumption of Theorem 3.11, we can also apply the mirror theorem to the associated bundle $\mathcal{E}_j$; hence, we have

$$I_{\mathcal{E}_j}(y, z) = J_{\mathcal{E}_j}(\tau^{(j)}(y), z),$$

where

$$\tau^{(j)}(y) = \tau^{(j)}_{0,2} + \tau^{(j)}_{tw}(y) \in H^2(\mathcal{E}_j) \oplus H^{\leq 2}_{tw}(\mathcal{E}_j) = H^{\leq 2}_{\mathrm{orb}}(\mathcal{E}_j).$$

For $\tau^{(j)}_{0,2}(y) = \sum_{a=0}^r p_a \log q_a \in H^2(\mathcal{E}_j)$, therefore,

$$\log q_i = \log y_i + g_i^{(j)}(y_0, \ldots, y_{r+l}) \quad \text{for } i = 0, \ldots, r,$$

where $g_i^{(j)}$ is a (fractional) power series in $y_0, y_1, \ldots, y_{r+l}$, which is homogeneous of degree zero with respect to the degree $\deg y^\beta = 2\langle \rho^S_{\mathcal{E}_j}, \beta\rangle$.

3.3. Seidel Elements and Mirror Maps

PROPOSITION 3.13. *The function $g_i^{(j)}$ does not depend on y_0, and we have*

$$g_i^{(j)}(y_0, \ldots, y_{r+l}) = g_i(y_1, \ldots, y_{r+l}) \quad \textit{for } i = 1, \ldots, r.$$

Proof. The functions g_i are the coefficients of $z^{-1} p_i$ in the expansion of $I_{\mathcal{X}}$:

$$I_{\mathcal{X}}(y, z) = e^{\sum_{i=1}^r p_i \log y_i / z} \left(1 + z^{-1} \left(\sum_{i=1}^r g_i(y) p_i + \tau_{tw} \right) + O(z^{-2}) \right).$$

The functions $g_i^{(j)}$ are the coefficients of $z^{-1} p_i$ in the expansion of $I_{\mathcal{E}_j}$:

$$I_{\mathcal{E}_j}(y, z) = e^{\sum_{i=0}^r p_i \log y_i / z} \left(1 + z^{-1} \left(\sum_{i=0}^r g_i^{(j)}(y) p_i + \tau^{(j)}_{tw} \right) + O(z^{-2}) \right).$$

Following the proof of Lemma 3.5 of [5], we obtain the conclusion of this proposition. ∎

We will prove that $\tau_{tw}^{(j)}(y)$ is also independent from y_0. To begin with, the following lemma implies that $\tau_{tw}^{(j)}(y)$ is an (integer) power series in y_0.

LEMMA 3.14. *For any* $\beta \in \mathbb{K}_{\mathcal{E}_j}$, *we have* $\langle p_0^S, \beta\rangle \in \mathbb{Z}$. *Furthermore, for any* $\beta \in \mathbb{K}_{\mathrm{eff},\mathcal{E}_j}$, *we have* $\langle p_0^S, \beta\rangle \in \mathbb{Z}_{\geq 0}$.

Proof. Any cone $\sigma \in \Sigma_j$ containing both $\hat{b}_{m+1}$ and $\hat{b}_{m+2}$ should also contain $\hat{b}_j$, but this is impossible since the fan Σ_j is simplicial and $\hat{b}_{m+1}$, $\hat{b}_{m+2}$, and $\hat{b}_j$ lie in the same plane. Hence, by the definition of $\mathbb{K}_{\mathcal{E}_j}$ (resp. $\mathbb{K}_{\mathrm{eff},\mathcal{E}_j}$) at least one of $\langle \hat{D}_{m+1}^S, \beta\rangle$ and $\langle \hat{D}_{m+2}^S, \beta\rangle$ has to be integer (resp. nonnegative integer) for any $\beta \in \mathbb{K}_{\mathcal{E}_j}$ (resp. $\beta \in \mathbb{K}_{\mathrm{eff},\mathcal{E}_j}$). On the other hand, we have

$$\langle p_0^S, \beta\rangle = \langle \hat{D}_{m+1}^S, \beta\rangle = \langle \hat{D}_{m+2}^S, \beta\rangle.$$

Therefore, we must have $\langle p_0^S, \beta\rangle \in \mathbb{Z}$ (resp. $\langle p_0^S, \beta\rangle \in \mathbb{Z}_{\geq 0}$). $\square$

As a direct consequence of the lemma, $\tau_{tw}^{(j)}(y)$ can only contain nonnegative integer powers of y_0.

PROPOSITION 3.15. *Let* $\tau_{tw}^{(j)}(y) = \sum_{n=0}^{\infty} P_n^{(j)}(y) y_0^n$, *where* $P_n^{(j)}(y)$ *is a (fractional) power series in* $y_1, \ldots, y_n$. *Then*

$$P_n^{(j)}(y) = 0 \quad \textit{for } n \geq 1,$$

that is, $\tau_{tw}^{(j)}(y)$ *is independent from* y_0. *Moreover, we have*

$$\iota^* \tau_{tw}^{(j)}(y) = \tau_{tw}(y).$$

Proof. Recall that $\tau_{tw}^{(j)}(y)$ is the coefficient of z^{-1} in

$$\begin{aligned} &e^{-\sum_{i=0}^r p_i \log y_i / z} I_{\mathcal{E}_j}(y, z) \\ &\quad = \sum_{\beta \in \mathbb{K}_{\mathrm{eff},\mathcal{E}_j}} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta\rangle \rceil}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta\rangle - k) z)}{\prod_{k=0}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta\rangle - k) z)} \right) y^{\beta} \mathbf{1}_{v(\beta)}, \end{aligned} \tag{29}$$

