Michigan Math. J. 63 (2014), 253–273

# Dirichlet Series Associated to Cubic Fields with Given Quadratic Resolvent

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Abstract. Let $k$ be a quadratic field. We give an explicit formula for the Dirichlet series $\sum_K |\operatorname{Disc}(K)|^{-s}$, where the sum is over isomorphism classes of all cubic fields whose quadratic resolvent field is isomorphic to $k$.

Our work is a sequel to [14] (see also [22]), where such formulas are proved in a more general setting, in terms of sums over characters of certain groups related to ray class groups. In the present paper we carry the analysis further and prove explicit formulas for these Dirichlet series over $\mathbb{Q}$, and in a companion paper we do the same for quartic fields having a given cubic resolvent.

As an application, we compute tables of the number of $S_3$-sextic fields $E$ with $|\operatorname{Disc}(E)| < X$ for $X$ ranging up to $10^{23}$. An accompanying PARI/GP implementation is available from the second author's website.

## 1. Introduction

A classical problem in algebraic number theory is that of *enumerating number fields* by discriminant. Let $N_d^{\pm}(X)$ denote the number of isomorphism classes of number fields $K$ with $\deg(K) = d$ and $0 < \pm\operatorname{Disc}(K) < X$. The quantity $N_d^{\pm}(X)$ has seen a great deal of study; see (e.g.) [12; 4; 30] for surveys of classical and more recent work.

It is widely believed that $N_d^{\pm}(X) = C_d^{\pm} X + o(X)$ for all $d \geq 2$. For $d = 2$, this is classical, and the case $d = 3$ was proved in 1971 work of Davenport and Heilbronn [19]. The cases $d = 4$ and $d = 5$ were proved much more recently by Bhargava [3; 6]. In addition, Bhargava [5] also conjectured a value of the constants $C_{d,S_d}^{\pm}$ for $d > 5$, where the additional index $S_d$ means that one counts only degree $d$ number fields with Galois group of the Galois closure isomorphic to $S_d$.

Related questions have also seen recent attention. For example, Belabas [1] developed and implemented a fast algorithm to compute large tables of cubic fields, which has proved essential for subsequent numerical computations (including one to be carried out in this paper!). Based on Belabas's data, Roberts [26] conjectured the existence of a secondary term of order $X^{5/6}$ in $N_3^{\pm}(X)$, and this was proved (independently and using different methods) by Bhargava, Shankar, and

Received January 15, 2013. Revision received March 1, 2014.

The second author was partially supported by the National Science Foundation under Grant DMS-0802967.

Tsimerman [7] and by Taniguchi and the second author [27]. Further details and references can be found in the survey papers above.

In the present paper we study cubic fields from a different angle. In 1954 Cohn [18] studied *cyclic* cubic fields and proved that

$$\sum_{K \text{ cyclic}} \frac{1}{\mathrm{Disc}(K)^s} = -\frac{1}{2} + \frac{1}{2}\left(1 + \frac{1}{3^{4s}}\right) \prod_{p \equiv 1 \pmod 6} \left(1 + \frac{2}{p^{2s}}\right). \tag{1.1}$$

Generalizations to cyclic and abelian extensions were proved by Mäki [21], Wright [32], and the first author, Diaz y Diaz, and Olivier [11], among others, and we refer to [11] for a more extensive description of past results.

Here we obtain an analogue of Cohn's result for noncyclic fields, building upon previous work of Morra and the first author [14]. Given a noncyclic cubic field $K$, its Galois closure $\widetilde{K}$ has Galois group $S_3$ and hence contains a unique quadratic subfield $k$, called the *quadratic resolvent.* We begin by fixing a fundamental discriminant $D$ and let $\mathcal{F}(\mathbb{Q}(\sqrt{D}))$ be the set of all cubic fields $K$ whose quadratic resolvent field is $\mathbb{Q}(\sqrt{D})$. For any $K \in \mathcal{F}(\mathbb{Q}(\sqrt{D}))$, we have $\mathrm{Disc}(K) = Df(K)^2$ for some positive integer $f(K)$, and we define

$$\Phi_D(s) := \frac{1}{2} + \sum_{K \in \mathcal{F}(\mathbb{Q}(\sqrt{D}))} \frac{1}{f(K)^s}, \tag{1.2}$$

where the constant $1/2$ is added to simplify the final formulas.

Motivated by Cohn's formula (1.1), we may ask if $\Phi_D(s)$ can be given an explicit form.

The answer is yes, as was essentially shown by Morra and the first author in [14], using Kummer theory. They proved a very general formula enumerating relative cubic extensions of any base field. However, this formula is rather complicated, and it is not in a form that is immediately conducive to applications. In the present paper, we will show that this formula can be put in such a form when the base field is $\mathbb{Q}$; our formula (Theorem 2.5) is similar to (1.1) but involves one additional Euler product for each cubic field of discriminant $-D/3$, $-3D$, or $-27D$.

Related results have also been obtained for other non-abelian Galois groups. The $D_4$, $A_4$, and $S_4$ cases were all handled by the first author, Diaz y Diaz, and Olivier [10; 13; 9], and in a companion to the present paper [15], we further analyze the $A_4$ and $S_4$ cases, obtaining explicit Dirichlet series representations for the analogues of $\Phi_D(s)$. In a second companion paper [16] we study the $D_\ell$ case (for $\ell$ an odd prime) and again obtain analogues of most of the same results.

### 1.1. *An Application*

One application of our result is to enumerating $S_3$-sextic field extensions, that is, sextic field extensions $\widetilde{K}$ that are Galois over $\mathbb{Q}$ with Galois group $S_3$. Suppose that $\widetilde{K}$ is such a field, where $K$ and $k$ are the cubic and quadratic subfields, respectively, the former being defined only up to isomorphism. Then $k$ is the quadratic resolvent of $K$, and in addition to the formula $\mathrm{Disc}(K) = \mathrm{Disc}(k) f(K)^2$, we have

$$\mathrm{Disc}(\widetilde{K}) = \mathrm{Disc}(K)^2 \, \mathrm{Disc}(k) = \mathrm{Disc}(k)^3 f(K)^4, \tag{1.3}$$

so that our formulas may be used to count all such $\widetilde{K}$ of bounded discriminant, starting from Belabas's tables [1] of cubic fields. Ours is not the only way to enumerate such $\widetilde{K}$, but it is straightforward to implement, and it seems to be (roughly) the most efficient.

We implemented this algorithm using PARI/GP [25] to compute counts of $S_3$-sextic $\widetilde{K}$ with $|\mathrm{Disc}(\widetilde{K})| < 10^{23}$. In Section 6 we present our data, and the accompanying code is available from the second author's website.

### 1.2. Outline of the Paper

In Section 2 we introduce our notation and give the main results. In Section 3 we summarize the work of Morra and the first author [14] and prove several propositions that will be needed for the proof of the main result. Our work relies on work of Ohno [24] and Nakagawa [23], establishing an identity for binary cubic forms. In the same section, we give a result (Proposition 3.9) that controls the splitting type of the prime 3 in certain cubic extensions and illustrates an application of Theorem 2.5. Finally, in Section 4 we prove Theorem 2.5, using the main theorem of [14], recalled as Theorem 3.2, as a starting point. In Section 5 we give some numerical examples which were helpful in double-checking our results, and in Section 6 we describe our computation of $S_3$-sextic fields.

### 1.3. Notation

We recall some standard notation. If $K$ is a number field, we write $\mathbb{Z}_K$ for its ring of integers, $\mathrm{Cl}(K)$ for its class group, and $h(K)$ for its class number, and if $\mathfrak{a}$ is an ideal of $K$, then we write $\mathrm{Cl}_{\mathfrak{a}}(K)$ for the associated ray class group. When $K$ is a quadratic field of discriminant $D$, we also write $\mathrm{Cl}(D)$, $h(D)$, and $\mathrm{Cl}_{\mathfrak{a}}(D)$ for the above.

For the convenience of the reader, we also list some of the notation introduced in this paper and say where it is introduced: $\omega_E(p)$, $D^*$, $\mathrm{rk}_3(D)$, $\mathcal{L}_N$, $\mathcal{F}(K_2)$, $\mathcal{L}_3(K_2)$, $\Phi_D(s)$ in Definitions 2.1 through 2.4; $c_D$ in Theorem 2.5; $K$, $L$, $N$, $\tau$, $\tau_2$, and $T$ at the beginning of Section 3; $S_\ell(k)$ in Definition 3.1; $G_{\mathfrak{b}}$ and $\mathcal{B}$ in (3.2); $F(\mathfrak{b}, \chi, s)$ and $\omega_\chi(p)$ in Theorem 3.2; $H_{\mathfrak{a}}$ in (3.3).

## 2. Statement of Results

We begin by introducing some notation. In what follows, by abuse of language we use the term "cubic field" to mean "isomorphism class of cubic number fields."

DEFINITION 2.1. Let $E$ be a cubic field. For a prime number $p$, we set

$$\omega_E(p) = \begin{cases} -1 & \text{if } p \text{ is inert in } E, \\ 2 & \text{if } p \text{ is totally split in } E, \\ 0 & \text{otherwise.} \end{cases}$$

REMARKS 2.2. (1) We have $\omega_E(p) = \chi(\sigma_p)$ for unramified $p$, where $\chi$ is the character of the standard representation of $S_3$, and $\sigma_p$ is the Frobenius element of the Galois closure of $E$ at $p$.

