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Local Dynamics of Holomorphic Maps in $\mathbf{C}^2$ with a Jordan Fixed Point

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1. Introduction

Many authors have studied the local dynamics of holomorphic maps in $\mathbf{C}^n$ around a fixed point; see, for example, [4; 6] for an introduction to this field and known results. Most of the results are obtained under the assumption that the linear part of the map at the fixed point is diagonalizable. There are few results in the non-diagonalizable case. In [8], Coman and Dabija studied a special map with a Jordan fixed point and described its stable and unstable manifolds. In [1], Abate provided a systematic way of diagonalizing a map with a Jordan fixed point and proved several results under certain assumptions. In [2], Abate showed the existence of "parabolic curves" for holomorphic maps in $\mathbf{C}^2$ with an isolated Jordan fixed point. In [3], Abate studied a special map with a Jordan fixed point and proved the existence of an attracting domain under certain conditions. The aim of this paper is to provide a detailed study of the local dynamics of holomorphic maps in $\mathbf{C}^2$ with a Jordan fixed point.

Let f be a holomorphic map in $\mathbf{C}^2$ with a Jordan fixed point. In suitable local coordinates (z, w), f can be written as

$$\begin{aligned} z_1 &= \lambda(z + w + p_2(z,w) + p_3(z,w) + \cdots),\\ w_1 &= \lambda(w + q_2(z,w) + q_3(z,w) + \cdots) \end{aligned} \tag{1.1}$$

if $\lambda \neq 0$ and as

$$\begin{aligned} z_1 &= w + p_2(z,w) + p_3(z,w) + \cdots,\\ w_1 &= q_2(z,w) + q_3(z,w) + \cdots \end{aligned} \tag{1.2}$$

if $\lambda = 0$. Here $p_i(z,w)$ and $q_i(z,w)$ are homogeneous polynomials of degree i.

If $|\lambda| \neq 1$ then we say that f has a *hyperbolic* Jordan fixed point. If $|\lambda| = 1$ and λ is not a root of unity, then we say that f has an *elliptic* Jordan fixed point. If λ is a root of unity, then we say that f has a *parabolic* Jordan fixed point and we can consider a suitable iteration of f instead. Thus we will assume that $\lambda = 1$ in the parabolic case.

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In analogy to the one-dimensional case, many authors have studied the existence of "parabolic curves" in higher dimensions (see Section 2 for the definition). Our detailed study of the local dynamics of holomorphic maps in $\mathbf{C}^2$ with a Jordan fixed point can be summarized as follows.

THEOREM 1.1. *Let f be a holomorphic map in $\mathbf{C}^2$ with a Jordan fixed point. Then the following statements hold.*

(1) *If f has an elliptic Jordan fixed point, then f has no parabolic curves.*
(2) *If f has a nonisolated parabolic Jordan fixed point, then f may or may not have parabolic curves and f has no attracting domains.*
(3) *If f has an isolated parabolic Jordan fixed point, then f always has parabolic curves but f may or may not have attracting domains.*
(4) *If f has a hyperbolic Jordan fixed point, then f or f^{-1} always has an attracting domain.*

In Section 2 we recall some basic definitions and results in local holomorphic dynamics. The hyperbolic case is quite easy and is dealt with in Section 6. The elliptic case is taken care of in Section 5. The parabolic case is quite involved; we study that case in Sections 3 and 4, where we also make more precise the statements (2) and (3) in Theorem 1.1.

REMARK 1.2. If $\lambda \neq 0$, then the iteration of f reads as

$$z_n = \lambda^n(z + nw + \tilde{p}_2(z,w) + \tilde{p}_3(z,w) + \cdots),$$
$$w_n = \lambda^n(w + \tilde{q}_2(z,w) + \tilde{q}_3(z,w) + \cdots).$$

From the linear part it is easy to see that if (z_n, w_n) goes to $(0,0)$ then $[z_n : w_n]$ goes to $[1:0]$. Thus the "attracting" dynamics of f is concentrated in the direction $[z:w] = [1:0]$. In particular, this justifies blowing up f in the direction $[1:0]$ in the nonhyperbolic case.