valued in $H_{tw}^{\leq 2}(\mathcal{E}_j)$. Hence, we only need to consider terms with $v(\beta) \neq 0$. On the other hand, by Lemma 3.14 we have

$$\langle p_0^S, \beta\rangle = \langle \hat{D}_{m+1}^S, \beta\rangle = \langle \hat{D}_{m+2}^S, \beta\rangle \in \mathbb{Z} \quad \text{for } \beta \in \mathbb{K}_{\mathcal{E}_j}.$$

Hence, we obtain

$$v(\beta) = \sum_{i=1}^{m+2} \{-\langle \hat{D}_i^S, \beta\rangle\} \hat{b}_i = \left(\sum_{i=1}^{m} \{-\langle D_i^S, d\rangle\} b_i, 0 \right) = (v(d), 0) \in \mathbf{N} \oplus \mathbb{Z},$$

where d is the natural projection of β onto $\mathbb{K}_{\mathrm{eff},\mathcal{X}}$. Therefore, $v(\beta) \neq 0$ is equivalent to $v(d) \neq 0$.

It remains to examine the product factor:

$$\begin{aligned}
&\prod_{i=1}^{m+l+2}\left(\frac{\prod_{k=\lceil\langle\hat{D}_i^S,\beta\rangle\rceil}^{\infty}(\hat{D}_i+(\langle\hat{D}_i^S,\beta\rangle-k)z)}{\prod_{k=0}^{\infty}(\hat{D}_i+(\langle\hat{D}_i^S,\beta\rangle-k)z)}\right)\\
&\quad=\frac{\prod_{i:\langle\hat{D}_i^S,\beta\rangle<0}\prod_{\langle\hat{D}_i^S,\beta\rangle\le k<0}(\hat{D}_i+(\langle\hat{D}_i^S,\beta\rangle-k)z)}{\prod_{i:\langle\hat{D}_i^S,\beta\rangle>0}\prod_{0\le k<\langle\hat{D}_i^S,\beta\rangle}(\hat{D}_i+(\langle\hat{D}_i^S,\beta\rangle-k)z)}\\
&\quad=C_\beta z^{-(\sum_{i=1}^{m+l+2}\lceil\langle\hat{D}_i^S,\beta\rangle\rceil+\#\{i:\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}_{<0}\})}\prod_{i:\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}_{<0}}\hat{D}_i+h.o.t., \qquad (30)
\end{aligned}$$

where

$$\begin{aligned}
C_\beta=&\prod_{i:\langle\hat{D}_i^S,\beta\rangle<0}\ \prod_{\langle\hat{D}_i^S,\beta\rangle<k<0}(\langle\hat{D}_i^S,\beta\rangle-k)\\
&\times\prod_{i:\langle\hat{D}_i^S,\beta\rangle>0}\ \prod_{0\le k<\langle\hat{D}_i^S,\beta\rangle}(\langle\hat{D}_i^S,\beta\rangle-k)^{-1}. \qquad (31)
\end{aligned}$$

By assumption we need to have

$$\sum_{i=1}^{m+l+2}\lceil\langle\hat{D}_i^S,\beta\rangle\rceil\ge\sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle\ge 0.$$

The equality holds if and only if

$$\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}\quad\text{for all } 1\le i\le m+l+2\quad\text{and}\quad\sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle=0.$$

However, this would imply $v(\beta)=0$, and hence we cannot have $\sum_{i=1}^{m+l+2}\lceil\langle\hat{D}_i^S,\beta\rangle\rceil=0$. Therefore, expansion (30) would contribute to $P_n^{(j)}$ only when

$$\sum_{i=1}^{m+l+2}\lceil\langle\hat{D}_i^S,\beta\rangle\rceil=1\quad\text{and}\quad\#\{i:\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}_{<0}\}=0.$$

In this case, if $\langle p_0^S,\beta\rangle\ge 1$, then

$$\sum_{i=1}^{m+l+2}\lceil\langle\hat{D}_i^S,\beta\rangle\rceil\ge\sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil+1,$$

and, therefore, we have

$$0\ge\sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil\ge\sum_{i=1}^{m+l}\langle D_i^S,d\rangle=0.$$

This implies that when $\langle p_0^S,\beta\rangle\ge 1$, we must have

$$\langle D_i^S,d\rangle\in\mathbb{Z}\quad\text{for } 1\le i\le m+l.$$

It is a contradiction since $\tau_{tw}^{(j)}(y) \in H_{tw}^{\leq 2}(\mathcal{E}_j)$ implies $v(\beta) \neq 0$, equivalently, $v(d) \neq 0$. Hence,

$$P_n^{(j)} = 0 \quad \text{for all } n > 0,$$

and $\tau_{tw}^{(j)}(y)$ is independent from y_0. Moreover, by the expression of I-functions and the identity

$$\iota^* I_{\mathcal{E}_j}|_{y_0=0} = I_{\mathcal{X}}$$

we have $\iota^* \tau_{tw}^{(j)}(y) = \tau_{tw}(y)$. □

Since $\tilde{S}_j(\tau^{(j)}(y))$ does not depend on y_0 or q_0, we use the following notation for the Seidel element:

$$\tilde{S}_j(\tau(y)) := \tilde{S}_j(\tau^{(j)}(y)). \tag{32}$$

3.4. Seidel Elements in Terms of I-Functions

We can rewrite the I-function of the associated bundle $\mathcal{E}_j$ as follows:

$$e^{\sum_{i=0}^r p_i \log y_i/z}\Bigg(1 + z^{-1}\Bigg(\sum_{i=0}^r g_i^{(j)}(y) p_i + \tau_{tw}^{(j)}(y)\Bigg) + z^{-2}\Bigg(\sum_{n=0}^2 G_n^{(j)}(y) y_0^n\Bigg) + O(z^{-3})\Bigg). \tag{33}$$

