Michigan Math. J. 64 (2015), 3–38

Merging Divisorial with Colored Fans

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Abstract. Given a spherical homogeneous space G/H of minimal rank, we provide a simple procedure to describe its embeddings as varieties with torus action in terms of divisorial fans. The torus in question is obtained as the identity component of the quotient group N/H, where N is the normalizer of H in G. The resulting Chow quotient is equal to (a blowup of) the simple toroidal compactification of $G/(HN^{\circ})$. In the horospherical case, for example, it is equal to a flag variety, and the slices (coefficients) of the divisorial fan are merely shifts of the colored fan along the colors.

1. Introduction

We are working over the base field $\mathbb{C}$. Normal varieties X coming with an effective action of an algebraic torus $\mathbb{T}$, also called $\mathbb{T}$-varieties, can be encoded by divisorial fans $\mathcal{S}^X = \sum_{D\subseteq Y} \mathcal{S}_D^X \otimes D$ on algebraic varieties Y of dimension equal to the complexity of the torus action. In this notation, $D \subseteq Y$ runs through all prime divisors on Y, and $\mathcal{S}_D^X$ denotes a combinatorial object associated to D (being nontrivial for finitely many summands only). Let N denote the lattice of one-parameter subgroups of $\mathbb{T}$. Every $\mathcal{S}_D^X$ stands for a polyhedral subdivision of $N_{\mathbb{Q}}$ together with a prescribed labeling of its cells referring to the set of affine charts covering X.

The $\mathbb{T}$-variety X in question is then given as a contraction of a toric fibration over Y, and the data D and $\mathcal{S}_D^X$ describe exactly where and how this fibration degenerates, respectively. Vice versa, X can be reconstructed explicitly from $\mathcal{S}^X$ in two steps. First, one glues certain relative spectra over Y; the result of this procedure is called $\widetilde{\mathbb{TV}}(\mathcal{S}^X)$. Finally, we obtain X as $\mathbb{TV}(\mathcal{S}^X)$, which denotes a certain birational contraction of $\widetilde{\mathbb{TV}}(\mathcal{S}^X)$. See Section 2 for further details.

1.1. The Comparison Theorem

Let G be a connected reductive group, and $H \subset G$ a spherical subgroup such that the spherical homogenous space G/H is of *minimal rank* (see Definition 3.5). The

Received November 21, 2012. Revision received January 12, 2015.
The second author was supported by Dynasty foundation, AG Laboratory NRU HSE, RScF grant 14-21-00053 (Sections 3 and 7), MESRF grants ag. 11.G34.31.0023, MK-983.2013.1, and by RFBR grants 12-01-31429-mol-a and 12-01-33101-mol-a-ved. This study was carried out within "The National Research University Higher School of Economics" Academic Fund Program in 2013–2014, research grant No. 12-01-0194.

goal of this paper is to describe spherical embeddings $X \supseteq G/H$ by a divisorial fan $\mathcal{S}$, that is, $X = \mathbb{TV}(\mathcal{S})$, on a modification Y of the simple, toroidal, and hence often wonderful compactification $\overline{Y} \supseteq G/H'$ with $H' := H \cdot N_G(H)^\circ$, see (3.1) and (3.2). The latter spaces are very well understood; for any G, there are only finitely many of them, and the modification $Y \to \overline{Y}$ is given by a certain fan Σ_Y refining the valuation cone $\mathcal{V}_{H'} \cong \mathbb{Q}^\ell_{\geq 0}$ of $\overline{Y}$, where l denotes the rank of G/H'. The basic tool for this construction will be the Tits fibration $\phi : G/H \to G/H'$. Its central fiber is the torus $\mathbb{T} := H'/H$. It acts on X from the right, which turns the spherical variety X into a $\mathbb{T}$-variety with $\mathcal{S} = \mathcal{S}^X$. See Section 3 for further details about spherical varieties.

Fixing a Borel subgroup $B \subseteq G$ such that $B \cdot H$ is open and dense in G, denote by $\mathbb{C}(G/H)^{(B)} \twoheadrightarrow \mathcal{X}(G/H)$ the sets of B-semiinvariant functions and their character lattice within $\mathcal{X}_B := \operatorname{Hom}(B, \mathbb{C}^*)$, respectively. The dual lattices are connected by an exact sequence

$$0 \to N \to \mathcal{X}^*(G/H) \xrightarrow{p} \mathcal{X}^*(G/H') \to 0;$$

see [Bri97, Théorème 4.3(ii)] and Proposition 4.3. Let $\mathcal{C}(G/H')$ denote the set of colors of G/H', that is, the set of B-invariant prime divisors of G/H'. After fixing a splitting of the above exact sequence, our main result is the following:

THEOREM 1.1. *Let $X \supseteq G/H$ be a spherical embedding of minimal rank given by a colored fan Σ^X inside $\mathcal{X}^*_{\mathbb{Q}}(G/H)$. Denote by $\mathcal{V}_H \subseteq \mathcal{X}^*_{\mathbb{Q}}(G/H)$ the valuation cone. Then $X = \mathbb{TV}(\mathcal{S})$, where $\mathcal{S}$ is a divisorial fan on (Y, N) with*:

(1) *The base space Y is the toroidal spherical embedding of G/H' given by the (un-)colored fan $(\Sigma_Y, \emptyset)$ arising as the image fan* (see Definition 4.4) *of $\Sigma^X \cap \mathcal{V}_H$ via the map p. Its rays $a \in \Sigma_Y(1)$ correspond to the G-invariant divisors $D_a \subseteq Y$.*

(2) *The maximal cells of the divisorial fan $\mathcal{S} = \mathcal{S}^X$ describing X as a $\mathbb{T}$-variety are labeled by the maximal colored cones $C = (C, \mathcal{F}_C) \in \Sigma^X$ and the elements $w \in W$ of the Weyl group of G. The part of $\mathcal{S}^X$ with label (C, w) is equal to*

$$\mathcal{S}^X(C, w) = \sum_{a \in \Sigma_Y(1)} \mathcal{S}^X_a(C) \otimes D_a + \sum_{D' \in \mathcal{C}(G/H')} (\overline{\rho}(D') + \mathcal{S}^X_0(C)) \otimes \overline{D'}$$
$$+ \sum_{D' \in \mathcal{C}(G/H') \setminus \mathcal{F}_C} \emptyset \otimes w\overline{D'},$$

where $\mathcal{S}^X_a(C) := C \cap p^{-1}(a)$ is considered as an element of $p^{-1}(a) \cong N_{\mathbb{Q}}$.

Note that the cells of the special fiber form a fan $\mathcal{S}^X_0$. Since it exhibits the asymptotic behavior of all other fibers $\mathcal{S}^X_a$, we will sometimes also call it the tail fan $\operatorname{tail}(\mathcal{S}^X)$. Let us furthermore point out that the coefficients of the colors D' are just shifts of $\operatorname{tail}(\mathcal{S}^X)$. The shift vectors $\overline{\rho}(D') \in N$ are defined as projections to N of the valuations $\rho_D \in \mathcal{X}^*(G/H)$ corresponding to the colors of G/H'; see (4.4). Finally, we would like to remark that the labeling of the maximal cells is not quite bijective. In (7.1) we will see that for a given $C = (C, \mathcal{F}_C) \in \Sigma^X$, the accompanying $w \in W$ are rather parameterized by W/W_C for some subgroup $W_C \subseteq W$ depending on $\mathcal{F}_C$.

It would be interesting to generalize Theorem 1.1 to other spherical varieties. As Example 7.3 shows, the divisorial fan $\mathcal{S}^X$ should contain additional maximal cells apart from those listed in Theorem 1.1.

1.2. *Possible Applications*

We believe that merging these two partially combinatorial descriptions via divisorial and colored fans may help to obtain further results and insights into the realm of spherical varieties, in particular, concerning their deformation theory (see e.g. [AB04]), and the computation of their Cox rings. As an example for the latter subject, we would like to mention that the Cox ring of a horospherical variety is known to be a polynomial ring over the Cox ring of a flag variety ([Bri07b, Theorem 4.3.2] or [Gag, Theorem 3.8]). However, an alternative way to understand this might be to combine our Theorem 1.1 with Theorem 1.2 from [HS10].

1.3. *Content of the Paper*

The present paper is organized as follows. In Sections 2 and 3, we shortly review polyhedral divisors and spherical varieties, respectively. Section 4 then introduces the $\mathbb{T}$-action on a spherical variety, which is relevant for our purposes and was already announced at the beginning of this section. Moreover, the toroidal part of our main Theorem 1.1 appears there as Theorem 4.6.

Sections 5 and 6 contain the proof of Theorem 4.6. The main idea is to reinterpret well-known facts from the spherical context within the context of divisorial fans. Using the language of p-divisors allows us to recover the encoded spherical variety directly from the given combinatorial data.

Section 7 finally deals with the nontoroidal case, and we conclude by presenting several examples in Section 8.

2. P-Divisors and Divisorial Fans

The upshot of [AH06; AHS08] is that normal varieties X with a complexity-k action of an algebraic torus correspond to p-divisors $\mathcal{D}^X$ (for affine X) or divisorial fans $\mathcal{S}^X$ (for general X) on a k-dimensional variety Y. The latter variety is the so-called Chow quotient $Y = X /\!\!/^{\mathrm{ch}} \mathbb{T}$ and defined as a GIT-limit quotient of the $\mathbb{T}$-action on X. But any modification of $X /\!\!/^{\mathrm{ch}} \mathbb{T}$ could be taken as well. Both data $\mathcal{D}^X$ and $\mathcal{S}^X$ induce a diagram like

$$Y \xleftarrow{\ \pi\ } \widetilde{X} \xrightarrow{\ r\ } X,$$

where r is a $\mathbb{T}$-equivariant proper birational contraction resolving the indeterminacies of the rational quotient map $\pi : X \dashrightarrow Y$. Whereas X is obtained as $\mathbb{TV}(\mathcal{S}^X)$, the auxiliary $\mathbb{T}$-variety $\widetilde{X}$ shows up as $\widetilde{\mathbb{TV}}(\mathcal{S}^X)$. We are now going to recall this language in more detail.

2.1. *Polyhedral Divisors*

Let $N \cong \mathbb{Z}^n$ be a free Abelian group of rank n, and denote its dual by $M := \operatorname{Hom}(N, \mathbb{Z})$. These data give rise to the torus $\mathbb{T} = N \otimes_{\mathbb{Z}} \mathbb{C}^*$, and one can recover M and N as its lattice of characters and 1-parameter subgroups, respectively. Let us furthermore consider convex polyhedra $\Delta \subseteq N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^n$. The *tail cone* of a polyhedron Δ is defined as

$$\operatorname{tail}(\Delta) := \{a \in N_{\mathbb{Q}} \mid a + \Delta \subseteq \Delta\}.$$

Note that the set of polyhedra with fixed tail cone σ forms a semigroup $\operatorname{Pol}^+(N, \sigma)$ with cancelation property (the addition is given by the Minkowski sum).

DEFINITION 2.1. Let Y be a normal, semiprojective (i.e., projective over an affine) variety and fix a polyhedral, pointed cone $\sigma \subseteq N_{\mathbb{Q}}$. A finite formal sum $\mathcal{D} = \sum_D \Delta_D \otimes D$ is called a *polyhedral divisor* on (Y, N) with $\operatorname{tail}(\mathcal{D}) = \sigma$ if

(1) all D are prime divisors on Y,
(2) all $\Delta_D \subseteq N_{\mathbb{Q}}$ are convex polyhedra with $\operatorname{tail}(\Delta_D) = \sigma$, and
(3) for every $u \in \sigma^\vee \cap M$, the evaluation $\mathcal{D}(u) := \sum_D \min\langle \Delta_D, u\rangle \cdot D$ is an element of the group of rational Cartier divisors $\operatorname{CaDiv}_{\mathbb{Q}}(Y)$ on Y.

REMARK 2.2. (i) The tail cone σ serves as the neutral element in $\operatorname{Pol}^+(N, \sigma)$; hence, summands of the form $\sigma \otimes D$ may be added or suppressed without having any impact on $\mathcal{D}$.

(ii) On the other hand, we will also allow $\emptyset$ as a possible coefficient. Whereas we define $\emptyset + \Delta := \emptyset$, the summand $\emptyset \otimes D$ indicates that the remaining sum is to be considered on $Y \setminus D$ instead of Y. This allows us to always ask for projective Y, although $\mathcal{D}$ is only defined on its *locus* $\operatorname{loc}(\mathcal{D}) := Y \setminus \bigcup_{\Delta_D = \emptyset} D$.

(iii) Condition (3) is automatically fulfilled for $\mathbb{Q}$-factorial, in particular, for smooth base varieties Y.

Concavity of the min function, that is, $\min\langle \Delta_D, u\rangle + \min\langle \Delta_D, v\rangle \le \min\langle \Delta_D, u + v\rangle$, implies that the M-graded sheaf

$$\mathcal{A} := \bigoplus_{u \in \operatorname{tail}(\mathcal{D})^\vee \cap M} \mathcal{O}_Y(\mathcal{D}(u))$$

carries the structure of an $\mathcal{O}_Y$-algebra that induces the following scheme over Y or, actually, over $\operatorname{loc}(\mathcal{D})$.

DEFINITION 2.3. (1) Let $\mathcal{D}$ be a polyhedral divisor on (Y, N). Then we call

$$\widetilde{\mathbb{TV}}(\mathcal{D}) := \operatorname{Spec}_Y \mathcal{A} \xrightarrow{\pi} \operatorname{loc}(\mathcal{D}) \hookrightarrow Y$$

the *relative* $\mathbb{T}$-variety associated to $\mathcal{D}$. It is affine if and only if $\operatorname{loc}(\mathcal{D})$ is.

(2) $\mathcal{D}$ is called *positive* (or short "p-divisor") if $\mathcal{D}(u)$ is semiample and big on $\mathrm{loc}(\mathcal{D})$ for every $u \in \sigma^\vee \cap M$ or $u \in \mathrm{int}\,\sigma^\vee \cap M$, respectively. If this is the case, then we define its associated *absolute* $\mathbb{T}$-variety $\mathbb{TV}(\mathcal{D}) := \mathrm{Spec}\,\Gamma(\mathrm{loc}(\mathcal{D}), \mathcal{A})$.

We would like to remark that also on a possibly noncomplete variety Y a rational Cartier divisor D is called semiample if it admits a basepoint-free multiple. If Y is semiprojective, then the spaces of sections $\Gamma(Y, \mathcal{O}_Y(mD))$ are finitely generated $\Gamma(Y, \mathcal{O}_Y)$-modules and, hence, define maps to projective spaces over $\Gamma(Y, \mathcal{O}_Y)$. Their images do not depend on the choice of generators, and we call a divisor D big if $|mD|$ induces a birational morphism for $m \gg 0$. It follows then from [AH06, Theorem 3.1] that $\widetilde{\mathbb{TV}}(\mathcal{D}) \to \mathbb{TV}(\mathcal{D})$ are normal varieties with the function field $\mathrm{Quot}\,\mathbb{C}(Y)[M]$. Moreover, the M-grading of $\mathcal{A}$ translates into a $\mathbb{T}$-action on both varieties, and π is a good quotient. Finally, all normal affine $\mathbb{T}$-varieties arise this way.

2.1.1. Toric Picture. Affine toric varieties X are $\mathbb{T}$-varieties of complexity 0, that is, $Y = \mathrm{pt}$. The notion of a polyhedral divisor collapses to its tail cone, that is, a polyhedral cone $\sigma \in N_{\mathbb{Q}}$ with $X = \mathbb{TV}(\sigma) = \widetilde{\mathbb{TV}}(\sigma)$.