2. Preliminaries

Let $f(z,w) = (f_1(z,w), f_2(z,w))$ be a holomorphic map tangent to the identity at the origin O; that is, $f_1(z,w) = z + p_2(z,w) + \cdots$ and $f_2(z,w) = w + q_2(z,w) + \cdots$. The *order* ν of f at O is by definition the minimum of i such that either $p_i(z,w) \not\equiv 0$ or $q_i(z,w) \not\equiv 0$. A direction $[z:w]$ is called a *characteristic direction* of f if there exists a $\lambda \in \mathbf{C}$ such that $p_\nu(z,w) = \lambda z$ and $q_\nu(z,w) = \lambda w$. If $\lambda \neq 0$ then $[z:w]$ is said to be *nondegenerate,* and otherwise it is said to be *degenerate*. If $p_\nu(z,w) = zr(z,w)$ and $q_\nu(z,w) = wr(z,w)$ for some $r(z,w)$ then f is said to be *dicritical* at O, and otherwise it is said to be *nondicritical.*

A *parabolic curve* for f at O is the image of an injective analytic disc $\varphi\colon \Delta \to \mathbf{C}^2$ (where Δ is the unit disc in $\mathbf{C}$) such that φ is continuous up to the boundary of Δ, $O = \varphi(1)$, $f(\varphi(\Delta)) \subset \varphi(\Delta)$, and for any $p \in \varphi(\Delta)$ we have $\lim_{n\to\infty} f^n(p) = O$. Moreover, φ is said to be *tangent to a direction* $[v] \in \mathbf{P}^1$ at O if $[\varphi(t)] \to [v]$ for $t \to 1$. Here $[\cdot]$ denotes the canonical projection of $\mathbf{C}^2$ onto $\mathbf{P}^1$.

THEOREM 2.1 [9, Thm. 1.3]. *Let f be a holomorphic map in $\mathbf{C}^2$ tangent to the identity at O of order ν. If $[v]$ is a nondegenerate characteristic direction of f, then there exist at least $\nu - 1$ parabolic curves at O that are tangent to the direction $[v]$.*

Assume that f has a nondegenerate characteristic direction $[v]$. After a linear transformation, we can assume that $[v] = [1 : 0]$ and f can be written as

$$z_1 = z + P_\nu(z, w) + O(\nu + 1),$$
$$w_1 = w + Q_\nu(z, w) + O(\nu + 1)$$

with $[P(1,0) : Q(1,0)] = \lambda[1 : 0]$ for some $\lambda \neq 0$. Under the blow-up $\{z = u, w = uv\}$, the blow-up map is given by

$$u_1 = u + u^\nu P_\nu(1, v) + O(u^{\nu+1}),$$
$$v_1 = v + u^{\nu-1} R(v) + O(u^\nu),$$

where $R(v) = Q_\nu(1, v) - v P_\nu(1, v)$. The *director* of f in the direction $[v]$ is defined to be $R'(0)/P_\nu(1,0)$, which is an invariant (cf. [9, Prop. 2.4]).

Write $R(v) = \sum_{i=1}^{\nu+1} a_i v^i$. Note that $a_i = 0$ for all $1 \leq i \leq \nu + 1$ if and only if f is dicritical at O. If f is nondicritical at O, then the *nondicritical order* of f at O is defined to be

$$\mu := \min\{i - 1 : a_i \neq 0, 1 \leq i \leq \nu + 1\},$$

which is also an invariant (cf. [13, Lemma 2.1]).

THEOREM 2.2 [9, Thm. 5.1; 13, Thm. 1.5]. *Let f be a holomorphic map in $\mathbf{C}^2$ tangent to the identity at O, let $[v]$ be a nondegenerate characteristic direction of f, and let α be the director in the direction $[v]$. Then f admits an attracting domain at O tangent to the direction $[v]$ if and only if* $\operatorname{Re}\alpha > 0$ *or* $\mu \geq 1$ (*in which case* $\alpha = 0$).

3. Nonisolated Parabolic Jordan Fixed Point

If f has a nonisolated parabolic Jordan fixed point, then under the normal form (1.1) the curve of fixed points is given by $\{w + g(z, w) = 0\}$, where $g(z, w) = O(2)$, and f can be written as

$$z_1 = z + (w + g(z, w))(1 + p(z, w)),$$
$$w_1 = w + (w + g(z, w))q(z, w).$$

Note that the curve of fixed points is nonsingular. Under the transformation $w \to w + g(z, w)$, the curve of fixed points becomes $\{w = 0\}$ and f can be written in the following normal form:

$$\begin{aligned} z_1 &= z + w(1 + P(z, w)), \\ w_1 &= w + wQ(z, w). \end{aligned} \tag{3.1}$$

Under the blow-up $\{z = u, w = uv\}$, the blow-up map $\tilde{f}$ is given by

$$\begin{aligned} u_1 &= u + uv(1 + u\tilde{P}(u, v)), \\ v_1 &= v - v^2 + uv(\tilde{Q}(u, v) - v\tilde{R}(u, v)), \end{aligned} \tag{3.2}$$

where $\tilde{P}(u, v) = P(u, uv)/u$, $\tilde{Q}(u, v) = Q(u, uv)/u$, and $\tilde{R}(u, v) = \tilde{P}(u, v) + \tilde{Q}(u, v) + u\tilde{P}(u, v)\tilde{Q}(u, v)$.

