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The Symplectic Mapping Class Group of $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ with $n \le 4$

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Abstract. In this paper, we prove that the Torelli part of the symplectomorphism groups of the n-point ($n \le 4$) blow-ups of the projective plane is trivial. Consequently, we determine the symplectic mapping class group. It is generated by reflections on K_ω-spherical class with zero ω area.

1. Introduction

A symplectic manifold (X, ω) is an even-dimensional manifold X with a closed, nondegenerate two-form ω. The symplectomorphism group of (X, ω), denoted by $\mathrm{Symp}(X, \omega)$, is the group of diffeomorphisms ϕ of M that preserve ω and is given the C^∞-topology. $\mathrm{Symp}(X, \omega)$ is an infinite-dimensional Fréchet Lie group.

For a closed four-dimensional symplectic manifold (X, ω), since Gromov's work [Gro85], the homotopy type of $\mathrm{Symp}(X, \omega)$ has attracted much interest over the past 30 years. For the special case of some monotone 4-manifolds, the (rational) homotopy of $\mathrm{Symp}(X, \omega)$ was fully computed in [Gro85; AM99; Eva11]. However, for an arbitrary symplectic 4-manifold, the complication grows drastically: see [Abr98; AM99; Anj02] for $S^2 \times S^2$ and [AP12] for other instances.

The goal of this note is modest: for some rational 4-manifolds, we compute $\pi_0(\mathrm{Symp}(X, \omega))$, which is the symplectic mapping class group (denoted as SMC for short). In the cases we consider, the homological action of $\mathrm{Symp}(X, \omega)$ is already known in [LW11]. Therefore, it suffices to describe $\pi_0(\mathrm{Symp}_h(X, \omega))$, which is the subgroup of $\mathrm{Symp}(X, \omega)$ acting trivially on homology, namely, its Torelli part.

Theorem 1.1. *$\mathrm{Symp}_h(X, \omega)$ is connected for $X = \mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$ with arbitrary symplectic form ω.*

The cases $S^2 \times S^2$ and $(\mathbb{C}P^2\#k\overline{\mathbb{C}P^2})$ with $k \le 3$ are known before. Our approach actually works in a uniform way for all $k \le 4$ (see discussions in Remark 3.5). We also note that Theorem 1.1 is not true in general for $k \ge 5$; see Seidel's famous example in [Sei08].

Our strategy is based on Evans' beautiful approach in [Eva11] by systematically exploring the geometry of certain stable configuration of symplectic spheres (a related approach first appeared in Abreu's paper [Abr98]). It is summarized by

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the following diagram:

$$\begin{array}{ccccccccc} \mathrm{Symp}_c(U) & \longrightarrow & \mathrm{Stab}^1(C) & \longrightarrow & \mathrm{Stab}^0(C) & \longrightarrow & \mathrm{Stab}(C) & \longrightarrow & \mathrm{Symp}_h(X) \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & \mathcal{G}(C) & & \mathrm{Symp}(C) & & \mathcal{C}_0 \end{array} \tag{1}$$

Here $\mathcal{C}_0$ is the space of a full stable standard configuration of fixed homological type. Every other term in diagram (1) is a group associated to $C \in \mathcal{C}_0$, and $U = X \setminus C$. Now we give the definition of stable standard spherical configurations, and the groups will be discussed later in Section 2.1.

Definition 1.2. Given a symplectic 4-manifold (X, ω), we call an ordered finite collection of symplectic spheres $\{C_i, i = 1, \dots, n\}$ a spherical symplectic configuration, or simply a *configuration,* if

1. for any pair i, j with $i \neq j$, $[C_i] \neq [C_j]$ and $[C_i] \cdot [C_j] = 0$ or 1;
2. they are simultaneously J-holomorphic for some $J \in \mathcal{J}_\omega$;
3. $C = \bigcup C_i$ is connected.

We will often use C to denote the configuration. The homological type of C refers to the set of homology classes $\{[C_i]\}$.

Further, a configuration is called

- *standard* if the components intersect ω-orthogonally at every intersection point of the configuration; denote by $\mathcal{C}_0$ the space of standard configurations having the same homology type as C;
- *stable* if $[C_i] \cdot [C_i] \geq -1$ for each i;
- *full* if $H^2(X, C; \mathbb{R}) = 0$.

It is shown in [LW11] that for a rational manifold, the homological action of $\mathrm{Symp}(X, \omega)$ is generated by Lagrangian Dehn twists. Therefore, Theorem 1.1 implies the following:

Corollary 1.3. *For a rational manifold with Euler number up to 7, the SMC is a finite group generated by Lagrangian Dehn twists. Moreover, a generating set corresponds to a finite set of K_ω-null spherical classes with zero ω-area. In particular, SMC is trivial for generic choice of ω.*

It is shown in [BLW12] that the following proposition holds.

Proposition 1.4. *Suppose that (X^4, ω) is a symplectic rational manifold. Then $\mathrm{Symp}_h(X, \omega)$ acts transitively on the space of*

- *homologous Lagrangian spheres,*
- *homologous symplectic -2-spheres,*
- *$\mathbb{Z}_2$-homologous Lagrangian $\mathbb{R}P^2$ and homologous symplectic -4-spheres if $b_2^-(X) \leq 8$.*

Hence, we also have the following corollary.

COROLLARY 1.5. *For a rational manifold with Euler number up to* 7, *the space of*

- *homologous Lagrangian spheres*,
- $\mathbb{Z}_2$*-homologous Lagrangian* $\mathbb{R}P^2$,
- *homologous* -2 *symplectic spheres*,
- *homologous* -4 *symplectic spheres*

is connected.