2.1.2. $\mathbb{C}^* \curvearrowright \mathbb{C}^2$. For example, let us consider three different types of $\mathbb{C}^*$-actions on the affine plane $\mathrm{Spec}\,\mathbb{C}[x, y]$. The latter are specified by their weights on the variables x and y, respectively. It is easy to check directly that these actions correspond to the following polyhedral divisors $\mathcal{D}^\bullet = \Delta_0^\bullet \otimes 0 + \Delta_\infty^\bullet \otimes \infty$ on $\mathbb{P}^1$ such that $\mathbb{C}^* \curvearrowright \mathbb{C}^2 = \mathbb{TV}(\mathcal{D}^\bullet)$:

deg x	deg y	type of action
1	0	parabolic
1	1	elliptic
1	-1	hyperbolic

	$\Delta_0^\bullet$	$\Delta_\infty^\bullet$	tail cone	locus
$\mathcal{D}^p$	$[0, \infty)$	$\emptyset$	$[0, \infty)$	$\mathbb{P}^1 \setminus \infty$
$\mathcal{D}^e$	$[1, \infty)$	$[0, \infty)$	$[0, \infty)$	$\mathbb{P}^1$
$\mathcal{D}^h$	$[0, 1]$	$\emptyset$	$\{0\}$	$\mathbb{P}^1 \setminus \infty$

2.2. *Equivariant Morphisms and Divisorial Fans*

Let $\mathcal{D}'$ and $\mathcal{D}$ be p-divisors on (Y', N) and (Y, N), respectively. By [AH06, Section 8], $\mathbb{T}$-equivariant maps $\widetilde{\mathbb{TV}}(\mathcal{D}') \to \widetilde{\mathbb{TV}}(\mathcal{D})$ and $\mathbb{TV}(\mathcal{D}') \to \mathbb{TV}(\mathcal{D})$ can be provided by a dominant map $\psi : Y' \to Y$ and a plurifunction $\mathfrak{f} \in N \otimes \mathbb{C}(Y')$ such that $\mathcal{D}' \subseteq \psi^*\mathcal{D} + \mathrm{div}(\mathfrak{f})$. Here, both operators ψ^* and div are supposed to be applied to the divisors on Y occurring in $\mathcal{D}$ or the elements of $\mathbb{C}(Y')$ from $\mathfrak{f}$, respectively. The inclusion sign is to be understood separately for each of the polyhedral coefficients on both sides. Moreover, it was shown in [AH06, Sect. 8] that all $\mathbb{T}$-equivariant maps $\mathbb{TV}(\mathcal{D}') \to \mathbb{TV}(\mathcal{D})$ arise in this way when one allows to replace $\mathcal{D}'$ on Y' by $p^*\mathcal{D}'$ with an appropriate proper birational map $p : Y'' \to Y'$ implying $\mathbb{TV}(p^*\mathcal{D}') = \mathbb{TV}(\mathcal{D}')$.

There is a special case that will play an important role later on. Consider two p-divisors $\mathcal{D}' = \sum_D \Delta'_D \otimes D$ and $\mathcal{D} = \sum_D \Delta_D \otimes D$ on (Y, N) that satisfy $\Delta'_D \subseteq \Delta_D$ for each D. Then we have a $\mathbb{T}$-equivariant open embedding

$\widetilde{\mathbb{TV}}(\mathcal{D}') \hookrightarrow \widetilde{\mathbb{TV}}(\mathcal{D})$ if and only if the polyhedra $\Delta'_y := \sum_{D \ni y} \Delta'_D$ are faces of the corresponding $\Delta_y := \sum_{D \ni y} \Delta_D$ for all $y \in Y$; see [AHS08, Prop 3.4, Rem. 3.5(ii)]. Moreover, it was also shown in loc. cit. that the condition of $\mathbb{TV}(\mathcal{D}') \hookrightarrow \mathbb{TV}(\mathcal{D})$ being an open embedding implies this condition. If $\mathbb{TV}(\mathcal{D}')$ is an open subset of $\mathbb{TV}(\mathcal{D})$, then we will call $\mathcal{D}'$ a *face* of $\mathcal{D}$.

DEFINITION 2.4 [AHS08, Def. 5.2]. A finite collection $\mathcal{S}$ of p-divisors on (Y, N) is called a *divisorial fan* if for all $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$, their intersection $\mathcal{D} \cap \mathcal{D}'$ (taken via the polyhedral coefficients) is again a p-divisor, a face of both $\mathcal{D}$ and $\mathcal{D}'$, and belongs to $\mathcal{S}$.

Gluing all affine pieces together, the divisorial fan $\mathcal{S}$ gives rise to the global $\mathbb{T}$-variety

$$\mathbb{TV}(\mathcal{S}) := \varinjlim_{\mathcal{D} \in \mathcal{S}} \mathbb{TV}(\mathcal{D}).$$

Moreover, since all coefficients $\Delta_D^{\mathcal{D}}$ of $\mathcal{D}$ ($\mathcal{D} \in \mathcal{S}$) fit into a polyhedral subdivision $\mathcal{S}_D$ of $N_{\mathbb{Q}}$, we may write the divisorial fan as $\mathcal{S} = \sum_D \mathcal{S}_D \otimes D$. In particular, all tail cones $\operatorname{tail}(\Delta_\bullet^{\mathcal{D}})$ form a fan $\operatorname{tail}(\mathcal{S})$. The latter encodes the asymptotic behavior of the *slices* $\mathcal{S}_D$. However, to store all contents of $\mathcal{S}$, it is still necessary to keep in mind which cell of $\mathcal{S}$ belongs to which p-divisor $\mathcal{D} \in \mathcal{S}$. This is what we previously referred to as the *labeling*. Note, however, that only the maximal elements of $\mathcal{S}$ matter for this kind of information.

2.2.1. Toric Picture. Open embeddings in the toric world correspond to inclusions of faces on the level of polyhedral cones. Since divisorial fans coincide with their polyhedral tail fans in this particular setting, face relations of polyhedral divisors turn out to be the usual face relations for polyhedral cones.

2.2.2. $\mathbb{C}^* \curvearrowright \mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(n))$. For example, let us consider the geometric line bundle $p : \mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(n)) \to \mathbb{P}^1$ associated to $\mathcal{O}(n)$ over $\mathbb{P}^1$. We assume that $\mathbb{C}^*$ acts with weight 1 on the fibers of $\mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(n))$ and trivially on its zero section $\mathbb{P}^1 \to \mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(n))$. This action is given by the following two maximal polyhedral divisors:

$$\mathcal{D}^1 = [n, \infty) \otimes 0 + \emptyset \otimes \infty \quad \text{and} \quad \mathcal{D}^2 = \emptyset \otimes 0 + [0, \infty) \otimes \infty.$$

They correspond to affine charts $p^{-1}(\mathbb{P}^1 \setminus \{\infty\})$ and $p^{-1}(\mathbb{P}^1 \setminus \{0\})$, respectively, and are glued along the polyhedral divisor $\mathcal{D}^1 \cap \mathcal{D}^2 = \emptyset \otimes 0 + \emptyset \otimes \infty$ using the plurifunctions $n \otimes z_1/z_0$ and $0 \otimes 1$ on $\mathbb{P}^1$.

2.2.3. $\mathbb{C}^* \curvearrowright \mathbb{P}^2$. Let us consider $\mathbb{P}^2$ as a $\mathbb{C}^*$-variety with the following action on its homogeneous coordinates: $\deg z_0 = 1$, $\deg z_1 = 0$, and $\deg z_2 = 2$. Using the corresponding toric downgrade (see Section 5.2) yields a divisorial fan $\mathcal{S}$ with three maximal elements $\mathcal{D}^i = \Delta_0^i \otimes 0 + \Delta_\infty^i \otimes \infty$:

	Δ_0^i	Δ_∞^i	tail cone	locus
$\mathcal{D}^1$	$[-1,0]$	$\emptyset$	$\{0\}$	$\mathbb{P}^1 \setminus \infty$
$\mathcal{D}^2$	$[0,\infty)$	$[1/2,\infty)$	$[0,\infty)$	$\mathbb{P}^1$
$\mathcal{D}^3$	$(-\infty,-1]$	$(-\infty,1/2]$	$(-\infty,0]$	$\mathbb{P}^1$

2.3. Compatible Group Actions

Let X be a $\mathbb{T}$-variety together with another group G acting on it. We say that $\mathbb{T}$ *normalizes* the G-action if $\mathbb{T} \subseteq N_{\mathrm{Aut}(X)}(G)$. This means that $\mathbb{T}$ acts on both X and G and, moreover, the G-action $m : G \times X \to X$ is $\mathbb{T}$-equivariant (with respect to the diagonal action of $\mathbb{T}$ on the left-hand side). In particular, this morphism can be understood in terms of (2.2). If X is given by a p-divisor $\mathcal{D}$ on some variety $Y = X/\!\!/^{\mathrm{ch}}\mathbb{T}$, then $G \times X$ is given by a p-divisor on $(G \times X)/\!\!/^{\mathrm{ch}}\mathbb{T}$, which looks like the familiar G-bundle $X \times^{\mathbb{T}} G$ over Y.

The actions of G and $\mathbb{T}$ even commute if and only if the $\mathbb{T}$-action on G is trivial. If this is the case, then G acts on Y, too, and the diagram

$$\begin{array}{ccccc} G & \times & X & \xrightarrow{m} & X \\ \| & & \downarrow & & \downarrow \\ G & \times & Y & \xrightarrow{m} & Y \end{array}$$

commutes. In the language of (2.2), this means that the p-divisors $G \times \mathcal{D}$ and $m^*\mathcal{D}$ only differ by some polyhedral principal divisor $\mathrm{div}(\mathfrak{f})$. If $\mathcal{D} = \sum_D \Delta_D \otimes D$, then the two p-divisors equal $\sum_D \Delta_D \otimes (G \times D)$ and $\sum_D \Delta_D \otimes m^*D$, respectively. Since $\mathrm{div}(\mathfrak{f})$ can only shift the polyhedral coefficients by integral vectors, this means that the Δ_D for non-G-invariant prime divisors D have to be almost trivial, that is, shifted tail cones. This occurs, for example, for the coefficients of the colors as pointed out in Theorem 1.1.

3. Spherical Varieties

In this section, we provide background on spherical varieties and colored fans. Spherical varieties are natural generalizations of toric varieties. They appear when a torus action is replaced by an action of an arbitrary connected reductive group G.

A normal variety X with a G-action is called *spherical* if a Borel subgroup $B \subset G$ has an open dense orbit in X. Similarly to toric varieties, every spherical variety contains only finitely many G-orbits and even finitely many B-orbits [Kno91, Rem. 2.2]. Well-known examples of spherical varieties include horospherical varieties (e.g. toric and flag varieties) [Pas] and symmetric varieties (e.g. complete collineations and complete quadrics) [DCP83; DCP85].

A spherical G-variety X can be regarded as a partial G-equivariant compactification of a spherical homogeneous space G/H (isomorphic to the open G-orbit of X). In what follows, by an *embedding* of a spherical homogenous space G/H we mean a spherical G-variety X together with a point $x \in X$ such that the G-orbit of x is open in X and the isotropy subgroup of x equals H. By a *compactification* of G/H we mean a complete embedding of G/H. The classification of spherical varieties consists of two parts. The first part amounts to classifying all G-equivariant embeddings of a given spherical homogeneous space G/H. Similarly to toric varieties, embeddings of G/H can be classified by fans together with an extra structure provided by colors [LV83]. We further shortly recall this classification following [Kno91].

The second part amounts to the classification of all spherical homogeneous spaces, which was finished only recently using D. Luna's program. An exposition of the main steps of this program can be found, for example, in [Bra10]. The classification of spherical homogeneous spaces is based on the classification of *wonderful varieties*. Recall that a smooth complete G-variety with an open dense orbit is called *wonderful* (of *rank r*) if

(1) the complement to the orbit is the union of r smooth irreducible divisors $D_1, \dots, D_r$ with normal crossings;
(2) for any $I \subset \{1, \dots, r\}$, the intersection $\bigcap_{i \in I} D_i$ is a nonempty G-orbit closure.

In particular, there is a unique closed G-orbit $D_1 \cap \dots \cap D_r$. Wonderful varieties are spherical (see [Bra10] for references).

3.1. Colored Fans

We now introduce definitions needed to formulate classification results. Let G/H be a spherical homogeneous space. As in (1.1), we fix a Borel subgroup B such that $1 \in G/H$ belongs to the dense orbit, that is, we assume that $B \cdot H$ is open and dense in G. A *color* is a B-invariant irreducible divisor in G/H. Let $\mathcal{C} = \mathcal{C}(G/H)$ denote the set of colors of G/H. The *weight lattice* $\mathcal{X} := \mathcal{X}(G/H)$ of G/H is the set of all characters of $\mathcal{X}_B = \mathrm{Hom}(B, \mathbb{C}^*)$ that occur as weights of eigenvectors for the natural action of B on the field of rational functions $\mathbb{C}(G/H)$. The rank of the weight lattice $\mathcal{X}$ is called the *rank* of G/H. Since G/H is spherical, for each weight in $\mathcal{X}$, there exists a unique (up to scalars) B-semiinvariant rational function with this weight [Kno91, S.6]. These functions are regular on $B \cdot H$; hence, there is an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathbb{C}(G/H)^{(B)} \longrightarrow \mathcal{X}(G/H) \longrightarrow 0.$$
$$f \text{ (with } f(1) = 1) \longmapsto \chi(f) = f|_B^{-1}$$

Thus a valuation v on $\mathbb{C}(G/H)$ with values in $\mathbb{Z}$ gives rise to a linear function ρ_v on $\mathcal{X}$. In particular, colors D give rise to elements $\rho_D \in \mathcal{X}^* := \mathrm{Hom}(\mathcal{X}, \mathbb{Z})$. Let $\mathcal{V}$ denote the set of all G-invariant $\mathbb{Z}$-valued valuations. It turns out that the map $\rho : \mathcal{V} \to \mathcal{X}^*$, $v \mapsto \rho_v$ is injective. The convex hull of the image of $\mathcal{V}_{\mathbb{Q}}$ in $\mathcal{X}^*_{\mathbb{Q}} := \mathcal{X}^* \otimes_{\mathbb{Z}} \mathbb{Q}$ is called the *valuation cone*. In what follows, we identify $\mathcal{V}_{\mathbb{Q}}$ with its image.

By a result of Brion and Knop [Kno91, Theorem 6.4], there exists a root system in $\mathcal{X}$ such that its simple roots $\alpha_1, \ldots, \alpha_r$ give linear equations on the facets of $\mathcal{V}$, that is,

$$\mathcal{V} = \{x \in \mathcal{X}^*_{\mathbb{Q}} \mid x(\alpha_i) \leq 0, i = 1, \ldots, r\}.$$

In particular, the valuation cone is always cosimplicial.

Definition 3.1. Let $\mathcal{F}$ be a subset (possibly empty) of $\mathcal{C}$ such that $\rho(\mathcal{F})$ does not contain 0, and let $C \subseteq \mathcal{X}^*_{\mathbb{Q}}$ be a strictly convex polyhedral cone. The pair $(C, \mathcal{F})$ is called a *colored cone* with the set of colors $\mathcal{F}$ if

(1) C is generated by $\rho(\mathcal{F})$ and some elements of $\mathcal{V}$ and if
(2) the relative interior of C intersects the valuation cone.

For instance, if G/H is a torus, then $\mathcal{V} = \mathcal{X}^*_{\mathbb{Q}}$ and $\mathcal{C} = \emptyset$. So every strictly convex polyhedral cone C of full dimension is a colored cone $(C, \emptyset)$. The face relation among colored cones is defined as

$$(C_1, \mathcal{F}_1) < (C_2, \mathcal{F}_2) \quad :\Leftrightarrow \quad C_1 \text{ is a face of } C_2 \text{ and } \mathcal{F}_1 = \mathcal{F}_2 \cap \rho^{-1}(C_1).$$

A finite nonempty set Σ of colored cones forms a *colored fan* if, first, every face of $(C, \mathcal{F}) \in \Sigma$ belongs to Σ and, second, every $v \in \mathcal{V}$ belongs to the interior of at most one cone C with $(C, \mathcal{F}) \in \Sigma$. This implies in particular that the intersection of two cones inside $\mathcal{V}$ is a common face of both.

Every spherical variety X with an open dense orbit G/H gives rise to a colored fan Σ^X. Namely, X can be covered by a finite number of simple spherical varieties. Recall that a spherical variety is *simple* if it contains a unique closed G-orbit. Every simple spherical variety X_0 defines a colored cone $(C(X_0), \mathcal{F}(X_0))$ as follows. The set $\mathcal{F}(X_0)$ is the set of all colors whose closure in X_0 contains the closed orbit. The cone $C(X_0)$ is spanned by

$$C(X_0) = \langle \rho(\mathcal{F}(X_0)), \rho(D_1), \ldots, \rho(D_r) \rangle,$$

where $D_1, \ldots, D_r$ are irreducible G-invariant divisors on X_0. The colored fan Σ^X is then the union of colored cones $(C(X_0), \mathcal{F}(X_0))$ over all simple G-invariant subvarieties $X_0 \subset X$. By results of [LV83] the map $X \mapsto \Sigma^X$ is a bijection between isomorphism classes of spherical varieties with an open dense orbit G/H and colored fans in $\mathcal{X}^*(G/H)_{\mathbb{Q}}$.