Let $Q(z, w) = q_1(z, w) + q_2(z, w) + \cdots$ be the homogeneous expansion of $Q(z, w)$ and write $q_i(z, w) = \alpha_i z^i + \cdots$, $i \geq 1$. If $\alpha_i = 0$ for $i < k$ and $\alpha_k \neq 0$ for some $k \geq 1$, then we can blow up k times and the blow-up map $\tilde{f}$ takes the form

$$\begin{aligned} u_1 &= u + u^{k-1}v(u + O(2)), \\ v_1 &= v + u^{k-1}v(-kv + \alpha_k u + O(2)). \end{aligned} \tag{3.3}$$

Equations (3.3) have one admissible characteristic direction $[(k + 1) : \alpha_k]$, which is nondegenerate. Thus (by Theorem 2.1) f admits a parabolic curve if $\alpha_k \neq 0$ for some k, which is the case if and only if $w \nmid Q(z, w)$.

REMARK 3.1. If $w \mid Q(z, w)$ then we blow up only once and the blow-up map $\tilde{f}$ can be written as

$$\begin{aligned} u_1 &= u + vu(1 + O(u)), \\ v_1 &= v + v^2(-1 + O(u)). \end{aligned} \tag{3.4}$$

Thus $\tilde{f}$ is *tangential* along $S := \{v = 0\}$ (cf. [5]). Obviously $\tilde{f}$ admits a parabolic curve contained in the exceptional divisor $\{u = 0\}$, which is the parabolic curve predicted by [5, Prop. 7.7] (since the "residual index" $\operatorname{Ind}(\tilde{f}, S, O)$ is -1). However, such a parabolic curve is not a parabolic curve for f.

In (3.1), write $P(z, w) = p(z) + wP_1(z, w)$ and $Q(z, w) = q(z) + wQ_1(z, w)$. Then, under the transformation $z \to z - z_0$, f becomes

$$\begin{aligned} z_1 &= z + w(1 + p(z_0) + O(1)), \\ w_1 &= w(1 + q(z_0) + O(1)). \end{aligned}$$

Thus $(z_0, 0)$ is also a parabolic Jordan fixed point of f for all z_0 near 0 if and only if $q(z) \equiv 0$—that is, $w \mid Q(z, w)$ or in other words $\tilde{f}$ is tangential along S.

If $w \nmid Q(z, w)$ then we can write $\tilde{f}$ as in (3.3) and so it is *nontangential* along S. By the previous paragraph, a result similar to [5, Prop. 7.8] does not even make sense in our case. This is essentially due to the fact that in [5] a map is nontangential along S if and only if $\mu = \nu \geq 2$ (cf. [5] for the notation), whereas in our case we have $\mu = \nu = 1$.

Assume that $\alpha_i = 0$ for $i < k$ and that $\alpha_k \neq 0$ for some $k \geq 1$. Then one readily checks that the director of $\tilde{f}$ in the direction $[(k+1) : \alpha_k]$ is $-(k+1)$. Therefore, by Theorem 2.2, $\tilde{f}$ does not admit attracting domains in the direction $[(k + 1) : \alpha_k]$; hence f has no attracting domains tangent to the direction $[1 : 0]$ with finite order. (Here "finite order" means that, after finitely many blow-ups, the strict transform of an attracting domain or a parabolic curve is no longer tangent to the direction $[1 : 0]$.)

Assume now that $\alpha_i = 0$ for all $i \geq 1$. Then $\tilde{f}$ can be written as (3.4). Assume that there exists a parabolic curve or an attracting domain tangent to the direction $[1:0]$ with infinite order. For (u_n, v_n) in the parabolic curve or the attracting domain, we have $v_n = o(u_n)$ and $u_n = o(1)$; hence the dynamics of $\tilde{f}$ is essentially the same as

$$u_1 = u + vu,$$
$$v_1 = v - v^2.$$

Thus we have $|v_n| \sim 1/n$, but then $|u_{n+1}| \sim |u_n||1 + 1/n| > |u_n|$, a contradiction.

