

Counting Genus One Fibered Knots in Lens Spaces

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ABSTRACT. The braid axis of a closed 3-braid lifts to a genus one fibered knot in the double cover of S^3 branched over the closed braid. Every genus one fibered knot in a 3-manifold may be obtained in this way. Using this perspective, we answer a question of Morimoto about the number of genus one fibered knots in lens spaces. We determine the number of genus one fibered knots up to homeomorphism and up to isotopy in any given lens space. This number is 3 in the case of the lens space $L(4, 1)$, 2 for the lens spaces $L(m, 1)$ with $m > 0$ and $m \neq 4$, and at most 1 otherwise. Furthermore, each homeomorphism equivalence class in a lens space is realized by at most two isotopy classes.

1. Introduction

Let M be a 3-manifold. We say that a knot K in M is a *genus one fibered knot*, GOF-knot for short, if $M - N(K)$ is a once-punctured torus bundle over the circle and the boundary of a fiber is a longitude of K . In particular, we will always consider a GOF-knot to be null homologous.

As begun by Burde and Zieschang in [BZ67], González-Acuña [GAn70] shows that the trefoil (and its mirror) and the figure-eight knot are the only GOF-knots in S^3 . Morimoto shows that up to homeomorphism each lens space $L(m, 1)$ contains at least two GOF-knots if $m > 0$ and exactly two if $m \in \{1, 2, 3, 5, 19\}$, $L(4, 1)$ contains exactly three GOF-knots, each of $L(0, 1)$, $L(5, 2)$, and $L(19, 3)$ contains exactly one GOF-knot, and each of $L(19, 2)$, $L(19, 4)$, and $L(19, 7)$ contains no GOF-knots, [Mor89]. Morimoto then asks the following question.

QUESTION [Mor89]. Are the numbers of GOF-knots in all lens spaces bounded?

In this article we use double branched covers of two-bridge links represented as closed 3-braids to address this question.

THEOREM 4.3. *Up to homeomorphisms, the lens space $L(\alpha', \beta')$ contains exactly*

- *three distinct GOF-knots if and only if $L(\alpha', \beta') \cong L(4, 1)$,*
- *two distinct GOF-knots if and only if $L(\alpha', \beta') \cong L(\alpha, 1)$ for $\alpha > 0$ and $\alpha \neq 4$,*

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- one distinct GOF-knot if and only if $L(\alpha', \beta') \cong L(\alpha, \beta)$ either for $\alpha = 0$ or for $0 < \beta < \alpha$, where either
 - $\alpha = 2pq + p + q$ and $\beta = 2q + 1$ for some integers $p, q > 1$, or
 - $\alpha = 2pq + p + q + 1$ and $\beta = 2q + 1$ for some integers $p, q > 0$, and
- zero GOF-knots otherwise.

REMARK 1.1. In Theorem 4.5 we further show that each of these homeomorphism classes of a GOF-knot in a lens space splits into at most two isotopy classes; indeed, we determine which homeomorphism classes represent two isotopy classes. In particular, the homeomorphism classes of the trefoil in S^3 and the single GOF-knot in $S^1 \times S^2$ each divide into two isotopy classes related by an orientation reversing homeomorphism. Furthermore “most” of the lens spaces with one homeomorphism class of GOF-knots actually have two isotopy classes related by an involution of the lens space that is not isotopic to the identity.

Theorem 4.3 is proven in Section 4. It is a consequence of the classification of 3-braid representations of two-bridge knots up to homeomorphisms given in Theorem 4.2 and the correspondence between the braid axes of closed 3-braids and GOF-knots given in Proposition 2.1. In Section 3 we blend results of Murasugi [Mur91] and Stoimenow [Sto03] with results of Birman and Menasco [BM93] to prepare for our proof of Theorem 4.2.

REMARK 1.2. The original preprint of this article was written in 2005. The text and content here is largely the same though we have updated our closing remarks of Section 5. In particular, we highlight a result from Lott’s thesis [Lot09] in Theorem 5.3 and point out that we have since answered one of the questions asked at the end.

1.1. Preliminaries

Throughout this article we will be classifying links in 3-manifolds up to homeomorphism. In particular, we consider homeomorphisms without regards to orientation. Therefore, for instance, we regard the right-handed trefoil and left-handed trefoil in S^3 as equivalent. In particular, if h is a homeomorphism between 3-manifolds M and M' such that $h(K) = K'$ for knots $K \subset M$ and $K' \subset M'$, then we say the pairs (M, K) and (M', K') are *equivalent* or simply that the knots K and K' are (*homeomorphism*) *equivalent*. However, if $M = M'$ and there is an ambient isotopy that takes K to K' , then we say that K and K' are *isotopic*. All our isotopies of links may be regarded as ambient isotopies.

An unknot A in S^3 that is disjoint from a link L is a (*braid*) *axis* for the link L if L is braided about A , that is, if the exterior of A may be identified with $S^1 \times D^2$ so that L is transverse to each disk fiber. If h is a homeomorphism of S^3 such that $h(L) = L'$ and $h(A) = A'$ for links L and L' with axes A and A' giving closed braid representations of L and L' , respectively, then we say that the pairs (L, A) and (L', A') are (*homeomorphism*) *equivalent*. If $L = L'$, then we simply say that A and A' are equivalent axes for L even though A and A' might not be isotopic

in the complement of L ; see Lemma 3.8. We say that (L, A) and (L', A') are *isotopic* if there is an ambient isotopy of S^3 taking L to L' and A to A' . If $L = L'$, then we simply say A and A' are isotopic axes for L if the ambient isotopy of S^3 that relates A and A' fixes L . Regardless, because any homeomorphism of S^3 is isotopic to either the identity or mirroring [Gug53], if (L, A) and (L', A') are equivalent, then (L, A) is isotopic to either (L', A') or the mirror of (L', A') .