By the definition of Σ^X there is a bijective correspondence between G-orbits in X and colored cones in Σ^X. Closed orbits correspond to maximal colored cones. Some further properties of X can be read from the colored fan Σ^X, for example, X is complete if and only if the support $|\Sigma^X|$ of the colored fan contains the valuation cone.

For two G-equivariant embeddings X and X' of G/H, we say that X *dominates* X' if there exists a G-equivariant morphism $X \to X'$. This can also be read from the colored fans Σ^X and $\Sigma^{X'}$. Namely, X dominates X' if the fan Σ^X fits into the fan $\Sigma^{X'}$, that is, for every colored $(C, \mathcal{F})$ of X, there exists a colored cone $(C', \mathcal{F}')$ of $\Sigma^{X'}$ such that $C \subseteq C'$ and $\mathcal{F} \subseteq \mathcal{F}'$. Note that we use the word "dominate" here even for nonproper or nonsurjective maps.

3.2. Toroidal Embeddings

There is a special class of spherical embeddings, namely, toroidal embeddings, whose geometric properties are easier to study.

DEFINITION 3.2. A G-equivariant embedding X of G/H is *toroidal* if it has no colors, that is, the closure in X of any color of G/H does not contain a closed G-orbit.

In other words, all of the colored cones in the colored fan of Y have empty sets of colors. In particular, any toric variety is toroidal. Wonderful varieties are toroidal. Any embedding X is dominated by a smallest toroidal one X^{tor} obtained by replacing every colored cone $(C, \mathcal{F})$ by the (un-)colored cone $(C \cap \mathcal{V}, \emptyset)$.

Smooth toroidal embeddings (also called *regular*) are the closest relatives of smooth toric varieties. If they are complete, then they can also be covered by affine charts $\mathbb{A}^n$ (where $n = \dim G/H$) so that the closures of codimension one G-orbits intersect each chart by coordinate hyperplanes $D_1, \ldots, D_r$ (where $r = \operatorname{rk} G/H$), and all intersections $\bigcap_{i \in I} D_i$ for $I \subset \{1, \ldots, r\}$ are exactly the intersections of $\mathbb{A}^n$ with the closures of G-orbits. These affine charts are translates of those defined in Proposition 3.7. There is a more general notion of log-homogeneous varieties introduced in [Bri07a]. From a geometric viewpoint, these are the nicest possible varieties among all varieties with an almost homogeneous action of an algebraic group. It turns out that if the group is linear, then log-homogeneous varieties are exactly smooth toroidal varieties (in particular, they are spherical); see [Bri09, Sect. 4]. From a geometric point of view, it is thus sometimes more natural to consider toroidal embeddings rather than arbitrary spherical varieties.

If the valuation cone is strictly convex (hence, simplicial), then there is a special compactification $\overline{Y}_{\mathcal{V}}$ of G/H whose colored fan is given by the valuation cone and all of its faces. This compactification is called *standard*. Note that the valuation cone is strictly convex if and only if $N_G(H)/H$ is finite; see [Kno91, Thm. 7.1] or [Bri97, (4.4), Prop. 1]. Those subgroups H are called *sober*. The standard compactification $\overline{Y}_{\mathcal{V}}$ is a unique both simple and toroidal compactification of X and, hence, the only candidate for a wonderful compactification of G/H. To determine when $\overline{Y}_{\mathcal{V}}$ is wonderful is a difficult problem, which is not yet completely solved. It is known that if $N_G(H)/H$ acts on the set of colors effectively (e.g. $N_G(H) = H$), then $\overline{Y}_{\mathcal{V}}$ is wonderful [Tim11, Thm. 30.1]. The converse is not true. Note that $\overline{Y}_{\mathcal{V}}$ dominates any simple compactification of G/H and is dominated by any toroidal compactification of G/H.

3.3. Horospherical Varieties

We shortly discuss properties of horospherical varieties since they will play a major role in Section 8. For more details on this subject, the reader may consult [Tim11, Chap. 29] or [Pas].

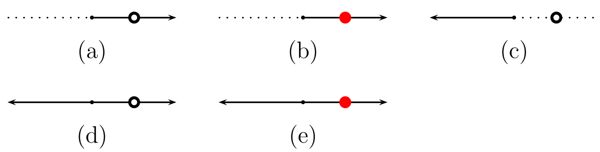


Figure 1 Colored fans associated to embeddings of SL_2/U. (a) $\mathrm{Bl}_0\mathbb{C}^2$, (b) $\mathbb{C}^2$, (c) $\mathbb{P}^2 \setminus \{0\}$, (d) $\mathrm{Bl}_0\mathbb{P}^2$, (e) $\mathbb{P}^2$

DEFINITION 3.3. A closed subgroup $H \subset G$ is called *horospherical* if it contains the unipotent radical of some Borel subgroup B^-. In this case, G/H is said to be a horospherical homogeneous space. Analogously, we call a normal G-variety X horospherical if it contains an open G-orbit that is isomorphic to a horospherical homogeneous space.

In particular, tori and complete rational homogeneous spaces are horospherical.

It follows from the Bruhat decomposition of G that horospherical varieties are also spherical, namely, the opposite Borel subgroup B has an open orbit on G/H. Moreover, for any horospherical subgroup $H \subset G$, there exists a unique parabolic subgroup $P \supset B^-$ such that H is the intersection of the kernels of the characters of P. Furthermore, we have $P = N_G(H)$. In more detail, given $H \subset G$ and the maximal torus $T = B \cap B^-$, there exists a subset I of the simple roots of G such that P is generated by W_I and B^-, that is, $P = P_I$. Here, W_I denotes the subgroup of the Weyl group $W = N_G(T)/T$ that is generated by the reflections associated to the elements of I. Even more, the lattice $\mathcal{X}(G/H)$ can be identified with the set of characters of P whose restrictions to H are trivial.

It turns out that any horospherical homogeneous space G/H is the total space of a torus fibration over the flag variety G/P where the fiber P/H equals the torus $\mathbb{T}$ with character lattice $\mathcal{X}(G/H)$. This fibration can be extended to the toroidal case, that is, any toroidal horospherical variety is of the form $G \times^P Y$ where $Y \supseteq \mathbb{T}$ is a toric variety. This feature may be regarded as the main reason for why horospherical varieties are more amenable to specific calculations than arbitrary spherical varieties. Note also that $\mathcal{X} = \mathcal{X}_B$ and $\mathcal{V} = \mathcal{X}(G/H)^*_{\mathbb{Q}}$ for a horospherical embedding $G/H \subset X$, which ensures that its colored fan is an honest polyhedral fan.

EXAMPLE 3.4. The simplest example of a noncompact horospherical homogeneous space is SL_2/U with

$$U = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \subset \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} = B^-.$$

Here $P = B^-$ and $I = \emptyset$. The homogeneous space SL_2/U is isomorphic to $\mathbb{C}^2 \setminus \{0\}$, and $\mathrm{SL}_2/P = \mathbb{P}^1$ with the usual projection. Apart from the trivial embedding $\mathbb{C}^2 \setminus \{0\}$ of SL_2/U, there are five nontrivial ones; see Figure 1 for their colored fans.

Note that the one-dimensional torus $H'/H = H \cdot N_G^\circ(H)/H = P/U$ acts on these embeddings by scalar matrices. Comparing (a) with the first example in (2.2.2) for $n = 1$ and (b) with the second example in (2.1.2), we can check that Theorem 1.1 holds for the horospherical embeddings $\mathrm{Bl}_0\mathbb{C}^2$ and $\mathbb{C}^2$, respectively.

3.4. Spherical Varieties of Minimal Rank

In what follows, we will mostly deal with spherical varieties of minimal rank. This class of varieties include horospherical varieties and embeddings of G (viewed as a homogeneous space under $G \times G$ acting by left and right multiplication). In general, the rank $\mathrm{rk}(G/H)$ of a spherical homogeneous space G/H satisfies the inequality

$$\mathrm{rk}(G/H) \geq \mathrm{rk}(G) - \mathrm{rk}(H),$$

where $\mathrm{rk}(G)$ and $\mathrm{rk}(H)$ denote the ranks of the groups G and H.

DEFINITION 3.5. A spherical homogeneous space G/H is *of minimal rank* if

$$\mathrm{rk}(G/H) = \mathrm{rk}(G) - \mathrm{rk}(H).$$

Spherical homogeneous spaces of minimal rank are classified in [Res]. They can be characterized by the following property:

PROPOSITION 3.6 [Res, Proposition 2.4]. *Let $T \subset G$ be a maximal torus. A spherical homogenous space G/H is of minimal rank if and only if for any toroidal embedding X of G/H, the T-fixed points of X lie in closed orbits.*

This property is important for us because it yields a covering of X by T-stable open affine subvarieties that can be explicitly described using the colored fan of X and the Weyl group of G. We now describe this covering. First, let $X(C, \mathcal{F}_C) \subset X$ be the simple spherical embedding corresponding to a maximal colored cone $(C, \mathcal{F}_C) \in \Sigma^X$. Then the open B-invariant subvariety

$$\widehat{X}_{\mathrm{id}}(C, \mathcal{F}_C) = X(C, \mathcal{F}_C) \setminus \bigcup_{D \in \mathcal{C}(G/H) \setminus \mathcal{F}_C} \overline{D}$$

is affine by [Kno91, Theorem 2.1]. For $w \in W$, put $\widehat{X}_w(C, \mathcal{F}_C) := w\widehat{X}_{\mathrm{id}}(C, \mathcal{F}_C)$. This is a T-invariant subvariety (we assume that $T \subset B$).

PROPOSITION 3.7. *If X is a spherical variety of minimal rank with the colored fan Σ^X, then*

$$X = \bigcup_{(C, \mathcal{F}_C) \in \Sigma^X, w \in W} \widehat{X}_w(C, \mathcal{F}_C),$$

where $(C, \mathcal{F}_C)$ runs through the maximal colored cones in Σ^X. This yields a covering of X by T-invariant open affine subvarieties labeled by the maximal colored cones and the elements of the Weyl group of G.

Proof. Without loss of generality, assume that X is complete. The variety

$$X' := \bigcup_{(C,\mathcal{F}_C)\in\Sigma^X, w\in W} \widehat{X}_w(C,\mathcal{F}_C)$$

is open and T-invariant. The intersection $X' \cap \mathcal{O}$ with any closed G-orbit $\mathcal{O} \subset X$ contains all T-fixed points in $\mathcal{O}$. Indeed, if $\mathcal{O}$ corresponds to a maximal cone C, then $\mathcal{O} \cap \widehat{X}_{\mathrm{id}}(C,\mathcal{F}_C)$ is the open dense B-orbit in $\mathcal{O}$ by [Kno91, Thm. 2.1 (c)]. Since $\mathcal{O}$ is isomorphic to a flag variety, the open dense B-orbit in $\mathcal{O}$ contains a unique T-fixed point x_0, and all other T-fixed points in $\mathcal{O}$ have the form wx_0 for $w \in W$.

Hence, if X is toroidal, then X' contains all T-fixed points by Proposition 3.6. It follows that the complement $X \setminus X'$ is empty. Indeed, every nonempty closed T-invariant subvariety of X must contain a T-fixed point by Borel's fixed point theorem.

The statement for a nontoroidal X follows at once from the corresponding statement for a toroidal resolution of X. □

We further give an example of a complete spherical space (not of minimal rank) for which Proposition 3.7 does not hold (this example was suggested to us by the referee).

Example 3.8. Let $G = \mathrm{GL}_2$. Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ as a G-variety under the diagonal action of G. Then Proposition 3.7 yields only two charts out of four standard affine charts for $\mathbb{P}^1 \times \mathbb{P}^1$.

We will also need the following description of B-orbits in G/H. For more details, see [Res; Bri01]. Let W_G and W_H denote the Weyl groups of G and H, respectively. For arbitrary spherical homogeneous spaces, F. Knop defined an action of W_G on B-orbits in G/H.

Proposition 3.9 [Res, Props. 2.1 and 2.2]. *The homogeneous space G/H is of minimal rank if and only if the W_G-action on its B-orbits is transitive. Then there exists a unique closed B-orbit, and its stabilizer in W_G is isomorphic to W_H.*

This description generalizes the description of Schubert cells in flag varieties to all spherical varieties of minimal rank. However, no such description is known for more general spherical varieties (see Example 3.8).

As in the case of flag varieties, there are two competing ways to label B-orbits by elements of W_G/W_H: the identity element of W_G labels either the closed (minimal) B-orbit or the open (maximal) B-orbit. We will use the former labeling, which we now define more precisely. Let $\mathcal{O}_{\mathrm{id}}$ denote the closed B-orbit in G/H. Note that this is not the B-orbit through the identity coset of G/H since the latter is open by our choice of B. In what follows, we denote by $\mathcal{O}_u$ (where $u \in W_G$) the B-orbit obtained from $\mathcal{O}_{\mathrm{id}}$ by the action of u. The action of W_G on B-orbits is related to the usual left action of W_G on G/H as follows. If α is a simple root and P_α and s_α are the associated minimal parabolic subgroup and the simple reflection, respectively, then two cases can occur:

(1) $P_\alpha \mathcal{O}_u = \mathcal{O}_u$, then $\mathcal{O}_u = \mathcal{O}_{s_\alpha u}$ and $s_\alpha(\mathcal{O}_u) = \mathcal{O}_u$;

(2) $P_\alpha \mathcal{O}_u = \mathcal{O}_u \sqcup \mathcal{O}_{s_\alpha u}$ and the natural map $P_\alpha \times^B \mathcal{O}_u \to P_\alpha \mathcal{O}_u$ is birational. If $\dim \mathcal{O}_u < \dim \mathcal{O}_{s_\alpha u}$ (i.e., $\dim \mathcal{O}_u + 1 = \dim \mathcal{O}_{s_\alpha u}$), then $s_\alpha(\mathcal{O}_u) \subset \mathcal{O}_{s_\alpha u}$.

There is an analog of the weak Bruhat order on B-orbits in G/H, namely, $\mathcal{O}_u \prec \mathcal{O}_v$ if $\dim \mathcal{O}_u + 1 = \dim \mathcal{O}_v$ and $v = s_\alpha u$ for a simple root α. In particular, $\mathcal{O}_u \subset \overline{\mathcal{O}_v}$. In what follows, $\Gamma(G/H)$ denotes the oriented graph associated with the relation $\prec$, that is, the vertices of $\Gamma(G/H)$ are B-orbits, and two vertices $\mathcal{O}_u$ and $\mathcal{O}_v$ are connected by an oriented edge labeled by α if $\mathcal{O}_u \prec \mathcal{O}_v$ and $v = s_\alpha u$. The graph $\Gamma(G/H)$ has a unique maximal element (the open dense B-orbit in G/H) and a unique minimal element (the closed orbit $\mathcal{O}_{\mathrm{id}}$). In particular, there exists a strictly increasing path from $\mathcal{O}_{\mathrm{id}}$ to any other orbit $\mathcal{O}_u$, and the number of edges in such a path is equal to $(\dim \mathcal{O}_u - \dim \mathcal{O}_{\mathrm{id}})$. Similarly, there exists a strictly decreasing path from the maximal orbit to $\mathcal{O}_u$, and the number of edges in such a path is equal to $\operatorname{codim} \mathcal{O}_u$.

REMARK 3.10. The colors of G/H are closures of codimension one B-orbits. In the above notation, if $\mathcal{O}_{u_0} = (G/H) \setminus \bigcup_{D \in \mathcal{C}} D$ is the maximal B-orbit in G/H that has stabilizer $W_H \subseteq W_G$, then the colors coincide with the closures of B-orbits $\mathcal{O}_{s_\alpha u_0}$ for simple reflections $s_\alpha \in W_G \setminus W_H$. Put $D(\alpha) := \overline{\mathcal{O}_{s_\alpha u_0}}$. Then $s_\alpha(D(\alpha)) \neq D$ for any color D since $s_\alpha(\mathcal{O}_{s_\alpha u_0}) \subset \mathcal{O}_{u_0}$. On the other hand, if $D(\beta) \neq D(\alpha)$ is any other color, then $s_\alpha D(\beta) = D(\beta)$. Indeed, $s_\alpha u_0 \neq s_\beta u_0$ implies $s_\alpha(s_\beta u_0) \neq u_0$ in W_G/W_H, that is, $\dim \mathcal{O}_{s_\alpha s_\beta u_0} < \dim \mathcal{O}_{s_\beta u_0}$ or $\mathcal{O}_{s_\alpha s_\beta u_0} = \mathcal{O}_{s_\beta u_0}$. In either case, $\mathcal{O}_{s_\alpha s_\beta u_0} \subseteq \overline{\mathcal{O}_{s_\beta u_0}} = D(\beta)$. Then, $s_\alpha \mathcal{O}_{s_\beta u_0} \subseteq P_\alpha \mathcal{O}_{s_\beta u_0} = \mathcal{O}_{s_\beta u_0} \cup \mathcal{O}_{s_\alpha s_\beta u_0} \subseteq D(\beta)$. The same argument shows that if $s_\alpha \in W_H$, then $s_\alpha D(\beta) = D(\beta)$, too.