In short, we have the following result.

Proposition 3.2. *Let f be a holomorphic map in $\mathbf{C}^2$ with a nonisolated parabolic Jordan fixed point. Then the following statements hold.*

(1) *The curve of fixed points S is always nonsingular, and f can be written in the normal form* (3.1).
(2) *Let $\tilde{f}$ be the blow-up map and denote the strict transform of S by $\tilde{S}$. Then S is a curve of parabolic Jordan fixed points of f if and only if $\tilde{f}$ is tangential along $\tilde{S}$.*
(3) *If $\tilde{f}$ is tangential along $\tilde{S}$, then f has no parabolic curves*; *if $\tilde{f}$ is nontangential along $\tilde{S}$, then f has a parabolic curve. In either case, f has no attracting domains.*

4. Isolated Parabolic Jordan Fixed Point

The first part of Theorem 1.1(3) is exactly [2, Cor. 3.2], so in this section we will focus on the study of attracting domains. In particular, we recover the attracting domain found by Abate in [3] (see Remark 4.6).

Let f be a holomorphic map in $\mathbf{C}^2$ with a parabolic Jordan fixed point. In suitable local coordinates (z, w), f can be written as

$$\begin{aligned} z_1 &= z + w + p_2(z,w) + p_3(z,w) + \cdots, \\ w_1 &= w + q_2(z,w) + q_3(z,w) + \cdots . \end{aligned} \tag{4.1}$$

Write $p_2(z,w) = a_{11}z^2 + a_{12}zw + a_{22}w^2$, $q_2(z,w) = b_{11}z^2 + b_{12}zw + b_{22}w^2$, and

$$\begin{aligned} p_i(z,w) &= \alpha_i z^i + \delta_i z^{i-1}w + \eta_i z^{i-2}w^2 + \cdots, \\ q_i(z,w) &= \gamma_i z^i + \beta_i z^{i-1}w + \rho_i z^{i-2}w^2 + \sigma_i z^{i-3}w^3 + \cdots \end{aligned} \tag{4.2}$$

for $i \geq 3$.

Under the blow-up $\{z = u, w = uv\}$, the blow-up map is given by

$$\begin{aligned} u_1 &= u + uv + u^2 p_2(1,v) + O(u^3), \\ v_1 &= v - v^2 + u r_2(v) + u^2 r_3(v) + O(u^3), \end{aligned} \tag{4.3}$$

where $r_i(v) = (1-v)q_i(1,v) - v p_i(1,v)$ for $i \geq 2$.

If $b_{11} \neq 0$ then, under an additional blow-up $\{u = st, v = s\}$ and the scaling $t \to t/b_{11}$, the blow-up map is given by

$$\begin{aligned} s_1 &= s - s^2 + st + O(3), \\ t_1 &= t - t^2 + 2st + O(3). \end{aligned} \tag{4.4}$$

Besides $[1:0]$ and $[0:1]$, (4.4) has one more characteristic direction, $[2:3]$, which is nondegenerate. One readily checks that the director of (4.4) in the direction $[2:3]$ is -6. Therefore, by Theorem 2.2, there is no attracting domain along the only admissible direction $[2:3]$ and so (4.4) does not admit attracting domains.

From now on, we assume that $b_{11} = 0$. Then (4.3) is a map tangent to the identity of the form

$$\begin{aligned} u_1 &= u + uv + a_{11}u^2 + O(3), \\ v_1 &= v - v^2 + (b_{12} - a_{11})uv + \gamma_3 u^2 + O(3). \end{aligned} \tag{4.5}$$

Case $a_{11} = 0$
In this case, (4.5) takes the form

$$\begin{aligned} u_1 &= u + uv + O(3), \\ v_1 &= v - v^2 + b_{12}uv + \gamma_3 u^2 + O(3). \end{aligned} \tag{4.6}$$

Subcase $b_{12} = \gamma_3 = 0$
In this subcase, (4.6) takes the form

$$\begin{aligned} u_1 &= u + uv + O(3), \\ v_1 &= v - v^2 + O(3). \end{aligned} \tag{4.7}$$

Besides $[1:0]$ and $[0:1]$, (4.7) has no other characteristic directions.

The only admissible direction $[1:0]$ is degenerate, and our next result is obvious.

LEMMA 4.1. (4.1) *admits an attracting domain when* $a_{11} = b_{11} = b_{12} = \gamma_3 = 0$ *if and only if* (4.7) *admits an attracting domain in the direction* $[1:0]$.

The following result shows that the statement of Lemma 4.1 is not empty.