2. Analyzing the Diagram

We analyze the diagram (1) and derive a criterion for the connectedness of $\mathrm{Symp}_h(X, \omega)$ in Corollary 2.10.

2.1. Groups Associated to a Configuration

Let C be a configuration in X. We first introduce the groups appearing in (1):

Subgroups of $\mathrm{Symp}_h(X, \omega)$. Recall that $\mathrm{Symp}_h(X, \omega)$ is the group of symplectomorphisms of (X, ω) that acts trivially on $H_*(X, \mathbb{Z})$.

- $\mathrm{Stab}(C) \subset \mathrm{Symp}_h(X, \omega)$ is the subgroup of symplectomorphisms fixing C setwise, but not necessarily pointwise.
- $\mathrm{Stab}^0(C) \subset \mathrm{Stab}(C)$ is the subgroup fixing C pointwise.
- $\mathrm{Stab}^1(C) \subset \mathrm{Stab}^0(C)$ is the subgroup fixing C pointwise and acting trivially on the normal bundles of its components.

$\mathrm{Symp}_c(U)$ *for the Complement* U. $\mathrm{Symp}_c(U)$ is the group of compactly supported symplectomorphisms of $(U, \omega|_U)$, where $U = X \setminus C$, and the form $\omega|_U$ is the inherited form on U from X. It is topologized in this way: let (U, ω) be a non-compact symplectic manifold, and let $\mathcal{K}$ be the set of compact subsets of U. For each $K \in \mathcal{K}$, let $\mathrm{Symp}_K(W)$ denote the group of symplectomorphisms of U supported in K, with the topology of $\mathcal{C}^\infty$-convergence. The group $\mathrm{Symp}_c(U, \omega)$ of compactly supported symplectomorphisms of (U, ω) is topologized as the direct limit of $\mathrm{Symp}_K(W)$ under inclusions.

$\mathrm{Symp}(C)$ *and* $\mathcal{G}(C)$ *for the Configuration* C. Given a configuration of embedded symplectic spheres $C = C_1 \cup \cdots \cup C_n \subset X$ in a 4-manifold, let I denote the set of intersection points among the components. Suppose that there is no triple intersection among components and that all intersections are transverse. Let k_i denote the cardinality of $I \cap C_i$, which is the number of intersection of points on C_i.

The group $\mathrm{Symp}(C)$ of symplectomorphisms of C fixing the components of C is the product $\prod_{i=1}^n \mathrm{Symp}(C_i, I \cap C_i)$. Here $\mathrm{Symp}(C_i, I \cap C_i)$ denotes the group of symplectomorphisms of C_i fixing the intersection points $I \cap C_i$. Since each C_i is a 2-sphere and $\mathrm{Symp}(S^2)$ acts transitivity on N-tuples of distinct points in S^2, we can write $\mathrm{Symp}(C_i, I \cap C_i)$ as $\mathrm{Symp}(S^2, k_i)$. Thus,

$$\mathrm{Symp}(C) \cong \prod_{i=1}^{n} \mathrm{Symp}(S^2, k_i). \tag{2}$$

As shown in [Eva11] we have:

$$\mathrm{Symp}(S^2, 1) \simeq S^1, \qquad \mathrm{Symp}(S^2, 2) \simeq S^1, \qquad \mathrm{Symp}(S^2, 3) \simeq \star, \tag{3}$$

where $\simeq$ means homotopy equivalence. And when $k = 1, 2$, the S^1 on the right can be taken to be the loop of a Hamiltonian circle action fixing the k points.

The symplectic gauge group $\mathcal{G}(C)$ is the product $\prod_{i=1}^n \mathcal{G}_{k_i}(C_i)$. Here $\mathcal{G}_{k_i}(C_i)$ denotes the group of symplectic gauge transformations of the symplectic normal bundle to $C_i \subset X$, which are equal to the identity at the k_i intersection points. As also shown in [Eva11],

$$\mathcal{G}_0(S^2) \simeq S^1, \qquad \mathcal{G}_1(S^2) \simeq \star, \qquad \mathcal{G}_k(S^2) \simeq \mathbb{Z}^{k-1}, \quad k > 1. \tag{4}$$

Since we assume that the configuration is connected, each $k_i \geq 1$. Thus, by (4) we have

$$\pi_0(\mathcal{G}(C)) = \bigoplus_{i=1}^n \pi_0(\mathcal{G}_{k_i}(S^2)) = \bigoplus_{i=1}^n \mathbb{Z}^{k_i-1}. \tag{5}$$

It is useful to describe a canonical set of k_i generators for $\mathcal{G}_{k_i}(C_i)$. For each intersection point $y \in I \cap C_i$, the evaluation map is the projection of the following homotopy fibration:

$$\mathcal{G}_{k_i}(C_i) \to \mathcal{G}_{k_i-1}(C_i) \stackrel{ev_y}{\to} SL(2, \mathbb{R}),$$

where the fiber $\mathcal{G}_{k_i-1}(C_i)$ is the gauge group fixing the other $k-1$ points except y. Inductively using this, we get the generators of $\mathcal{G}_{k_i}(C_i)$ marked by all k_i intersection points. Hence, it induces a map $\mathbb{Z} = \pi_1(SL(2, \mathbb{R})) \to \pi_0(\mathcal{G}_{k_i}(C_i))$. Let $g_{C_i}(y) \in \pi_0(\mathcal{G}_{k_i}(C_i))$ denote the image of $1 \in \mathbb{Z}$.

2.2. *Reduction to the Connectedness of* Stab(C)

The aim of this subsection is to show the following:

Proposition 2.1. $\mathrm{Symp}_h(X, \omega)$ *is connected if there is a full, stable, standard configuration C with connected* $\mathrm{Stab}(C)$.