(2) Note that we have $\omega_E(p) = 0$ if and only if $(\frac{\mathrm{Disc}(E)}{p}) \neq 1$, and since in all cases that we will use we have $\mathrm{Disc}(E) = -D/3$, $-3D$, or $-27D$ for some fundamental discriminant $D$; for $p \neq 3$, this is true if and only if $(\frac{-3D}{p}) \neq 1$. Thus, in Euler products involving the quantities $1 + \omega_E(p)/p^s$, we can either include all $p \neq 3$ or restrict to $(\frac{-3D}{p}) = 1$.

DEFINITION 2.3. (1) Let $D$ be a fundamental discriminant (including 1). We let $D^*$ be the discriminant of the *mirror field* $\mathbb{Q}(\sqrt{-3D})$, so that $D^* = -3D$ if $3 \nmid D$ and $D^* = -D/3$ if $3 \mid D$.

(2) For any fundamental discriminant $D$, we denote by $\mathrm{rk}_3(D)$ the 3-rank of the class group of the field $\mathbb{Q}(\sqrt{D})$.

(3) For any integer $N$, we let $\mathcal{L}_N$ be the set of cubic fields of discriminant $N$. We will use the notation $\mathcal{L}_N$ only for $N = D^*$ or $N = -27D$, with $D$ a fundamental discriminant.

(4) If $K_2 = \mathbb{Q}(\sqrt{D})$ with $D$ fundamental, then we denote by $\mathcal{F}(K_2)$ the set of cubic fields with resolvent field equal to $K_2$ or, equivalently, with discriminant of the form $Df^2$.

(5) With a slight abuse of notation, we let

$$\mathcal{L}_3(K_2) = \mathcal{L}_3(D) = \mathcal{L}_{D^*} \cup \mathcal{L}_{-27D}.$$

REMARK. Scholz's theorem tells us that for $D < 0$, we have $0 \leq \mathrm{rk}_3(D) - \mathrm{rk}_3(D^*) \leq 1$ (or, equivalently, that for $D > 0$, we have $0 \leq \mathrm{rk}_3(D^*) - \mathrm{rk}_3(D) \leq 1$) and gives also a necessary and sufficient condition for $\mathrm{rk}_3(D) = \mathrm{rk}_3(D^*)$ in terms of the fundamental unit of the real field.

DEFINITION 2.4. As in the Introduction, for any fundamental discriminant $D$, we define the Dirichlet series

$$\Phi_D(s) := \frac{1}{2} + \sum_{K \in \mathcal{F}(\mathbb{Q}(\sqrt{D}))} \frac{1}{f(K)^s}. \tag{2.1}$$

Our main theorem is as follows.

THEOREM 2.5. *For any fundamental discriminant $D$, we have*

$$\begin{aligned} c_D \Phi_D(s) = \frac{1}{2} M_1(s) \prod_{(-3D/p)=1} \left(1 + \frac{2}{p^s}\right) \\ + \sum_{E \in \mathcal{L}_3(D)} M_{2,E}(s) \prod_{(-3D/p)=1} \left(1 + \frac{\omega_E(p)}{p^s}\right), \end{aligned} \tag{2.2}$$

*where $c_D = 1$ if $D = 1$ or $D < -3$, $c_D = 3$ if $D = -3$ or $D > 1$, and the 3-Euler factors $M_1(s)$ and $M_{2,E}(s)$ are given in Table 1.*

**Table 1**

| Condition on $D$ | $M_1(s)$ | $M_{2,E}(s)$, $E \in \mathcal{L}_{D^*}$ | $M_{2,E}(s)$, $E \in \mathcal{L}_{-27D}$ |
|---|---|---|---|
| $3 \nmid D$ | $1+2/3^{2s}$ | $1+2/3^{2s}$ | $1-1/3^{2s}$ |
| $D \equiv 3 \pmod 9$ | $1+2/3^s$ | $1+2/3^s$ | $1-1/3^s$ |
| $D \equiv 6 \pmod 9$ | $1+2/3^s+6/3^{2s}$ | $1+2/3^s+3\omega_E(3)/3^{2s}$ | $1-1/3^s$ |

REMARKS 2.6. (1) When $D \equiv 3 \pmod 9$, we have $D^* \equiv 2 \pmod 3$, so 3 is partially split in any cubic field of discriminant $D^*$. It follows that when $E \in \mathcal{L}_{D^*}$, we have $M_{2,E}(s) = 1 + 2/3^s + 3\omega_E(3)/3^{2s}$ for all $D$ such that $3 \mid D$.

(2) When $3 \nmid D$, there are no terms for $1/3^s$, in accordance with Proposition 3.8.

(3) In the terms involving $E \in \mathcal{L}_{D^*}$ the condition $\left(\frac{-3D}{p}\right) = 1$ can be replaced by $p \neq 3$ and even omitted altogether if $3 \nmid D$, and in the terms involving $E \in \mathcal{L}_{-27D}$ it can be omitted.

(4) The case $D = 1$ is formula (1.1) of Cohn mentioned previously, and the case $D = -3$ was proved by Morra and the first author [14]. The paper [14] also proves (2.2) when $D < 0$ and $3 \nmid h(D)$, in which case $\mathcal{L}_3(D) = \emptyset$. In her thesis [22], Morra also proves some special cases of an analogue of (2.2) for cubic extensions of imaginary quadratic fields. Finally, one additional case of (2.2) was proved in [29], with an application to Shintani zeta functions.

## 3. Preliminaries

We briefly summarize the work of [14] and introduce some further notation, which will be needed in the proof. We assume from now on that $D \neq 1, -3$; these cases are similar but simpler and are already handled in [14; 22] (and the case $D = 1$ in [18]).

Suppose that $K/\mathbb{Q}$ is a cubic field of discriminant $Dn^2$, where $D \notin \{1, -3\}$ is a fundamental discriminant, and let $N$ be the Galois closure of $K$. Then $N(\sqrt{-3})$ is a cyclic cubic extension of $L := \mathbb{Q}(\sqrt{D}, \sqrt{-3})$, and Kummer theory implies that $N(\sqrt{-3}) = L(\alpha^{1/3})$ for some $\alpha \in L$. We write (following [14, Remark 2.2])

$$\mathrm{Gal}(L/\mathbb{Q}) = \{1, \tau, \tau_2, \tau\tau_2\}, \tag{3.1}$$

where $\tau$, $\tau_2$, $\tau\tau_2$ fix $\sqrt{D}$, $\sqrt{-3}$, $\sqrt{-3D}$, respectively.

The starting point of [14] is a correspondence between such fields $K$ and such elements $\alpha$. In particular, isomorphism classes of such $K$ are in bijection with equivalence classes of elements $1 \neq \overline{\alpha} \in L^\times/(L^\times)^3$, with $\alpha$ identified with its inverse, such that $\alpha\tau'(\alpha) \in (L^\times)^3$ for $\tau' \in \{\tau, \tau_2\}$. We express this by writing (as in [14, Definition 2.3]) $\overline{\alpha} \in (L^\times/(L^\times)^3)[T]$, where $T \subseteq \mathbb{F}_3[\mathrm{Gal}(L/\mathbb{Q})]$ is defined by $T = \{\tau + 1, \tau_2 + 1\}$, and the notation $[T]$ means that $\overline{\alpha}$ is annihilated by $T$.

To go further, we introduce the following definition.

DEFINITION 3.1. Let $k$ be a number field, and $\ell$ be a prime.

(1) We say that an element $\alpha \in k^*$ is an $\ell$-virtual unit if $\alpha\mathbb{Z}_k = \mathfrak{q}^\ell$ for some ideal $\mathfrak{q}$ of $k$ or, equivalently, if $v_\mathfrak{p}(\alpha) \equiv 0 \pmod{\ell}$ for all prime ideals $\mathfrak{p}$, and we denote by $V_\ell(k)$ the group of $\ell$-virtual units.
(2) We define the $\ell$-Selmer group $S_\ell(k)$ of $k$ as $S_\ell(k) = V_\ell(k)/k^{*\ell}$.

Using this definition, it is immediate to see that the bijection described above induces a bijection between fields $K$ as above, and triples $(\mathfrak{a}_0, \mathfrak{a}_1, \overline{u})$ (up to equivalence with the triple $(\mathfrak{a}_1, \mathfrak{a}_0, 1/\overline{u})$), satisfying the following:

(1) $\mathfrak{a}_0$ and $\mathfrak{a}_1$ are coprime integral squarefree ideals of $L$ such that $\mathfrak{a}_0\mathfrak{a}_1^2 \in (I/I^3)[T]$ (where $I$ is the group of fractional ideals of $L$) and $\overline{\mathfrak{a}_0\mathfrak{a}_1^2} \in \mathrm{Cl}(L)^3$.
(2) $\overline{u} \in S_3(L)[T]$, and $\overline{u} \neq 1$ if $\mathfrak{a}_0 = \mathfrak{a}_1 = \mathbb{Z}_L$.