Let ω be the braid word in the standard generators of a braid whose closure is the link L with braid axis A . Then the absolute value of the exponent sum of ω is an invariant of the homeomorphism equivalence class of the pair (L, A) , whereas the exponent sum itself is an invariant of the isotopy class of the pair (L, A) . This may be seen as follows. First, fixing A , any isotopy between two braided configurations of L may be achieved by braid isotopy; thus, their braid words are conjugate and have the same exponent sum. Next, because the axis A is an unknot, its symmetry group is $\text{Sym}(S^3, A) = \text{Homeo}(S^3, A)/\text{Homeo}_0(S^3, A) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which is generated by a mirroring of S^3 and an isotopy of S^3 that flips over a sphere containing A . The mirroring that preserves the orientation on A takes the inverse of ω and thus negates the exponent sum. The flip reverses the orientation of A while inverting the order the word ω is read and thus preserves the exponent sum.

We follow the conventions of Burde and Zieschang [BZ03] in regards to continued fractions, two-bridge links, and lens spaces and further refer the reader there for background regarding fibered knots, braids, double branched coverings, etc. In this convention, a continued fraction expansion

$$\beta/\alpha = [a_1, a_2, \dots, a_n] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}$$

for coprime integers α and β gives a geometric description of the two-bridge link $b(\alpha, \beta)$ whose double branched cover is the lens space $L(\alpha, \beta)$. By flipping the signs on both α and β , we may always take $\alpha \geq 0$. This link $b(\alpha, \beta)$ has Conway [Con70] notation (a_1, a_2, \dots, a_n) . Figure 1 shows the standard diagrams of the two-bridge link $b(2pq + p + q + 1, 2q + 1)$ for two of its corresponding continued fraction expansions $[p, 1, 1, q] = \frac{2q+1}{2pq+p+q+1} = [p, 2, -q - 1]$. Due to the choice of continued fraction convention, the twist regions corresponding to the coefficients with even index twist in the direction opposite the sign of the coefficient. We orient the link $b(\alpha, \beta)$ with $\beta/\alpha = [a_1, \dots, a_n]$ so that from the top, the third strand of the diagram goes downward, as does the second if the link has two components.

Consider two two-bridge links $b(\alpha_1, \beta_1)$ and $b(\alpha_2, \beta_2)$ with $\alpha_1, \alpha_2 > 0$ with the corresponding lens spaces $L(\alpha_1, \beta_1)$ and $L(\alpha_2, \beta_2)$. Schubert shows that the two-bridge links are isotopic as unoriented links and the lens spaces are orientation-preserving homeomorphic if and only if $\alpha_1 = \alpha_2$ and $\beta_1 \equiv \beta_2^{\pm 1}$

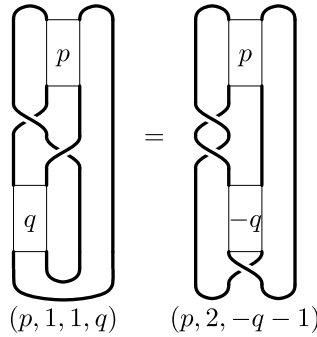


Figure 1 The two bridge link $(p, 1, 1, q)$ is equivalent to $(p, 2, -q - 1)$

mod α_0 . They are mirrors if and only if $\alpha_1 = \alpha_2$ and $\beta_1 \equiv -\beta_2^{\pm 1} \pmod{\alpha_0}$. As oriented links, the two links are isotopic if and only if $\alpha_1 = \alpha_2$ and $\beta_1 \equiv \beta_2^{\pm 1} \pmod{\alpha_0}$. See Theorem 12.6 [BZ03] for example.

2. GOF-Knots via Double Branched Covers of Closed 3-Braids

PROPOSITION 2.1. *The homeomorphism classes of the pairs (M, K) of a GOF-knot K in a 3-manifold M are in one-to-one correspondence with the homeomorphism classes of the pairs (L, A) of a link L in S^3 and a braid axis A giving a closed 3-braid representation of L . The pair (M, K) corresponds to the pair (L, A) if and only if M is the double cover of S^3 branched over L and K is the lift of A .*

Proof. This proposition appears to be known to the experts, and we offer a sketch here. For further details, see [Bal08, Section 2].

Each orientable once-punctured torus bundle over the circle is the double cover of a solid torus branched over a closed 3-braid. Moreover, a Dehn filling of such a once-punctured torus bundle along a slope that intersects each fiber once is the double cover of S^3 branched over a closed 3-braid. This may be seen as follows. The once-punctured torus T admits an involution τ with three fixed points. Any orientation-preserving homeomorphism of T is isotopic rel- ∂ to one invariant under τ . The involution τ then extends across each fiber of the mapping torus of such a homeomorphism. The involution further extends across the solid torus of the Dehn filling described above (cf. Sections 4 and 5 of [MR97]).

A GOF-knot in a 3-manifold M is then the lift of the braid axis of some closed 3-braid in S^3 where M is the double cover of S^3 branched over the closed 3-braid. This also becomes evident by considering the genus 2 Heegaard splitting of M induced by the GOF-knot and the corresponding involution on M (arising from the hyper-elliptic involution on the genus 2 Heegaard surface that extends across the two handlebodies) that acts freely on the GOF-knot (cf. Section 5 of [BH75]).

Quotienting by the involution, the fixed set becomes the closed braid, and the GOF-knot becomes the braid axis. \square

LEMMA 2.2. *The isotopy classes of the pairs (M, K) of a GOF-knot K in a 3-manifold M are in one-to-one correspondence with the isotopy classes of the pairs (L, A) of a link L in S^3 whose double branched cover is M and a braid axis A that gives a closed 3-braid presentation of L .*

Proof. Let K and K' be GOF-knots in M corresponding respectively to the pairs (L, A) and (L', A') as in Proposition 2.1. An isotopy from (L, A) to (L', A') lifts through double branched covers (branched over the trace of the isotopy from L to L') to an ambient isotopy of M taking K to K' . On the other hand, an ambient isotopy of M taking K to K' provides an isotopy of their once-punctured torus bundle exteriors, framed with their meridians. The isotopy may be further taken to relate the involutions τ and τ' on these once-punctured torus bundles as constructed above. These involutions and the isotopy between them then extends across the knots to all of M . The isotopy of involutions from τ to τ' is then an involution of the isotopy from (M, K) to (M, K') whose quotient is an isotopy from the quotient by τ to the quotient by τ' relating the images of their fixed sets (the links L and L') and the images of K and K' (the axes A and A'). In other words, the isotopy descends to an isotopy from (L, A) to (L', A') . \square