Then, $\log q_i = \log y_i + g_i^{(j)}(y)$ implies

$$I_{\mathcal{E}_j}(y,z) = e^{\sum_{i=0}^r p_i \log q_i/z}\Bigg(1 + z^{-1}\tau_{tw}^{(j)}(y) + z^{-2}\Bigg(\sum_{n=0}^2 G_n^{(j)}(y) y_0^n\Bigg) + O(z^{-3})\Bigg), \tag{34}$$

where $G_n^{(j)}(y)$ is a (fractional) power series in $y_1, \ldots, y_{r+l}$ taking values in $H^*_{\mathrm{orb}}(\mathcal{E}_j)$.

By Proposition 2.6 the Seidel element $\tilde{S}_j(\tau^{(j)}(y))$ is the coefficient of q_0/z^2 in

$$\exp\Bigg(-\sum_{i=0}^r p_i \log q_i/z\Bigg) J_{\mathcal{E}_j}(\tau^{(j)}(y), z);$$

hence, $J_{\mathcal{E}_j}(\tau^{(j)}(y), z) = I_{\mathcal{E}_j}(y,z)$ and $\log q_0 = \log y_0 + g_0^{(j)}(y)$ imply the following result.

THEOREM 3.16. *The Seidel element S_j associated to the toric divisor D_j is given by*

$$S_j(\tau^{(j)}(y)) = \iota^*(G_1^{(j)}(y) y_0). \tag{35}$$

Furthermore, we have

$$\tilde{S}_j(\tau(y)) = \tilde{S}_j(\tau^{(j)}(y)) = \exp(-g_0^j(y))\iota^*(G_1^{(j)}(y)). \tag{36}$$

3.5. Computation of $g_0^{(j)}$

The computation is essentially the same as the proof of Lemma 3.16 of [5]. Consider the product factors in $I_{\mathcal{E}_j}$:

$$\prod_{i=1}^{m+l+2}\left(\frac{\prod_{k=\lceil\langle\hat{D}_i^S,\beta\rangle\rceil}^{\infty}(\hat{D}_i+(\langle\hat{D}_i^S,\beta\rangle-k)z)}{\prod_{k=0}^{\infty}(\hat{D}_i+(\langle\hat{D}_i^S,\beta\rangle-k)z)}\right)y^\beta\mathbf{1}_{v(\beta)}.$$

These factors contribute to $g_i^{(j)}$ if

$$v(\beta)=\sum_{i=1}^{m+l+2}\{-\langle\hat{D}_i^S,\beta\rangle\}\hat{b}_i=0.$$

Then, by the definition of $\mathbb{K}_{\text{eff}}$ we must have

$$\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}\quad\text{for all } 1\le i\le m+l+2.$$

In this case, the product factors can be rewritten as

$$\begin{aligned}
&\prod_{i=1}^{m+l+2}\left(\frac{\prod_{k=\lceil\langle\hat{D}_i^S,\beta\rangle\rceil}^{\infty}(\hat{D}_i+(\langle\hat{D}_i^S,\beta\rangle-k)z)}{\prod_{k=0}^{\infty}(\hat{D}_i+(\langle\hat{D}_i^S,\beta\rangle-k)z)}\right)y^\beta\mathbf{1}_{v(\beta)}\\
&=\prod_{i=1}^{m+l+2}\frac{\prod_{k=-\infty}^{0}(\hat{D}_i+kz)}{\prod_{k=-\infty}^{\langle\hat{D}_i^S,\beta\rangle}(\hat{D}_i+kz)}y^\beta\\
&=\left(C_\beta z^{-\sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle-\#\{i:\langle\hat{D}_i^S,\beta\rangle<0\}}\prod_{i:\langle\hat{D}_i^S,\beta\rangle<0}\hat{D}_i+h.o.t.\right)y^\beta,
\end{aligned}\tag{37}$$

where $h.o.t.$ stands for higher-order terms in z^{-1}, and

$$C_\beta=\prod_{i:\langle\hat{D}_i^S,\beta\rangle<0}(-1)^{-\langle\hat{D}_i^S,\beta\rangle-1}(-\langle\hat{D}_i^S,\beta\rangle-1)!\prod_{i:\langle\hat{D}_i^S,\beta\rangle\ge0}(\langle\hat{D}_i^S,\beta\rangle!)^{-1}.\tag{38}$$

They contribute to the z^{-1} term if

$$\sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle+\#\{i:\langle\hat{D}_i^S,\beta\rangle<0\}\le1.$$

Since by assumption $\rho_{\mathcal{X}}^S\in\operatorname{cl}(C_{\mathcal{X}}^S)$, we have $\rho_{\mathcal{E}_j}^S\in\operatorname{cl}(C_{\mathcal{E}_j}^S)$. So there are the following three cases:

- $\begin{cases}\sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle=0,\\ \#\{i:\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}_{<0}\}=0;\end{cases}$
- $\begin{cases}\sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle=1,\\ \#\{i:\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}_{<0}\}=0;\end{cases}$
- $\begin{cases}\sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle\rceil=0,\\ \#\{i:\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}_{<0}\}=1.\end{cases}$

In the first case, we have $\langle \hat{D}_i^S, \beta \rangle = 0$ for all i, and hence $\beta = 0$; the second case cannot happen, since β has to satisfy $\langle \hat{D}_i^S, \beta \rangle = 0$ except for one i, and this implies $\beta = 0$. Therefore, the coefficient of z^{-1} is from the third case, where

$$\sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle = 0 \quad \text{and} \quad \#\{i : \langle \hat{D}_i^S, \beta \rangle < 0\} = 1. \tag{39}$$

By the assumption $\rho^S_{\mathcal{X}} \in \mathrm{cl}(C^S_{\mathcal{X}})$ we must have $\sum_{i=1}^{m+l} \langle D_i^S, d \rangle = 0$ and $\langle p_0^S, \beta \rangle = 0$. Moreover, $\langle D_i^S, d \rangle < 0$ for exactly one i in $\{1, \dots, m\}$. (Note that $\langle D_i^S, d \rangle \geq 0$ for $i \in \{m+1, \dots, m+l\}$.)

Now $g_0^{(j)}$ is the coefficient corresponding to p_0, and $\hat{D}_j = \langle D_j, -1 \rangle = D_j - p_0$ is the only one, among $\hat{D}_1, \dots, \hat{D}_m$, which contains p_0. By expression (37) we must have $\langle D_j^S, d \rangle < 0$ and $\langle D_i^S, d \rangle \geq 0$ for $i \neq j$. Hence, we have

LEMMA 3.17. *The coefficient $g_0^{(j)}$ is given by*

$$g_0^j(y_1, \dots, y_{r+l}) = \sum_{\substack{\langle D_i^S, d \rangle \in \mathbb{Z}, 1 \leq i \leq m+l \\ \langle \rho^S_{\mathcal{X}}, d \rangle = 0 \\ \langle D_j^S, d \rangle < 0 \\ \langle D_i^S, d \rangle \geq 0, \forall i \neq j}} \frac{(-1)^{-\langle D_j^S, d \rangle} (-\langle D_j^S, d \rangle - 1)!}{\prod_{i \neq j} \langle D_i^S, d \rangle !} y^d. \tag{40}$$

4. Batyrev Elements

In this section, we extend the definition of Batyrev elements in [5] to toric Deligne–Mumford stacks and explore their relationships with Seidel elements. Batyrev elements satisfy the multiplicative and linear Batyrev relations as in Batyrev's presentation of quantum cohomology ring for toric manifolds in [1].

4.1. Batyrev Elements

Following [6], consider the mirror coordinates $y_1, \dots, y_{r+l}$ of the toric Deligne–Mumford stacks $\mathcal{X}$ with $\rho^S_{\mathcal{X}} \in \mathrm{cl}(C^S_{\mathcal{X}})$. Set $\mathbb{C}[y^\pm] = \mathbb{C}[y_1^\pm, \dots, y_{r+l}^\pm]$.

DEFINITION 4.1 ([6]). The Batyrev ring $B(\mathcal{X})$ of $\mathcal{X}$ is a $\mathbb{C}[y^\pm]$-algebra generated by the variables $\lambda_1, \dots, \lambda_{r+l}$ with the following two Batyrev relations:

$$\begin{aligned} &\text{(multiplicative):} \quad y^d \prod_{i : \langle D_i^S, d \rangle < 0} \omega_i^{-\langle D_i^S, d \rangle} = \prod_{i : \langle D_i^S, d \rangle > 0} \omega_i^{\langle D_i^S, d \rangle}, \quad d \in \mathbb{L}^S; \\ &\text{(linear):} \quad \omega_i = \sum_{a=1}^{r+l} m_{ai} \lambda_a, \end{aligned} \tag{41}$$

where ω_i is invertible in $B(\mathcal{X})$.

DEFINITION 4.2. Let $\mathbb{Q}\langle\langle y_1, \dots, y_{r+l}\rangle\rangle$ be the field of fractional power series of $y_1, \dots, y_{r+l}$. We define the element $\tilde{p}_i^S \in H^{\leq 2}_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{Q}\langle\langle y_1, \dots, y_{r+l}\rangle\rangle$ as

$$\tilde{p}_i^S = \frac{\partial \tau(y)}{\partial \log y_i}, \quad i = 1, \dots, r+l.$$

Recall that

$$D_j^S = \sum_{i=1}^{r+l} m_{ij} p_i^S, \quad \text{for } 1 \leq j \leq m+l,$$

Then, the Batyrev element associated to D_j^S is defined by

$$\tilde{D}_j^S = \sum_{i=1}^{r+l} m_{ij} \tilde{p}_i^S \quad \text{for } 1 \leq j \leq m+l.$$

PROPOSITION 4.3. *The Batyrev elements $\tilde{D}_1^S, \dots, \tilde{D}_{m+l}^S$ satisfy the multiplicative and linear Batyrev relations for $\omega_j = \tilde{D}_j^S$.*

Proof. We consider the differential operator $\mathcal{P}_d \in \mathbb{C}[z, y^{\pm}, zy(\partial/\partial y)]$ for $d \in \mathbb{L}^S$, introduced by Iritani in [6], Section 4.2:

$$\mathcal{P}_d := y^d \prod_{i:\langle D_i^S, d\rangle < 0} \ \prod_{k=0}^{-\langle D_i^S, d\rangle - 1} (\mathcal{D}_i - kz) - \prod_{i:\langle D_i^S, d\rangle > 0} \ \prod_{k=0}^{\langle D_i^S, d\rangle - 1} (\mathcal{D}_i - kz), \tag{42}$$

where $\mathcal{D}_i := \sum_{j=1}^{r+l} m_{ij} z y_j \partial/\partial y_j$.