LEMMA 4.2. *Write* (4.7) *as*

$$\begin{aligned} u_1 &= u + uv - p(u) + vO(2), \\ v_1 &= v - v(v + q(u)) + v^2 O(1) + O(u^k), \end{aligned} \tag{4.8}$$

where $p(u) = \alpha u^m + O(u^{m+1})$ *for* $m \geq 3$, $q(u) = \beta u^{m-1} + O(u^m)$, *and* $k \geq 2m$. *If* $\operatorname{Re}\alpha > 0$ *and* $\operatorname{Re}(\beta - (m-1)\alpha) > 0$, *then* (4.8) *admits an attracting domain tangent to the direction* $[1:0]$.

Proof. Blowing up (4.8) $m-1$ times yields

$$\begin{aligned} s_1 &= s - \alpha s^m + O(m+1), \\ t_1 &= t - (\beta - (m-1)\alpha)s^{m-1}t + O(m+1). \end{aligned} \tag{4.9}$$

The lemma then follows from Theorem 2.2. □

Subcase $b_{12} \neq 0$, $\gamma_3 = 0$
In this subcase, (4.6) takes the form (after scaling $u \to u/b_{12}$)

$$\begin{aligned} u_1 &= u + uv + O(3), \\ v_1 &= v - v^2 + uv + O(3). \end{aligned} \tag{4.10}$$

Besides $[1:0]$ and $[0:1]$, (4.10) has one more characteristic direction, $[2:1]$, which is nondegenerate. One readily checks that the director of (4.10) in the direction $[2:1]$ is -2. Therefore, by Theorem 2.2, there is no attracting domain along $[2:1]$.

The other admissible direction, $[1:0]$, is degenerate. Under the linear transformation $\{x = v - u,\ y = v\}$, (4.10) is transformed as

$$\begin{aligned} x_1 &= x - y^2 + O(3), \\ y_1 &= y - xy + O(3). \end{aligned} \tag{4.11}$$

Our next statement follows from the preceding paragraph.

LEMMA 4.3. (4.1) *admits an attracting domain when* $a_{11} = b_{11} = \gamma_3 = 0$ *and* $b_{12} \neq 0$ *if and only if* (4.10), *or equivalently* (4.11), *admits an attracting domain in the direction* $[1:0]$.

The next result shows that the statement of Lemma 3 is not empty.

LEMMA 4.4. *Write* (4.11) *as*

$$\begin{aligned} x_1 &= x - y^2 - p(x) + yO(m), \\ y_1 &= y - xy - q(x) + yO(m), \end{aligned} \tag{4.12}$$

where $p(x) = \alpha x^{m+1} + O(x^{m+1})$ *and* $q(x) = \beta x^{m+1} + O(x^{m+2})$, $m \geq 2$. *If* $\operatorname{Re} \alpha > 0$, *then* (4.12) *admits an attracting domain tangent to the direction* $[1:0]$.

Proof. By scaling $x \to x/a$ with $a = \sqrt[m]{m\alpha}$, (4.12) becomes

$$\begin{aligned} x_1 &= x - y^2 - \frac{1}{m}x^{m+1} + O(x^{m+2}) + yO(m), \\ y_1 &= y - \frac{1}{a}xy - \frac{\beta}{a^{m+1}}x^{m+1} + O(x^{m+2}) + yO(m). \end{aligned} \tag{4.13}$$

For $0 < \delta \ll \varepsilon$ small enough, set $V_{\varepsilon,\delta} = \{t \in \mathbf{C} : 0 < |t| < \varepsilon, |\arg t| < \delta\}$. Denote by D the open set $\{(x, y) \in \mathbf{C}^2 : x \in V_{\varepsilon,\delta}, |y| < |x|^{(m+1)/2}\}$. We first show that $\tilde{f}(D) \subset D$, where $\tilde{f}$ is as in (4.13).