REMARK 2.3. A meridian of the braid axis is a longitudinal curve on the solid torus containing the closed 3-braid and lifts to two meridians of the GOF-knot in the double cover. More generally, let V be the solid torus neighborhood of the braid axis, and \tilde{V} be the solid torus neighborhood of the GOF-knot that is the lift of V . The meridian of the solid torus $S^3 - \text{Int}(V)$ is the longitude of V and lifts to the longitude of \tilde{V} . Since \tilde{V} double covers V , simple closed curves of slope p/q on ∂V lift to curves of slope $2p/q$ on $\partial\tilde{V}$. Assuming that p and q are coprime, if q is even, then the slope $2p/q$ is to be interpreted as two parallel curves of slope $p/(q/2)$. It follows that $1/n$ surgery on GOF-knot corresponds to inserting $2n$ full twists (right-handed if $n < 0$, left-handed if $n > 0$) into the 3-braid.

3. Two-Bridge Links and Closed 3-Braids

The main two ingredients for our proof of Theorem 4.2 are the classification of two-bridge links with closed 3-braid representations and the classification of braid axes giving closed 3-braid representations of a link. Murasugi [Mur91, Proposition 7.2] and later Stoimenow [Sto03, Corollary 8] determine which oriented two-bridge links have representations as closed 3-braids. The Classification Theorem of Birman and Menasco [BM93] then permits us to count the number of braid axes representing an oriented two-bridge link as a closed 3-braid that are not isotopic in the complement of the two-bridge link. In Lemma 3.8 we show when these braid axes paired with the link are equivalent by a homeomorphism of S^3 . Theorem 4.2 is then proved by determining which orientations of which two-bridge links admit closed 3-braid representations.



Figure 2 The standard generators σ_1 and σ_2 for the three-strand braid group B_3

Let $\mathbf{b}(L)$ denote the braid index of the link L .

PROPOSITION 3.1 [Mur91, Proposition 7.2]. *Let L be an oriented two-bridge link of type $\mathbf{b}(\alpha, \beta)$, where $0 < \beta < \alpha$ and β is odd. Then*

- (1) $\mathbf{b}(L) = 2$ if and only if $\beta = 1$.
- (2) $\mathbf{b}(L) = 3$ if and only if either
 - (a) for some $p, q > 1$, $\alpha = 2pq + p + q$ and $\beta = 2q + 1$, or
 - (b) for some $q > 0$, $\alpha = 2pq + p + q + 1$ and $\beta = 2q + 1$.

Note that in (b) together $0 < \beta < \alpha$, $q > 0$, and $\alpha = p\beta + q + 1$ imply $p > 0$.

COROLLARY 3.2 [Sto03, Corollary 8]. *If L is a two-bridge link of braid index at most 3, then L has Conway notation $(p, 1, 1, q)$ or $(p, 2, q)$ for some $p, q > 0$.*

REMARK 3.3. The link with Conway notation $(p, 2, q)$ corresponds to type (a) in Proposition 3.1. The link $(p, 1, 1, q)$ corresponds to type (b) and is equivalent to the link $(p, 2, -q - 1)$. See Figure 1. Observe then that up to mirror equivalence the links $(p, 2, q)$ for any $p \in \mathbb{Z}_+$ and $q \in \mathbb{Z}$ contain all oriented two-bridge links of braid index at most 3 and that every such link has braid index at most 3.

Let σ_1 and σ_2 be the standard generators of the 3-braid group as depicted in Figure 2.

THEOREM 3.4 (The Classification Theorem, [BM93]). *An oriented link L represented by a closed 3-braid admits a unique conjugacy class of 3-braid representatives, with the following exceptions:*

- (1) L is the unknot which has three conjugacy classes of 3-braid representatives, namely the classes of $\sigma_1\sigma_2$, $\sigma_1^{-1}\sigma_2^{-1}$, and $\sigma_1\sigma_2^{-1}$.
- (2) L is a type $(2, k)$ torus link, $k \neq \pm 1$, which has two conjugacy classes of 3-braid representatives, namely the classes of $\sigma_1^k\sigma_2$ and $\sigma_1^k\sigma_2^{-1}$.
- (3) L is one of a special class of links of braid index 3 that have 3-braid representatives admitting “braid-preserving flypes.” These links have at most two conjugacy classes of 3-braid representatives, namely the classes of $\sigma_1^p\sigma_2^r\sigma_1^q\sigma_2^\delta$ and $\sigma_1^p\sigma_2^\delta\sigma_1^q\sigma_2^r$, where p, q, r are distinct integers having absolute value at least 2 and where $\delta = \pm 1$.

REMARK 3.5. The links in the lower-left and lower-right of Figure 3 are related by an explicit flype, a flipping of a rational tangle that passes a crossing on one

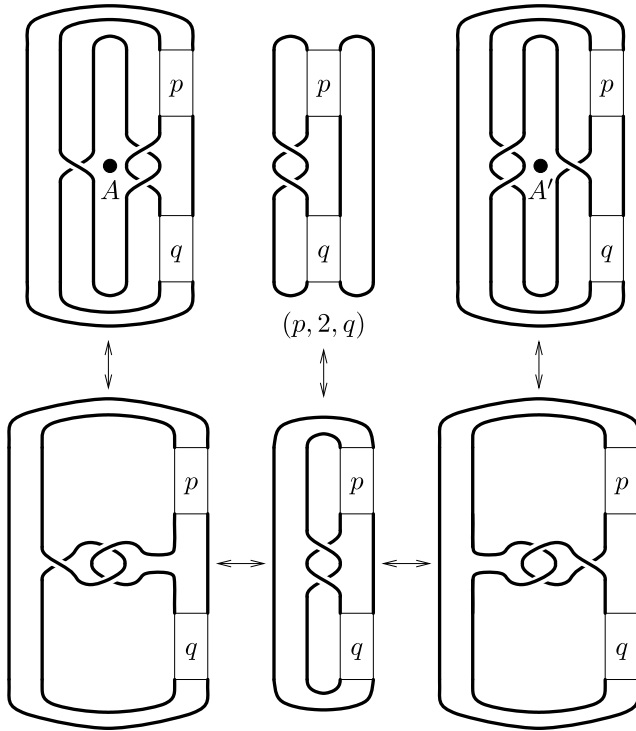


Figure 3 The two bridge link $(p, 2, q)$ and its two typically distinct closed 3-braid representatives are shown in the top row

side to the other; see [Con70]. The links shown above them contain the corresponding braid axes, thereby justifying Birman and Menasco’s terminology of “braid-preserving flype.”