By [6, Lemma 4.6], we have

$$\mathcal{P}_d I(y, z) = 0, \quad d \in \mathbb{L}^S.$$

Hence,

$$0 = \mathcal{P}_d(z, y, zy\partial/\partial y) I(y, z) = \mathcal{P}_d(z, y, zy\partial/\partial y) J(\tau(y), z).$$

This implies that

$$\mathcal{P}_d(z, y, z\tau^* \nabla)\mathbf{1} = 0,$$

where $\tau^*\nabla_i := \nabla_{\tau_*(y_i(\partial/\partial y_i))}$. Since

$$\tau(y) = \sum_{i=1}^{r} p_i \log y_i + \tau_{tw}(y)$$

and

$$\nabla_{\tau_*(y_i(\partial/\partial y_i))} = \tau_*(y_i(\partial/\partial y_i)) + \frac{1}{z} y_i \frac{\partial \tau(y)}{\partial y_i} \circ_\tau,$$

by setting $z = 0$ we proved that the Batyrev elements satisfy the multiplicative relation.

It is straightforward from the definition that the Batyrev elements satisfy the linear relation. $\square$

Consider the I-function for the bundle $\mathcal{E}_j$ associated to the toric divisor D_j^S for $1 \leq j \leq m$:

$$I_{\mathcal{E}_j}(y,z) = e^{\sum_{i=0}^r p_i \log y_i/z} \times \sum_{\beta \in \mathbb{K}_{\mathcal{E}_j}} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta \rangle \rceil}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k)z)}{\prod_{k=0}^{\infty} (\hat{D}_i + (\langle \hat{D}_i^S, \beta \rangle - k)z)} \right) y^\beta \mathbf{1}_{v(\beta)},$$

where $y^\beta = y_0^{\langle p_0^S, \beta, \rangle} y_1^{\langle p_1^S, \beta \rangle} \cdots y_{r+l}^{\langle p_{r+l}^S, \beta \rangle}$. The following lemma is a generalization of Lemma 3.11 in [5].

LEMMA 4.4. *The I-function $I_{\mathcal{E}_j}$ of the bundle $\mathcal{E}_j$, associated to the toric divisor D_j^S, satisfies the following partial differential equation*:

$$z \frac{\partial}{\partial y_0} \left(y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} = \left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j}. \tag{43}$$

Proof. It follows from applying the operator $\mathcal{P}_\beta$ defined in equation (42) to the bundle $\mathcal{E}_j$ with $\beta = [\sigma_0]$. □

Using the expansion of $I_{\mathcal{E}_j}$, we have

$$I_{\mathcal{E}_j}(y,z) = e^{\sum_{i=0}^r p_i \log y_i/z} \left(1 + z^{-1} \left(\sum_{i=0}^r g_i^{(j)}(y) p_i + \tau_{tw}^{(j)} \right) + z^{-2} \left(\sum_{n=0}^2 G_n^{(j)}(y) y_0^n \right) + O(z^{-3}) \right),$$

where $G_n^{(j)}$ is a (fractional) power series in $y_1, \dots, y_{r+l}$ taking values in $H^*_{\text{orb}}(\mathcal{E}_j)$. Therefore, we obtain

$$\begin{aligned} y_0 \frac{\partial}{\partial y_0} I_{\mathcal{E}_j} = \frac{p_0}{z} e^{\sum_{i=0}^r p_i \log y_i/z} & \left(1 + z^{-1} \left(\sum_{i=0}^r g_i^{(j)}(y) p_i + \tau_{tw}^{(j)} \right) \right. \\ & \left. + z^{-2} \left(\sum_{n=0}^2 G_n^{(j)}(y) y_0^n \right) + O(z^{-3}) \right) \\ & + e^{\sum_{i=0}^r p_i \log y_i/z} \left(z^{-2} \left(\sum_{n=1}^2 G_n^{(j)}(y) n y_0^n \right) + O(z^{-3}) \right). \end{aligned}$$