From (4.13) we have

$$x_1^{m+1} = x^{m+1}\left(1 - \frac{m+1}{m}x^m - (m+1)\frac{y^2}{x} + O(x^{m+1}) + \frac{y}{x}O(m)\right) \tag{4.14}$$

and

$$y_1^2 = y^2\left(1 - \frac{2}{a}x - \frac{2\beta}{a^{m+1}}\frac{x^{m+1}}{y} + \frac{1}{y}O(x^{m+2}) + O(m)\right). \tag{4.15}$$

For $(x, y) \in D$ we can write $|y| = |x|^\gamma$ for some $\gamma = \gamma(x, y) > (m+1)/2$. If $\gamma < m$ then $x^{m+1} = o(xy)$ and so it follows from (4.14), (4.15), and $\operatorname{Re} a > 0$ that

$$\frac{|y_1|^2}{|x_1|^{m+1}} = \frac{|y|^2}{|x|^{m+1}}\left|1 - \frac{2}{a}x + \frac{m+1}{m}x^m + o(x^m)\right| < \frac{|y|^2}{|x|^{m+1}} < 1.$$

If $|y| = |x|^\gamma$ for some $m+1 > \gamma \geq m$, then from (4.14) and (4.15) we obtain

$$\frac{|y_1|^2}{|x_1|^{m+1}} = \frac{|y|^2}{|x|^{m+1}}|1 + o(1)| = |x|^{2\gamma-m-1}|1 + o(1)| < 1.$$

If $|y| = |x|^\gamma$ for some $\gamma \geq m+1$ then, by (4.14) and (4.15),

$$\frac{|y_1|^2}{|x_1|^{m+1}} = |O(y)||1 + o(1)| < 1.$$

Write $x = \varepsilon(x)e^{i\delta(x)}$ with $0 < \varepsilon(x) < \varepsilon$ and $|\delta(x)| < \delta$. Put

$$z = 1 - x^m\left(\frac{1}{m} + o(1)\right).$$

Then it is easy to see that $|z| < 1$ and that $\arg z$ and $\delta(x)$ are of different signs, where $|\arg z| < |\delta(x)|$. For $(x, y) \in D$, from (4.13) we have

$$|x_1| = \varepsilon(x)|z| < \varepsilon(x) < \varepsilon, \qquad |\arg x_1| = |\delta(x) + \arg z| < |\delta(x)| < \delta.$$

Thus we have shown that $\tilde{f}(D) \subset D$.

By (4.13),

$$\frac{1}{x_1^m} = \frac{1}{x^m} + 1 + m\frac{y^2}{x^{m+1}} + O(x) + \frac{y}{x^{m+1}}O(m),$$

which for $(x, y) \in D$ yields

$$x_n \sim \frac{1}{n^{1/m}}. \tag{4.16}$$

For the estimation of $|y_n|$, we rewrite y_1 as

$$y_1 = y\left(1 - \frac{1}{a}x + O(m)\right) + dx^s + O(x^{s+1}), \tag{4.17}$$

where $s = m+1$ and $d = -\beta/a^{m+1}$ if $\beta \neq 0$ (otherwise $s \geq m+2$). Set $c := \mathrm{Re}(1/a)$.

Set $b_k = 1 - (1/a)x_k + O(m)$. Then we have

$$y_n = y\prod_{k=0}^{n-1} b_k + d\sum_{l=0}^{n-1} x_l^s \prod_{j=l+1}^{n-1} b_j + \text{higher-order terms}. \tag{4.18}$$

For $(x, y) \in D$, we have

$$\prod_{j=l+1}^{n-1} b_j = \exp\left\{\sum_{j=l+1}^{n-1} \log b_j\right\} \sim \exp\left\{-\frac{1}{a}\sum_{j=l+1}^{n-1} x_j\right\}. \tag{4.19}$$

From (4.16) and (4.19) it follows that

$$\prod_{j=l+1}^{n-1} |b_j| \sim \exp\left\{-c\sum_{j=l+1}^{n-1} j^{-1/m}\right\} \sim \exp\{-c(n^{1/m} - l^{1/m})\}. \tag{4.20}$$

Therefore

$$\sum_{l=0}^{n-1} |x_l|^s \prod_{j=l+1}^{n-1} |b_j| \sim \sum_{l=1}^{n-1} l^{-s/m} \exp\{-c(n^{1/m} - l^{1/m})\}$$
$$= \exp\{-cn^{1/m}\} \sum_{l=1}^{n-1} l^{-s/m} \exp\{cl^{1/m}\}. \qquad (4.21)$$

For n large, we have

$$\sum_{l=1}^{n-1} l^{-s/m} \exp\{cl^{1/m}\} \sim \int_1^n t^{-s/m} \exp\{ct^{1/m}\}\, dt \sim \exp\{cn^{1/m}\} n^{-(s-1)/m}. \qquad (4.22)$$

From (4.18), (4.20), (4.21), and (4.22), we finally obtain the estimate

$$|y_n| \sim \frac{1}{n^{(s-1)/m}} \quad (s < \infty); \qquad |y_n| \sim \exp\{-cn^{(m-1)/m}\} \quad (s = \infty). \qquad (4.23)$$