Indeed, given $\alpha$ such that $N(\sqrt{-3}) = L(\alpha^{1/3})$, we can write uniquely $\alpha = \mathfrak{a}_0\mathfrak{a}_1^2\mathfrak{q}^3$ with $\mathfrak{a}_0$ and $\mathfrak{a}_1$ coprime integral squarefree ideals, and since $\overline{\alpha} \in (L^\times/(L^\times)^3)[T]$ and the ideal class of $\mathfrak{a}_0\mathfrak{a}_1^2$ is equal to that of $\mathfrak{q}^{-3}$, the conditions on the ideals are satisfied. Conversely, given a triple as above, we write $\mathfrak{a}_0\mathfrak{a}_1^2\mathfrak{q}_0^3 = \alpha_0\mathbb{Z}_L$ for some $\alpha_0 \in (L^*/(L^*)^3)[T]$ and some ideal $\mathfrak{q}_0$. Then $K$ is the cubic subextension of $L(\sqrt[3]{\alpha_0 u})$, for any lift $u$ of $\overline{u}$.

It is easy to see that $\mathfrak{a}_0\mathfrak{a}_1 = \mathfrak{a}_\alpha\mathbb{Z}_L$ for some ideal $\mathfrak{a}_\alpha$ of $\mathbb{Q}(\sqrt{D})$, and the conductor $\mathfrak{f}(K(\sqrt{D})/\mathbb{Q}(\sqrt{D}))$ is equal to $\mathfrak{a}_\alpha$ apart from a complicated 3-adic factor. Furthermore, $\mathfrak{f}(K(\sqrt{D})/\mathbb{Q}(\sqrt{D})) = f(K)\mathbb{Z}_{\mathbb{Q}(\sqrt{D})}$, and the Dirichlet series for $\Phi_D(s)$ consists of a sum involving the norms of ideals $\mathfrak{a}_0$ and $\mathfrak{a}_1$ satisfying the conditions above. The condition $\overline{\mathfrak{a}_0\mathfrak{a}_1^2} \in \mathrm{Cl}(L)^3$ may be detected by summing over characters of $\mathrm{Cl}(L)/\mathrm{Cl}(L)^3$, suggesting that cubic fields $K$ can be counted in terms of unramified abelian cubic extensions of $L$.

Due to the 3-adic complications, the formula (Theorem 6.1 of [14]) in fact involves a sum over characters of the group

$$G_\mathfrak{b} := \frac{\mathrm{Cl}_\mathfrak{b}(L)}{(\mathrm{Cl}_\mathfrak{b}(L))^3}[T] \tag{3.2}$$

for $\mathfrak{b} \in \mathcal{B} := \{(1), (\sqrt{-3}), (3), (3\sqrt{-3})\}$. More precisely, in the case considered here where the base field is $\mathbb{Q}$, Theorem 6.1 of [14] specializes to the following (see also [22]; note that we slightly changed the definition of $F(\mathfrak{b}, \chi, s)$ given in [14] when $\mathfrak{b} = (1)$ and $3 \mid D$):

THEOREM 3.2. *If $D \notin \{1, -3\}$ we have*

$$\Phi_D(s) = \frac{3}{2c_D}\sum_{\mathfrak{b}\in\mathcal{B}} A_\mathfrak{b}(s) \sum_{\chi\in\widehat{G_\mathfrak{b}}} \omega_\chi(3)F(\mathfrak{b}, \chi, s),$$

*where $c_D = 1$ if $D < 0$, $c_D = 3$ if $D > 0$, and $A_\mathfrak{b}(s)$ are given by Table 2,*

$$F(\mathfrak{b}, \chi, s) = \prod_{(-3D/p)=1}\left(1 + \frac{\omega_\chi(p)}{p^s}\right),$$

**Table 2**

| Condition on $D$ | $A_{(1)}(s)$ | $A_{(\sqrt{-3})}(s)$ | $A_{(3)}(s)$ | $A_{(3\sqrt{-3})}(s)$ |
|---|---|---|---|---|
| $3 \nmid D$ | $3^{-2s}$ | $0$ | $-3^{-2s-1}$ | $1/3$ |
| $D \equiv 3 \pmod 9$ | $0$ | $3^{-3s/2}$ | $3^{-s} - 3^{-3s/2}$ | $(1-3^{-s})/3$ |
| $D \equiv 6 \pmod 9$ | $3^{-2s}$ | $3^{-3s/2}$ | $3^{-s} - 3^{-3s/2}$ | $(1-3^{-s})/3$ |

*where if we write* $p\mathbb{Z}_L = \mathfrak{c}\tau(\mathfrak{c})$ (*with* $\mathfrak{c}$ *not necessarily prime*), *we set*[1] *for* $p \neq 3$:

$$\omega_\chi(p) = \begin{cases} 2 & \text{if } \chi(\tau(\mathfrak{c})/\mathfrak{c}) = 1, \\ -1 & \text{if } \chi(\tau(\mathfrak{c})/\mathfrak{c}) \neq 1, \end{cases}$$

*and, for* $p = 3$:

$$\omega_\chi(3) = \begin{cases} 1 & \text{if } \mathfrak{b} \neq (1) \text{ or } \mathfrak{b} = (1) \text{ and } 3 \nmid D, \\ 2 & \text{if } \mathfrak{b} = (1),\ 3 \mid D, \text{ and } \chi(\tau(\mathfrak{c})/\mathfrak{c}) = 1, \\ -1 & \text{if } \mathfrak{b} = (1),\ 3 \mid D, \text{ and } \chi(\tau(\mathfrak{c})/\mathfrak{c}) \neq 1. \end{cases}$$

*Proof.* We briefly explain how this follows from Theorem 6.1 of [14]. Warning: in the present proof we use the notation of [14] which conflicts somewhat with that of the present paper. All the definition, proposition, and theorem numbers are those of [14].

- We have $k = \mathbb{Q}$, so $[k:\mathbb{Q}] = 1$, so $3^{(3/2)[k:\mathbb{Q}]s} = 3^{3s/2}$.
- By Definition 3.6 we have $\mathcal{P}_3 = \{3\}$ if $3 \nmid D$ and $\emptyset$ if $3 \mid D$, so $\prod_{p \in \mathcal{P}_3} p^{s/2} = 3^{s/2}$ if $3 \nmid D$ and 1 if $3 \mid D$.
- We have $k_z = \mathbb{Q}(\sqrt{-3})$, $K_2 = \mathbb{Q}(\sqrt{D})$, and $L = \mathbb{Q}(\sqrt{-3}, \sqrt{D})$, so by Lemma 5.4 we have $|(U/U^3)[T]| = 3^{r(U)}$ with $r(U) = 2 + 0 - 1 - \delta_{D>0}$, where $\delta$ is the Kronecker symbol; hence, $3^{r(U)} = 3/c_D$ with the notation of our theorem.
- By Definition 4.4, if $3 \nmid D$, then we have $\lceil \mathcal{N}(\mathfrak{b}) \rceil = (1, *, 3, 3^2)$, whereas if $3 \mid D$, then we have $\lceil \mathcal{N}(\mathfrak{b}) \rceil = (1, 3^{1/2}, 3, 3^{3/2})$ for $\mathfrak{b} = ((1), (\sqrt{-3}), (3), (3\sqrt{-3}))$, respectively. Note that we use the convention of Definition 4.1, so that $\lceil \mathcal{N}(\mathfrak{b}) \rceil$ can be the square root of an integer.
- By Definition 4.4 we have $\mathcal{N}(\mathfrak{r}^e(\mathfrak{b})) = 1$ unless $\mathfrak{b} = (1)$ and $3 \mid D$, in which case $\mathcal{N}(\mathfrak{r}^e(\mathfrak{b})) = 3^{1/2}$ (where we again use the convention of Definition 4.1).
- By Proposition 2.10 we have $\mathcal{D}_3 = \emptyset$ (hence $\mathfrak{d}_3 = 1$) unless $D \equiv 6 \pmod 9$, in which case $\mathcal{D}_3 = \{3\}$ (hence $\mathfrak{d}_3 = 3$), and for $p \neq 3$, we have $p \in \mathcal{D}$ if and only if $\left(\frac{-3D}{p}\right) = 1$. In particular, $\mathfrak{r}^e(\mathfrak{b}) \nmid \mathfrak{d}_3$ if and only if $\mathfrak{b} = (1)$ and $D \equiv 3 \pmod 9$.

[1] Note that this fixes a small mistake in the statement of Theorem 6.1 of [14], where the condition is described as $\chi(\mathfrak{c}) = \chi(\tau'(\mathfrak{c}))$. The conditions are equivalent whenever $p$ is a cube modulo $\mathfrak{b}$; if $\mathfrak{b} = (3\sqrt{-3})$ and $p \not\equiv \pm 1 \pmod 9$, then $p$ and $p\tau'(p)$ are not cubes in $\mathrm{Cl}_{\mathfrak{b}}(L)$ for $\tau' \in \{\tau, \tau_2\}$, and so the class of $p$ is not in $G_{\mathfrak{b}}$.