This is derived from the right end of diagram (1) for a full, stable, standard configuration C. More explicitly, we consider the fibration

$$\mathrm{Stab}(C) \to \mathrm{Symp}_h(X, \omega) \to \mathcal{C}_0. \tag{6}$$

Recall that $\mathcal{C}_0$ is the space of standard configurations having the homology type of C. We will show that (1) is a homotopy fibration and $\mathcal{C}_0$ is connected.

We first review certain general facts regarding these configurations, which are well known to experts. By [LW11] we have the following fact.

Lemma 2.2. *Let (M, ω) be a symplectic 4-manifold, and C a stable configuration $\bigcup_i C_i$. Let $d(C_i)$ be the nonnegative integer given by $[C_i] \cdot [C_i] + c_1(X, \omega) \cdot [C_i]$. Then there is a path-connected Baire subset $\mathcal{T}_D$ of $\mathcal{J}_\omega \times \prod_i M^{d(C_i)}$ such that a pair $(J, \Omega = \prod_i \Omega_i)$, where $\Omega_i \in M^{d(C_i)}$, lies in $\mathcal{T}_D$ if and only if there is a unique* *embedded* J*-holomorphic configuration having the same homological type as* C *with the* i*th component containing* Ω_i.

LEMMA 2.3. *Assume that* C *is a stable, standard configuration. The space* $\mathcal{C}_0$ *of standard configurations having the homology type of* C *is path connected.*

Proof. Consider $\mathcal{C}$, the space of configurations as in Definition 1.2. By Lemma 2.2 we see that the space $\mathcal{C}$ is connected. Using a Gompf isotopy argument, it is shown in [Eva11] that the inclusion $\iota : \mathcal{C}_0 \to \mathcal{C}$ is a weak homotopy equivalence. Therefore, $\mathcal{C}_0$ is also connected. □

With C being full, the following lemma holds.

LEMMA 2.4. *If the stable, standard configuration* C *is also full, then* $\mathrm{Symp}_h(X,\omega)$ *acts transitively on* $\mathcal{C}_0$. *In particular,* (6) *is a homotopy fibration.*

Proof. By Lemma 2.3 any $C_1, C_2 \in \mathcal{C}_0$ are isotopic through standard configurations. The property that the configurations are *symplectically orthogonal* where they intersect, together with the *vanishing* of $H^2(X, C; \mathbb{R})$, allows us to extend such an isotopy to a global homologically trivial symplectomorphism of X (by Banyaga's symplectic isotopy extension theorem; see [MS05], Theorem 3.19). So we have shown that the action of $\mathrm{Symp}_h(X,\omega)$ on the connected space $\mathcal{C}_0$ is transitive by establishing the one-dimensional homotopy lifting property of the map $\mathrm{Symp}_h(X,\omega) \to \mathcal{C}_0$. By a finite-dimensional version of this argument (or Theorem A in [Pai60]) we conclude that (6) is a homotopy fibration. □

Proof of Proposition 2.1. Since (6) is a homotopy fibration by Lemma 2.4, we have the associated homotopy long exact sequence. Because of the connectedness of $\mathcal{C}_0$ as shown in Lemma 2.3, the connectedness of $\mathrm{Stab}(C)$ implies the connectedness of $\mathrm{Symp}_h(X,\omega)$. Therefore, we have 2.1 as the reduction of our problem. □

2.3. *Reduction to the Surjectivity of* $\psi : \pi_1(\mathrm{Symp}(C)) \to \pi_0(\mathrm{Stab}^0(C))$

To investigate the connectedness of $\mathrm{Stab}(C)$, considering the action of $\mathrm{Stab}(C)$ on C and the following portion of diagram (1), which appeared in [Eva11] and [AP12]:

$$\mathrm{Stab}^0(C) \to \mathrm{Stab}(C) \to \mathrm{Symp}(C). \tag{7}$$

The following lemma already appeared in [Eva11] and was explained to the authors by J. D. Evans.[1] We here include more details for readers' convenience.

LEMMA 2.5. *Diagram* (7) *is a homotopy fibration when* C *is a simply connected standard configuration.*

[1]Private communications.

Proof. First, we show that $\mathrm{Stab}(C) \to \mathrm{Symp}(C)$ is surjective.

Recall that at each intersection point between two different components $\{x_{ij}\} = C_i \cap C_j$, the two components are symplectically orthogonal to each other in a Darboux chart containing x_{ij}. For convenience of exposition, define the *level* of components as follows: let C_1 be the unique component of level 1, and the level k components are defined as those that intersect components in level $k-1$ but do not belong to any lower levels. This is well defined again because of the simply connectedness assumption.

An element in $\mathrm{Symp}(C)$ is the composition of Hamiltonian diffeomorphism ϕ_i on each component C_i because of the simply connectedness of a sphere. We start with endowing C_1 with a Hamiltonian function f_1 generating ϕ_1. Let C_i^2 be curves on level 2. Because C_i^2 intersects C_1 ω-orthogonally, we can find a symplectic neighborhood U_1 of C_1, identified as a neighborhood of the zero section of the normal bundle, so that $U_1 \cap C_i$ consists of finitely many fibers. Pull-back f_1 by the projection π of the normal bundle and multiply a cut-off function $\rho(r)$ such that $\rho(r) = 1$, $r \leq \epsilon \ll 1$; $\rho(r) = 0$, $r \geq 2\epsilon$. Here r is the radius in the fiber direction. Denote by $\bar{\phi}_1$ the symplectomorphism generated by this cut-off. Notice that $\bar{\phi}_1$ creates an extra Hamiltonian diffeomorphism ϵ_j on each component C_j of level 2, and we denote $\phi'_j = \phi_j \circ \epsilon_j^{-1}$ for C_j belonging to level 2.