Since $s \geq m+1$, from (4.16) and (4.23) we have $[x_n : y_n] \to [1 : 0]$ as $n \to \infty$. □

Subcase $\gamma_3 \neq 0$

In this subcase, (4.6) takes the form $\big($after scaling $u \to u/\sqrt{2\gamma_3}\,\big)$

$$\begin{aligned} u_1 &= u + uv + O(3), \\ v_1 &= v - v^2 + cuv + \tfrac{1}{2}u^2 + O(3), \end{aligned} \qquad (4.24)$$

where $c = b_{12}/\sqrt{2\gamma_3}$. Besides $[0 : 1]$, (4.24) has two more characteristic directions, $[c_\pm : 1]$, which are nondegenerate; here $c_\pm = -c \pm \sqrt{c^2+4}$. One readily checks that the director of (4.24) in the direction $[c_\pm : 1]$ is $cc_\pm - 4$. If $cc_\pm - 4 = 0$, then it is easy to check that c is equal to $\pm 2i$ or $\pm\sqrt{3}i$ and that the nondicritical order is 1. Therefore, by Theorem 2.2, there is an attracting domain along the admissible direction $[c_\pm : 1]$ if and only if $\mathrm{Re}(cc_\pm - 4) > 0$ or c is equal to $\pm 2i$ or $\pm\sqrt{3}i$. In particular, if $b_{12} = 0$ then $cc_\pm - 4 = -4$ and so (4.24) does not admit attracting domains.

Case $a_{11} \neq 0$

In this case, (4.5) takes the form (after scaling $u \to u/a_{11}$)

$$\begin{aligned} u_1 &= u + u^2 + uv + O(3), \\ v_1 &= v - v^2 + auv + bu^2 + O(3), \end{aligned} \qquad (4.25)$$

where $a = b_{12}/a_{11} - 1$ and $b = \gamma_3/a_{11}^2$.

Subcase $\gamma_3 = 0$

In this subcase, (4.25) takes the form

$$\begin{aligned} u_1 &= u + u^2 + uv + O(3), \\ v_1 &= v - v^2 + auv + O(3). \end{aligned} \qquad (4.26)$$

Besides $[1 : 0]$ and $[0 : 1]$, (4.26) has one more characteristic direction, $[2 : a - 1]$ for $a \neq 1$, which is nondegenerate if $a \neq -1$ and is degenerate if $a = -1$.

The admissible direction $[1:0]$ is nondegenerate. One readily checks that the director of (4.26) in the direction $[1:0]$ is $a-1$ and that when $a=1$, the nondicritical order is 1. Hence by Theorem 2.2 there is an attracting domain along $[1:0]$ if and only if $\operatorname{Re}(a-1)>0$ or $a=1$.

So assume for now that $a\neq 1$. Under the linear transformation $\left\{x=u+\frac{2}{1-a}v,\ y=v\right\}$, (4.26) is transformed as

$$\begin{aligned} x_1 &= x+x^2+\tfrac{3-a}{a-1}xy+O(3),\\ y_1 &= y+\tfrac{a+1}{a-1}y^2+axy+O(3). \end{aligned} \tag{4.27}$$

If $a\neq -1$, then the director of (4.26) in the direction $[2:a-1]$ is $\frac{2-2a}{a+1}$. Therefore, by Theorem 2.2, there is an attracting domain along the admissible direction $[2:a-1]$ if and only if $\operatorname{Re}\left(\frac{2-2a}{a+1}\right)>0$. In particular, if $b_{12}=a_{11}$ then $\frac{2-2a}{a+1}=2$ and so (4.26) admits an attracting domain.

If $a=-1$, then (4.27) takes the form

$$\begin{aligned} x_1 &= x+x^2-2xy+O(3),\\ y_1 &= y-xy+O(3). \end{aligned} \tag{4.28}$$

Note that, under the linear transformation $\{u=-2y,\ v=-x\}$, (4.28) becomes (4.10). Our next lemma is a consequence of the foregoing discussion.

LEMMA 4.5. (4.1) *admits an attracting domain when* $b_{11}=b_{12}=\gamma_3=0$ *and* $a_{11}\neq 0$ *if and only if* (4.10), *or equivalently* (4.11), *admits an attracting domain in the direction* $[1:0]$.