LEMMA 3.6. *A two-bridge link with braid index 3, denoted up to mirror equivalence with Conway notation $(p, 2, q)$, belongs to the third type of links in Theorem 3.4 where $r = 2$ and $\delta = -1$. Such a description gives two braid axes presenting the link as a 3-braid that are distinct up to isotopy in the complement of the link if and only if $p, q \in \mathbb{Z} - \{-1, 0, 1, 2\}$ and $p \neq q$.*

REMARK 3.7. Note that in Theorem 3.4 the unknot and the $(2, k)$ torus links of the first two types may all also be written as members of the third type with, say, $r = 0$. When presented as such, the braid-preserving flypes do not give rise to their various braid axes. But of course these links have braid index less than 3.

Proof of Lemma 3.6. By Corollary 3.2 and Remark 3.3, up to mirror equivalence we may assume that a two-bridge link with braid index 3 has Conway notation $(p, 2, q)$. Figure 3 shows the passage from the link with Conway notation $(p, 2, q)$

to the two closed 3-braids $\sigma_1^p \sigma_2^q \sigma_1^q \sigma_2^{-1}$ and $\sigma_1^p \sigma_2^{-1} \sigma_1^q \sigma_2^q$. These presentations as closed 3-braids give two braid axes A and A' for the two-bridge link. By Theorem 3.4 the braid axes A and A' are not isotopic in the complement of the two-bridge link if and only if $p, q \in \mathbb{Z} - \{-1, 0, 1, 2\}$ and $p \neq q$. \square

LEMMA 3.8. *An unoriented link L that may be represented by a closed 3-braid admits at most one equivalence class of braid axes giving 3-braid representatives for a given orientation of L and its reverse, with the following exception: L or its mirror is a type $(2, k)$ torus link with $k > 0$, which has two equivalence classes of 3-braid axes corresponding to the conjugacy classes of $\sigma_1^k \sigma_2$ and $\sigma_1^k \sigma_2^{-1}$ when coherently oriented.*

Proof. This lemma is perhaps suggested by The Classification Theorem of [BM93], Theorem 3.4. We only need consider the oriented links L with at least two conjugacy classes of 3-braid representatives as described in Theorem 3.4.

If L is the unknot, the braid axes A and \bar{A} that correspond to the conjugacy classes of $\sigma_1 \sigma_2$ and $\sigma_1^{-1} \sigma_2^{-1}$, respectively, are equivalent by an orientation-reversing homeomorphism of S^3 . They are not equivalent, however, to the braid axis A' that corresponds to the conjugacy class of $\sigma_1 \sigma_2^{-1}$ since the absolute values of the exponent sums of the braid words $\sigma_1 \sigma_2$ and $\sigma_1 \sigma_2^{-1}$ are not equal. Note that we may consider the unknot as a type $(2, 1)$ torus link.

If L is a type $(2, k)$ torus link with $k \neq \pm 1$, then let A and A' be the two braid axes that correspond to the conjugacy classes of $\sigma_1^k \sigma_2$ and $\sigma_1^k \sigma_2^{-1}$. If $k = 0$, then these axes are equivalent since there is an orientation-reversing homeomorphism of S^3 taking L to L and A to A' . If $|k| \geq 2$, then these axes are not equivalent since the absolute values of the exponent sums of the braid words $\sigma_1^k \sigma_2$ and $\sigma_1^k \sigma_2^{-1}$ are not equal. Since the $(2, -k)$ torus link is the mirror of the $(2, k)$ torus link, we may assume that $k > 0$.

If L is a link that admits a braid-preserving flype, that is, one of the third type in Theorem 3.4, then let A and A' be the braid axes that give the two corresponding conjugacy classes of 3-braid representatives. These axes are typically not isotopic in the complement of the link as noted in Lemma 3.6 for our special case where L is a two-bridge link. Nevertheless, there is a homeomorphism of S^3 that takes L to itself and exchanges the axes. Figure 4 illustrates this homeomorphism by presenting an axis of involution for the link $L \cup A \cup A'$. (The figure also shows how each of A and A' becomes the stated braid axis for L when the other is dropped.) If ι is the involution of S^3 about this axis, then $\iota(L) = L$, $\iota(A) = A'$, and $\iota(A') = A$. Hence, the braid axes A and A' give equivalent 3-braid representatives of L . Note that ι reverses any orientation on L . \square

REMARK 3.9. Recall that two oriented two-bridge links $b(\alpha_1, \beta_1)$ and $b(\alpha_2, \beta_2)$ with $\alpha_1, \alpha_2 > 0$ are isotopic equivalent as oriented links if and only if

$$\alpha_1 = \alpha_2 \quad \text{and} \quad \beta_1^{\pm 1} \equiv \beta_2 \pmod{2\alpha_1},$$

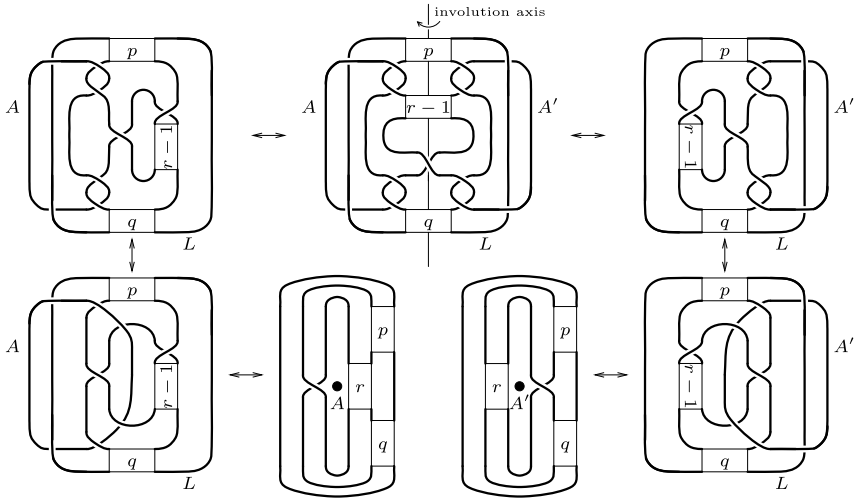


Figure 4 The (unoriented) link L of the third type in Theorem 3.4 (shown here with $\delta = -1$) admits an involution that exchanges its two braid axes A and A'

whereas the unoriented two-bridge links are equivalent if and only if the second condition is taken simply mod α_1 .