- By Definition 4.5, if $3 \nmid D$, then we have $P_{\mathfrak{b}}(s) = (1, *, -3^{-s}, 1)$, whereas if $3 \mid D$, then we have $P_{\mathfrak{b}}(s) = (1, 3^{-s/2}, 3^{-s/2} - 3^{-s}, 1 - 3^{-s})$ for $\mathfrak{b} = ((1), (\sqrt{-3}), (3), (3\sqrt{-3}))$, respectively.[2]
- By Lemma 5.6, if $3 \nmid D$, then we have $|(Z_{\mathfrak{b}}/Z_{\mathfrak{b}}^3)[T]| = (1, *, 3, 3)$, whereas if $3 \mid D$, then we have $|(Z_{\mathfrak{b}}/Z_{\mathfrak{b}}^3)[T]| = (1, 1, 1, 3)$ for $\mathfrak{b} = ((1), (\sqrt{-3}), (3), (3\sqrt{-3}))$, respectively.

(In the above we put $*$ whenever $3 \nmid D$ and $\mathfrak{b} = (\sqrt{-3})$ since this case is impossible.)

The theorem now follows immediately from Theorem 6.1 of [14]. □

For future reference, note the following lemma, whose trivial proof is left to the reader.

LEMMA 3.3. *Let $\chi$ be a cubic character, and as above write $p\mathbb{Z}_L = \mathfrak{c}\tau(\mathfrak{c})$. The following conditions are equivalent*:

(1) $\chi(\tau(\mathfrak{c})/\mathfrak{c}) = 1$.
(2) $\omega_\chi(p) = 2$.
(3) $\chi(p\mathfrak{c}) = 1$.

*If these conditions are not satisfied, then we have $\omega_\chi(p) = -1$.*

To proceed further, we need to compute the size of the groups $G_{\mathfrak{b}}$ and to reinterpret the conditions involving $\chi$ as conditions involving the cubic field associated to each pair $(\chi, \overline{\chi})$.

In what follows we write

$$H_{\mathfrak{a}} := \frac{\mathrm{Cl}_{\mathfrak{a}}(D^*)}{(\mathrm{Cl}_{\mathfrak{a}}(D^*))^3}[1+\tau], \qquad H'_{\mathfrak{a}} := \frac{\mathrm{Cl}_{\mathfrak{a}}(D)}{(\mathrm{Cl}_{\mathfrak{a}}(D))^3}[1+\tau'], \tag{3.3}$$

where $\tau$ and $\tau'$ are the nontrivial elements of $\mathrm{Gal}(\mathbb{Q}(\sqrt{D^*})/\mathbb{Q})$ and $\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})$, respectively. This $\tau$ is the restriction of $\tau \in \mathrm{Gal}(L/\mathbb{Q})$ (see (3.1)) to $\mathbb{Q}(\sqrt{D^*})$, and we regard $\tau$ as an automorphism of both $L$ and of $\mathbb{Q}(\sqrt{D^*})$ (and $\tau'$ is the restriction of $\tau_2$, but we prefer calling it $\tau'$).

PROPOSITION 3.4. *We have*

$$G_{\mathfrak{b}} \simeq H_{(a)}, \tag{3.4}$$

*where*

- $a = 1$ *if* $\mathfrak{b} = (1)$ *or* $(\sqrt{-3})$, *or if* $\mathfrak{b} = (3)$ *and* $3 \mid D$;
- $a = 3$ *if* $\mathfrak{b} = (3)$ *or* $(3\sqrt{-3})$, *and* $3 \nmid D$;
- $a = 9$ *if* $\mathfrak{b} = (3\sqrt{-3})$ *and* $3 \mid D$.

REMARKS 3.5. (1) Later we will associate a cubic field of discriminant $-D/3$, $-3D$, or $-27D$ to each pair of conjugate nontrivial characters of $G_{\mathfrak{b}}$. Propositions 3.4 and 3.6 will show that we obtain all such fields in this manner.

[2] Note that there is a misprint in Definition 4.5 of [14]: when $e(\mathfrak{p}_z/\mathfrak{p}) = 1$ and $b = 0$, we must set $Q((p\mathbb{Z}_{K_2})^b, s) = 1$ and not 0. The vanishing of certain terms in the final sum comes from the condition $\mathfrak{r}^e(\mathfrak{b}) \mid \mathfrak{d}_3$ of the theorem.

(2) Propositions 3.4, 3.6, and 3.8 imply equalities among $|H_{(3^n)}|$ for different values of $n$. In particular, $|H_{(3^n)}| = |H_{(9)}|$ for $n > 2$, and if $3 \mid D$, then $|H_{(3)}| = |H_{(1)}|$ as well.

*Proof of Proposition 3.4.* For $\mathfrak{b} = (1)$, $G_{\mathfrak{b}}$ is just $(\mathrm{Cl}(L)/\mathrm{Cl}(L)^3)[T]$. This case may be handled with the others, but for clarity, we describe it first.

By arguments familiar in the proof of the Scholz reflection principle (see, e.g., [31, p. 191]) we have

$$\frac{\mathrm{Cl}(L)}{(\mathrm{Cl}(L))^3} \simeq \frac{\mathrm{Cl}(D)}{(\mathrm{Cl}(D))^3} \oplus \frac{\mathrm{Cl}(D^*)}{(\mathrm{Cl}(D^*))^3} \tag{3.5}$$

as $\mathrm{Gal}(L/\mathbb{Q})$-modules. Since $\tau$ acts trivially on $\mathbb{Q}(\sqrt{D})$, we have $(\mathrm{Cl}(D)/\mathrm{Cl}(D)^3)[1+\tau] = 1$; hence, $(\mathrm{Cl}(D)/\mathrm{Cl}(D)^3)[T] = 1$, and since $\tau_2$ acts nontrivially on $\mathrm{Cl}(D^*)/\mathrm{Cl}(D^*)^3$, $1+\tau_2$ acts as the norm, which annihilates the class group, so $(\mathrm{Cl}(D^*)/\mathrm{Cl}(D^*)^3)[T] = \mathrm{Cl}(D^*)/\mathrm{Cl}(D^*)^3[1+\tau]$, so finally

$$\frac{\mathrm{Cl}(L)}{(\mathrm{Cl}(L))^3}[T] = \frac{\mathrm{Cl}(D^*)}{(\mathrm{Cl}(D^*))^3}[1+\tau] = H_{(1)} \tag{3.6}$$

as desired (note that since $\tau$ also acts nontrivially, we have in fact $\mathrm{Cl}(D^*)/\mathrm{Cl}(D^*)^3[1+\tau] = \mathrm{Cl}(D^*)/\mathrm{Cl}(D^*)^3$).

Now suppose that $\mathfrak{b}$ is equal to $(\sqrt{-3})$, $(3)$, or $(3\sqrt{-3})$. Since $\tau$ and $\tau_2$ both act nontrivially on $G_{\mathfrak{b}}$, $\tau\tau_2$ acts trivially. Moreover, $(1+\tau\tau_2)/2 \in \mathbb{F}_3[\mathrm{Gal}(L/\mathbb{Q})]$ is an idempotent, so the elements of $G_{\mathfrak{b}}$ are those that may be represented by an ideal of the form $\mathfrak{a}\tau\tau_2(\mathfrak{a})$, which is necessarily of the form $\mathfrak{a}'\mathbb{Z}_L$ for an ideal $\mathfrak{a}'$ of $\mathbb{Q}(\sqrt{D^*})$. When $\mathfrak{b} = (1)$, this yields an isomorphism $G_{(1)} \xrightarrow{\sim} H_{(1)}$, following (3.5). When $\mathfrak{b} \neq (1)$, this yields an isomorphism $G_{\mathfrak{b}} \xrightarrow{\sim} H_{\mathfrak{a}'}$, where $\mathfrak{a}' := \mathfrak{b} \cap \mathbb{Z}_{\mathbb{Q}(\sqrt{D^*})}$. In this case the $(1+\tau)$-invariance is no longer automatic.

For convenience, we write $F := \mathbb{Q}(\sqrt{D^*})$ in the remainder of the proof. The ideal $\mathfrak{b} \cap \mathbb{Z}_F$ is simple to determine. If $3 \mid D$, then 3 is unramified in $F$, and so $\mathfrak{b} \cap \mathbb{Z}_F$ is equal to $(1)$, $(3)$, $(3)$, $(9)$ for $\mathfrak{b} = (1)$, $(\sqrt{-3})$, $(3)$, $(3\sqrt{-3})$, respectively, as desired. Moreover, in this case Propositions 3.6 and 3.8 imply that $H_{(3)} \simeq H_{(1)}$ (there are no cubic fields whose discriminants have 3-adic valuation 2).