We will also need the following result describing the orbits inside a given color.

LEMMA 3.11. *Let $\mathcal{O}_u$ be a B-orbit in G/H such that an increasing path $(\alpha_{i_1}, \dots, \alpha_{i_\ell})$ from $\mathcal{O}_u$ to $\mathcal{O}_{u_0}$ contains an edge labeled by α. Then $\mathcal{O}_u \subset D(\alpha)$.*

Proof. Proceed by induction on $\ell = \operatorname{codim} \mathcal{O}_u$. If $\ell = 1$, then we have even $\overline{\mathcal{O}_u} = D$. Choose j such that $\alpha_{i_j} = \alpha$. By replacing u with $s_{\alpha_{i_{j-1}}} \cdots s_{\alpha_{i_1}} u$ we may assume that $j = 1$. Put $\beta = \alpha_{i_2}$ and $v = s_\alpha s_\beta s_\alpha u$. Then either $\mathcal{O}_v \prec \mathcal{O}_{s_\alpha v} = \mathcal{O}_{s_\beta s_\alpha u}$ (and $\dim \mathcal{O}_v = \dim \mathcal{O}_u + 1$) or $\mathcal{O}_{s_\beta s_\alpha u} \prec \mathcal{O}_v$ (and $\dim \mathcal{O}_v = \dim \mathcal{O}_u + 3$). In the latter case, $\mathcal{O}_{s_\beta s_\alpha u}$ satisfies the induction hypothesis for $(l-2)$ and $\mathcal{O}_u \subset \overline{\mathcal{O}_{s_\beta s_\alpha u}}$. In the former case, $\mathcal{O}_v$ satisfies the induction hypothesis for $(l-1)$, so it remains to show that $\mathcal{O}_u \subset \overline{\mathcal{O}_v}$.

The braid relation $(s_\alpha s_\beta)^{m(\alpha,\beta)} = \mathrm{id}$ within W_G yields a closed path Π in $\Gamma(G/H)$ that goes through $\mathcal{O}_u$ and $\mathcal{O}_v$. Note that any such path contains a unique *locally maximal* vertex $\mathcal{O}_w$, that is, $\mathcal{O}_{s_\alpha w} \prec \mathcal{O}_w$ and $\mathcal{O}_{s_\beta w} \prec \mathcal{O}_w$. Indeed, $\overline{\mathcal{O}_w}$ is both P_α- and P_β-invariant in this case; hence, $\overline{\mathcal{O}_w}$ contains all the other orbits in the path. This implies that Π also has a unique minimal vertex

$\mathcal{O}_{v_0}$ with respect to $\prec$. Indeed, let $\mathcal{O}_{w_1} \in \Pi$ be a *locally minimal* vertex, that is, $\mathcal{O}_{s_\alpha w_1} \succ \mathcal{O}_{w_1}$ and $\mathcal{O}_{s_\beta w_1} \succ \mathcal{O}_{w_1}$. Let $\mathcal{O}_{w_2} \in P$ be another locally minimal vertex. Unless $\mathcal{O}_{w_1} = \mathcal{O}_{w_2}$, we can represent Π as the union of two distinct paths Π_1 and Π_2 where Π_1 goes from $\mathcal{O}_{w_1}$ to $\mathcal{O}_{w_2}$, Π_2 goes from $\mathcal{O}_{w_2}$ to $\mathcal{O}_{w_1}$, and Π_1 and Π_2 intersect only at the endpoints. For $i = 1, 2$, the first edge in Π_i goes upward, and the last edge goes downward; hence, Π_i contains a locally maximal vertex. This contradicts to the uniqueness of a locally maximal vertex.

Let Π_u and Π_v be strictly increasing paths inside the loop from $\mathcal{O}_{v_0}$ to $\mathcal{O}_u$ and $\mathcal{O}_v$, respectively. The labels on both paths look like $(\dots, \alpha, \beta, \alpha, \beta, \dots)$. Denote by P the parabolic group corresponding to the path Π_u. Moreover, we may assume that $\Pi_v = (s_\beta, \Pi_u)$. Then, on the one hand, $P_\beta \mathcal{O}_{v_0} = \mathcal{O}_{v_0} \sqcup \mathcal{O}_{s_\beta v_0}$. On the other hand, $\mathcal{O}_u$ is the open orbit in $P \cdot \mathcal{O}_{v_0}$, and $\mathcal{O}_v$ is the open orbit in $P \cdot \mathcal{O}_{s_\beta v_0}$. Thus, $\mathcal{O}_u \subseteq P \cdot \mathcal{O}_{v_0} \subseteq P \cdot P_\beta \mathcal{O}_{v_0} \subseteq P \cdot \overline{\mathcal{O}_{s_\beta v_0}} \subseteq \overline{\mathcal{O}_v}$. □

4. Toward the Toroidal Case

4.1. A New Torus Action

We now introduce a torus action on embeddings of the spherical homogeneous space G/H. Note that this will not be the restriction of the G-action to a maximal torus $T \subseteq G$. Instead, we use the fact that $N_G(H)/H$ is a subgroup of a torus [Bri09, Prop. 5.2]. In particular, if we put $H' = H \cdot N_G^\circ(H)$, then $\mathbb{T} := H'/H$ is a torus, too. Note that the group H' is the smallest sober subgroup that contains H [Tim11, Lemma 30.1]. The maximal linear subspace contained in the valuation cone $\mathcal{V}$ has dimension $\dim \mathbb{T}$ [Kno91, Thm. 7.1]. Note that $\mathbb{T}$ acts on G/H from the right and hence commutes with the left action of G.

Lemma 4.1. *The right action of $\mathbb{T}$ on G/H extends to any G-equivariant embedding of G/H.*

Proof. The group $\widetilde{G} := G \times \mathbb{T}$ acts on G/H by left and right multiplications as

$$(g, t) : x \mapsto gxt^{-1} \quad (g \in G, t \in \mathbb{T}, x \in G/H).$$

Hence, G/H (so far only being considered as a homogeneous G-space) may also be regarded as the homogeneous $\widetilde{G}$-space $\widetilde{G}/\widetilde{H}$, where $\widetilde{H} := \{(th, t) \mid t \in \mathbb{T}, h \in H\} \cong H \times \mathbb{T}$. Recall that we fixed a Borel subgroup $B \subset G$ such that $B \cdot H$ is open and dense in G. It follows that $N_G(H)$ is the stabilizer of BH from the right; see [Bri97, Thm. 4.3(iii)]. In particular, we obtain that $BH \supset H'$ (indeed, $bH \cdot h' = bh'H = bb_1h_1H \in BH$). Note that $\widetilde{B} := B \times \mathbb{T} \subset \widetilde{G}$ is a Borel subgroup in $\widetilde{G}$.

We now show that G/H and $\widetilde{G}/\widetilde{H}$ have the same colors and isomorphic weight lattices. Indeed, the open B-orbit in G/H is $\mathbb{T}$-invariant since $BH \supseteq H'$; hence, any B-invariant irreducible divisor in G/H is also $B \times \mathbb{T}$-invariant. Next, since the actions of $B \subseteq G$ and $\mathbb{T}$ commute, each B-eigenvector $f \in \mathbb{C}(G/H)^{(B)}_\chi$ remains in this one-dimensional space after applying the $\mathbb{T}$-action. Hence, f becomes a $B \times \mathbb{T}$-eigenvector of some weight $\widetilde{\chi} \in \mathcal{X}_{\widetilde{B}}$ lifting $\chi \in \mathcal{X}_B$.

Since the resulting dual isomorphism $\mathcal{X}^*(G/H) \xrightarrow{\sim} \mathcal{X}^*(\widetilde{G}/\widetilde{H})$ is compatible with the identification $\mathcal{C}(G/H) = \mathcal{C}(\widetilde{G}/\widetilde{H})$ we stated before, we obtain a bijection between the sets of colored fans for G/H and $\widetilde{G}/\widetilde{H}$, respectively. Thus, every G-equivariant embedding of G/H extends to a $\widetilde{G}$-equivariant one of $\widetilde{G}/\widetilde{H}$. □

Since H' is sober, the homogeneous space G/H' admits the standard compactification $\overline{Y}_{\mathcal{V}'}$ (see Sect. 3.2), which is wonderful in many cases of interest. Moreover, blow-ups of the latter will serve as base varieties for polyhedral divisors describing spherical embeddings of G/H as $\mathbb{T}$-varieties.

4.2. *Comparing H and H'*

Let M denote the character lattice of $\mathbb{T}$, and N its dual, that is, the lattice of one-parameter subgroups of $\mathbb{T}$.

LEMMA 4.2. *Each $f \in \mathbb{C}(G/H)^{(B)}$ with $f(1) = 1$ is partially multiplicative, that is, it satisfies $f(gh') = f(g) \cdot f(h')$ for $g \in G$ and $h' \in H'$. In particular, restriction to H' gives the vertical homomorphism $\mathcal{X}(G/H) \to M$ in the diagram*

$$\begin{array}{ccccl} & \overset{\sim}{\nearrow} & \mathcal{X}(G/H) & \hookrightarrow & \mathcal{X}_B \quad (\textit{by restriction to } B) \\ \mathbb{C}(G/H)^{(B)}_{1\mapsto 1} & & \downarrow & & \\ & \searrow & M & = & \mathcal{X}_{\mathbb{T}} \quad (\textit{by restriction to } H'). \end{array}$$

Proof. From Lemma 4.1 it follows that f is an eigenvector of $B \times \mathbb{T}$ and hence of $B \times H'$. Alternatively, one has the following direct argument: For each $h' \in H'$, we define a new rational function $f'(g) := f(gh')$. It is not hard to see that f' and f transform in the same ways with respect to the left B-action. Moreover, since H' normalizes H, we see that f' also is H-invariant (for the action from the right). Hence, f and f' differ multiplicatively by a constant, that is, $f'(g) = f(g) \cdot f'(1)$. □

The following result is derived from the *Tits fibration* $\phi : G/H \to G/H'$.

PROPOSITION 4.3. *The dual of the vertical homomorphism from Lemma 4.2 fits into the short exact sequence*

$$0 \to N \to \mathcal{X}^*(G/H) \xrightarrow{p} \mathcal{X}^*(G/H') \to 0.$$

The valuation cone $\mathcal{V}$ of G/H is the full preimage of the (strictly convex) valuation cone $\mathcal{V}'$ of G/H' under $p_{\mathbb{Q}}$. Moreover, there is a natural identification of colors $\phi : \mathcal{C}(G/H) \xrightarrow{\sim} \mathcal{C}(G/H')$ compatible with $p = \phi_*$.

Proof. For the exactness of

$$0 \to \mathcal{X}(G/H') \to \mathcal{X}(G/H) \to M \to 0,$$

see [Bri97, Thm. 4.3(ii)]. Since $H' \subseteq BH$, we know that $\phi^{-1}(BH'/H') = BH/H$, that is, the two dense B-orbits correspond to each other via ϕ. The map ϕ is a locally trivial fibration with fiber $\mathbb{T}$, cf. the beginning of Section 6. Hence, for every color $D' \in \mathcal{C}(G/H')$, we know that $\phi^{-1}(D')$ equals a single color $D \in \mathcal{C}(G/H)$, and no colors from G/H can be sent via ϕ to a variety of larger codimension.

The equality of cones $\mathcal{V} = p^{-1}(\mathcal{V}')$ is also established in [Bri97, 4.3] by representing the valuation cone as the dual of the cone generated by some negative roots; see (3.1), right before Definition 3.1. Moreover, it is clear that for colors $D = \phi^{-1}(D')$, the associated valuations v_D and v'_D coincide on $\mathbb{C}(G/H')$ understood as a subfield of $\mathbb{C}(G/H)$, that is, $p(\rho_D) = \rho_{D'}$ inside $\mathcal{X}^*(G/H')$. □

Note that the previous exact sequences imply that G/H is of minimal rank if and only if G/H' has the same property.

4.3. Introducing New Fans

Let X, X' be embeddings of G/H and of G/H', respectively. Generalizing a remark at the end of (3.1) to the situation of now two different subgroups H and H', we quote from [Kno91, Thm. 4.1] that there exists a G-equivariant map $X \to X'$ if and only if the fan Σ^X maps to the fan $\Sigma^{X'}$, that is, for every colored $(C, \mathcal{F})$ of X, there exists a colored cone $(C', \mathcal{F}')$ of $\Sigma^{X'}$ such that $p(C) \subseteq C'$ and $p(\mathcal{F}) \subseteq \mathcal{F}'$. For example, every toroidal embedding X maps to the simple toroidal compactification $\overline{Y} = \overline{Y}_{\mathcal{V}'}$ already mentioned in (4.1).

Definition 4.4. Let Σ^X denote the colored fan that is associated with the G/H-embedding X. Now we define the following "ordinary" fans:

(i) $\Sigma_X := \{C \cap \mathcal{V} \mid (C, \mathcal{F}) \in \Sigma^X\}$ is called the underlying *uncolored* fan.
(ii) Let $\Sigma_Y = p(\Sigma_X)$ denote the *image fan* of Σ_X via p, that is, the coarsest subdivision of the pointed cone $\mathcal{V}'$ that refines all images $p(C)$ of cones $C \in \Sigma_X$.
(iii) Finally, let $\Sigma_{\widetilde{X}}$ be the coarsest common refinement of Σ_X and $p^*\Sigma_Y := \{p^{-1}(C') \mid C' \in \Sigma_Y\}$.

Let X^{tor} and $\widetilde{X}$ denote the toroidal G/H-embeddings corresponding to the fans Σ_X and $\Sigma_{\widetilde{X}}$, respectively. Similarly, we call Y the toroidal G/H'-embedding corresponding to the fan Σ_Y. Invoking the remark on G-equivariant maps of spherical varieties at the beginning of this section, we have the following G-equivariant diagram connecting all these varieties:

$$\begin{array}{ccccc} \widetilde{X} & \longrightarrow & X^{\mathrm{tor}} & \longrightarrow & X \\ \downarrow & & \downarrow & & \\ Y & \longrightarrow & \overline{Y}. & & \end{array}$$

Whereas the varieties and the maps of the first row carry the natural $\mathbb{T}$-action provided by Lemma 4.1, the torus $\mathbb{T}$ acts trivially on the second row. Recall that the rays $a \in \Sigma_Y(1)$ correspond to the G-invariant divisors D_a of Y.

4.4. *Statement of the Result*

In what follows, we assume that G/H is of minimal rank. Assume that X is a spherical embedding of G/H and consider the varieties shown in the diagram of (4.3). Fix a maximal torus $T \subseteq B$. Let $W := N(T)/T$ denote the Weyl group. The action of the Weyl group on T by conjugations induces the action of W on $\mathcal{X}_T$. It is easy to check that $w\mathcal{X}(G/H) = \mathcal{X}(G/H^w)$ where $H^w := wHw^{-1}$ and $B^w := wBw^{-1}$. As indicated in

$$
\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{X}(G/H') & \longrightarrow & \mathcal{X}(G/H) & \hookrightarrow & & \mathcal{X}_B & = & \mathcal{X}_T \\
& \downarrow w & & \downarrow w & \searrow & M \longrightarrow 0 & \downarrow w & & \downarrow w \\
0 \longrightarrow & w\mathcal{X}(G/H') & \longrightarrow & w\mathcal{X}(G/H) & \hookrightarrow & & \mathcal{X}_{B^w} & = & \mathcal{X}_T \\
& & & & \searrow & M^w \longrightarrow 0, & & &
\end{array}
$$

the W-action induces the isomorphism θ_w between M and the character lattice M^w of $\mathbb{T}^w := {H'}^w/H^w$. We will use W to build the divisorial fan $\mathcal{S}^X$ and $\widetilde{\mathcal{S}}^X$ on (Y, N). Note that the affine covering defined in Proposition 3.7 is $\mathbb{T}$-invariant. Moreover, multiplication by $w \in W$ gives an isomorphism between the $\mathbb{T}$-varieties $\widehat{X}_{\mathrm{id}}(C, \mathcal{F}_C)$ and $\widehat{X}_w(C, \mathcal{F}_C)$. As the above diagram shows, the isomorphism does not affect the coefficients of the corresponding p-divisors in $N_{\mathbb{Q}}$. Indeed, multiplication by w gives the isomorphism between X regarded as a G/H-embedding and X regarded as a G/H^w-embedding, which takes the $\mathbb{T}$-variety $\widehat{X}_{\mathrm{id}}(C, \mathcal{F}_C)$ to the $\mathbb{T}^w$-variety $\widehat{X}_w(C, \mathcal{F}_C)$. Hence, if $\sum S_D \otimes D$ and $\sum S_{D'} \otimes D'$ are their respective p-divisors, then $\sum S_{D'} \otimes D' = \sum \theta_w^*(S_D) \otimes wD$. If we now regard $\widehat{X}_w(C, \mathcal{F}_C)$ not as $\mathbb{T}^w$- but as $\mathbb{T}$-variety, then we have to apply the inverse of θ_w^* to the coefficients of its p-divisor, thus getting $\sum S_D \otimes wD$.