We proceed by induction on the level k. Notice that we can always choose a Hamiltonian function f_i on a component C_i on level k that generates ϕ'_i with the property that $f_i(x_{il}) = 0$. Here C_l is the component of level $k-1$ intersecting C_i. We emphasize that this can be done because the component C_l on level $k-1$ that intersects C_i is unique (and that the intersection is a single point) due to the simply connectedness assumption, and we do not restrict the value on any other intersections of C_i and components of level $k+1$. Therefore, we only fix the value of f_i at a single point.

We then again use the pull-back on the symplectic neighborhood and cut-off along the fiber direction to get a Hamiltonian function H_i that generates a diffeomorphism $\bar{\phi}_i$ supported on the neighborhood of C_i. We note that $d(\pi^* f_1 \cdot \rho(r))|_{F_x} = 0$ whenever $f_1(x) = 0$, where F_x is the normal fiber over the point $x \in C_1$. Hence, $dH_i|_{C_l} = 0$ since $f_i(x_{il}) = 0$ as prescribed earlier, which means that the action of $\bar{\phi}_i$ on C_l is trivial. Taking the composition ϕ of all these $\bar{\phi}_i$, ϕ is supported on a neighborhood of C and equals ϕ_i when restricted to C_i.

The transitivity of the action of $\mathrm{Stab}(C)$ on $\mathrm{Symp}(C)$ follows easily. For any two maps $\phi_1, \phi_2 \in \mathrm{Symp}(C)$, $\phi_2\phi_1^{-1} \in \mathrm{Symp}(C)$. We can extend $\phi_2\phi_1^{-1}$ to $\mathrm{Stab}(C)$. Then this extended $\phi_2\phi_1^{-1}$ maps ϕ_1 to ϕ_2.

Now the symplectic isotopy theorem (or Theorem A in [Pai60]) for the surjective map $\mathrm{Stab}(C) \to \mathrm{Symp}(C)$ proves that diagram (7) is a fibration. □

Now we can establish the connectedness of $\mathrm{Stab}(C)$ under certain assumptions.

Proposition 2.6. *Let (X, ω) be a symplectic 4-manifold, and C a simply connected, full, stable, standard configuration. If each component of C has no* *more than three intersection points, then the surjectivity of the connecting map* $\psi : \pi_1(\mathrm{Symp}(C)) \to \pi_0(\mathrm{Stab}^0(C))$ *implies the connectedness of* $\mathrm{Stab}(C)$.

Proof. Since we assume that each component of C has no more than three intersection points, it follows from (3) and (2) that $\pi_0(\mathrm{Symp}(C)) = 1$.

By Lemma 2.5 we have the homotopy long exact sequence associated to (7),

$$\cdots \to \pi_1(\mathrm{Symp}(C)) \xrightarrow{\psi} \pi_0(\mathrm{Stab}^0(C)) \to \pi_0(\mathrm{Stab}(C)) \to \pi_0(\mathrm{Symp}(C)).$$

Then the surjectivity of ψ implies that $\mathrm{Stab}(C)$ is connected. □

2.4. Three Types of Configurations

Next, we investigate when the map $\psi : \pi_1(\mathrm{Symp}(C)) \to \pi_0(\mathrm{Stab}^0(C))$ is surjective. For this purpose, we observe that an element of $\mathrm{Stab}^0(C)$ induces an automorphism of the normal bundle of C. Thus, we further have the following homotopy fibration appeared in [Eva11] and [AP12]:

$$\mathrm{Stab}^1(C) \to \mathrm{Stab}^0(C) \to \mathcal{G}(C). \tag{8}$$

In particular, there is the associated map $\iota : \pi_0(\mathrm{Stab}^0(C)) \to \pi_0(\mathcal{G})(C)$. Consider the composition map

$$\bar{\psi} = \iota \circ \psi : \pi_1(\mathrm{Symp}(C)) \to \pi_0(\mathrm{Stab}^0(C)) \to \pi_0(\mathcal{G}(C)).$$

Notice that $\pi_0(\mathcal{G}(C))$ inherits a group structure from $\mathcal{G}(C)$ and $\bar{\psi}$ is a group homomorphism. As shown in [Eva11], $\bar{\psi}$ can be computed explicitly.

When $k_i = 3$, $\pi_1(\mathrm{Symp}(S^2, k))$ is trivial by (3). When $k_i = 1, 2$, $\mathrm{Symp}(C_i, I \cap C_i)$ is homotopic to the loop of a Hamiltonian circle action on C_i fixing the k_i points. Denote such a loop by $(\phi_i)_t$. Observe that $(\phi_i)_t$ is a generator of $\pi_1(\mathrm{Symp}(C_i, I \cap C_i)) = \mathbb{Z}$. Recall that for each component C_j, there is a canonical set of generators $\{g_{C_j}(y), y \in I \cap C_j\}$ for $\mathcal{G}_{k_j}(C_j)$, introduced at the end of Section 2.1. The following is Lemma 4.1 in [Eva11].