If $3 \nmid D$, then write $(3) = \mathfrak{p}^2$ in $\mathbb{Z}_F$, and for $\mathfrak{b} = (1)$, $(\sqrt{-3})$, $(3)$, $(3\sqrt{-3})$, $\mathfrak{b} \cap \mathbb{Z}_F$ is equal to $(1)$, $\mathfrak{p}$, $(3)$, $3\mathfrak{p}$, respectively. We write down the *ray class group exact sequence*

$$1 \to \mathbb{Z}_F^\times/\mathbb{Z}_F^{\mathfrak{a}'} \to (\mathbb{Z}_F/\mathfrak{a}')^\times \to \mathrm{Cl}_{\mathfrak{a}'}(F) \to \mathrm{Cl}(F) \to 1, \tag{3.7}$$

where $\mathbb{Z}_F^{\mathfrak{a}'}$ is the subgroup of units congruent to 1 modulo $\mathfrak{a}'$. We take 3-Sylow subgroups and take negative eigenspaces for the action of $\tau$ (i.e., write each 3-Sylow subgroup $A$ as $A^+ \oplus A^-$, where $A^\pm := \{x \in A : \ \tau(x) = x^{\pm 1}\}$, and take $A^-$); these operations preserve exactness. If $A$ is the 3-Sylow subgroup of $\mathrm{Cl}_{\mathfrak{a}'}(F)$, then $H_{\mathfrak{a}'}$ is isomorphic to $(A/A^3)^- \simeq A^-/(A^-)^3$.

For $\mathfrak{b} = (1)$ or $(3)$, this finishes the proof. For $\mathfrak{b} = (\sqrt{-3})$, the 3-part of $(\mathbb{Z}_F/\mathfrak{p})^\times$ is trivial; hence, the 3-Sylow subgroups of $\mathrm{Cl}(F)$ and $\mathrm{Cl}_{\mathfrak{p}}(F)$ are isomorphic, so $H_{\mathfrak{p}} \simeq H_{(1)}$.

For $\mathfrak{b} = (3\sqrt{-3})$, the 3-Sylow subgroup of $(\mathbb{Z}_F/\mathfrak{p}^3)^\times$ is larger than that of $(\mathbb{Z}_F/3)^\times$ by a factor of 3; however, the same is true for the positive eigenspace and hence not for the negative eigenspace. Therefore, $H_{\mathfrak{p}^3} \simeq H_{(3)}$. □

We now state a well-known formula for counting cubic fields in terms of ray class groups.

PROPOSITION 3.6. *If $D$ is a fundamental discriminant other than* 1, *$c$ is any nonzero integer, and $H'_{\mathfrak{a}}$ is defined as in* (3.3), *then we have*

$$\sum_{d\mid c} |\mathcal{L}_{Dd^2}| = \frac{1}{2}(|H'_{(c)}| - 1). \tag{3.8}$$

*Proof.* This is a combination of (1.3) and Lemma 1.10 of Nakagawa [23], and it is also a fairly standard fact from class field theory. The idea is that cubic extensions of $\mathbb{Q}(\sqrt{D^*})$ whose conductor divides $(c)$ correspond to subgroups of $\mathrm{Cl}_{(c)}(D^*)$ of index 3, and that such a cubic extension descends to a cubic extension of $\mathbb{Q}$ if and only if it is in the kernel of $1+\tau$. □

We also have the following counting result, which relies on the deeper part of the work of Nakagawa [23] and Ohno [24].

PROPOSITION 3.7. *Let $D$ be a fundamental discriminant, and set $r = \mathrm{rk}_3(D^*)$.*

(1) *Assume that $D < -3$. Then we have*

$$(|\mathcal{L}_{D^*}|, |\mathcal{L}_{-27D}|) = \begin{cases} ((3^r-1)/2, 3^r) & \text{if } \mathrm{rk}_3(D) = r+1, \\ ((3^r-1)/2, 0) & \text{if } \mathrm{rk}_3(D) = r. \end{cases}$$

*In either case,* $|\mathcal{L}_3(D)| = (3^{\mathrm{rk}_3(D)} - 1)/2$.

(2) *Assume that $D > 1$. Then we have*

$$(|\mathcal{L}_{D^*}|, |\mathcal{L}_{-27D}|) = \begin{cases} ((3^r-1)/2, 0) & \text{if } \mathrm{rk}_3(D) = r-1, \\ ((3^r-1)/2, 3^r) & \text{if } \mathrm{rk}_3(D) = r. \end{cases}$$

*In either case,* $|\mathcal{L}_3(D)| = (3^{\mathrm{rk}_3(D)+1} - 1)/2$.

*Proof.* This is essentially Theorem 0.4 of Nakagawa [23]. We elaborate somewhat:

The formulas for $|\mathcal{L}_{D^*}|$ follow from class field theory, as these count unramified cyclic cubic extensions of $\mathbb{Q}(\sqrt{-3D})$, which are in bijection with subgroups of $\mathrm{Cl}(-3D)$ of index 3. It therefore suffices to prove the stated formulas for $|\mathcal{L}_3(D)|$. Recalling the notation in (3.3), Nakagawa proved [23, Theorem 0.4] that if $D < 0$, then

$$|H'_{(1)}| = |H_{(a)}|, \tag{3.9}$$

where $a = 3$ if $3 \nmid D$ and $a = 9$ if $3 \mid D$; and if $D > 0$, then

$$3|H'_{(1)}| = |H_{(a)}| \tag{3.10}$$

with the same $a$. By Proposition 3.6, these formulas are equivalent to the stated formulas for $|\mathcal{L}_3(D)|$. □

REMARK. In a follow-up paper [17] joint with Rubinstein-Salzedo, we demonstrate that the machinery of this section suffices to *prove* Nakagawa's Theorem 0.4, without appealing to his careful study of binary cubic forms and their Hessians. The main additional ingredients are the perfect pairing

$$G_{(3\sqrt{-3})} \times S_3(K_z)[\tau_2 + 1, \tau - 1] \mapsto \boldsymbol{\mu}_3,$$

induced from the Kummer pairing, and a further study of the sizes of the groups $G_{\mathfrak{b}}$.

We refer to [17] for more details and for a generalization relating counts of $D_\ell$ and $F_\ell$ field discriminants for any odd prime $\ell$.

PROPOSITION 3.8. *If $K$ is a cubic field, then $v_3(\mathrm{Disc}(K))$ can only be equal to* 0, 1, 3, 4, *and* 5 *in relative proportions* 81/117, 27/117, 6/117, 2/117, *and* 1/117 *when the fields are ordered by increasing absolute value of their discriminant.*

*Proof.* The proof that $v_3(\mathrm{Disc}(K))$ can take only the given values is classical; see Hasse [20]. The proportions follow from the proof of the Davenport–Heilbronn theorem. A convenient reference is Section 6.2 of [27], where a table of these proportions is given in the context of "local specifications"; these proportions also appear (in a slightly less explicit form) in the earlier related literature. $\square$

Before proceeding to the proof of Theorem 2.5 we give the following application.

PROPOSITION 3.9. (1) *If $D < 0$ and $(\mathrm{rk}_3(D), \mathrm{rk}_3(D^*)) = (2, 1)$, or $D > 0$ and $(\mathrm{rk}_3(D), \mathrm{rk}_3(D^*)) = (1, 1)$, then there exist a unique cubic field of discriminant $D^*$ and three cubic fields of discriminant $-27D$.*

(2) *If $D < 0$ and $(\mathrm{rk}_3(D), \mathrm{rk}_3(D^*)) = (2, 2)$, or $D > 0$ and $(\mathrm{rk}_3(D), \mathrm{rk}_3(D^*)) = (1, 2)$, then there exist four cubic fields of discriminant $D^*$ and no cubic field of discriminant $-27D$.*

*In addition, if $3 \nmid D$, then 3 is partially ramified in the four cubic fields, if $D \equiv 3 \pmod 9$, then 3 is partially split in the four cubic fields, and if $D \equiv 6 \pmod 9$, then 3 is totally split in one of the four cubic fields and inert in the three others.*

*Proof.* The first statements are special cases of Proposition 3.7, the behavior of 3 when $3 \nmid D$ is classical (see [20]), and when $D \equiv 3 \pmod 9$, the last statement is trivial since $D^* \equiv 2 \pmod 3$.

For the case $D \equiv 6 \pmod 9$, we use Theorem 2.5. Writing out the 3-part of the theorem for the discriminant $D$, we see that

$$|\mathcal{L}_{81D}| = 3\left(1 + \sum_E \omega_E(3)\right), \tag{3.11}$$

where the sum ranges over the cubic fields $E$ of discriminant $-D/3$, and $\omega_E(3)$ is equal to 2 if 3 is totally split in $E$ and $-1$ if 3 is inert in $E$. Therefore, if 3 is totally split in 0, 1, 2, 3, or 4 of these fields, then $|\mathcal{L}_{81D}|$ is equal to $-9$, 0, 9, 18, or 27. Obviously, we can rule out the first possibility.

We first observe that $|\mathcal{L}_{9D}| = 9$, again by Theorem 2.5. By Proposition 3.6,

$$|\mathcal{L}_{9D}| = 9 = \frac{1}{2}(H'_{(3)} - H'_{(1)}), \qquad |\mathcal{L}_{81D}| = \frac{1}{2}(H'_{(9)} - H'_{(3)}). \tag{3.12}$$

By assumption, $|H'_{(1)}| = 9$ and so $|H'_{(3)}| = 27$. Therefore, either $|H'_{(9)}| = 81$ and $|\mathcal{L}_{81D}| = 27$ or $|H'_{(3)}| = 27$ and $|\mathcal{L}_{81D}| = 0$.