We are going to define p-divisors corresponding to the affine charts in this covering. To do so, fix a splitting of the exact sequence

$$0 \to N \to \mathcal{X}^*(G/H) \xrightarrow{p} \mathcal{X}^*(G/H') \to 0$$

from Proposition 4.3 by choosing a cosection $s^* : \mathcal{X}^*(G/H) \twoheadrightarrow N$. We use this projection to define $\overline{\rho}(D') \in N$ as $s^*(\rho_D)$ where D is the color of G/H corresponding to D' via the bijection $\phi : \mathcal{C}(G/H) \xrightarrow{\sim} \mathcal{C}(G/H')$ also established in Proposition 4.3, that is, satisfying $p(\rho_D) = \rho_{D'}$. Moreover, we will use the splitting to always identify the fibers $p^{-1}(a)$ with $p^{-1}(0) = N_{\mathbb{Q}}$.

DEFINITION 4.5. The maximal elements of $\widetilde{\mathcal{S}}^X$ are p-divisors $\widetilde{\mathcal{S}}^X(C, w)$ on (Y, N) labeled by pairs of maximal colored cones $(C, \mathcal{F}) \in \Sigma^X$ (or, equivalently, by maximal ordinary cones $C \cap \mathcal{V} \in \Sigma_X$) and elements $w \in W$. They are defined as

$$\widetilde{\mathcal{S}}^X(C, w) := \sum_a (C \cap p^{-1}(a)) \otimes D_a + \sum_{D'} (\overline{\rho}(D') + (C \cap N_{\mathbb{Q}})) \otimes \overline{D'} + \sum_{D'} \emptyset \otimes w\overline{D'},$$

where $a \in \Sigma_Y(1)$ runs through the (primitive generators of the) rays of Σ_Y, and $D' \in \mathcal{C}(G/H')$ runs through the colors of G/H'. Note that it makes the difference between $\widetilde{\mathcal{S}}^X$ and the $\mathcal{S}^X$ defined earlier in Theorem 1.1 in (1.1) that now D' runs through *all* the colors even in the third summand. However, $\widetilde{\mathcal{S}}^X = \mathcal{S}^X$ for toroidal X. Note further that for $w = 1$, the second summand will be annihilated by the third one.

The following theorem covers Theorem 1.1 for toroidal X, that is, for the case $X = X^{\mathrm{tor}}$.

THEOREM 4.6. *The divisorial fan $\widetilde{\mathcal{S}}^X$ on (Y, N) describes $\widetilde{X}$ and X^{tor} as $\mathbb{T}$-varieties, namely $\widetilde{X} = \widetilde{\mathbb{TV}}(\widetilde{\mathcal{S}}^X) = \widetilde{\mathbb{TV}}(\mathcal{S}^X)$ and $X^{\mathrm{tor}} = \mathbb{TV}(\widetilde{\mathcal{S}}^X)$.*

The proof of this statement consists of a local part (Section 5), and a global one (Section 6).

REMARK 4.7. The $\widetilde{\mathbb{TV}}$ construction is a local one, and hence it yields the same result for the arguments $\widetilde{\mathcal{S}}$ and $\mathcal{S}$. The description of the divisorial fan providing $\widetilde{X}$ can be even more simplified, namely

$$\widetilde{X} = \widetilde{\mathbb{TV}}\left(\sum_a (\Sigma^X \cap p^{-1}(a)) \otimes D_a + \sum_{D'} (\overline{\rho}(D') + \mathrm{tail}) \otimes \overline{D'}\right).$$

Note that no labels via cells of Σ^X or elements of W are necessary. This is due to the fact that Definition 2.3 of $\widetilde{\mathbb{TV}}(\mathcal{D})$ does not make any positivity assumptions on $\mathcal{D}$. However, the latter are necessary for the definition of $\mathbb{TV}(\mathcal{D})$. Thus, we cannot expect simplifications for the descriptions of X^{tor} or X.

5. Toric Downgrades Give the Local Picture

5.1. The Toric Skeleton

The first step toward the proof of Theorem 4.6 is a local understanding of X^{tor} with respect to the right $\mathbb{T}$-action. Denoting by Δ_X^{tor} the union of the closures in X^{tor} of all colors of G/H, the stabilizer of Δ_X^{tor} is given by a parabolic subgroup $P := P(\Delta_X^{\mathrm{tor}})$ of G, which is actually independent of the particular toroidal embedding. Furthermore, it comes with a Levi decomposition $P = P_u \rtimes L$ such that

$$X^{\mathrm{tor}} \setminus \Delta_X^{\mathrm{tor}} \cong P \times^L \mathbb{TV}(\Sigma_X) \cong P_u \times \mathbb{TV}(\Sigma_X),$$

where $\mathbb{TV}(\Sigma_X)$ denotes the ordinary toric variety associated to the fan Σ_X; see [Bri97, Sect. 2.4] and [Tim11, Thm. 29.1]. The accompanying torus is equal to a quotient of L, and its character lattice equals $\mathcal{X}(G/H)$. Moreover, we may consider $\mathbb{T} = H'/H$ as a subtorus that turns $\mathbb{TV}(\Sigma_X)$ into a $\mathbb{T}$-variety; see loc. cit.

The very same procedure also works for $\widetilde{X}$ and Y. Moreover, it is compatible with the morphisms shown in (4.3). Hence, denoting the union of the closures of all colors of G/H and G/H' in the respective varieties by $\widetilde{\Delta}_X$ and Δ_Y, we obtain the following commutative diagrams:

$$\begin{array}{ccccc}
\widetilde{X} \setminus \widetilde{\Delta}_X & \xrightarrow{\sim} & & P_u \times \mathbb{TV}(\Sigma_{\widetilde{X}}) & \\
& \searrow & & & \searrow \\
& & X^{\text{tor}} \setminus \Delta_X^{\text{tor}} & \xrightarrow{\sim} & P_u \times \mathbb{TV}(\Sigma_X) \\
\downarrow & & & \downarrow & \\
Y \setminus \Delta_Y & \xrightarrow{\sim} & & P_u \times \mathbb{TV}(\Sigma_Y). &
\end{array}$$

Of particular interest to us is the right-hand side of the diagram. There we have two maps between three toric varieties multiplied with the unipotent group P_u. In the next section, we will show how such a diagram between toric varieties can be understood in the context of $\mathbb{T}$-varieties and polyhedral divisors.

5.2. Toric Downgrades

There is a very prominent procedure that gives rise to polyhedral divisors and divisorial fans. This construction plays a fundamental role in the proof of Theorem 4.6, so we shortly recall it from [AH03, Sect. 8].

Let $\mathbb{T} \subseteq \widetilde{T}$ be a subtorus, and assume that we have fixed a splitting of the corresponding exact sequence of 1-parameter subgroups

$$0 \to N \to \widetilde{N} \xrightarrow{p} N_Y \to 0.$$

Now, whenever $Z = \mathbb{TV}(\Sigma)$ is a toric variety given by a fan Σ in $\widetilde{N}_{\mathbb{Q}}$, then we define the fans $\Sigma_Y := p(\Sigma)$ and $\widetilde{\Sigma} := \{C \cap p^{-1}(C') \mid C \in \Sigma, C' \in \Sigma_Y\}$ as in Definition 4.4(ii) and (iii), respectively. They give rise to toric varieties $\widetilde{Z} := \mathbb{TV}(\widetilde{\Sigma})$ and $\mathbb{TV}(\Sigma_Y)$. Similarly to the situation in (5.1), these varieties fit into the diagram

$$\begin{array}{ccc}
\mathbb{TV}(\widetilde{\Sigma}) & \longrightarrow & \mathbb{TV}(\Sigma) \\
\downarrow p & & \\
\mathbb{TV}(\Sigma_Y). & &
\end{array}$$

The embedding $\mathbb{T} \hookrightarrow \widetilde{T}$ turns $\widetilde{Z}$ and Z of the upper row into $\mathbb{T}$-varieties. They can be described by a divisorial fan $\mathcal{S}$ on $(\mathbb{TV}(\Sigma_Y), N)$. Let $T_Y := N_Y \otimes_{\mathbb{Z}} \mathbb{C}^*$ denote the torus of the toric variety $\mathbb{TV}(\Sigma_Y)$. It turns out that the divisors occurring in $\mathcal{S}$ are T_Y-invariant, that is, they are closures $\overline{\text{orb}}(a)$ of T_Y-orbits of codimension one parameterized by the rays $a \in \Sigma_Y(1)$. We define

$$\mathcal{S} := \sum_{a \in \Sigma_Y(1)} \mathcal{S}_a \otimes \overline{\text{orb}}(a) \quad \text{with } \mathcal{S}_a = \Sigma \cap p^{-1}(a),$$

that is, all $\mathcal{S}_a$ become polyhedral subdivisions of $p^{-1}(a) = N_{\mathbb{Q}}$ with a naturally defined labeling.

PROPOSITION 5.1 [AH03, Sect. 8]. *The $\mathbb{T}$-structure of $\widetilde{Z} \to Z$ is given by the divisorial fan $\mathcal{S}$ on $(\mathbb{TV}(\Sigma_Y), N)$, that is, this morphism is equal to $\widetilde{\mathbb{TV}}(\mathcal{S}) \to \mathbb{TV}(\mathcal{S})$.*

5.3. *The* $\mathbb{T}$*-Variety* $X^{\mathrm{tor}} \setminus \Delta_X^{\mathrm{tor}}$

Combining results from (5.1) and (5.2), we deduce that the $\mathbb{T}$-equivariant map $(\widetilde{X} \setminus \widetilde{\Delta}_X) \to (X^{\mathrm{tor}} \setminus \Delta_X^{\mathrm{tor}})$ is equal to $\widetilde{\mathbb{TV}}(\widetilde{\mathcal{S}}_1^X) \to \mathbb{TV}(\widetilde{\mathcal{S}}_1^X)$, where $\widetilde{\mathcal{S}}_1^X$ consists of the p-divisors $\widetilde{\mathcal{S}}^X(\bullet, \mathrm{id}_W)$ introduced in Definition 4.5:

The first summand is literally built by the recipe of the toric downgrade of (5.2); the former divisors $\overline{\mathrm{orb}}(a)$ have just been replaced by $P_u \times \overline{\mathrm{orb}}(a) = D_a \setminus \Delta_Y$. The second summand in $\widetilde{\mathcal{S}}_1^X$ is void because of $w = 1$, and the presence of the last one just means that the divisorial fan is supposed to be evaluated on $Y \setminus \Delta_Y$ instead of the entire complete Y.

5.4. *The Action of the Weyl Group*

Both spherical varieties $\widetilde{X} \to X^{\mathrm{tor}}$ are covered by the open subsets $(\widetilde{X} \setminus w\widetilde{\Delta}_X) \to (X^{\mathrm{tor}} \setminus w\Delta_X^{\mathrm{tor}})$, where $w \in W$ runs through all elements of the Weyl group. Since these charts arise from $(\widetilde{X} \setminus \widetilde{\Delta}_X) \to (X^{\mathrm{tor}} \setminus \Delta_X^{\mathrm{tor}})$ by applying w, they are equal to $\widetilde{\mathbb{TV}}(\widetilde{\mathcal{S}}_w^X) \to \mathbb{TV}(\widetilde{\mathcal{S}}_w^X)$ with $\widetilde{\mathcal{S}}_w^X := w(\widetilde{\mathcal{S}}_1^X)$, that is,

$$\widetilde{\mathcal{S}}_w^X := \sum_{a \in \Sigma_Y} (C \cap p^{-1}(a)) \otimes D_a + \sum_{D' \in \mathcal{C}'} \emptyset \otimes w\overline{D'}.$$

Gluing the charts of $\widetilde{X}$ (and similarly of X^{tor}) leads to isomorphisms φ_w:

$$\widetilde{X} \setminus (\widetilde{\Delta}_X \cap w\widetilde{\Delta}_X) \hookrightarrow \widetilde{X} \setminus \widetilde{\Delta}_X, \qquad \widetilde{X} \setminus (\widetilde{\Delta}_X \cap w\widetilde{\Delta}_X) \hookrightarrow \widetilde{X} \setminus w\widetilde{\Delta}_X$$

$$\begin{array}{ccc} \widetilde{\mathbb{TV}}(\widetilde{\mathcal{S}}_1^X + \emptyset \otimes w\Delta_Y) & \hookrightarrow & \widetilde{\mathbb{TV}}(\widetilde{\mathcal{S}}_1^X) \\ \downarrow{\scriptstyle \sim\, \varphi_w} & & \\ \widetilde{\mathbb{TV}}(\widetilde{\mathcal{S}}_w^X + \emptyset \otimes \Delta_Y) & \hookrightarrow & \widetilde{\mathbb{TV}}(\widetilde{\mathcal{S}}_w^X). \end{array}$$

Note that we use $\emptyset \otimes \Delta_Y$ as an abbreviation for $\sum_{D' \in \mathcal{C}'} \emptyset \otimes \overline{D'}$ and recall from (2.2) what equivariant maps between $\mathbb{T}$-varieties look like in terms of p-divisors or divisorial fans. Since φ_w induces the identity map id_Y on Y, it corresponds to a plurifunction $\mathfrak{f}_w$ with

$$\widetilde{\mathcal{S}}_1^X + (\emptyset \otimes w\Delta_Y) \subseteq \widetilde{\mathcal{S}}_w^X + (\emptyset \otimes \Delta_Y) + \mathrm{div}(\mathfrak{f}_w).$$

We cannot expect to have $\mathrm{div}(\mathfrak{f}_w) = 0$. Otherwise, all local isomorphisms $\widetilde{X} \setminus w\widetilde{\Delta}_X \cong wP_u \times \mathbb{TV}(\widetilde{\Sigma})$ would glue to a global one and thus expose $\mathbb{TV}(\widetilde{\Sigma})$ as a factor of $\widetilde{X}$. On the other hand, $\mathrm{div}(\mathfrak{f}_w)$ clearly has to vanish on those slices where both divisorial fans already agreed in the first place. This observation shows that $\mathrm{supp}(\mathrm{div}\, \mathfrak{f}_w) \subseteq \Delta_Y \cup w\Delta_Y$. Furthermore, we see that coefficients of the principal polyhedral divisors $\mathrm{div}\, \mathfrak{f}_w$ are just shifts of the tail fan. Using this "hint", we correct the previous definition by

$$\widetilde{\mathcal{S}}_w^X := \widetilde{\mathcal{S}}_w^X + \mathrm{div}(\mathfrak{f}_w).$$

Then we still have that $\widetilde{X} \setminus w\widetilde{\Delta}_X \cong \widetilde{\mathbb{TV}}(\widetilde{\mathcal{S}}_w^X)$. But the gluing of $\widetilde{X} \setminus \widetilde{\Delta}_X$ and $\widetilde{X} \setminus w\widetilde{\Delta}_X$ now simply corresponds to the inclusion $\widetilde{\mathcal{S}}_1^X + (\emptyset \otimes w\Delta_Y) \subseteq \widetilde{\mathcal{S}}_w^X + (\emptyset \otimes \Delta_Y)$. Thus, the corrected $\widetilde{\mathcal{S}}_w^X$ fit into a huge common divisorial fan $\widetilde{\mathcal{S}}_{\mathrm{pre}}^X$ with $\widetilde{\mathcal{S}}_{\mathrm{pre}}^X(\bullet, w) = \widetilde{\mathcal{S}}_w^X$. Up to now, we have proven that $\widetilde{X} = \widetilde{\mathbb{TV}}(\widetilde{\mathcal{S}}_{\mathrm{pre}}^X)$ and $X^{\mathrm{tor}} =$

$\mathbb{TV}(\widetilde{\mathcal{S}}^X_{\mathrm{pre}})$. Yet, in contrast to the definition of $\widetilde{\mathcal{S}}^X$ in Definition 4.5 in (4.4), we have that

$$\widetilde{\mathcal{S}}^X_{\mathrm{pre}}(C,w) = \sum_a (C \cap p^{-1}(a)) \otimes D_a + \sum_{D'} (l_{D',w} + (C \cap N_{\mathbb{Q}})) \otimes \overline{D'} + \sum_{D'} \emptyset \otimes w\overline{D'}$$

for certain elements $l_{D',w} \in N$. To complete the proof of Theorem 4.6, it remains to check that these elements do not depend on w and are equal to $\overline{\rho}(D')$.