Since oriented two-bridge links are invertible, there is no distinction between the orientations of a two-bridge knot. However, switching the orientation on one component of the oriented link $b(\alpha, \beta)$ yields the link $b(\alpha, \beta - \alpha)$, which is mirror equivalent to $b(\alpha, \alpha - \beta)$. Thus, up to mirror equivalence, every oriented two-bridge link except the two-component unlink $b(0, 1)$ and the unknot $b(1, 1)$ is represented by $b(\alpha, \beta)$ for some $0 < \beta < \alpha$ with β odd.

REMARK 3.10. If β and β' are odd integers with $0 < \beta < \alpha$ and $0 < |\beta'| < \alpha$ such that $\beta\beta' \equiv 1 \pmod{2\alpha}$, then for positive integers p and q ,

- if $\alpha = 2pq + p + q$ and $\beta = 2p + 1$, then $\beta' = 2q + 1$, and
- if $\alpha = 2pq + p + q + 1$ and $\beta = 2p + 1$, then $\beta' = -(2q + 1)$.

Therefore, up to mirror equivalence, the oriented two-bridge link $b(2pq + p + q + \delta, 2p + 1)$ is equivalent to $b(2pq + p + q + \delta, 2q + 1)$ where $\delta \in \{0, 1\}$.

LEMMA 3.11. *Among two-bridge links up to mirror equivalence, only the link $b(4, 1)$ has two distinct orientations that each admits a closed 3-braid representation.*

Proof. Assume that the unoriented two-bridge link L has two distinct orientations and a closed 3-braid representative for each. Then, as noted in Remark 3.9, L is necessarily a link of two components. Also observe that the unlink has only

one orientation up isotopy, so L is not the unlink. Thus, up to mirror equivalence, the two orientations of L may be denoted as $b(\alpha, \beta)$ and $b(\alpha, \alpha - \beta)$ where $0 < \beta < \alpha$. Since α is necessarily even, both β and $\alpha - \beta$ are odd.

Case 1 ($\beta = 1$ or $\alpha - \beta = 1$).

We may assume that $\beta = 1$. If $\alpha = 2$, then the two orientations on $b(2, 1)$ are mirror equivalent. Hence, we may further assume that $\alpha > 2$.

Theorem 3.4 shows that $b(\alpha, 1)$ has a closed 3-braid representative. Since $\alpha > 2$, a 3-braid representative of $b(\alpha, \alpha - 1)$ must be of the second type in Proposition 3.1. Therefore, in accordance with Remark 3.10, we only need check if

$$\alpha = 2pq + p + q + \delta \quad \text{and} \quad \alpha - 1 = 2p + 1$$

for some integers $p, q > 0$ and $\delta \in \{0, 1\}$. It follows that $\alpha = 2p + 2$ and hence $p = (2 - q - \delta)/(2q - 1)$. The only valid solution is $p = 1 = q$ with $\delta = 0$. Thus, $\alpha = 4$. Because $1 \cdot 3 \equiv 3 \pmod{2 \cdot 4}$, the two oriented links $b(4, 1)$ and $b(4, 3)$ are not mirror equivalent. Therefore, each orientation of the unoriented two-bridge link $L = b(4, 1)$ has a 3-braid representative.

Case 2 ($\beta > 1$ and $\alpha - \beta > 1$).

Any 3-braid representative of L must be of the second type in Proposition 3.1. Therefore, again in accordance with Remark 3.10, we only need check if

$$\alpha = 2pq + p + q + \delta \quad \text{and} \quad \beta = 2p + 1$$

and

$$\alpha = 2rs + r + s + \varepsilon \quad \text{and} \quad \alpha - \beta = 2r + 1$$

for some integers $p, q, r, s > 0$ and $\delta, \varepsilon \in \{0, 1\}$. Eliminating β , we have the three equations

$$\alpha = 2pq + p + q + \delta = (2p + 1)q + p + \delta, \tag{1}$$

$$\alpha = 2rs + r + s + \varepsilon = (2r + 1)s + r + \varepsilon, \tag{2}$$

$$\alpha = (2p + 1) + (2r + 1). \tag{3}$$

Combining equation (3) with (1) and (2), we obtain

$$(2r + 1) = (2p + 1)(q - 1) + p + \delta, \tag{4}$$

$$(2p + 1) = (2r + 1)(s - 1) + r + \varepsilon. \tag{5}$$

By examining equation (4), if $q = 1$, then $p > r$, and if $q > 1$, then $p < r$. Similarly, equation (5) implies that if $s = 1$, then $r > p$, and if $s > 1$, then $p > r$. Hence, either $q = 1$ and $s > 1$ (in which case $p > r$) or $q > 1$ and $s = 1$ (in which case $r > p$). These two cases are symmetric.

Assume that $q = 1$ and $s > 1$. Then equation (4) gives

$$2r + 1 = p + \delta. \tag{6}$$

Substituting this into equation (5) yields

$$\begin{aligned} 2p + 1 &= (p + \delta)(s - 1) + r + \varepsilon \\ &= (s - 1)p + (s - 1)\delta + r + \varepsilon, \end{aligned}$$

and thus

$$(3 - s)p + 1 = (s - 1)\delta + r + \varepsilon. \tag{7}$$

Since the right-hand side is necessarily positive, $s = 2$ or $s = 3$.