LEMMA 2.7. *Suppose that C_i is a component with $k_i = 1, 2$. The image of $[(\phi_i)_t] \in \pi_1(\mathrm{Symp}(C_i, I \cap C_i))$ under $\bar{\psi}$ is described as follows.*

- *If $k_i = 1$ and C_j is the only component intersecting C_i with $\{x\} = C_i \cap C_j$, then $(\phi_i)_{2\pi}$ is sent to*
$$g_{C_j}(x)$$
in the factor subgroup $\pi_0(\mathcal{G}_{k_j}(C_j))$ of $\pi_0(\mathcal{G}(C))$.
- *If $k_i = 2$ and $x \in C_i \cap C_j$, $y \in C_i \cap C_l$, then $(\phi_i)_{2\pi}$ is sent to*
$$(g_{C_j}(x), g_{C_l}(y))$$
in the factor subgroup $\pi_0(\mathcal{G}_{k_j}(C_j)) \times \pi_0(\mathcal{G}_{k_l}(C_l))$ of $\pi_0(\mathcal{G}(C))$.

Using Lemma 2.7, we will show that $\bar{\psi}$ is surjective for the following configurations.

DEFINITION 2.8. Introduce three types of configurations (see Figure 1 for examples).

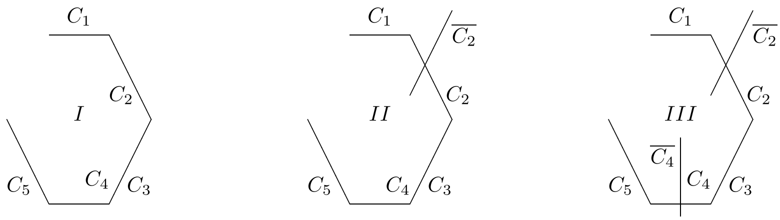

Figure 1

- (type I) $C = \bigcup_1^n C_i$ is called a chain, or a type I configuration, if $k_1 = k_n = 1$ and $k_j = 2, 2 \leq j \leq n-1$.
- (type II) Suppose that $C = \bigcup_1^n C_i$ is a chain. $C' = C \cup \overline{C_p}$ is called a type II configuration if the sphere $\overline{C_p}$ is attached to C_p at exactly one point for some p with $2 \leq p \leq n-1$.
- (type III) Suppose that $C' = C \cup \overline{C_p}$ is a type II configuration. $C'' = C' \cup \overline{C_q}$ is called a type III configuration if the sphere $\overline{C_q}$ is attached to C_q at exactly one point for some q with $2 \leq q \leq n-1$ and $q \neq p$.

Lemma 2.9. *$\bar{\psi}$ is surjective for a type I or II configuration and an isomorphism for a type III configuration.*

Proof. We first prove the surjectivity for a type I configuration $C = \bigcup_1^n C_i$. In this case, there are $n-1$ intersection points $x_1, \dots, x_{n-1}$ in total with

$$I \cap C_1 = \{x_1\}, \qquad I \cap C_n = \{x_{n-1}\}, \qquad I \cap C_i = \{x_{i-1}, x_i\}, \quad i = 2, \dots, n.$$

Notice that $\pi_1(\mathrm{Symp}(C_i, k_i)) = \mathbb{Z}$ for each $i = 1, \dots, n$. Notice also that $\pi_0(\mathcal{G}_{k_i}(C_i)) = \mathbb{Z}$ for each i for $i = 2, \dots, n-1$, and $\pi_0(\mathcal{G}_{k_1}(C_1))$ and $\pi_0(\mathcal{G}_{k_n}(C_n))$ are trivial. Thus, the homomorphism $\bar{\psi}_C$ associated to C is of the form $\mathbb{Z}^n \to \mathbb{Z}^{n-2}$.

For each $i = 1, \dots, n$, denote the generator $(\phi_i)_t$ of $\pi_1(\mathrm{Symp}(C_i, k_i)) = \mathbb{Z}$ by $\mathrm{rot}(i)$. For each $i = 2, \dots, n-1$, denote by $g_i(i-1)$ and $g_i(i)$ the generators $g_{C_i}(x_{i-1})$ and $g_{C_i}(x_i)$ of $\pi_0(\mathcal{G}_2(C_i)) = \mathbb{Z}$.

Then by Lemma 2.7 the homomorphism $\bar{\psi}_C$ is described by

$$\bar{\psi}_C: \begin{array}{l} \mathrm{rot}(1) \to g_2(1), \\ \mathrm{rot}(2) \to (0, g_3(2)), \\ \mathrm{rot}(j) \to (g_{j-1}(j-1), g_{j+1}(j)), \quad 3 \leq j \leq n-2, \\ \mathrm{rot}(n-1) \to (g_{n-2}(n-2), 0), \\ \mathrm{rot}(n) \to g_{n-1}(n-1). \end{array} \tag{9}$$

Choose the bases of $\pi_1(\mathrm{Symp}(C_i))$ and $\pi_0(\mathcal{G}(C))$ to be

$$\{\mathrm{rot}(1), \dots, \mathrm{rot}(n)\}$$

and

$$\{g_2(2), g_3(3), g_4(4), \ldots, g_{n-1}(n-1)\},$$

respectively. Notice that $g_i(i-1) = \pm g_i(i)$, then by (9) $\bar{\psi}_C$ is represented by the following $(n-2) \times n$ matrix if we drop the possible negative sign for each entry:

$$\begin{bmatrix} 1 & 0 & 1 & & & & & & \\ 0 & 1 & 0 & 1 & & & & & \\ 0 & 0 & 1 & 0 & 1 & 0 & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 1 & 0 & 1 & 0 & 0 \\ & & & & & 1 & 0 & 1 & 0 \\ & & & & & & 1 & 0 & 1 \end{bmatrix}.$$

Observe that the first $n-2$ minor as a $(n-2) \times (n-2)$ is a upper triangular matrix whose determinant is ± 1. This shows that $\bar{\psi}_C$ is surjective.