6. Concluding the Toroidal Case

Restricting the map $\phi : \widetilde{X} \to Y$ introduced in (4.3) to $\phi^{-1}(G/H') \to G/H'$, we obtain a locally trivial fibration. This is well known (following from G-homogeneity), and it is also visible in the description of $\widetilde{X}$ as $\widetilde{\mathbb{TV}}(\widetilde{\mathcal{S}}^X_{\mathrm{pre}})$ in (5.4)—it is reflected by the fact that, after restricting the divisorial fan $\widetilde{\mathcal{S}}^X_{\mathrm{pre}}$ to G/H', all its remaining polyhedral coefficients are shifted tail fans only.

The map $\phi^{-1}(G/H') \to G/H'$ extends the classical Tits fibration $\phi : G/H \to G/H'$. In particular, both share the same twist, which is encoded in the lattice elements $l_{D',w} \in N$ introduced at the end of (5.4). The only difference between the divisorial fans describing $\phi^{-1}(G/H')$ and G/H can be found in their tail fans, which are $\mathrm{tail}(\mathcal{S})$ and $\{0\}$, respectively.

We exploit this relation to determine the shift vectors $l_{D',w} \in N$ by presenting a polyhedral divisor $\mathcal{D}^{G/H}$ supported on the colors $\mathcal{C}(G/H')$ on G/H' such that $G/H \cong \widetilde{\mathbb{TV}}(\mathcal{D}^{G/H})$ under the right action of the torus $\mathbb{T} = H'/H$. In particular, in this section, we will forget about the embeddings $\widetilde{X}$, X^{tor}, and X discussed before—we just focus on the original Tits fibration.

6.1. *The Tits Fibration*

By abuse of notation, let ϕ also denote the $\mathbb{Q}$-linear extension $\mathbb{Q}^{\mathcal{C}(G/H)} \to \mathbb{Q}^{\mathcal{C}(G/H')}$ of the natural identification of colors $\phi : \mathcal{C}(G/H) \xrightarrow{\sim} \mathcal{C}(G/H')$. Recall further from Proposition 4.3 and its proof in (4.2) that we have an exact sequence

$$0 \to \mathcal{X}(G/H') \to \mathcal{X}(G/H) \to M \to 0$$

together with a splitting associated to a section $s : M \to \mathcal{X}(G/H)$. Moreover, given a character $\chi \in M$, we fix an associated eigenfunction $f_{s(\chi)} \in \mathbb{C}(G/H)^{(B)}_{s(\chi)}$ on G/H. In other words, it satisfies $f_{s(\chi)}(b^{-1}gH) = s(\chi)(b) \cdot f_{s(\chi)}(gH)$. We now define

$$\mathcal{L}(\chi) := \mathcal{O}_{G/H'}(\phi(\mathrm{div}\, f_{s(\chi)})) = \mathcal{O}_{G/H'}\Big(\sum_{D' \in \mathcal{C}(G/H')} \langle s(\chi), \rho_{\phi^{-1}(D')}\rangle D'\Big),$$

where, as before, $\rho_D = \rho_{\phi^{-1}(D')} \in \mathcal{X}^*(G/H)_{\mathbb{Q}}$ denotes the restriction of the valuation associated to the color $D = \phi^{-1}(D') \in \mathcal{C}(G/H)$. Note also that by

$\phi(\sum_{D\in\mathcal{C}(G/H)} a_D D)$ we mean $\sum_{D'\in\mathcal{C}(G/H')} a_D D'$ by using our identification of colors $D = \phi^{-1}(D')$ in G/H and G/H'.

On the other hand, choosing a basis $\mathcal{B}_M$ of M, we may embed $\mathbb{T} = \mathrm{Hom}_{\mathrm{group}}(M, \mathbb{C}^*)$ inside $\mathbb{C}^m$ with $m := \mathrm{rk}\, M$. Note that the action of $\mathbb{T}$ on itself extends to an action on $\mathbb{C}^m$ such that $\mathbb{C}^m = \bigoplus_{\chi\in\mathcal{B}_M} \mathbb{C}_\chi$ as a $\mathbb{T}$-module where $\mathbb{T}$ acts on $\mathbb{C}_\chi$ via the character $\chi \in \mathcal{B}_M \subset M$. Hence, we obtain the following embedding of $\mathbb{T}$-varieties:

$$G/H = G \times^{H'} \mathbb{T} \subset G \times^{H'} \left(\bigoplus_{\chi\in\mathcal{B}_M} \mathbb{C}_\chi \right) =: E.$$

Let $\mathcal{E}$ denote the sheaf of sections of E. Note that it is equal to $\bigoplus_{\chi\in\mathcal{B}_M} \mathcal{E}(\chi)$, where $\mathcal{E}(\chi)$ denotes the sheaf of sections of $G \times^{H'} \mathbb{C}_\chi$.

LEMMA 6.1. *$\mathcal{L}(\chi) \cong \mathcal{E}(-\chi)$, namely $f_{s(\chi)} \cdot \mathcal{L}(\chi) = \mathcal{E}(-\chi)$ as subsheaves of $\mathbb{C}(G/H)$.*

Proof. Given an open subset $U \subset G/H'$, we have that

$$\Gamma(U, \mathcal{E}(\chi)) = \mathrm{Mor}_{H'}(\pi^{-1}(U), \mathbb{C}_\chi) = \{\eta \in \mathcal{O}_G(\pi^{-1}(U)) \mid \eta \cdot h' = \chi(h')\eta\},$$

where π denotes the projection $G \to G/H'$ (which factors through ϕ), and χ is considered a character on H' that is trivial on H; see [Tim11, Prop. 2.1]. The function $f = f_{s(\chi)}$ was introduced as a B-eigenfunction for $s(\chi) \in \mathcal{X}_B$; we may assume that $f(1) = 1$. According to Lemma 4.2, this implies that $f(bHh') = f(bH)f(h') = f(bH)\chi(h')^{-1}$. Since BH is dense inside G, this means that $f_{s(\chi)}$ is a χ-eigenfunction for the right $\mathbb{T}$-action, too. Hence, if we multiply the elements of

$$\begin{aligned}\Gamma(U, \mathcal{L}(\chi)) = \{\zeta \in \mathbb{C}(G/H') \mid \mathrm{div}\,\zeta + \phi(\mathrm{div}\, f_{s(\chi)})|_U \geq 0\} \subset \mathbb{C}(G/H')\\ = \mathbb{C}(G)^{H'}\end{aligned}$$

with $f_{s(\chi)} \in \mathbb{C}(G)^H$, then we obtain regular functions on $\pi^{-1}(U) \subset G$ that are H'-semiinvariant with eigenvalue χ, namely,

$$f_{s(\chi)} \cdot \Gamma(U, \mathcal{L}(\chi)) = \{\eta \in \mathcal{O}_G(\pi^{-1}(U))^H \mid \eta \cdot h' = \chi(h')^{-1}\eta\} = \Gamma(U, \mathcal{E}(-\chi)).$$

□

6.2. *The Shift Vectors*

Recall that in (4.4) we already had met the section $s : M \to \mathcal{X}(G/H)$ mentioned in (6.1), but there it was used via the dual cosection

$$s^* : \mathcal{X}^*(G/H) \twoheadrightarrow N.$$

In other words, $(p, s^*) : \mathcal{X}^*(G/H) \xrightarrow{\sim} \mathcal{X}^*(G/H') \oplus N$ establishes a splitting of the exact sequence from Proposition 4.3. Note that this proposition also states that $p(\rho_D) = \rho_{D'}$ for colors $D \in \mathcal{C}(G/H)$ and $D' = \phi(D) \in \mathcal{C}(G/H')$, where $\rho_\bullet$ refers to the elements of $\mathcal{X}^*(G/H)$ and $\mathcal{X}^*(G/H')$ induced from the valuations associated to these colors, respectively.

DEFINITION 6.2. Using the notation from above, we define for every color $D' = \phi(D)$ its associated *shift vector*

$$\overline{\rho}(D') = \overline{\rho}(D) := s^*(\rho_D) \in N.$$

That is, for $\chi \in M$, $\langle \chi, \overline{\rho}(D') \rangle = \langle \chi, s^* \rho_{\phi^{-1}(D')} \rangle = \langle s(\chi), \rho_{\phi^{-1}(D')} \rangle$.

The choice of a basis $\mathcal{B}_M$ of M in (6.1) allows us to define the polyhedral cone $\sigma \subset N_{\mathbb{Q}} := N \otimes \mathbb{Q} \cong \mathbb{Q}^n$ as the positive orthant in the latter space. This will become the tail cone for the following important polyhedral divisor.

PROPOSITION 6.3. *The vector bundle $E \to G/H'$ from* (6.1) *is $\mathbb{T}$-equivariantly isomorphic to $\widetilde{\mathbb{TV}}(\mathcal{D}^E)$, where*

$$\mathcal{D}^E = \sum_{D' \in \mathcal{C}(G/H')} (\overline{\rho}(D') + \sigma) \otimes D'.$$

Proof. By Lemma 6.1 we can describe the vector bundle E as

$$E = \operatorname{Spec}_{G/H'} \operatorname{Sym}^{\bullet} \mathcal{E}^{\vee} = \operatorname{Spec}_{G/H'} \bigoplus_{\chi \in \sigma^{\vee} \cap M} \mathcal{E}(-\chi)$$
$$= \operatorname{Spec}_{G/H'} \bigoplus_{\chi \in \sigma^{\vee} \cap M} \mathcal{L}(\chi).$$

However, the very same result is obtained when we analyze the evaluation of the polyhedral divisor $\mathcal{D}^E$ on a multidegree $\chi \in \sigma^{\vee} \cap M$, namely

$$\mathcal{D}^E(\chi) = \sum_{D'} \langle \chi, \overline{\rho}(D') \rangle \cdot D' = \sum_{D'} \langle s(\chi), \rho_{\phi^{-1}(D')} \rangle \cdot D' = \mathcal{L}(\chi).$$

□

As explained in (2.2), the $\mathbb{T}$-equivariant open embedding $G/H \subset E$ translates into a face relation of the corresponding polyhedral divisors. Since this embedding is induced by $\mathbb{T} \subset \mathbb{C}^m$, it arises from the face relation $0 \preceq \sigma$ among the tail cones. So, as a corollary, we obtain a description of the polyhedral divisor $\mathcal{D}^{G/H}$. Note that it depends on the choice of the section s (hidden in the shift vectors $\overline{\rho}(D')$). However, in contrast to E and $\mathcal{D}^E$, it does not depend on the choice of a basis of M.

COROLLARY 6.4. *The $\mathbb{T}$-variety G/H is equal to $\widetilde{\mathbb{TV}}(\mathcal{D}^{G/H})$, where*

$$\mathcal{D}^{G/H} = \sum_{D' \in \mathcal{C}(G/H')} \overline{\rho}(D') \otimes D'.$$

In particular, its tail cone is equal to 0.

This completes the proof of Theorem 4.6.

7. The General Case

7.1. The Weyl Group

Recall that we have fixed $T \subseteq B \subseteq G$ such that $BH \subseteq G$ is open and dense. Let ∇ denote a basis of the positive roots R^+ that correspond to the choice of B. In particular, $W = W_G$ is generated by simple reflections $\{s_\alpha \mid \alpha \in \nabla\}$. For every subset $I \subset \nabla$ of simple roots, let $W_I \subset W$ denote the subgroup generated by simple reflections $\{s_\alpha \mid \alpha \in I\}$. The subgroup W_I comes with a distinguished set $W^I \subseteq W$ consisting of the representatives of minimal length of the left cosets of W_I. In particular, $W^I \times W_I \xrightarrow{\sim} W$ preserves the minimal representations as products of simple reflections. For proofs and further details, see [Spr].

Note that if G/H is of minimal rank then for every simple reflection $s_\alpha \in W$, there exists at most one color $D = D(\alpha) \in \mathcal{C}(G/H)$ such that $s_\alpha(D) \neq D$. Namely, $D = \overline{\mathcal{O}_{s_\alpha u_0}}$ unless $s_\alpha \in W_H$ (see Remark 3.10). For an arbitrary subset $\mathcal{F} \subseteq \mathcal{C}(G/H)$ of colors, define the subgroup $W_{I(\mathcal{F})} \subset W$ by taking $I(\mathcal{F}) = \{\alpha \in \nabla \mid D(\alpha) \in \mathcal{F} \text{ or } s_\alpha \in W_H\}$. In other words, $W_{I(\mathcal{F})}$ is generated by all simple reflections that leave $\bigcup_{D \in \mathcal{C}(G/H) \setminus \mathcal{F}_C} D$ invariant. In particular, $W_{I(\mathcal{F})} = \{w \in W \mid w(\bigcup_{D \in \mathcal{C} \setminus \mathcal{F}_C} D) = \bigcup_{D \in \mathcal{C} \setminus \mathcal{F}_C} D\}$. Indeed, the subgroup $\{g \in G \mid g(\bigcup_{D \in \mathcal{C} \setminus \mathcal{F}_C} D) = \bigcup_{D \in \mathcal{C} \setminus \mathcal{F}_C} D\}$ is parabolic; hence, its Weyl group is generated by simple reflections. For example, $W_{I(\emptyset)} = W_H$ is the Weyl group of the parabolic subgroup $P = P(\Delta_X^{\mathrm{tor}})$ mentioned in (5.1). The other extremal case is $W_{I(\mathcal{C}(G/H))} = W$, that is, $I(\mathcal{C}(G/H)) = \nabla$.

7.2. Contracting the Spherical Side

Let $X \mathrel{\widehat{=}} (C, \mathcal{F}_C)$ be a simple spherical embedding of minimal rank. Recall that there is an open affine $\mathbb{T}$-invariant covering $X = \bigcup_{w \in W} \widehat{X}_w$ (see Proposition 3.7). Note that some of these charts may be identical. To obtain a nonredundant description, we exploit the subgroup $W_C := W_{I(\mathcal{F}_C)}$ of W associated to $(C, \mathcal{F}_C)$. Summing things up, we obtain $X = \bigcup_{w \in W^{I(\mathcal{F}_C)}} \widehat{X}_w$, where the $\widehat{X}_w$ are now pairwise distinct.

Let $\pi : X^{\mathrm{tor}} \longrightarrow X$ (see diagram from (4.3)) be the map corresponding to $(C \cap \mathcal{V}, \emptyset) \to (C, \mathcal{F}_C)$. Then we have a similar covering $X^{\mathrm{tor}} = \bigcup_{w \in W} \widehat{X}_w^{\mathrm{tor}}$.

LEMMA 7.1. *For every $w \in W$, the map $\pi_w : \bigcup_{w' \in wW_C} \widehat{X}_{w'}^{\mathrm{tor}} \to \widehat{X}_w$ is the full preimage of $\widehat{X}_w$ under $\pi : X^{\mathrm{tor}} \longrightarrow X$. In particular, π_w is birational and proper.*

Proof. We may assume that $w = \mathrm{id}$ (renaming w' into w afterward). It remains to show that $\pi_{\mathrm{id}} : \bigcup_{w \in W_C} \widehat{X}_w^{\mathrm{tor}} \to \widehat{X}_{\mathrm{id}}$ is a full preimage, that is,

$$\bigcap_{w \in W_C} w \cdot \Delta_X^{\mathrm{tor}} = \bigcup_{D \in \mathcal{C} \setminus \mathcal{F}_C} \pi^{-1}(\overline{D}), \tag{1}$$

where we had defined $\Delta_X^{\mathrm{tor}} := \bigcup_{D\in\mathcal{C}} \overline{D} \subseteq X^{\mathrm{tor}}$. First, let us compare the intersections with the open orbit G/H, that is, show

$$\Omega := \bigcap_{w\in W_C} w\Big(\bigcup_{D\in\mathcal{C}} D\Big) = \bigcup_{D\in\mathcal{C}\setminus\mathcal{F}_C} D. \tag{2}$$

Let $\mathcal{O}_u$ be a B-orbit in G/H that contains a given point $x \in \Omega$ (here we use notation of Section 3.4). Take an increasing path in the graph $\Gamma(G/H)$ from $\mathcal{O}_u$ to the maximal B-orbit $\mathcal{O}_{u_0}$. Let $(\alpha_{i_1}, \dots, \alpha_{i_\ell})$ be the labels on its edges leading to $wx \in w(\mathcal{O}_u) \subset \mathcal{O}_{u_0} \neq \bigcup_{D\in\mathcal{C}} D$ for $w = s_{\alpha_{i_\ell}} \cdots s_{\alpha_{i_1}} \in W_C$. Then there exists an i_j such that $i_j \notin I(\mathcal{F}_C)$ since otherwise we would have $w \in W_C$. Put $\alpha_{i_j} = \alpha$. Then Lemma 3.11 implies that $\mathcal{O}_u \subset D$ for some $D \in \mathcal{C} \setminus \mathcal{F}_C$, namely, for $D(\alpha) = \overline{\mathcal{O}_{s_\alpha u_0}}$.