To rule out the former possibility, we again consider the exact sequence (3.7), with $F = \mathbb{Q}(\sqrt{D^*})$ replaced by $\mathbb{Q}(\sqrt{D})$ and $\tau$ replaced by $\tau'$, and take 3-Sylow subgroups and $(1+\tau')$-invariants (preserving exactness). The 3-rank of $(\mathbb{Z}_{\mathbb{Q}(\sqrt{D})}/(9))^\times[1+\tau]$ is equal to 1, and so the 3-rank of $\mathrm{Cl}_{(9)}(D)$ is at most 1 more than that of $\mathrm{Cl}(D)$. In other words, $|H'_{(9)}| \le 3|H'_{(1)}|$, but we saw previously that $|H'_{(3)}| = 3|H'_{(1)}|$, so $|H'_{(9)}| = 3|H'_{(1)}|$ and $|\mathcal{L}_{81D}| = 0$, as desired. □

## 4. Proof of Theorem 2.5

Theorem 2.5 follows from a more general result of Morra and the first author (Theorem 6.1 of [14] and Theorem 1.6.1 of [22]), given above as Theorem 3.2 in our case where the base field is $\mathbb{Q}$. To each character of the groups $G_{\mathfrak{b}}$ we use class field theory and Galois theory to uniquely associate a cubic field $E$. Some arithmetic involving discriminants, as well as a comparison to our earlier counting formulas, proves that these fields $E$ range over all fields in $\mathcal{L}_3(D)$. Finally, we apply Theorem 3.2 to obtain the correct Euler product for each $E$.

This section borrows from the first author's work in [29], which established a particular case of Theorem 2.5 for an application to Shintani zeta functions.

### *4.1. Construction of the Fields E*

We refer to the beginning of Section 3 for our notation and setup. The contribution of the trivial characters occurring in Theorem 3.2 being easy to compute (see below; it has also been computed in [14]), we must handle the *nontrivial* characters.

We relate these characters to cubic fields by means of the following.

PROPOSITION 4.1. *For each $\mathfrak{b} \in \mathcal{B}$, there is a bijection between the set of pairs of nontrivial characters $(\chi, \overline{\chi})$ of $G_{\mathfrak{b}}$ and the following sets of cubic fields $E$:*

- *If $\mathfrak{b} = (1)$ or $(\sqrt{-3})$, or if $\mathfrak{b} = (3)$ and $3 \mid D$, then all $E \in \mathcal{L}_{D^*}$.*
- *If $\mathfrak{b} = (3\sqrt{-3})$, or if $\mathfrak{b} = (3)$ and $3 \nmid D$, then all $E \in \mathcal{L}_3(D) = \mathcal{L}_{D^*} \cup \mathcal{L}_{-27D}$.*

*Moreover, for each prime $p$ with $\left(\frac{-3D}{p}\right) = 1$, write $p\mathbb{Z}_L = \mathfrak{c}\tau(\mathfrak{c})$ as in [14], where $\mathfrak{c}$ is not necessarily prime, and recall from Theorem 3.2 and Lemma 3.3 the definition of $\omega_\chi(p)$. Under our bijection, the following conditions are equivalent:*

- $\chi(p\mathfrak{c}) = 1$.
- *The prime $p$ splits completely in $E$.*
- $\omega_\chi(p) = 2$.

*If these conditions are not satisfied, then $p$ is inert in $E$, and $\omega_\chi(p) = -1$.*

*Proof.* Define[3] $G'_{\mathfrak{b}} := \mathrm{Cl}_{\mathfrak{b}}(L)/\mathrm{Cl}_{\mathfrak{b}}(L)^3$, so that $G'_{\mathfrak{b}}$ is a 3-torsion group containing $G_{\mathfrak{b}}$. We have a canonical decomposition of $G'_{\mathfrak{b}}$ into four eigenspaces for the actions of $\tau$ and $\tau_2$, and we write

$$G'_{\mathfrak{b}} \simeq G_{\mathfrak{b}} \times G''_{\mathfrak{b}}, \tag{4.1}$$

where $G''_{\mathfrak{b}}$ is the direct sum of the three eigenspaces other than $G_{\mathfrak{b}}$. Note that $G''_{\mathfrak{b}}$ will contain the classes of all principal ideals generated by rational integers coprime to 3; any such class in the kernel of $T$ will necessarily be in $\mathrm{Cl}_{\mathfrak{b}}(L)^3$.

For any nontrivial character $\chi$ of $G_{\mathfrak{b}}$, let $B$ be its kernel that has index 3. We extend $\chi$ to a character $\chi'$ of $G'_{\mathfrak{b}}$ by setting $\chi(G''_{\mathfrak{b}}) = 1$ and write $B' := \mathrm{Ker}(\chi') = B \times G''_{\mathfrak{b}} \subseteq G'_{\mathfrak{b}}$, so that $B'$ has index 3 in $G'_{\mathfrak{b}}$ and is uniquely determined by $\mathfrak{b}$ and $\chi$. By class field theory, there is a unique abelian extension $E_1/L$ for which the Artin map induces an isomorphism $G'_{\mathfrak{b}}/B' \simeq \mathrm{Gal}(E_1/L)$, and it must be cyclic cubic since $G'_{\mathfrak{b}}/B'$ is. The uniqueness forces $E_1$ to be Galois over $\mathbb{Q}$ since the groups $G'_{\mathfrak{b}}$ and $B'$ are preserved by $\tau$ and $\tau_2$ and hence by all of $\mathrm{Gal}(L/\mathbb{Q})$. For each fixed $\mathfrak{b}$, we obtain a unique such field $E_1$ for each distinct pair of characters $\chi, \overline{\chi}$, but we may obtain the same field $E_1$ for different values of $\mathfrak{b}$.

We have $\mathrm{Gal}(E_1/\mathbb{Q}) \simeq S_3 \times C_2$: $\tau$ and $\tau_2 \in \mathrm{Gal}(E_1/L)$ both act nontrivially (elementwise) on $G_{\mathfrak{b}}$ and preserve $B$ and hence act nontrivially on $G_{\mathfrak{b}}/B$ and $G'_{\mathfrak{b}'}/B'$. Under the Artin map, this implies that $\tau$ and $\tau_2$ both act nontrivially on $\mathrm{Gal}(E_1/L)$ by conjugation, so that $\tau\tau_2$ commutes with $\mathrm{Gal}(E_1/L)$. Since $\tau\tau_2$ fixes $\mathbb{Q}(\sqrt{-3D})$, this implies that $E_1$ contains a cubic extension $E/\mathbb{Q}$ with quadratic resolvent $\mathbb{Q}(\sqrt{-3D})$, which is unique up to isomorphism. Any prime $p$ that splits in $\mathbb{Q}(\sqrt{-3D})$ must either be inert or totally split in $E$.

We now prove that the fields $E$ that occur in this construction are those described by the proposition. Since the quadratic resolvent of any $E$ is $\mathbb{Q}(\sqrt{-3D}) = \mathbb{Q}(\sqrt{D^*})$ with $D^*$ fundamental, we have $\mathrm{Disc}(E) = r^2 D^*$ for some integer $r$ divisible only by 3 and prime divisors of $D^*$.

No prime $\ell > 3$ can divide $r$ because $\ell^3$ cannot divide the discriminant of any cubic field. Similarly 2 cannot divide $r$ since if $2 \mid D$, then $4 \mid D$, but 16 cannot divide the discriminant of a cubic field. Therefore, $r$ must be a power of 3. Since the 3-adic valuation of a cubic field discriminant is never larger than 5, $r$ must be 1, 3, or 9, and by Proposition 3.8 we cannot have $r = 3$ if $3 \nmid D^*$ or, in other words, if $3 \mid D$. It follows that $E$ must have discriminant $D^*$, $-27D$, or $-243D$.

We further claim that $\mathrm{Disc}(E) \neq -243D$. To see this, we apply the formula

$$\mathrm{Disc}(E_1) = \pm \mathrm{Disc}(L)^3 \mathcal{N}_{L/\mathbb{Q}}(\mathfrak{d}(E_1/L)) \tag{4.2}$$

and bound $v_3(\mathrm{Disc}(E_1))$. We see that $v_3(\mathrm{Disc}(L)) = 2$ (by direct computation or by a formula similar to (4.2)). Moreover, the conductor of $E_1/L$ divides $(3\sqrt{-3})$, and therefore $\mathfrak{d}(E_1/L)$ divides (27). The norm of this ideal is $3^{12}$, and putting all of this together, we see that $v_3(\mathrm{Disc}(E_1)) \leq 18$.

[3] We have followed the notation of [14] where practical, but the notations $G'_{\mathfrak{b}}$, $G''_{\mathfrak{b}}$, $E$, $E_1$ are used for the first time here and do not appear in [14].

We also have

$$\mathrm{Disc}(E_1) = \pm \mathrm{Disc}(E)^4 \mathcal{N}_{E/\mathbb{Q}}(\mathfrak{d}(E_1/E)), \tag{4.3}$$

so that $v_3(\mathrm{Disc}(E)) < \frac{18}{4}$, and in particular $v_3(\mathrm{Disc}(E)) \neq 5$, as desired.

Therefore, in all cases, $E$ must have discriminant $D^*$ or $-27D$. Moreover, similar comparisons of (4.2) and (4.3) show that if $\mathfrak{b} \in \{(1), (\sqrt{-3})\}$ or if $\mathfrak{b} = (3)$ and $3 \mid D$, then $E$ must have discriminant $D^*$.