If $s = 2$, then

$$\begin{aligned} p + 1 &= r + \delta + \varepsilon && \text{by equation (7),} \\ 2r + 1 - \delta + 1 &= r + \delta + \varepsilon && \text{by equation (6),} \\ r &= 2\delta + \varepsilon - 2. \end{aligned}$$

Since $r > 0$, $\delta = \varepsilon = 1$, $r = 1$, and $p = 2$. Thus, $\alpha = 8$, $\beta = 5$, and $\alpha - \beta = 3$. Because $3 \cdot 5 \equiv -1 \pmod{2 \cdot 8}$, the oriented links $b(8, 3)$ and $b(8, 5)$ are mirror equivalent.

If $s = 3$, then by equation (7),

$$1 = 2\delta + r + \varepsilon.$$

Since $r > 0$, $\delta = \varepsilon = 0$, $r = 1$, and $p = 3$. Thus, $\alpha = 10$, $\beta = 7$, and $\alpha - \beta = 3$. Because $3 \cdot 7 \equiv 1 \pmod{2 \cdot 10}$, the oriented links $b(10, 3)$ and $b(10, 7)$ are equivalent. □

4. Counting Genus One Fibered Knots

LEMMA 4.1. *An unoriented link L has at most four equivalence classes of braid axes that represent L as a closed 3-braid.*

Proof. By Lemma 3.8, up to reversal each orientation of a link L admits at most one equivalence class of braid axes representing the oriented link as a closed 3-braid except when L is a type $(2, k)$ torus link with $k > 0$. When L is oriented as a type $(2, k)$ torus link with $k > 0$, it has two distinct equivalence class of braid axes representing it as a closed 3-braid.

Since a $(2, k)$ torus link has at most two components, it has at most two distinct orientations up to reversal. Therefore, it may have at most four distinct equivalence classes of braid axes representing it as a closed 3-braid.

A link L that may be represented as a closed 3-braid has at most three components and so has at most four distinct orientations up to reversal. Assuming that L is not a $(2, k)$ torus link, each orientation up to reversal admits at most one equivalence class of braid axes representing it as a closed 3-braid. Therefore, L may have at most four equivalence classes of braid axes representing it as a closed 3-braid. □

For any given 3-manifold M , there may be several different links in S^3 with M as their double branched covers ([BGAM76; Vir72; Bed84], among others). In particular, there may be several different links with representations as closed 3-braids that have M as their double branched covers.

The lens space $L(\alpha, \beta)$ is the double cover of S^3 branched along the (unoriented) two-bridge link $b(\alpha, \beta)$; see, for example, [BZ03]. By classifying involutions on lens spaces Hodgson and Rubinstein show that if the lens space $L(\alpha, \beta)$

is the double cover of S^3 branched over a link L , then L is the two-bridge link $b(\alpha, \beta)$, [HR85, Corollary 4.12].

THEOREM 4.2. *Up to homeomorphism, an unoriented two-bridge link L admits exactly*

- *three equivalence classes of 3-braid representatives if and only if L is equivalent to $b(4, 1)$,*
- *two equivalence classes of 3-braid representatives if and only if L is equivalent to $b(\alpha, 1)$ for $\alpha > 0$ and $\alpha \neq 4$,*
- *one equivalence class of 3-braid representatives if and only if L is equivalent to $b(\alpha, \beta)$ either for $\alpha = 0$ or for $0 < \beta < \alpha$, where either*
 - *$\alpha = 2pq + p + q$ and $\beta = 2q + 1$ for some integers $p, q > 1$, or*
 - *$\alpha = 2pq + p + q + 1$ and $\beta = 2q + 1$ for some integers $p, q > 0$,*
- *no 3-braid representatives otherwise.*

Proof. By Lemma 4.1, an unoriented link L has at most four equivalence classes of braid axes that represent L as a closed 3-braid. By Lemma 3.11, only the two-bridge link $b(4, 1)$ has two inequivalent orientations that each admit closed 3-braid representatives. Thus, $b(4, 1)$ is the only two-bridge link that a priori could have more than two inequivalent closed 3-braid representatives.

Theorem 3.4 implies that, as an oriented two-bridge link, $b(4, 1)$ has two closed 3-braid representatives $\sigma_1^4\sigma_2$ and $\sigma_1^4\sigma_2^{-1}$ with axes A_1 and A_2 , respectively. By Lemma 3.8, these two axes are inequivalent. The other orientation, $b(4, 3)$, has Conway notation $(1, 2, 1)$ and a 3-braid representative $\sigma_1\sigma_2^2\sigma_1\sigma_2^{-1}$. By Theorem 3.4 (and Lemma 3.8), $b(4, 3)$ has just one closed 3-braid representative with braid axis A_3 . The axes A_1 and A_3 are not equivalent since the absolute value of the exponent sums of their 3-braid representatives are distinct. The axes A_2 and A_3 are not equivalent since A_3 cobounds an embedded annulus with a component of L , whereas SnapPea [Wee] reports $L \cup A_2$ as a hyperbolic link. Thus, the unoriented link $b(4, 1)$ admits a total of three equivalence classes of closed 3-braid axes.

Every unoriented two-bridge link of braid index at most 2 is equivalent to $b(\alpha, 1)$ (for $\alpha \geq 0$) as noted in Proposition 3.1. By Lemma 3.11 and Lemma 3.8, if $\alpha \neq 0$ or 4, then such links have exactly two equivalence classes of braid axes giving closed 3-braid representatives. If $\alpha = 0$, then Lemma 3.8 implies that the link has just one equivalence class of braid axes representing L as a closed 3-braid.

By Proposition 3.1 a two-bridge link of braid index 3 is equivalent to $b(\alpha, \beta)$ with $0 < \beta < \alpha$ if and only if α and β satisfy either (a) or (b) of the proposition. By Lemma 3.11 and Lemma 3.8 these links have just one equivalence class of braid axes representing L as a closed 3-braid.

By Proposition 3.1 a two-bridge link has no closed 3-braid representatives if it is not equivalent to some $b(\alpha, \beta)$ where either $\beta = 1$, (a) is satisfied, or (b) is satisfied. Thus, such a link has no equivalence classes of braid axes representing L as a closed 3-braid. \square

Now we may pull together the proof of our main theorem.