For a type II configuration $C' = C \cup \overline{C_p}$, let $\bar{x}_p$ be the intersection of C_p and $\overline{C_p}$. Notice that $\pi_1(\mathrm{Symp}(C')) = \mathbb{Z}^n$ as in the case of C, with the $\mathbb{Z}$ summand from C_p replaced by a $\mathbb{Z}$ summand from $\overline{C_p}$. Notice also that $\pi_0(\mathcal{G}(C')) = \mathbb{Z}^{n-1}$ with the extra $\mathbb{Z}$ summand coming from the new intersection point $\bar{x}_p$ in C_p. Denote by $\mathrm{rot}(\bar{p})$ the generator of $\pi_1(\mathrm{Symp}(\overline{C_p}, \bar{x}_p))$. Denote by $g'_p(p)$ the generator $g_{C_p}(\bar{x}_p)$ of $\pi_0(\mathcal{G}_3(C_p))$. By Lemma 2.7 the homomorphism $\bar{\psi}_{C'}$ is of the form $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$, and it differs from $\bar{\psi}_C$ as in (9):

$$\begin{aligned} &\mathrm{rot}(p) = 0, \\ &\mathrm{rot}(\bar{p}) \to g'_p(p). \end{aligned} \tag{10}$$

It is not hard to see that $\bar{\psi}_{C'}$ is again surjective. We illustrate by the type II configuration in Figure 1. With respect to the bases

$$\{\mathrm{rot}(1), \mathrm{rot}(\bar{2}), \mathrm{rot}(3), \mathrm{rot}(4), \mathrm{rot}(5)\} \quad \text{and} \quad \{g_2(2), g'_2(2), g_3(3), g_4(4)\},$$

$\bar{\psi}_{C'}$ is represented by the following 4×5 matrix (if we drop the possible negative sign):

$$\begin{bmatrix} 1 & 0 & 1 & & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 1 \end{bmatrix}.$$

For a type III configuration $C'' = C' \cup \overline{C_q} = C \cup \overline{C_p} \cup \overline{C_q}$, observe first that $\pi_1(\mathrm{Symp}(C'')) = \mathbb{Z}^n$ and $\pi_0(\mathcal{G}(C')) = \mathbb{Z}^n$. By Lemma 2.7 we can describe $\bar{\psi}_{C''} : \mathbb{Z}^n \to \mathbb{Z}^n$ similarly to the case of the type II configuration C'. Precisely, $\bar{\psi}_{C''}$ differs from $\bar{\psi}_C$ in (9) as follows:

$$\begin{aligned} \mathrm{rot}(p) &= \mathrm{rot}(q) = 0, \\ \mathrm{rot}(\bar{p}) &\to g'_p(p), \\ \mathrm{rot}(\bar{q}) &\to g'_q(q). \end{aligned} \tag{11}$$

It is easy to see that $\bar{\psi}_{C''}$ is an isomorphism in this case. We illustrate by the type III configuration in Figure 1. With respect to the bases

$$\{\mathrm{rot}(1), \mathrm{rot}(\bar{2}), \mathrm{rot}(3), \mathrm{rot}(\bar{4}), \mathrm{rot}(5)\} \quad \text{and} \quad \{g_2(2), g_2'(2), g_3(3), g_4'(4), g_4(4)\},$$

$\bar{\psi}_{C''}$ is represented by the following square matrix (if we drop the possible negative sign):

$$\begin{bmatrix} 1 & 0 & 1 & & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 1 \end{bmatrix}.$$

□

2.5. Criterion

Finally, we arrive at the following criterion for the connectedness of $\mathrm{Symp}_h(X, \omega)$.

Corollary 2.10. *Suppose that a stable, standard configuration C is type I, II, or III and that it is full. If $\mathrm{Symp}_c(U)$ is connected, then $\mathrm{Symp}_h(X, \omega)$ is connected.*

Proof. By Lemma 5.2 in [Eva11], $\mathrm{Symp}_c(U)$ is weakly homotopy equivalent to $\mathrm{Stab}^1(C)$. So by our assumption that $\mathrm{Symp}_c(U)$ is connected, $\mathrm{Stab}^1(C)$ is also connected. Therefore, the map $\iota : \pi_0(\mathrm{Stab}^0(C)) \to \pi_0(\mathcal{G})(C)$ associated to the homotopy fibration (8) is a group isomorphism. Now we have $\psi_C = \bar{\psi}_C$.

Since C is type I, II, or III, by Lemma 2.9 ψ_C is surjective. Notice that any type I, II, or III configuration is simply connected. By the assumption of C being full, we can apply Proposition 2.6 and Proposition 2.1 to conclude that $\mathrm{Symp}_h(X, \omega)$ is connected. □

3. Proof in the Case of $\mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$

3.1. The Configuration for $\mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$

Let $X = \mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$, and ω an arbitrary symplectic form on X. We consider a configuration C in [Eva11], consisting of symplectic spheres in homology classes $S_{12} = H - E_1 - E_2$, $S_{34} = H - E_3 - E_4$, E_1, E_2, E_3, and E_4. Here $\{H, E_i\}$ is the standard basis of $H_2(X; \mathbb{Z})$ with positive pairing with ω. In Figure 2, we label the spheres by their homology classes.

To apply the criterion in Corollary 2.10, we need to check that we can always find a configuration C of such a homology type, so that

- C is stable,
- C is a type I, II, or III configuration,
- C is full,
- $\mathrm{Symp}_c(U)$ is connected.