We have therefore associated a unique $E$ to each pair $(\chi, \overline{\chi})$ as in the proposition, and it follows from Propositions 3.6 and 3.4 that we obtain all such $E$ in this manner.

Finally, we prove the second part of the proposition. The statements concerning $\omega_\chi(p)$ and $\chi(p\mathfrak{c})$ follow from Lemma 3.3. We show the equivalence of the first and second statements.

We extend $\chi$ to a character $\chi'$ of $G'_{\mathfrak{b}}$ as defined previously, so that $\chi'(p)$ is defined and equal to 1. Therefore, $\chi(p\mathfrak{c}) = 1$ if and only if $\chi'(\mathfrak{c}) = 1$, and we must show that this is true if and only if $p$ splits completely in $E$.

Suppose first that $\mathfrak{c}$ is prime in $\mathbb{Z}_L$. By class field theory, $\chi'(\mathfrak{c}) = 1$ if and only if $\mathfrak{c}$ splits completely in $E_1/L$, which happens if and only if $(p)$ splits into six ideals in $E_1$, which happens if and only if $p$ splits completely in $E$.

Suppose now that $\mathfrak{c} = \mathfrak{p}\tau\tau_2(\mathfrak{p})$ in $L$. Since $\mathfrak{p}$ and $\tau\tau_2(\mathfrak{p})$ represent the same element of $G'_{\mathfrak{b}}/B'$, they have the same Frobenius element in $E_1/L$; hence, since $\chi'$ is a cubic character, it follows that $\chi'(\mathfrak{c}) = 1$ if and only if $\chi'(\mathfrak{p}) = 1$. By class field theory this is true if and only if $\mathfrak{p}$ splits completely in $E_1/L$, in which case $(p)$ splits into twelve ideals in $E_1$; for this, it is necessary and sufficient that $p$ splits completely in $E$. □

### 4.2. *Putting It All Together*

Applying Proposition 4.1, we may regard the formula of Theorem 3.2 as a sum over cubic fields. We now divide into cases $3 \nmid D$ and $D \equiv 3, 6 \pmod 9$. The main terms, corresponding to the trivial characters of $G_{\mathfrak{b}}$, contribute

$$\frac{3}{2c_D} \sum_{\mathfrak{b} \in \mathcal{B}} \omega_1(3) A_{\mathfrak{b}}(s) = \frac{1}{2c_D} M_1(s) \prod_{(-3D/p)=1} \left(1 + \frac{2}{p^s}\right) \tag{4.4}$$

of equation (2.2). These terms have also been given in [14] and [22].

It remains to handle the contribution of the nontrivial characters.

Assume first that $3 \nmid D$. In this case, by Theorem 3.2,

$$c_D \Phi_D(s) = \frac{3}{2} \cdot \left[ 3^{-2s} \sum_{\chi \in \widehat{G_{(1)}}} F((1), \chi, s) - 3^{-2s-1} \sum_{\chi \in \widehat{G_{(3)}}} F((3), \chi, s) \right.$$
$$\left. + \frac{1}{3} \sum_{\chi \in \widehat{G_{(3\sqrt{-3})}}} F\big((3\sqrt{-3}), \chi, s\big) \right], \tag{4.5}$$

and the calculations above show that

$$F(\mathfrak{b}, \chi, s) = \prod_{(-3D/p)=1} \left(1 + \frac{\omega_E(p)}{p^s}\right), \tag{4.6}$$

where $E$ is the cubic field associated to $\chi$. Each field $E$ of discriminant $D^*$ contributes twice (each character yields the same field as its inverse) to each of the three sums above, and each field of discriminant $-27D$ contributes twice to each of the last two. We obtain a contribution of $1 + 2 \cdot 9^{-s}$ for each field of discriminant $D^* = -3D$, and of $1 - 9^{-s}$ for each field of discriminant $-27D$. This is the assertion of the theorem.

Assume now that $D \equiv 3 \pmod 9$. Then

$$\begin{aligned} c_D \Phi_D(s) = \frac{3}{2} \cdot \Bigg[ & 3^{-3s/2} \sum_{\chi \in \widehat{G_{(\sqrt{-3})}}} F\big((\sqrt{-3}), \chi, s\big) \\ & + (3^{-s} - 3^{-3s/2}) \sum_{\chi \in \widehat{G_{(3)}}} F((3), \chi, s) \\ & + \frac{1}{3}(1 - 3^{-s}) \sum_{\chi \in \widehat{G_{(3\sqrt{-3})}}} F\big((3\sqrt{-3}), \chi, s\big) \Bigg]. \end{aligned} \tag{4.7}$$

The first two sums are over fields of discriminant $D^* = -D/3$, and the last sum also includes fields of discriminant $-27D$. We obtain a contribution of $1 + 2 \cdot 3^{-s}$ for each field of discriminant $D^*$, and of $1 - 3^{-s}$ for each field of discriminant $-27D$, in accordance with the theorem.

Finally, assume that $D \equiv 6 \pmod 9$. Then

$$\begin{aligned} c_D \Phi_D(s) = \frac{3}{2} \cdot \Bigg[ & 3^{-2s} \sum_{\chi \in \widehat{G_{(1)}}} \omega_\chi(3) F((1), \chi, s) \\ & + 3^{-3s/2} \sum_{\chi \in \widehat{G_{(\sqrt{-3})}}} F\big((\sqrt{-3}), \chi, s\big) \\ & + (3^{-s} - 3^{-3s/2}) \sum_{\chi \in \widehat{G_{(3)}}} F((3), \chi, s) \\ & + \frac{1}{3}(1 - 3^{-s}) \sum_{\chi \in \widehat{G_{(3\sqrt{-3})}}} F\big((3\sqrt{-3}), \chi, s\big) \Bigg]. \end{aligned} \tag{4.8}$$

The first three sums are over fields of discriminant $D^* = -D/3$, and the last sum also includes fields of discriminant $-27D$. For the same reasons as discussed for $p \neq 3$, we have $\omega_\chi(3) = \omega_E(3)$, where $E$ is the cubic field associated to $\chi$.

We thus obtain a contribution of $1 + 2 \cdot 3^{-s} + 3\omega_E(3) \cdot 3^{-2s}$ for each field of discriminant $D^*$ and of $1 - 3^{-s}$ for each field of discriminant $-27D$, in accordance with the theorem.

## 5. Numerical Examples

We present some numerical examples of our main results.

Suppose first that $D < 0$.

If $(\mathrm{rk}_3(D), \mathrm{rk}_3(D^*)) = (0, 0)$, then there are no cubic fields of discriminant $D^*$ or $-27D$.

$$\Phi_{-4}(s) = \frac{1}{2}\left(1 + \frac{2}{3^{2s}}\right) \prod_{(12/p)=1} \left(1 + \frac{2}{p^s}\right). \tag{5.1}$$

If $(\mathrm{rk}_3(D), \mathrm{rk}_3(D^*)) = (1, 0)$, there are no cubic fields of discriminant $D^*$ and a unique cubic field of discriminant $-27D$.

$$\begin{aligned}\Phi_{-255}(s) &= \frac{1}{2}\left(1 + \frac{2}{3^s} + \frac{6}{3^{2s}}\right) \prod_{(6{,}885/p)=1} \left(1 + \frac{2}{p^s}\right)\\ &\quad + \left(1 - \frac{1}{3^s}\right) \prod_p \left(1 + \frac{\omega_{L6885}(p)}{p^s}\right),\end{aligned} \tag{5.2}$$

where $L6885$ is the cubic field determined by $x^3 - 12x - 1 = 0$.

If $(\mathrm{rk}_3(D), \mathrm{rk}_3(D^*)) = (1, 1)$, there is a unique cubic field of discriminant $D^*$ and no cubic fields of discriminant $-27D$.

$$\begin{aligned}\Phi_{-107}(s) &= \frac{1}{2}\left(1 + \frac{2}{3^{2s}}\right) \prod_{(321/p)=1} \left(1 + \frac{2}{p^s}\right)\\ &\quad + \left(1 + \frac{2}{3^{2s}}\right) \prod_p \left(1 + \frac{\omega_{L321}(p)}{p^s}\right),\end{aligned} \tag{5.3}$$

where $L321$ is the field determined by $x^3 - x^2 - 4x + 1$.

If $(\mathrm{rk}_3(D), \mathrm{rk}_3(D^*)) = (2, 1)$, there is a unique cubic field of discriminant $D^*$ and three cubic fields of discriminant $-27D$.

$$\begin{aligned}\Phi_{-8751}(s) &= \frac{1}{2}\left(1 + \frac{2}{3^s} + \frac{6}{3^{2s}}\right) \prod_{(26{,}253/p)=1} \left(1 + \frac{2}{p^s}\right)\\ &\quad + \left(1 + \frac{2}{3^s} - \frac{3}{3^{2s}}\right) \prod_{p\neq 3} \left(1 + \frac{\omega_{L2917}(p)}{p^s}\right)\\ &\quad + \left(1 - \frac{1}{3^s}\right) \sum_{1\le i\le 3} \prod_p \left(1 + \frac{\omega_{L236277_i}(p)}{p^s}\right),\end{aligned}$$

where the four fields above are defined in Table 3.