THEOREM 4.3. *Up to homeomorphisms, the lens space $L(\alpha', \beta')$ contains exactly*

- *three distinct GOF-knots if and only if $L(\alpha', \beta') \cong L(4, 1)$,*
- *two distinct GOF-knots if and only if $L(\alpha', \beta') \cong L(\alpha, 1)$ for $\alpha > 0$ and $\alpha \neq 4$,*
- *one distinct GOF-knot if and only if $L(\alpha', \beta') \cong L(\alpha, \beta)$ either for $\alpha = 0$ or for $0 < \beta < \alpha$, where either*
 - *$\alpha = 2pq + p + q$ and $\beta = 2q + 1$ for some integers $p, q > 1$, or*
 - *$\alpha = 2pq + p + q + 1$ and $\beta = 2q + 1$ for some integers $p, q > 0$, and*
- *zero GOF-knots otherwise.*

Proof. By Corollary 4.12 of [HR85], the two-bridge link $b(\alpha, \beta)$ is the only link in S^3 for which the double branched cover is the lens space $L(\alpha, \beta)$. Therefore, by Proposition 2.1, a GOF-knot in $L(\alpha, \beta)$ up to homeomorphism corresponds exactly to the braid axis of a representation of $b(\alpha, \beta)$ as a closed 3-braid up to equivalence. Hence, the classification of equivalence classes of closed 3-braid representatives of two-bridge links given in Theorem 4.2 yields the desired result. □

REMARK 4.4. Since a link that may be represented as a closed 3-braid has bridge number at most 3, the full strength of [HR85, Corollary 4.12] is not necessary to obtain Theorem 4.3. Independently, Viro [Vir72] and Birman and Hilden [BH75, Theorem 5] show that the double cover of S^3 branched over a link of bridge number $b \leq 3$ is a 3-manifold of Heegaard genus $b - 1$. Since, with the exception of S^3 , lens spaces (including $S^1 \times S^2$) are the 3-manifolds of Heegaard genus 1, this implies that the only links with representations as closed 3-braids that have a lens space as their double branched cover are two-bridge links and the unknot.

THEOREM 4.5. *In a given lens space $L(\alpha', \beta')$ each homeomorphism equivalence class of GOF-knots is realized by a single isotopy equivalence class with the following exceptions:*

- *the trefoil in $L(\alpha', \beta') \cong S^3$,*
- *the sole GOF-knot in $L(\alpha', \beta') \cong S^1 \times S^2$,*
- *the sole GOF-knot in $L(\alpha', \beta') \cong L(\alpha, \beta)$ with $0 < \beta < \alpha$, where for distinct $p, q \in \mathbb{Z} - \{-1, 0, 1, 2\}$, either*
 - *$\alpha = 2pq + p + q$ and $\beta = 2q + 1$, or*
 - *$\alpha = 2pq + p + q + 1$ and $\beta = 2q + 1$.*

Each of these homeomorphism classes divides into two isotopy classes. These homeomorphic pairs of isotopy classes in S^3 and $S^1 \times S^2$ are related by an orientation-reversing homeomorphism. The homeomorphic pairs of isotopy classes in the remaining lens spaces are related by an involution of the lens space not isotopic to the identity.

Proof. By Lemma 2.2, Theorem 3.4 may be interpreted as a classification of isotopy classes of GOF-knots. On the other hand, Theorem 4.3 by way of Lemma 3.8

determines the homeomorphism classes of GOF-knots. We may now observe when distinct isotopy classes are in the same homeomorphism class.

If A and A' are not isotopic in the complement of L yet are equivalent by a homeomorphism of S^3 taking (L, A) to (L, A') , then this homeomorphism cannot be isotoped to be the identity on a neighborhood of L while still taking A to A' . (Orientation-reversing homeomorphisms of S^3 and orientation-preserving involutions of S^3 taking L to itself that reverse any orientation on L are examples of such homeomorphisms.)

The proof of Lemma 3.8 shows: (1) the axes A and \bar{A} presenting the unknot as the closed 3-braids $\sigma_1\sigma_2$ and $\sigma_1^{-1}\sigma_2^{-1}$, respectively, are equivalent by an orientation-reversing homeomorphism, (2) the axes A and A' presenting the 2-component unlink as the closed 3-braids σ_2 and σ_2^{-1} , respectively, are equivalent by an orientation-reversing homeomorphism, and (3) the axes A and A' presenting a two-bridge link admitting a braid-preserving flype as the closed 3-braids $\sigma_1^p\sigma_2^r\sigma_1^q\sigma_2^\delta$ and $\sigma_1^p\sigma_2^\delta\sigma_1^q\sigma_2^r$ where $r = \pm 2$ and $\delta = -\text{sgn}(r)$ are equivalent by an involution that reverses any orientation on the two-bridge link. (By an orientation-reversing homeomorphism of S^3 we may assume that $r = 2$ and $\delta = -1$.) By Theorem 3.4 all pairs of these braid axes are in distinct isotopy classes (in the complement of L) so long as in the third case, p, q , and r are distinct integers having absolute value at least 2. Therefore, the pairs of GOF-knots corresponding to each of these pairs of axes, though in homeomorphism equivalence classes, are not isotopic in their lens space.

Lemma 3.8 further shows that there are no other homeomorphism equivalences among the remaining braid axes presenting two-bridge links as closed 3-braids. Hence, these equivalence classes correspond to isotopy classes. \square

5. Remarks

REMARK 5.1. One may obtain explicit pictures of the fiber surface of these GOF-knots in lens spaces like those in [Mor89] by carrying a disk that both is bounded by the braid axis and intersects the 3-braid minimally through the sequence of steps done to obtain a presentation of a lens space as surgery on the unknot from its corresponding two-bridge link.

REMARK 5.2. As Morimoto notes [Mor89, Remark 1], his knot K_2 in the lens space $L(5, 1)$ has two meridians. That is, K_2 has a *cosmetic surgery* [BHW99], a nontrivial Dehn surgery to a homeomorphic manifold. Indeed, a $+1$ surgery on K_2 produces the manifold $L(5, 4)$, which is equivalent to $L(5, 1)$ by an orientation-reversing homeomorphism. This knot (at least its exterior) is also known as the figure-eight sister.