Finally, note that the above argument goes through if G/H is replaced by any other G-orbit $\mathcal{O} \subset X^{\mathrm{tor}}$ because $\mathcal{O}$ is also of minimal rank by [Res, Lemma 2.1]. More precisely, let $\mathcal{F}_C(\mathcal{O})$ be the set of all colors of $\mathcal{O}$ that are not contained in $\bigcup_{D\in\mathcal{C}\setminus\mathcal{F}_C} \pi^{-1}(\overline{D})$. Identity (2) for $\mathcal{O}$ and $\mathcal{F}_C(\mathcal{O})$ takes the form

$$\bigcap_{w\in W_C} w\Big(\bigcup_{D\in\mathcal{C}(\mathcal{O})} D\Big) = \bigcup_{D\in\mathcal{C}(\mathcal{O})\setminus\mathcal{F}_C(\mathcal{O})} D. \tag{3}$$

We have

$$\mathcal{O} \cap \Delta_X^{\mathrm{tor}} = \bigcup_{D\in\mathcal{C}(\mathcal{O})} D \quad \text{and} \quad \mathcal{O} \cap \bigcup_{D\in\mathcal{C}\setminus\mathcal{F}_C} \pi^{-1}(\overline{D}) = \bigcup_{D\in\mathcal{C}(\mathcal{O})\setminus\mathcal{F}_C(\mathcal{O})} D.$$

Hence, both sides of (3) coincide with the respective sides of (1) intersected with $\mathcal{O}$. This implies equality (1).

Now, the birationality and properness of π_w follow directly from the corresponding properties of $\pi : X^{\mathrm{tor}} \to X$; see [Kno91, Theorem 4.2]. □

7.3. *Contracting the $\mathbb{T}$-Variety Side*

Theorem 1.1 states that the simple spherical variety X can be described by a divisorial fan $\mathcal{S}^X$ whose maximal elements are indexed by elements of W; more precisely,

$$\mathcal{S}_w^X = \sum_{a\in\Sigma_Y} (C \cap p^{-1}(a)) \otimes D_a + \sum_{D'\in\mathcal{C}'} (\overline{\rho}(D') + (C \cap N_{\mathbb{Q}})) \otimes \overline{D'} + \sum_{D'\in\mathcal{C}'\setminus\mathcal{F}_C} \emptyset \otimes w\overline{D'}.$$

The only difference with respect to $\widetilde{\mathcal{S}}^X = \mathcal{S}^{X^{\mathrm{tor}}}$ is that the last sum runs over $\mathcal{C}' \setminus \mathcal{F}_C$ instead of the entire $\mathcal{C}'$. In other words, we have

$$\widetilde{\mathcal{S}}_w^X = \mathcal{S}_w^{X^{\mathrm{tor}}} = \mathcal{Z} \quad \text{on } V_w^{\mathrm{tor}} := Y \setminus \bigcup_{D'\in\mathcal{C}'} w\overline{D'},$$

$$\mathcal{S}_w^X = \mathcal{Z} \quad \text{on } V_w := Y \setminus \bigcup_{D'\in\mathcal{C}'\setminus\mathcal{F}_C} w\overline{D'},$$

where $\mathcal{Z} = \sum_{a\in\Sigma_Y}(C\cap p^{-1}(a))\otimes D_a + \sum_{D'\in\mathcal{C}'}(\overline{\rho}(D') + (C\cap N_{\mathbb{Q}}))\otimes\overline{D'}$ does not depend on $w\in W$. Since Y is also of minimal rank (see (4.2)), both $\mathcal{V}^{\text{tor}} := \{V_w^{\text{tor}} \mid w\in W\}$ and $\mathcal{V} := \{V_w \mid w\in W\}$ are coverings of Y.

LEMMA 7.2. *The covering $\mathcal{V}^{\text{tor}}$ is a refinement of $\mathcal{V}$. In detail, for every $w\in W$, we have $\bigcup_{w'\in wW_C} V_{w'}^{\text{tor}} = V_w$.*

Proof. Similarly to the proof of Lemma 7.1, we may assume that $w = \text{id}$ (renaming w' again into w afterward). It remains to show that $\bigcup_{w\in W_C} V_w^{\text{tor}} = V_{\text{id}}$, that is,

$$\bigcap_{w\in W_C} w\cdot\Delta_Y = \bigcup_{D'\in\mathcal{C}'\setminus\mathcal{F}_C}\overline{D'}$$

with $\Delta_Y := \bigcup_{D'\in\mathcal{C}'}\overline{D'}\subseteq Y$. However, this claim literally equals the equation we have shown in the proof of Lemma 7.1 for the X-level. Thus, the same arguments apply. □

7.4. Comparison of Both Sides

Our goal now is to compare the map $X^{\text{tor}}\longrightarrow X$ (see diagram from (4.3)) with the map $\mathbb{TV}(\widetilde{\mathcal{S}}^X)\to\mathbb{TV}(\mathcal{S}^X)$. The already proven Theorem 4.6 ensures that the sources of both maps coincide. Using $\mathcal{Z}$, we define $\mathcal{A} := \bigoplus_{u\in(C\cap N_{\mathbb{Q}})^\vee}\mathcal{O}(\mathcal{Z}(u))$ together with the following two affine $\mathbb{T}$-varieties:

$$X_w^{\text{tor}} := \operatorname{Spec}\Gamma(V_w^{\text{tor}},\mathcal{A}),\qquad X_w := \operatorname{Spec}\Gamma(V_w,\mathcal{A}).$$

They are open subsets of $\mathbb{TV}(\widetilde{\mathcal{S}}^X)$ and $\mathbb{TV}(\mathcal{S}^X)$, respectively. Everything fits now into the following commutative diagram:

$$\begin{array}{ccccccc}
X^{\text{tor}} & \hookleftarrow & \bigcup_{w'\in wW_C}\widehat{X}_{w'}^{\text{tor}} & \xrightarrow{\pi_w} & \widehat{X}_w & \hookrightarrow & X\\
\| & & \| & & & & \\
\mathbb{TV}(\widetilde{\mathcal{S}}^X) & \hookleftarrow & \bigcup_{w'\in wW_C} X_{w'}^{\text{tor}} & \xrightarrow{\psi_w} & X_w & \hookrightarrow & \mathbb{TV}(\mathcal{S}^X)
\end{array}$$

Whereas the vertical equalities $\widehat{X}_{w'}^{\text{tor}} = X_{w'}^{\text{tor}}$ come from Theorem 4.6, we have seen in Lemmas 7.1 and 7.2 that both central horizontal maps π_w and ψ_w are birational and proper. On the other hand, both $\widehat{X}_w$ and X_w are affine; hence, they have to coincide, too. This proves Theorem 1.1.

7.5. A Counterexample for the Nonminimal Rank Case

Note that Proposition 3.7 was essential for the proof. The assumption that G/H is of minimal rank in Theorem 1.1 was used both in Proposition 3.7 and in Lemma 7.1 and cannot be dropped as can be seen from the following example with a nontrivial $\mathbb{T}$-action.

EXAMPLE 7.3. Take $G = \mathrm{GL}_2$ and consider its action on $X = \mathbb{P}^1 \times \mathbb{P}^2$ by matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that X, together with the point $(1:0) \times (0:1:1)$, is a spherical embedding of G/H, where

$$H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, \lambda \in \mathbb{C}^* \right\}.$$

Note that G/H is not of minimal rank. The $\mathbb{T}$-variety given by Theorem 1.1 in this case does not coincide with the whole X since the former is not complete (note that the base space Y in this case is exactly the G-variety $\mathbb{P}^1 \times \mathbb{P}^1$ from Example 3.8). Namely, Theorem 1.1 yields only four out of six standard affine charts for $\mathbb{P}^1 \times \mathbb{P}^2$.

8. Examples

8.1. Horospherical Varieties

We use the notation introduced in (3.3). Also, recall from loc. cit. that $\mathcal{V} = \mathcal{X}(G/H)^*_{\mathbb{Q}}$ for any horospherical embedding $G/H \subset X$. Hence, our polyhedral divisors $\mathcal{S}^X$ will be defined on the flag variety G/P. Note also that for horospherical varieties, the exact sequence

$$0 \to N \to \mathcal{X}^*(G/H) \xrightarrow{p} \mathcal{X}^*(G/P) \to 0$$

reduces to the canonical isomorphism $N \simeq \mathcal{X}^*(G/H)$, that is, there is no need to choose a splitting. The following examples are taken from [Pas].

8.1.1. Embeddings of SL_2/U. This example continues Example 3.4 and relates the colored fans on Figure 1 to their corresponding divisorial fans. The Weyl group of SL_2 contains only two elements, id and w. Using Theorems 4.6 and 1.1 and identifying the color $D' = \{0\}$ and $wD' = \{\infty\}$ on $\mathbb{P}^1$, we obtain the following maximal elements of the respective divisorial fans:

$$\begin{aligned}
\mathcal{S}^{(a)}([0,\infty), \mathrm{id}) &= \emptyset \otimes 0,\\
\mathcal{S}^{(a)}([0,\infty), w) &= [1,\infty) \otimes 0 + \emptyset \otimes \infty,\\
\mathcal{S}^{(b)}(([0,\infty),\alpha), \{\mathrm{id}, w\}) &= [1,\infty) \otimes 0,\\
\mathcal{S}^{(c)}((-\infty,0], \mathrm{id}) &= \emptyset \otimes 0,\\
\mathcal{S}^{(c)}((-\infty,0], w) &= (-\infty,1] \otimes 0 + \emptyset \otimes \infty,\\
\mathcal{S}^{(d)}([0,\infty), \mathrm{id}) &= \emptyset \otimes 0,\\
\mathcal{S}^{(d)}([0,\infty), w) &= [1,\infty) \otimes 0 + \emptyset \otimes \infty,\\
\mathcal{S}^{(d)}((-\infty,0], \mathrm{id}) &= \emptyset \otimes 0,\\
\mathcal{S}^{(d)}((-\infty,0], w) &= (-\infty,1] \otimes 0 + \emptyset \otimes \infty,\\
\mathcal{S}^{(e)}(([0,\infty),\alpha), \{\mathrm{id}, w\}) &= [1,\infty) \otimes 0,
\end{aligned}$$

(a) (b)

Figure 2 Colored fans associated to complete embeddings of SL_3/H

$$\mathcal{S}^{(e)}((-\infty, 0], \mathrm{id}) = \emptyset \otimes 0,$$
$$\mathcal{S}^{(e)}((-\infty, 0], w) = (-\infty, 1] \otimes 0 + \emptyset \otimes \infty.$$

They are all toric, and it can easily be verified that the torus action is the action of a subtorus given by the following exact sequence of lattices of one-parameter subgroups:

$$0 \longrightarrow \mathbb{Z} \underset{\sigma}{\overset{\phi}{\rightleftarrows}} \mathbb{Z}^2 \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0$$

with

$$\phi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \pi = \begin{pmatrix} 1 & -1 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

So we are in fact in a toric downgrade situation as described in (5.2). The Chow quotient Y is $\mathbb{P}^1$, and one can check that applying the recipe from (5.2) yields the same divisorial fans as described above.

8.1.2. An Example of Rank 1 *and* $\mathbb{T}$*-Complexity* 3. Let $B^- \subset \mathrm{SL}_3$ denote the subgroup of lower-triangular matrices. We consider the subgroup $H \subset B^-$ of matrices whose second diagonal entry is 1. This yields a four-dimensional horospherical homogeneous space G/H of rank one over the full flag variety G/B^-. There are four complete embeddings, but we will only have a closer look at two of them, namely those whose colored fans are given in Figure 2.

Let α, β denote simple roots of SL_3. The Weyl group is isomorphic to S_3. It is generated by the reflections s_α and s_β and consists of six elements: $1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha$. Let D'_α and D'_β denote the colors of G/B. The W-action maps D'_α to $s_\alpha D'_\alpha$ or $s_\beta s_\alpha D'_\alpha$ (note that $s_\beta D'_\alpha = D'_\alpha$), and D'_β to $s_\beta D_\beta$ or $s_\alpha s_\beta D_\beta$ (again, $s_\alpha D'_\beta = D'_\beta$). The following tables in Figure 3 encode the maximal elements of the corresponding divisorial fans.

They are to be read as follows. Each row is indexed by a divisor $\overline{D'_\bullet}$. The corresponding one-dimensional slice is subdivided at v into two unbounded components. The labels of these components are given in columns 3 and 4, respectively. Note that we use a shorthand notation for the labels, that is, $\bullet = s_\bullet$.

8.2. $(\mathrm{GL}_2 \times \mathrm{GL}_2)$*-Equivariant Embeddings of* GL_2

These examples are classical yet we choose to discuss all details in order to give the reader the possibility to recall all notions that have been defined so far.

8.2.1. Basic Setup. Let $G := \mathrm{GL}_2 \times \mathrm{GL}_2$ act on GL_2 by left and right multiplications, that is, $(g_1, g_2) : g \mapsto g_1 g g_2^{-1}$. It follows that $H := \Delta(\mathrm{GL}_2)$, where Δ denotes the diagonal embedding of GL_2 to G. We fix the Borel subgroup

(a)	v	$(-\infty,0]$	$[0,\infty)$
D'_α	1	$\alpha,\alpha\beta,\beta\alpha,\alpha\beta\alpha$	$\alpha,\alpha\beta,\beta\alpha,\alpha\beta\alpha$
$s_\alpha D'_\alpha$	0	$1,\beta,\beta\alpha,\alpha\beta\alpha$	$1,\beta,\beta\alpha,\alpha\beta\alpha$
$s_\beta s_\alpha D'_\alpha$	0	$1,\alpha,\beta,\alpha\beta$	$1,\alpha,\beta,\alpha\beta$
D'_β	-1	$\beta,\alpha\beta,\beta\alpha,\alpha\beta\alpha$	$\beta,\alpha\beta,\beta\alpha,\alpha\beta\alpha$
$s_\beta D'_\beta$	0	$1,\alpha,\alpha\beta,\alpha\beta\alpha$	$1,\alpha,\alpha\beta,\alpha\beta\alpha$
$s_\alpha s_\beta D'_\beta$	0	$1,\alpha,\beta,\beta\alpha$	$1,\alpha,\beta,\beta\alpha$

(b)	v	$(-\infty,0]$	$[0,\infty)$
D'_α	1	$\alpha,\alpha\beta,\beta\alpha,\alpha\beta\alpha$	$1,\alpha,\beta,\alpha\beta,\beta\alpha,\alpha\beta\alpha$
$s_\alpha D'_\alpha$	0	$1,\beta,\beta\alpha,\alpha\beta\alpha$	$1,\alpha,\beta,\alpha\beta,\beta\alpha,\alpha\beta\alpha$
$s_\beta s_\alpha D'_\alpha$	0	$1,\alpha,\beta,\alpha\beta$	$1,\alpha,\beta,\alpha\beta,\beta\alpha,\alpha\beta\alpha$
D'_β	-1	$1,\alpha,\beta,\alpha\beta,\beta\alpha,\alpha\beta\alpha$	$\beta,\alpha\beta,\beta\alpha,\alpha\beta\alpha$
$s_\beta D'_\beta$	0	$1,\alpha,\beta,\alpha\beta,\beta\alpha,\alpha\beta\alpha$	$1,\alpha,\alpha\beta,\alpha\beta\alpha$
$s_\alpha s_\beta D'_\beta$	0	$1,\alpha,\beta,\alpha\beta,\beta\alpha,\alpha\beta\alpha$	$1,\alpha,\beta,\beta\alpha$

Figure 3 Divisorial fans associated to complete embeddings of SL_3/H

$B := B^+_{\mathrm{GL}_2} \times B^-_{\mathrm{GL}_2} \subset G$, where $B^+_{\mathrm{GL}_2}$ and $B^-_{\mathrm{GL}_2}$ consist of upper and lower triangular matrices, respectively. Furthermore, we fix the maximal torus $T \subseteq B$ given by the diagonal matrices $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2$. Hence, we have that $\mathcal{X}_B = \mathcal{X}_T = \mathbb{Z}^4$ with basis $\{e_1^+, e_2^+, e_1^-, e_2^-\}$. Finally, $U := B_u$ denotes the unipotent radical of B. As usual, elements of GL_2 are denoted by matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

One easily checks that $\mathbb{C}(G/H) = \mathbb{C}(\mathrm{GL}_2) = \mathbb{C}(a,b,c,d)$ and $\mathbb{C}(\mathrm{GL}_2)^U = \mathbb{C}(d, \det)$ with $\det = ad - bc$. Using the exact sequence from (3.1), we see that the weights of these generators are $\chi(d) = e_2^- - e_2^+$ and $\chi(\det) = (e_1^- - e_1^+) + (e_2^- - e_2^+)$. The Weyl group of G is $W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta\}$ with

$$s_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

that is, $s_\alpha : e_1^+ \leftrightarrow e_2^+$ and $s_\beta : e_1^- \leftrightarrow e_2^-$.