If $(\mathrm{rk}_3(D), \mathrm{rk}_3(D^*)) = (2, 2)$, there are four cubic fields (Table 4) of discriminant $D^*$ and none of discriminant $-27D$. Recall from Proposition 3.9 above that if $D \equiv 6 \pmod 9$, then 3 is totally split in one of them and inert in the other three, so one of the cubic fields of discriminant $D^*$, which we include first, is

**Table 3**

| Cubic field | Discriminant | Defining polynomial |
|---|---|---|
| $L2917$ | $8{,}751/3$ | $x^3 - x^2 - 13x + 20$ |
| $L236277_1$ | $3^3 \cdot 8{,}751$ | $x^3 - 138x + 413$ |
| $L236277_2$ | $3^3 \cdot 8{,}751$ | $x^3 - 129x - 532$ |
| $L236277_3$ | $3^3 \cdot 8{,}751$ | $x^3 - 90x - 171$ |

**Table 4**

| Cubic field | Discriminant | Defining polynomial |
|---|---|---|
| $L103809_1$ | $3 \cdot 34{,}603$ | $x^3 - x^2 - 84x + 261$ |
| $L103809_2$ | $3 \cdot 34{,}603$ | $x^3 - x^2 - 64x + 91$ |
| $L103809_3$ | $3 \cdot 34{,}603$ | $x^3 - x^2 - 92x - 204$ |
| $L103809_4$ | $3 \cdot 34{,}603$ | $x^3 - x^2 - 62x - 15$ |

distinguished by the fact that 3 is totally split:

$$\Phi_{-34{,}603}(s) = \frac{1}{2}\left(1 + \frac{2}{3^{2s}}\right) \prod_{(103{,}809/p)=1} \left(1 + \frac{2}{p^s}\right) + \left(1 + \frac{2}{3^{2s}}\right) \sum_{1 \le i \le 4} \prod_p \left(1 + \frac{\omega_{L103809_i}(p)}{p^s}\right). \tag{5.4}$$

The case $D > 1$ is very similar, so we will only give one example. If (e.g.) $(\mathrm{rk}_3(D), \mathrm{rk}_3(D^*)) = (1, 1)$, then there is a unique cubic field of discriminant $D^*$ and three cubic fields of discriminant $-27D$.

$$\begin{aligned} 3\Phi_{321}(s) = &\frac{1}{2}\left(1 + \frac{2}{3^s} + \frac{6}{3^{2s}}\right) \prod_{(-963/p)=1} \left(1 + \frac{2}{p^s}\right) \\ &+ \left(1 + \frac{2}{3^s} - \frac{3}{3^{2s}}\right) \prod_{p \neq 3} \left(1 + \frac{\omega_{LM107}(p)}{p^s}\right) \\ &+ \left(1 - \frac{1}{3^s}\right) \sum_{1 \le i \le 3} \prod_p \left(1 + \frac{\omega_{LM8667_i}}{p^s}\right), \end{aligned}$$

where the indicated cubic fields are given in Table 5.

## 6. Counting $S_3$-Sextic Fields of Bounded Discriminant

We thank the anonymous referee of [28] for (somewhat indirectly) suggesting this application.

**Table 5**

| Cubic field | Discriminant | Defining polynomial |
|---|---|---|
| $LM107$ | $-321/3$ | $x^3 - x^2 + 3x - 2$ |
| $LM8667_1$ | $-3^3 \cdot 321$ | $x^3 + 18x - 45$ |
| $LM8667_2$ | $-3^3 \cdot 321$ | $x^3 + 6x - 17$ |
| $LM8667_3$ | $-3^3 \cdot 321$ | $x^3 + 15x - 28$ |

Theorem 2.5 naturally lends itself to counting $S_3$-sextic fields, that is, fields that are Galois over $\mathbb{Q}$ with Galois group $S_3$. For any such $\widetilde{K}$ with cubic and quadratic subfields $K$ and $k$, respectively, we have the formula

$$\operatorname{Disc}(\widetilde{K}) = \operatorname{Disc}(K)^2 \operatorname{Disc}(k) = \operatorname{Disc}(k)^3 f(K)^4. \tag{6.1}$$

Let $N^{\pm}(X; S_3)$ denote the number of $S_3$-sextic fields $\widetilde{K}$ with $0 < \pm\operatorname{Disc}(\widetilde{K}) < X$. Then Theorem 2.5 may be used to compute $N^{\pm}(X; S_3)$: iterate over fundamental discriminants $D$ with $0 < \pm D < X^{1/3}$; compute the Dirichlet series $\Phi_D(s)$ to $f(K) < (X/D^3)^{1/4}$ and evaluate its partial sums; finally, sum the results.

We implemented this algorithm in PARI/GP [25], which can easily handle the various quantities occurring in (2.2). For a list of cubic fields, we relied on Belabas's `cubic` program [1].

We used the GP calculator, which has the advantage of simplicity. The disadvantage of this approach is that we were obliged to read the output of `cubic` from disk, limiting us by available disk space. One could probably compute $N^{\pm}(X; S_3)$ to at least $X = 10^{27}$ by directly implementing (2.2) within Belabas's code; alternatively, Belabas has informed us that an implementation of `cubic` within PARI/GP may be forthcoming. In any case we leave further computations for later.

This approach dictated a slight variant of the algorithm described above:

- We parsed Belabas's output into files readable by PARI/GP, using a Java program written for this purpose.
- Given a table of all cubic fields $K$ with $0 < \pm\operatorname{Disc}(K) < Y$ for some $Y$, it must contain all fields in $\mathcal{L}_3(D)$ with $0 < \mp\operatorname{Disc}(K) < Y/27$, allowing us to choose any $X \leq 3^{-9} Y^3$.
- Processing each cubic field in turn and ignoring those not in $\mathcal{L}_3(D)$ for some fundamental discriminant $D$ with $|D| \leq X^{1/3}$, we computed the associated Dirichlet series to a length of $\lfloor (X/D^3)^{1/4} \rfloor$, and its partial sum (less the $\frac{1}{2}$ term for $f(K) = 1$), and maintained a running total of the results.
- Finally, we added the main term of (2.2) for each $D$ with $0 < \pm D < X^{1/3}$.

Our algorithm would also allow for efficient computation of the $\Phi_D(s)$, given a *sorted* version of Belabas's output.

The implementation posed no particular difficulties, and our PARI/GP source code is available on the second author's website. To replicate our data for large $X$, one must also install and run Belabas's `cubic` program. With our source code

**Table 6**

| $X$ | $N_6^+(X; S_3)$ | $X$ | $N_6^-(X; S_3)$ |
|---|---|---|---|
| $10^{12}$ | 690 | $10^{12}$ | 2,809 |
| $10^{13}$ | 1,650 | $10^{13}$ | 6,315 |
| $10^{14}$ | 3,848 | $10^{14}$ | 14,121 |
| $10^{15}$ | 8,867 | $10^{15}$ | 31,276 |
| $10^{16}$ | 20,062 | $10^{16}$ | 68,972 |
| $10^{17}$ | 45,054 | $10^{17}$ | 151,877 |
| $10^{18}$ | 100,335 | $10^{18}$ | 333,398 |
| $10^{19}$ | 222,939 | $10^{19}$ | 729,572 |
| $10^{20}$ | 492,335 | $10^{20}$ | 1,592,941 |
| $10^{21}$ | 1,083,761 | $10^{21}$ | 3,470,007 |
| $10^{22}$ | 2,378,358 | $10^{22}$ | 7,550,171 |
| $10^{23}$ | 5,207,310 | $10^{23}$ | 16,399,890 |
| – | – | $3 \cdot 10^{23}$ | 23,738,460 |

we have also made available a modestly sized table of cubic fields, with which our PARI/GP program suffices alone to replicate our data for smaller values of $X$.

On a 2.1 GHz MacBook our computation took approximately 3 hours for negative discriminants $< 3 \cdot 10^{23}$ and 10 hours for positive discriminants $< 10^{23}$; it is to be expected from the shape of (2.2) that counts of negative discriminants may be computed more efficiently than positive, even though there are more of them. Our data is presented in Table 6.

This data may be compared to known theoretical results on $N^{\pm}(X; S_3)$. It was proved by Belabas and Fouvry [2] and Bhargava and Wood [8] (independently) that $N^{\pm}(X; S_3) \sim B^{\pm} X^{1/3}$ for explicit constants $B^{\pm}$ (with $B^- = 3B^+$), and in [28] Taniguchi and the second author obtained a power saving error term.

The authors of [28] also computed tables of $N^{\pm}(X; S_3)$ up to $X = 10^{18}$ using a different method, allowing us to double-check our work here. Based on this data and on [26; 7; 27], they guessed the existence of a secondary term of order $X^{5/18}$ and found that the data further suggested the existence of additional, unexplained lower-order terms. For more on this, we defer to [28].

Acknowledgments. The authors would like to thank Karim Belabas, Franz Lemmermeyer, Guillermo Mantilla-Soler, Simon Rubinstein-Salzedo, and an anonymous referee, among many others, for helpful discussions related to the topic of this paper. We would especially like to thank the anonymous referee of [28], who (indirectly) suggested the application to counting $S_3$-sextic fields.

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