We observe this in the context of closed 3-braids as follows. The knot K_2 in $L(5, 1)$ is the lift of the braid axis in the double branched cover of the closure of the braid $\sigma_1^5\sigma_2$, the two-bridge knot $\mathfrak{b}(5, 1)$. As noted in Remark 2.3, $+1$ surgery on K_2 corresponds to inserting two full left-handed twists (i.e., inserting $((\sigma_1\sigma_2)^3)^{-2}$) into the braid. Therefore, $+1$ surgery on K_2 corresponds to the

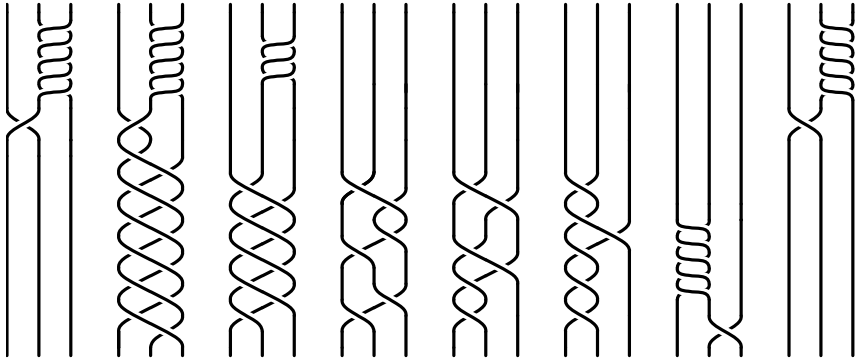


Figure 5 Inserting two full left-handed twists into the closed braid $\sigma_1^5\sigma_2$ produces the closed braid $\sigma_1^{-5}\sigma_2^{-1}$ after a sequence of braid isotopies and conjugations

double branched cover of the closure of the braid $(\sigma_1^5\sigma_2)((\sigma_1\sigma_2)^3)^{-2}$, which, as Figure 5 shows, is equivalent (by braid moves and conjugation) to the closure of $\sigma_1^{-5}\sigma_2^{-1}$, the two-bridge knot $b(5, 4)$ that is the mirror of the closure of $\sigma_1^5\sigma_2$.

Observe that $1/n$ surgery on a GOF-knot confers a GOF-knot in the surgered manifold. This figure-eight sister example above may be generalized to obtain GOF-knots in other manifolds that admit a $1/n$ surgery yielding a homeomorphic manifold with the opposite orientation, a cosmetic surgery. In his thesis, Lott classifies the hyperbolic once-punctured torus bundles with multiple lens space fillings [Lot09]. As a consequence, he obtains a classification of hyperbolic GOF-knots in lens spaces with nontrivial lens space fillings. In particular, Lott identifies a total of three pairs of hyperbolic GOF-knots that admit cosmetic surgeries and another pair of hyperbolic GOF-knots related by surgery in lens spaces of order 7.

THEOREM 5.3 (Lott [Lot09, Theorem 6.6.1]). *Assume that K is a hyperbolic GOF-knot in a lens space M such that $+1$ surgery on K yields a dual GOF-knot K' in a lens space M' . Then up to orientation-preserving homeomorphism, M and M' are the double branched covers of the closures of a pair of braids β and $\beta' = \beta((\sigma_1\sigma_2)^3)^{-2}$ in the table below, and the knots K and K' are the lifts of their braid axes.*

$M = K(\infty)$	$M' = K(+1)$	β	β'
$L(5, 1)$	$L(5, -1)$	$\sigma_1^5\sigma_2$	$\sigma_1^{-5}\sigma_2^{-1}$
$L(7, 1)$	$L(7, -3)$	$\sigma_1^7\sigma_2$	$\sigma_1\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-1}$
$L(7, 3)$	$L(7, -1)$	$\sigma_1^2\sigma_2^{-1}\sigma_1\sigma_2^2$	$\sigma_1^{-7}\sigma_2^{-1}$
$L(13, 3)$	$L(13, -3)$	$\sigma_1^4\sigma_2^{-1}\sigma_1\sigma_2^2$	$\sigma_1^{-4}\sigma_2^{-2}\sigma_1^{-1}\sigma_2$
$L(17, -5)$	$L(17, 5)$	$\sigma_1^{-4}\sigma_2^{-1}\sigma_1^{-3}\sigma_2^2$	$\sigma_1^4\sigma_2^{-2}\sigma_1^3\sigma_2$

REMARK 5.4. Our orientation conventions are opposite that of Lott. Though his Table 6.1 disregards orientations, his proof keeps track of them. Lott's work further implies that, when oriented, the GOF-knots K and K' associated to the last line of the above table are not isotopic to their reverse. In particular, the 3-braid $\sigma_1^{-4}\sigma_2^{-1}\sigma_1^{-3}\sigma_2^2$ is not conjugate to its reverse $\sigma_2^2\sigma_1^{-3}\sigma_2^{-1}\sigma_1^{-4}$.

REMARK 5.5. As Morimoto shows in [Mor89] and is further observed in Theorem 4.3, there are lens spaces that contain no GOF-knots. Nevertheless, there are knots in lens spaces representing a nontrivial element of homology whose exteriors are once-punctured torus bundles. For instance, since $-19/3$ surgery on the right-handed trefoil yields $L(19, 7)$, even though the core of the surgered solid torus is not a GOF-knot in $L(19, 7)$, its exterior is a once-punctured torus bundle. Indeed, Theorem 4.3 shows that $L(19, 7)$ contains no GOF-knots.

QUESTION. Must a lens space contain a knot whose exterior is a once-punctured torus bundle? If not, which do? [Update: Baldwin [Bal06] answered this question for lens spaces with prime order fundamental group. Thereafter the author fully answered this question and classified all such knots [Bak11]. See also Lott's thesis [Lot09].]

REMARK 5.6. Via double branched covers, we obtain a genus g fibered knot from the braid axis of a closed braid of braid index $2g + 1$. However, not all genus g fibered knots arise in this manner if $g > 1$. Again considering that there are lens spaces that contain no GOF-knots, we ask the following question.

QUESTION. What is the minimal genus among fibered knots in a given lens space $L(\alpha, \beta)$?

Murasugi [Mur91, Theorem B] shows that the braid index of a two-bridge link may be arbitrarily large. Perhaps then it is not too foolish to conjecture that there exist lens spaces whose minimal genus fibered knot has an arbitrarily large genus.

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