Therefore, the left-hand side of equation (43) is

$$\begin{aligned} & z \frac{\partial}{\partial y_0} \left(y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} \\ & \quad = \frac{\partial}{\partial y_0} \left(p_0 e^{\sum_{i=0}^r p_i \log y_i/z} \left(1 + z^{-1} \left(\sum_{i=0}^r g_i^{(j)}(y) p_i + \tau_{tw}^{(j)} \right) \right. \right. \end{aligned}$$

$$
\begin{aligned}
&+ z^{-2}\left(\sum_{n=0}^{2} G_n^{(j)}(y) y_0^n\right) + O(z^{-3})\Bigg)\Bigg) \\
&+ \frac{\partial}{\partial y_0}\left(e^{\sum_{i=0}^r p_i \log y_i / z}\left(z^{-1}\left(\sum_{n=1}^{2} G_n^{(j)}(y) n y_0^n\right) + O(z^{-2})\right)\right) \\
= {}& p_0 e^{\sum_{i=0}^r p_i \log y_i / z}(O(z^{-2})) \\
&+ \frac{p_0}{y_0 z} e^{\sum_{i=0}^r p_i \log y_i / z}\left(z^{-1}\left(\sum_{n=1}^{2} G_n^{(j)}(y) n y_0^n\right) + O(z^{-2})\right) \\
&+ e^{\sum_{i=0}^r p_i \log y_i / z}\left(z^{-1}\left(\sum_{n=1}^{2} G_n^{(j)} n^2 y_0^{n-1} + O(z^{-2})\right)\right) \\
= {}& e^{\sum_{i=0}^r p_i \log y_i / z}\left(z^{-1}\left(\sum_{n=1}^{2} G_n^{(j)} n^2 y_0^{n-1}\right) + O(z^{-2})\right).
\end{aligned}
$$

On the other hand, the pull-back of the right-hand side of equation (43) by ι^* is

$$
\begin{aligned}
&\iota^*\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i \frac{\partial}{\partial y_i}\right) - y_0 \frac{\partial}{\partial y_0}\right) I_{\mathcal{E}_j} \\
&\quad = \left(\sum_{i=1}^{r+l} m_{ij}\left(y_i \frac{\partial}{\partial y_i}\right) - y_0 \frac{\partial}{\partial y_0}\right) \iota^* I_{\mathcal{E}_j} \\
&\quad = \left(\sum_{i=1}^{r+l} m_{ij}\left(y_i \frac{\partial}{\partial y_i}\right)\right)(I_{\mathcal{X}} + O(y_0)) \\
&\quad = z^{-1}\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i \frac{\partial}{\partial y_i}\right)\tau(y)\right) + O(z^{-2}) + O(y_0).
\end{aligned}
$$

Hence, we conclude the following lemma.

LEMMA 4.5. *The Batyrev element $\tilde{D}_j(y)$ is given by*

$$
\tilde{D}_j(y) = \iota^* G_1^{(j)}(y) \quad \textit{for } 1 \le j \le m + l. \tag{44}
$$

Hence, the following theorem is a direct consequence of the last lemma and Theorem 3.16.

THEOREM 4.6. *The Seidel element $\tilde{S}_j$ corresponding to the toric divisor D_j is given by*

$$
\tilde{S}_j(\tau(y)) = \exp(-g_0^j(y)) \tilde{D}_j(y). \tag{45}
$$

4.2. The Computation of $\tilde{D}_j$

Using the expansion

$$\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)\right)I_{\mathcal{X}} = e^{\sum_{i=1}^{r} p_i \log y_i/z}(z^{-1}\tilde{D}_j + O(z^{-2})),$$

we see that $\tilde{D}_j$ is the coefficient of z^{-1} in the expansion of

$$e^{-\sum_{i=1}^{r} p_i \log y_i/z}\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)\right)I_{\mathcal{X}}.$$

By direct computation we get

$$\begin{aligned}&\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)\right)I_{\mathcal{X}} \\ &\quad = e^{\sum_{i=1}^{r} p_i \log y_i/z}\sum_{d\in\mathbb{K}_{\mathrm{eff},\mathcal{X}}}\prod_{i=1}^{m+l}\left(\frac{\prod_{k=\lceil\langle D_i^S,d\rangle\rceil}^{\infty}(D_i+(\langle D_i^S,d\rangle-k)z)}{\prod_{k=0}^{\infty}(D_i+(\langle D_i^S,d\rangle-k)z)}\right)\\ &\quad\quad\times\left(\frac{D_j}{z}+\langle D_j^S,d\rangle\right)y^d\mathbf{1}_{v(d)}.\end{aligned}$$

Hence, to compute the Batyrev element $\tilde{D}_j$, it remains to examine the expansion of the product factor

$$\begin{aligned}&\frac{\prod_{k=\lceil\langle D_i^S,d\rangle\rceil}^{\infty}(D_i+(\langle D_i^S,d\rangle-k)z)}{\prod_{k=0}^{\infty}(D_i+(\langle D_i^S,d\rangle-k)z)}\\ &\quad = C_d z^{-(\sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil+\#\{i:\langle D_i^S,d\rangle\in\mathbb{Z}_{<0}\})}\prod_{i:\langle D_i^S,d\rangle\in\mathbb{Z}_{<0}} D_i + h.o.t.,\end{aligned}$$

where

$$\begin{aligned}C_d = &\prod_{i:\langle D_i^S,d\rangle<0}\ \prod_{\langle D_i^S,d\rangle<k<0}(\langle D_i^S,d\rangle-k)\\ &\times\prod_{i:\langle D_i^S,d\rangle>0}\ \prod_{0\le k<\langle D_i^S,d\rangle}(\langle D_i^S,d\rangle-k)^{-1},\end{aligned}\tag{46}$$

The summand indexed by $d\in\mathbb{K}_{\mathrm{eff},\mathcal{X}}$ contributes to the coefficient of z^{-1} if and only if

$$\sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil+\#\{i:\langle D_i^S,d\rangle\in\mathbb{Z}_{<0}\}\le 1.$$

This happens only in the following three cases:

- $\sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil+\#\{i:\langle D_i^S,d\rangle\in\mathbb{Z}_{<0}\}=0$;
- $\begin{cases}\sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil=0,\\ \#\{i:\langle D_i^S,d\rangle\in\mathbb{Z}_{<0}\}=1;\end{cases}$

- $\begin{cases} \sum_{i=1}^{m+l} \lceil \langle D_i^S, d\rangle \rceil = 1, \\ \#\{i : \langle D_i^S, d\rangle \in \mathbb{Z}_{<0}\} = 0. \end{cases}$

The first case happens if and only if $d = 0$. If the second case happens, then

$$\sum_{i=1}^{m+l} \lceil \langle D_i^S, d\rangle \rceil = \sum_{i=1}^{m+l} \langle D_i^S, d\rangle = \langle \rho_{\mathcal{X}}^S, d\rangle = 0.$$

In particular,

$$\langle D_i^S, d\rangle \in \mathbb{Z}, \quad 1 \le i \le m + l.$$

Hence, we obtain the following lemma.

LEMMA 4.7. *For $1 \le j \le m + l$, the Batyrev element $\tilde{D}_j$ is given by*

$$\tilde{D}_j = D_j + \sum_{i=1}^{m} D_i \sum_{\substack{\langle \rho_{\mathcal{X}}^S, d\rangle = 0 \\ \langle D_i^S, d\rangle \in \mathbb{Z}_{<0} \\ \langle D_k^S, d\rangle \in \mathbb{Z}_{\ge 0}, \forall k \ne i}} C_d \langle D_j^S, d\rangle y^d + \sum_{\substack{\sum_{i=1}^{m+l} \lceil \langle D_i^S, d\rangle \rceil = 1 \\ \langle D_i^S, d\rangle \notin \mathbb{Z}_{<0}, \forall i}} C_d \langle D_j^S, d\rangle y^d \mathbf{1}_{v(d)}, \tag{47}$$

where C_d is given by equation (46).

5. Seidel Elements Corresponding to Box Elements

Consider the box element $s_j \in \mathrm{Box}(\mathbf{\Sigma})$ such that

$$\bar{s}_j = \sum_{i=1}^{m} c_{ji} \bar{b}_i \in \mathbf{N}_{\mathbb{Q}} \quad \text{for some } 0 \le c_{ji} < 1.$$

Note that $c_{ji} = 0$ for $i \in I_j^S$. We define a $\mathbb{C}^\times$-action on $\mathcal{U}^S \times (\mathbb{C}^2 \setminus \{0\})$ by

$$(z_1, \dots, z_{m+l}, u, v) \mapsto (z_1, \dots, t^{-1} z_{m+j}, \dots, z_{m+l}, tu, tv), \quad t \in \mathbb{C}^\times.$$

Hence, we have the associated bundle

$$\mathcal{E}_{m+j} = \mathcal{U}^S \times (\mathbb{C}^2 \setminus \{0\}) / G^S \times \mathbb{C}^\times$$

over $\mathbb{CP}^1$ with fiber $\mathcal{X}$.

We can identify $H^2(\mathcal{E}_{m+j}; \mathbb{Q})$ with $H^2(\mathcal{X}; \mathbb{Q}) \oplus \mathbb{Q}$ using the section $[\sigma_0]$. Note that this splitting given by the section $[\sigma_0]$ of $\mathcal{E}_{m+j}$ does not give a splitting of $\mathrm{Pic}(\mathcal{E}_{m+j})$ over $\mathbb{Z}$ since the intersection numbers of $[\sigma_0]$ with line bundles on $\mathcal{E}_{m+j}$ are fractional.

The fan of $\mathcal{E}_{m+j}$ is contained in $N_{\mathbb{Q}} \oplus \mathbb{Q}$. The 1-skeleton is given by

$$\hat{b}_i = (b_i, 0), \quad \text{for } 1 \le i \le m; \qquad \hat{b}_{m+1} = (0, 1); \qquad \hat{b}_{m+2} = (s_j, -1). \tag{48}$$

Then

$$\hat{D}_i^S = (D_i^S, -c_{ji}), \quad \text{for } 1 \le i \le m;$$

$$\hat{D}^S_{m+i} = (D^S_{m+i}, 0) \quad \text{for } 1 \le i \le l;$$
$$\hat{D}^S_{m+l+1} = \hat{D}^S_{m+l+2} = (0, 1).$$

The Seidel element is defined as in equation (5). Moreover, we define $\{p_0, p_1, \dots, p_r\}$ and $\{p^S_0, p^S_1, \dots, p^S_{r+l}\}$ in the same way as in Section 3.2.

As in the toric divisor case, we have the following expansion of the I-function:

$$I_{\mathcal{E}_{m+j}}(y, z) = e^{\sum_{i=0}^r p_i \log y_i / z}\Bigg(1 + z^{-1}\Bigg(\sum_{i=0}^r g_i^{(m+j)}(y) p_i + \tau_{tw}^{(m+j)}(y)\Bigg) + z^{-2}\Bigg(\sum_{n=0}^2 G_n^{(m+j)}(y) y_0^n\Bigg) + O(z^{-3})\Bigg), \tag{49}$$

and using the same argument as in Lemmas 3.13 and 3.15, we can show that $g_i^{(m+j)}(y)$ and $\tau_{tw}^{(m+j)}(y)$ are independent from y_0 for $1 \le i \le r$ and $1 \le j \le l$. Moreover, for each $j \in \{1, \dots, l\}$, we have

$$g_i^{(m+j)}(y_0, \dots, y_{r+l}) = g_i(y_1, \dots, y_{r+l}) \quad \text{for } i = 1, \dots, r$$

and

$$\iota^* \tau_{tw}^{(m+j)}(y) = \tau_{tw}(y).$$

We also obtain the following theorem.