The existence of such a configuration is a direct consequence of Gromov–Witten theory, and the first three statements follows from the definition. Note also

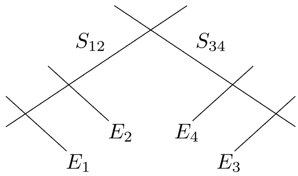

Figure 2

that the actual choice of configuration will not affect the last statement because $\mathrm{Symp}_h(X)$ acts transitively on $\mathcal{C}_0$, which means that U is well defined up to symplectomorphism for any choice of $C \in \mathcal{C}_0$.

It thus remains to prove the connectedness of $\mathrm{Symp}_c(U)$. In the next subsection, we will actually show that $\mathrm{Symp}_c(U)$ is weakly contractible.

3.2. Contractibility of $\mathrm{Symp}_c(U)$

Let us first recall the following result of Evans (Theorem 1.6 in [Eva11]).

THEOREM 3.1. *If* $\mathbb{C}^* \times \mathbb{C}$ *is equipped with the standard (product) symplectic form* ω_{std}, *then* $\mathrm{Symp}_c(\mathbb{C}^* \times \mathbb{C})$ *is weakly contractible.*

This is relevant since Evans observed in Section 4.2 in his thesis [Eva10] that if (ω, J_0) is Kähler with ω monotone and C holomorphic, then (U, J_0) has a finite-type Stein structure f with $\omega|_U = -dd^c f$, and there is a biholomorphism Ψ from (U, J_0) to $\mathbb{C}^* \times \mathbb{C}$ (in addition, Ψ satisfies $\Psi^*\omega_{\mathrm{std}} = \omega|_U$). We will generalize and prove this observation in nonmonotone cases in Proposition 3.3.

Let us also recall the next result of Evans (Proposition 2.2 in [Eva11]).

PROPOSITION 3.2. *If* (W, J_0) *is a complex manifold with two finite-type Stein structures* ϕ_1 *and* ϕ_2, *then* $\mathrm{Symp}_c(W, -dd^c\phi_1)$ *and* $\mathrm{Symp}_c(W, -dd^c\phi_2)$ *are weakly homotopy equivalent.*

Now we complete our proof of the connectedness of $\mathrm{Symp}_h(\mathbb{C}P^2\#4\overline{\mathbb{C}P^2}, \omega)$ for an arbitrary ω by proving the following:

PROPOSITION 3.3. $\mathrm{Symp}_c(U, \omega|_U)$ *is weakly contractible.*

Proof. We first choose a specific configuration C convenient for our purpose (as we explained in Section 3.1, this does not affect our result). According to [Li08, Proposition 4.8], we can always pick an integrable complex structure J_0 compatible with ω, so that (X, J_0) is biholomorphic to a generic blow up of four points on $\mathbb{C}P^2$ (the genericity here means that no three points lie on the same line, and indeed this can always be done for less than nine point blow ups). For such a generic holomorphic blow up, there is a unique smooth rational curve in each class in the homology type of C. Thus, we canonically obtain a configuration C associated to J_0. Observe that the complement $U = X \setminus C$ is biholomorphic to $\mathbb{C}^* \times \mathbb{C}$ because the configuration C is the total transformation of two lines blowing up at four points. Removing C gives us a biholomorphism from (U, J_0) to $\mathbb{C}P^2$ with two lines removed, which is $\mathbb{C}^* \times \mathbb{C}$.

Now we construct a Stein structure ϕ on (U, J_0) with $-dd^c\phi = \omega|_U$ whenever ω is a rational symplectic form on $\mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$. Since (U, J_0) is biholomorphic to $\mathbb{C}^* \times \mathbb{C}$ equipped with the standard finite-type Stein structure $(J_{\mathrm{std}}, \omega_{\mathrm{std}} = -dd^c|z|^2)$, we can then apply Proposition 3.2 and Theorem 3.1 in this case to conclude the weak contractibility of $\mathrm{Symp}_c(U, \omega|_U)$.

So we assume that $[\omega] \in H^2(X; \mathbb{Q})$. Up to rescaling, we can write $PD([l\omega]) = aH - b_1E_1 - b_2E_2 - b_3E_3 - b_4E_4$ with $a, b_i \in \mathbb{Z}^{\geq 0}$. Further, we assume that $b_1 \geq b_2, b_3 \geq b_4$. Since $H - E_1 - E_3$ is an exceptional class, we also have $\omega(H - E_1 - E_3) > 0$. This means that $a > b_1 + b_3$, namely, $2a \geq 2b_1 + 2b_3 + 2$. Rewrite

$$\begin{aligned} PD([2l\omega]) = {} & (2b_1+1)(H - E_1 - E_2) + E_1 + (2b_1 - 2b_2 + 1)E_2 \\ & + (2a - 2b_1 - 1)(H - E_3 - E_4) + (2a - 1 - 2b_1 - 2b_3)E_3 \\ & + (2a - 1 - 2b_1 - b_4)E_4. \end{aligned}$$

Notice that the coefficients are all in $\mathbb{Z}^{>0}$. In this way, we represent $PD([2l\omega])$ as a positive integral combination of all elements in the set $\{H - E_1 - E_2, H - E_3 - E_4, E_1, E_2, E_3, E_4\}$, which is the homology type of C.