8.2.2. Further Ingredients. The Bruhat decomposition of GL_2 with $W_{\mathrm{GL}_2} = \{1, s\}$ yields $\mathrm{GL}_2 = (B^+B^-) \sqcup (B^+ s B^-)$. The first double class is the open orbit and shows that $G/H = \mathrm{GL}_2$ is spherical, whereas the second double class corresponds to the unique color $D = V(d)$ in $\mathcal{C}$.

The normalizer H' is equal to $\Delta(\mathrm{GL}_2) \cdot (\mathbb{C}^* \times \mathbb{C}^*)$. The transitive G-action on GL_2 from (8.2.1) induces a transitive G-action on PGL_2 providing the isomorphism $G/H' = \mathrm{PGL}_2$. Its unique and wonderful compactification is $\mathbb{P}^3 = \mathrm{PGL}_2 \sqcup V(\det)$.

8.2.3. The Ambient Spaces for the (Colored) Fans. We identify $\mathbb{T} = H'/H$ with $\mathbb{C}^*$ via $(t, 1) \mapsto t$. In particular, $M = \mathbb{Z}$, and we derive from the commutative diagram of Lemma 4.2 that the sequence

$$0 \to \mathcal{X}(G/H') \to \mathcal{X}(G/H) \to M \to 0$$

sends $(e_i^+ - e_i^-) \mapsto 1$ for $i = 1, 2$; see the proof of Proposition 4.3. The kernel $\mathcal{X}(G/H')$ is generated by $(e_1^- - e_1^+) - (e_2^- - e_2^+)$ which is $\chi(\det/d^2)$. The dual sequence

$$0 \to (N = \mathbb{Z}) \to \mathcal{X}^*(G/H) \xrightarrow{p} \mathcal{X}^*(G/H') \to 0$$

sends $1 \mapsto -E^1 - E^2$ and $E^1 \mapsto E$, $E^2 \mapsto -E$, where $\{E^1, E^2\}$ and $\{E\}$ are the dual bases of $\{(e_1^- - e_1^+), (e_2^- - e_2^+)\} = \{\chi(\det/d), \chi(d)\}$ and $\{(e_1^- - e_1^+) - (e_2^- - e_2^+)\} = \{\chi(\det/d^2)\}$, respectively. Since the valuation $v_D = v_{V(d)}$ sends $\det/d \mapsto -1$ and $d \mapsto 1$, we obtain $\rho_D = E^2 - E^1$. We fix the splitting $E \mapsto E^1$. This induces the projection $\mathcal{X}^*(G/H) \twoheadrightarrow N = \mathbb{Z}$ with $E^1 \mapsto 0$ and $E^2 \mapsto -1$. In particular, the shift vector from (4.4) equals $\overline{\rho}(D) = -1$.

8.2.4. The Valuation Cone. Let us consider the GL_2-embeddings given in the following diagram:

$$\begin{array}{ccccc} & & \widetilde{\mathbb{C}^4} & \hookrightarrow & \widetilde{\mathbb{P}^4} \\ & \nearrow & \downarrow \pi & & \downarrow \pi \\ \mathrm{GL}_2 & \hookrightarrow & \mathbb{C}^4 & \hookrightarrow & \mathbb{P}^4 \end{array}$$

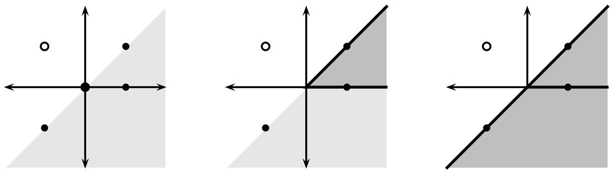
where the upper row consists of the blow ups at the origin of the corresponding varieties in the lower row. This picture provides us with three G-invariant divisors and their associated valuations $v_{\det} \mathrel{\widehat{=}} V(\det) = \mathbb{C}^4 \setminus \mathrm{GL}_2$, $v_E \mathrel{\widehat{=}} E = \pi^{-1}(0)$, and $v_\infty \mathrel{\widehat{=}} \mathbb{P}^4 \setminus \mathbb{C}^4$. They send the equations $\det/d = (x_1x_4 - x_2x_3)/(x_0x_4)$ and $d = x_4/x_0$ to $(1, 0)$, $(1, 1)$, and $(-1, -1)$, respectively. This means that $\rho_{\det} = E^1$, $\rho_E = E^1 + E^2$, and $\rho_\infty = -(E^1 + E^2)$. These elements span the valuation cone

$$\mathcal{V} = \{w_1E^1 + w_2E^2 \mid w_1 \geq w_2\} \subseteq \mathcal{X}^*(G/H),$$

that is, the lower half plane which is bounded by the line $\langle E^1 + E^2 \rangle$.

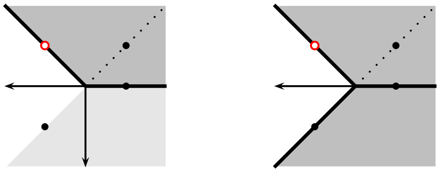
8.2.5. The Colored Fans. All upper embeddings $\mathrm{GL}_2 \subseteq \widetilde{\mathbb{C}^4} \subseteq \widetilde{\mathbb{P}^4}$ are toroidal. The uncolored cone of GL_2 is equal to $\{0\}$, the one corresponding to $\widetilde{\mathbb{C}^4}$ equals $\langle E^1, E^1 + E^2 \rangle$, whereas $\widetilde{\mathbb{P}^4}$ is given by the complete subdivision of $\mathcal{V}$ by the ray $\langle E^1 \rangle$. Hence, it consists of the two uncolored cones $\langle -(E^1 + E^2), E^1 \rangle$ and $\langle E^1, E^1 + E^2 \rangle$; see Figure 4.

Blowing down the exceptional divisor E via π gives us two nontoroidal spherical embeddings of GL_2, namely $\mathbb{C}^4$ and $\mathbb{P}^4$. All we have to do is to replace the uncolored cone $\langle E^1, E^1 + E^2 \rangle$ appearing in both blow ups by the colored one $(\langle E^1, E^2 - E_1 \rangle, \{D\})$; see Figure 5.



(A) GL_2 (B) $\widetilde{\mathbb{C}^4}$ (C) $\widetilde{\mathbb{P}^4}$

Figure 4 Toroidal $\mathrm{GL}_2 \times \mathrm{GL}_2$-equivariant embeddings of GL_2



(A) $\mathbb{C}^4$ (B) $\mathbb{P}^4$

Figure 5 Nontoroidal $\mathrm{GL}_2 \times \mathrm{GL}_2$-equivariant embeddings of GL_2

8.2.6. The Divisorial Fans. The induced action of $\mathbb{T} = \mathbb{C}^*$ on $\mathbb{C}^4$ corresponds, up to sign, to the standard $\mathbb{Z}$-grading of the affine coordinate ring $\mathbb{C}[a, b, c, d]$. Performing the usual downgrading procedure for the diagonal subtorus $\mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^4$, we see that the polyhedral divisor $\mathcal{D}$ for $\mathbb{C}^4$ is defined over $Y = \mathbb{P}^3$ and equal to $[1, \infty) \otimes H$, where $H = H_0 \subseteq \mathbb{P}^3$ denotes a hyperplane. Actually, the toric downgrade yields

$$\mathcal{D} = [1, \infty) \otimes H_0 + \sum_{i=1}^{3} [0, \infty) \otimes H_i$$

with $H_i = V(z_i)$. However, since one can omit trivial summands, that is, those having just the tail cone as their coefficient, we arrive at the description from above. The coefficients of H_i in the extended version arise as intersections of the four affine lines $\ell_i = e^i + \mathbb{Q} \cdot e$ (with $e := \sum_i e^i$) with the upper orthant $\mathbb{Q}^4_{\geq 0} = \langle e^0, \dots, e^3 \rangle$ that represents $\mathbb{C}^4$ as an affine toric variety.

Blowing up $0 \in \mathbb{C}^4$, we obtain $\widetilde{\mathbb{C}^4} = \widetilde{\mathbb{TV}}(\mathcal{D}) = \mathbb{TV}(\mathcal{S})$, where the four maximal elements of the divisorial fan $\mathcal{S} = \{\mathcal{D}_0, \dots, \mathcal{D}_3\}$ are given by $\mathcal{D}_i := \mathcal{D} + \emptyset \otimes H_i$. On the one hand, this simulates the relative Spec construction via the affine open covering $\{\mathbb{P}^3 \setminus H_i\}$. On the other hand, it arises naturally from the toric downgrade construction. Namely, for $\mathbb{C}^4$, we had intersected the four lines ℓ_i with a single polyhedral cone. But for $\widetilde{\mathbb{C}^4}$, we subdivide $\mathbb{Q}^4_{\geq 0}$ into four chambers C_i by inserting the new ray $e = \sum_i e^i$. These smaller cones correspond to the $\mathcal{D}_i$. Since each of the four lines ℓ_i misses exactly one of them, namely $\ell_i \cap C_i = \emptyset$, we obtain $\emptyset$ as the coefficient of H_i in $\mathcal{D}_i$.

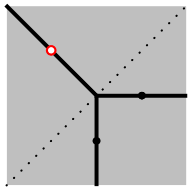


Figure 6 Colored fan of Grass(2, 4)

Representing $\mathbb{P}^4$ and $\widetilde{\mathbb{P}^4}$ as toric varieties involves four additional cones, respectively. Hence, our toric downgrade creates another four p-divisors $\{\mathcal{D}'_0, \dots, \mathcal{D}'_3\}$ following the same pattern as for the case of $\widetilde{\mathbb{C}^4}$. Their common tail cone becomes $(-\infty, 0]$:

$$\mathcal{D}'_i = (-\infty, 1] \otimes H_0 + \emptyset \otimes H_i.$$

8.2.7. The Grassmannian Grass(2, 4). Let $W_i := \mathbb{C}^2$ $(i = 1, 2)$ be two copies of the very same complex plane $\mathbb{C}^2$. By $G \subseteq \mathrm{GL}_4$ we see that G acts on $\mathbb{C}^4 = W_1 \oplus W_2$ and therefore also on Grass(2, 4). Since G respects the decomposition of $\mathbb{C}^4$, its orbits are given by $\mathrm{orb}(d_1, d_2) := \{V \in \mathrm{Grass}(2, 4) \mid \dim(V \cap W_i) = d_i\}$. The following list displays all pairs (d_1, d_2) which give a nonempty orbit:

dim	0	2	3	4
(d_1, d_2)	(2, 0), (0, 2)	(1, 1)	(1, 0), (0, 1)	(0, 0)

Let $V_0 := \{(v, v) \mid v \in \mathbb{C}^2 = W_i\} \in \mathrm{orb}(0, 0)$. Then $\mathrm{stab}_{V_0} = \Delta(\mathrm{GL}_2)$, that is, $1_G \mapsto V_0$ provides an embedding $\mathrm{GL}_2 \hookrightarrow \mathrm{Grass}(2, 4)$ of the usual type. Its colored fan is presented in Figure 6.

Using Plücker coordinates, the induced $\mathbb{T}$-action on Grass(2, 4) can be obtained from $\deg x_{01} = 1$, $\deg x_{23} = -1$, and $\deg = 0$ for the remaining variables. Then the resulting p-divisor for $\mathbb{C}^6$ and the divisorial fan $\mathcal{S}$ for $\mathbb{P}^5$ live on the four-dimensional weighted projective space $\mathbb{P}(2, 1, 1, 1, 1)$. Denoting its hyperplanes by $A, H_1, \dots, H_4$, the slices are given by

$$\mathcal{S}_A = (-\infty, 0, 1, \infty), \qquad \mathcal{S}_{H_i} = (-\infty, 0, \infty) \quad (i = 1, 2, 3), \quad \text{and}$$
$$\mathcal{S}_{H_4} = (-\infty, -1, \infty).$$

The divisorial fan $\mathcal{S}$ is generated by six p-divisors $\mathcal{D}^0, \dots, \mathcal{D}^5$ corresponding to the standard affine open covering of $\mathbb{P}^5$. They can be visualized as labels on the cells of the slices: All cells $(-\infty, \bullet]$ carry the label $\mathcal{D}^0$, and, similarly, all $[\bullet, \infty)$ belong to $\mathcal{D}^5$. All middle cells, namely $[0, 1]$ in $\mathcal{S}_A$ and the vertices in $\mathcal{S}_{H_i}$, are labeled with $\{\mathcal{D}^1, \dots, \mathcal{D}^4\} \setminus \mathcal{D}^i$, that is, the H_i-coefficient in $\mathcal{D}^i$ is $\emptyset$.

Finally, the embedding $\mathrm{Grass}(2, 4) \hookrightarrow \mathbb{P}^5$ corresponds to the embedding $\mathbb{P}^3 \hookrightarrow \mathbb{P}(2, 1, 1, 1, 1)$, $(a : b : c : d) \mapsto ((bc - ad) : a : b : c : d)$ on the level of Chow quotients. Hence, the divisorial fan of Grass(2, 4) is the restriction of $\mathcal{S}$ to

$\mathbb{P}^3$. In particular, $H_1, \ldots, H_4$ become the standard hyperplanes, and A turns into the quadric det $\subseteq \mathbb{P}^3$.

8.2.8. Comparison of Divisorial and Colored Fans. The slices of the divisorial fan of $\widetilde{\mathbb{P}^4}$ on $\mathbb{P}^3$ from (8.2.6) are either $(-\infty, 1, \infty)$ or $(-\infty, 0, \infty)$ with four separate labels for both negative and positive sides. This labeling together with the presence of empty coefficients corresponds exactly to the divisorial fan introduced in Definition 4.5 in (4.4): The first summand involves the only G-invariant divisor $V(\det) \subset \mathbb{P}^3$. Since its coefficient equals a shift of the tail fan $(-\infty, 0, \infty)$, this sum can be incorporated in the other summands, involving the only color $V(d) \subseteq \mathbb{P}^3$. Indeed, since the Weyl group has four elements, both top-dimensional cells appear exactly four times.

Comparing this with the divisorial fan of $\mathbb{P}^4$ on $\mathbb{P}^3$, we see in (8.2.6) that the four different labels on the one side merge into one common label. This reflects exactly the description of the divisorial fan from Definition 4.5: Since $\mathcal{C}' \setminus \mathcal{F} = \emptyset$, the last sum becomes void for these cells.

Finally, we consider the colored fan of Grass$(2,4)$; see Figure 6. It is induced from the subdivision of $\mathcal{V}$ by two rays, namely those spanned by E^1 and $-E^2$, respectively. Note that there are two maximal cones which are not contained in $\mathcal{V}$ since they contain the color as a generator. This means that $\mathcal{C}' \setminus \mathcal{F} = \emptyset$ occurs now on both sides—creating the simple labelings by $\mathcal{D}^0$ and $\mathcal{D}^5$ in (8.2.7). Moreover, the polyhedral coefficient $\mathcal{S}_A$ clearly is the intersection of the colored fan with an affine line within the valuation cone. However, this summand cannot be incorporated in the others as it was possible for $\mathbb{P}^4$. The reason is that it carries a richer structure as just being a shift of the tail fan.

ACKNOWLEDGMENT. We are grateful to the referee for valuable comments and suggestions and for correcting an error in the first version of this paper.

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