THEOREM 5.1. *The Seidel element $\tilde{S}_{m+j}$ associated to the box element s_j is given by*

$$\tilde{S}_{m+j}(\tau(y)) := \tilde{S}_{m+j}(\tau^{(m+j)}(y)) = \exp(-g_0^{(m+j)}(y))\iota^*(G_1^{(m+j)}(y)). \tag{50}$$

Using the same computation as in the toric divisor case, we can compute the correction coefficient $g_0^{(m+j)}$.

LEMMA 5.2. *The function $g_0^{(m+j)}$ is given by*

$$g_0^{(m+j)}(y_1, \dots, y_{r+l}) = \sum_{1 \le k \le m, k \notin I^S_j} \sum_{\substack{\langle D^S_i, d\rangle \in \mathbb{Z}, 1 \le i \le m+l \\ \langle \rho^S_{\mathcal{X}}, d\rangle = 0 \\ \langle D^S_k, d\rangle < 0 \\ \langle D^S_i, d\rangle \ge 0, \forall i \ne k}} c_{jk} \frac{(-1)^{-\langle D^S_k, d\rangle}(-\langle D^S_k, d\rangle - 1)!}{\prod_{i \ne k}\langle D^S_i, d\rangle!} y^d, \tag{51}$$

where I^S_j is the "anticone" of the cone containing s_j.

Proof. The argument is almost the same as the argument in Section 3.5. The only change we need to make is the paragraph before Lemma 3.17:

In this case, $g_0^{(m+j)}$ is the coefficient corresponding to p_0, and elements in $\{\hat{D}_1, \dots, \hat{D}_m\}$ that contain p_0 are precisely the following elements:

$$\hat{D}_k = \langle D_k, -c_{jk}\rangle = D_k - c_{jk} p_0 \quad \text{for } 1 \le k \le m \text{ and } k \notin I^S_j.$$

Therefore, by expressions (37) and (39) we must have $\langle D_k^S, d\rangle < 0$ for exactly one k in $\{k \in \mathbb{Z} \mid 1 \le k \le m \text{ and } k \notin I_j^S\}$. $\square$

Moreover, by mimicking the computation in Lemma 4.4 we have the following:

LEMMA 5.3. *The I-function of $\mathcal{E}_{m+j}$ satisfies the following differential equation*:

$$z\frac{\partial}{\partial y_0}\left(y_0\frac{\partial}{\partial y_0}\right)I_{\mathcal{E}_j} = y^{-D^{S\vee}_{m+j}}\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right) - y_0\frac{\partial}{\partial y_0}\right)I_{\mathcal{E}_j}, \tag{52}$$

where $D^{S\vee}_{m+j} \in \mathbb{L}^S \otimes \mathbb{Q}$ is defined by (18).

Proof. The proof is almost identical to the proof of Lemma 4.4, except, this time, we need to choose $\beta = [\sigma_0] - D^{S\vee}_{m+j}$. Then everything else follows. $\square$

Using the lemmas and following the argument in the toric divisor case, we conclude:

THEOREM 5.4. *The Seidel element $\tilde{S}_{m+j}$ corresponding to the box element s_j with*

$$\bar{s}_j = \sum_{i=1}^{m} c_{ji}\bar{b}_i \quad \textit{for some } 0 \le c_{ji} < 1$$

is given by

$$\tilde{S}_{m+j}(\tau^{(m+j)}(y)) = \exp(-g_0^{(m+j)})y^{-D^{S\vee}_{m+j}}\tilde{D}_{m+j}(y), \tag{53}$$

where $\tilde{D}_{m+j}(y)$ is the corresponding Batyrev element. Moreover,

$$\begin{aligned}\tilde{D}_{m+j} = \sum_{i=1}^{m} D_i &\sum_{\substack{\langle \rho^S_{\mathcal{X}}, d\rangle = 0\\ \langle D_i^S, d\rangle \in \mathbb{Z}_{<0}\\ \langle D_k^S, d\rangle \in \mathbb{Z}_{\ge 0}, \forall k\neq i}} C_d\langle D^S_{m+j}, d\rangle y^d\\ &+ \sum_{\substack{\sum_{i=1}^{m+l}\lceil\langle D_i^S, d\rangle\rceil = 1\\ \langle D_i^S, d\rangle \notin \mathbb{Z}_{<0}, \forall i}} C_d\langle D^S_{m+j}, d\rangle y^d \mathbf{1}_{v(d)},\end{aligned} \tag{54}$$

and

$$\begin{aligned}C_d = &\prod_{i:\langle D_i^S, d\rangle < 0}\ \prod_{\langle D_i^S, d\rangle < k < 0}(\langle D_i^S, d\rangle - k)\\ &\times \prod_{i:\langle D_i^S, d\rangle > 0}\ \prod_{0 \le k < \langle D_i^S, d\rangle}(\langle D_i^S, d\rangle - k)^{-1}.\end{aligned} \tag{55}$$

ACKNOWLEDGMENTS. The author wants to thank professor Hsian-Hua Tseng for his guidance and lots of helpful discussions. The author is also very grateful for the referees' numerous useful comments and suggestions.

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