Denote the symplectic sphere with homology class E_i in C by C_{E_i} and similarly for the two remaining spheres. Notice that each sphere is a smooth divisor. Consider the effective divisor

$$\begin{aligned} F = {} & (2b_1+1)C_{H-E_1-E_2} + C_{E_1} + (2b_1 - 2b_2 + 1)C_{E_2} \\ & + (2a - 2b_1 - 1)C_{H-E_3-E_4} + (2a - 1 - 2b_1 - 2b_3)C_{E_3} \\ & + (2a - 1 - 2b_1 - b_4)C_{E_4}. \end{aligned}$$

There is a holomorphic line bundle $\mathcal{L}$ with a holomorphic section s whose zero divisor is exactly F. Notice that

$$c_1(\mathcal{L}) = [F] = [2l\omega].$$

By [GH94, Section 1.2] we can take an Hermitian metric $|\cdot|$ and a compatible connection on $\mathcal{L}$ such that the curvature form is just $2l\omega$. Moreover, for the holomorphic section s, the function $\phi = -\log|s|^2$ is plurisubharmonic on the complement U with $-d(d\phi \circ J_0) = 2l\omega$. Notice that F and C have the same support, so that the complement of F is the same as U. Thus, we have endowed (U, J_0) with a finite-type Stein structure ϕ.

As argued before, this implies that $\mathrm{Symp}_c(U, \omega|_U) = \mathrm{Symp}_c(U, 2l\omega|_U)$ is weakly contractible when $[\omega] \in H^2(X, \mathbb{Q})$ by the biholomorphism from (U, J_0) to $(\mathcal{C}^* \times \mathcal{C}, J_{\mathrm{std}})$.

Finally, suppose that ω is not rational, but we assume that $\omega(H) \in \mathbb{Q}$ without loss of generality by rescaling. We take a base point $\varphi_0 \in \mathrm{Symp}_c(U, \omega|_U)$ and a S^n $(n \geq 0)$ family of symplectomorphisms determined by a based map $\iota : S^n \to$

$\mathrm{Symp}_c(U, \omega'|_U)$. Denote the union of support of this S^n family by V_ι, which is a compact subset of U.

Note the following fact.

CLAIM 3.4. *There exists an* ω' *symplectic on* X *such that*:

(1) $[\omega'] \in H^2(X, \mathbb{Q})$,

(2) $[\omega'](E_i) \geq [\omega](E_i)$, $[\omega'](H) = [\omega](H)$,

(3) *the configuration* C *is* ω'*-symplectic*,

(4) $(X \setminus C, \omega') \hookrightarrow (X \setminus C, \omega)$ *in such a way that the image contains* V_ι.

Proof. Recall that to blow up an embedded ball B in a symplectic manifold (M, ω), we remove the ball and collapse the boundary by a Hopf fibration that incurs an exceptional divisor. The reverse of this procedure is a blowdown.

Now take E_i in the configuration C and blow them down to get a disjoint union of balls B_i in the blown-down manifold, which is a symplectic $\mathbb{C}P^2$ with line area equal $\omega(H)$. We then enlarge B_i by a very small amount to B_i' so that the sizes of B_i' become rational numbers. After the enlargement, blow up B_i'. This produces a symplectic form on X, which clearly satisfies (1) and (2). Property (3) can be achieved as long as the enlarged ball has boundary intersecting proper transformation of S_{12} and S_{34} on a big circle. This is always possible: perturb S_{12} and S_{34} slightly so that they are symplectically orthogonal to E_i before blow-down. Then in a neighborhood of the resulting balls B_i, we have a Darboux chart where B_i is the standard ball, whereas the portion of S_{12} and S_{34} inside this chart is the $x_1 - x_2$ plane. This is guaranteed by the symplectic neighborhood theorem near E_i. Hence, (3) is obtained when the enlargement stays inside the Darboux chart. For more details, we refer to [MW96].

To see (4), we note that by the previous description $(X \setminus C, \omega')$ is symplectomorphic to the complement of $\bigcup_i B_i'$ union two lines (the proper transforms of S_{12} and S_{34}) in the symplectic $\mathbb{C}P^2$ from blowing down. The same thus applies to $(X \setminus C, \omega)$ with B_i' replaced by $B_i \subset B_i'$. Therefore, the statement regarding embedding holds in (4). Since V_ι is compact and embeds in $(X \setminus C, \omega)$, as long as the amount of enlargement from B_i to B_i' is small enough, the embedded image contains V_ι, as claimed. □

Therefore, we can find an isotopy in $\mathrm{Symp}_c(U, \omega'|_U) \hookrightarrow \mathrm{Symp}_c(U, \omega|_U)$ from the S^n family of maps to the base point φ_0 by the proved case where ω is rational. We emphasize in the previous proof that the choice of ω' depends on ι, but this is irrelevant for our purpose. This concludes that for arbitrary symplectic form ω on X, $\mathrm{Symp}_c(U, \omega|_U)$ is weakly contractible, and hence $\mathrm{Symp}_h(\mathbb{C}P^2\#4\overline{\mathbb{C}P^2})$ is connected for any symplectic form. □

REMARK 3.5. The approach we adopt in this note in fact provides a uniform way to establish the connectedness of the Torelli part of SMC for all symplectic rational 4-manifolds with $\chi \leq 7$. This can be viewed as a continuation of the techniques first introduced by Gromov [Gro85] and further developed by many others in [Abr98; AM99; LP04; Eva11; AP12], and so on.

Here we just list the configurations for the 1, 2, 3-point blow up of $\mathbb{C}P^2$ equipped with an arbitrary symplectic form:

- $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, $\{E_1, H - E_1(\text{with a marked point})\}$,
- $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$, $\{E_1, E_2, H - E_1 - E_2\}$,
- $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$, $\{E_1, E_2, H - E_1 - E_2, H - E_1 - E_3, H - E_2 - E_3\}$.

The configurations are all of type I. Combined with our argument verbatim, we can recover the connectedness of $\mathrm{Symp}_h(\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}, \omega)$, $n \leq 3$. However, such a result for these manifolds is not new; see [Abr98; AM99; LP04; Eva11].

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