Following the successive transformations from (4.1) to (4.11), the corresponding (4.13) takes the form

$$\begin{aligned} x_1 &= x-y^2-d/8x^3+O(x^4)+yO(2),\\ y_1 &= y-xy+e/8x^3+O(x^4)+yO(2), \end{aligned}$$

where

$$d=\frac{b_{22}}{a_{11}}-\frac{\alpha_3+\beta_3}{a_{11}^2}+\frac{\gamma_4}{a_{11}^3},\qquad e=\frac{b_{22}-2a_{12}}{a_{11}}+\frac{\alpha_3-\beta_3}{a_{11}^2}+\frac{\gamma_4}{a_{11}^3}. \tag{4.29}$$

By Lemma 4.4, f admits an attracting domain in this subcase if $\operatorname{Re} d>0$.

REMARK 4.6. In [3], Abate found an attracting domain for the map

$$\begin{aligned} z_1 &= z+w+\alpha z^2+\beta w^2,\\ w_1 &= w+w^2, \end{aligned} \tag{4.30}$$

under the condition $\operatorname{Re}\alpha>0$. Observe first that (4.30) is an instance of the current subcase, since $b_{11}=\gamma_3=0$ and $a_{11}=\alpha\neq 0$. Second, in (4.30), $\alpha_3=\beta_3=\gamma_4=0$ and $b_{22}=1$. Thus the criterion we obtained previously becomes $\operatorname{Re}(1/\alpha)>0$, which is equivalent to $\operatorname{Re}\alpha>0$. A more detailed analysis shows that the attracting domain we obtain is indeed the same as Abate's.

Subcase $\gamma_3 \neq 0$

Assume first that $\gamma_3 = a_{11}b_{12}$ (i.e., $b = a + 1$). Then besides $[0 : 1]$, (4.25) has one more characteristic direction, $[2 : b]$, which is nondegenerate if $b \neq -2$ and is degenerate if $b = -2$.

Under the linear transformation $\{x = u - (2/b)v,\ y = v\}$, (4.25) is transformed as

$$\begin{aligned} x_1 &= x - x^2 - \tfrac{b+2}{b}xy + O(3), \\ y_1 &= y + \tfrac{b+2}{b}y^2 + (b+3)xy + bx^2 + O(3). \end{aligned} \tag{4.31}$$

If $b \neq -2$, then the director of (4.25) in the direction $[2 : b]$ is -2. Therefore, by Theorem 2.2, there is no attracting domain along the only admissible direction $[2 : b]$ and so (4.25) does not admit attracting domains.

If $b = -2$, then (4.31) takes the form

$$\begin{aligned} x_1 &= x - x^2 + O(3), \\ y_1 &= y + xy - 2x^2 + O(3). \end{aligned} \tag{4.32}$$

Note that, under the linear transformation $\{u = y - x,\ v = x\}$, (4.32) becomes (4.7). As a consequence we have the following result.

Lemma 4.7. (4.1) *admits an attracting domain when* $b_{11} = 0$, $a_{11} \neq 0$, $b_{12} = -2a_{11}$, *and* $\gamma_3 = -2a_{11}$ *if and only if* (4.7) *admits an attracting domain in the direction* $[1 : 0]$.

Assume now that $\gamma_3 \neq a_{11}b_{12}$ (i.e., $b \neq a+1$). Then besides $[0 : 1]$, (4.25) has two more characteristic directions, $[e_\pm : d_\pm]$, which are nondegenerate. Here $e_\pm = -(a+3) \pm \sqrt{(a-1)^2 + 8b}$ and $d_\pm = 2(b-a-1) - e_\pm$. One readily checks that the director of (4.25) in the direction $[e_\pm : d_\pm]$ is $-4 + \frac{(a+3)e_\pm}{2(b-a-1)}$. If $-4 + \frac{(a+3)e_\pm}{2(b-a-1)} = 0$ then the nondicritical order is either 1 or 2. Hence, by Theorem 2.2, there is an attracting domain along $[e_\pm : d_\pm]$ if and only if $\mathrm{Re}\bigl(-4 + \frac{(a+3)e_\pm}{2(b-a-1)}\bigr) > 0$ or $-4 + \frac{(a+3)e_\pm}{2(b-a-1)} = 0$.

The previous discussion shows that, in degenerate directions, the map can always be transformed into either (4.7) or (4.11). If a map can have attracting domains only in degenerate directions, then we say the map is of *degenerate type I* (resp. *II*) if the map in that direction can be transformed into (4.7) (resp. (4.11)).

We summarize this section in the following formal proposition.

Proposition 4.8. *Let* f *be a holomorphic map in* $\mathbf{C}^2$ *with an isolated parabolic Jordan fixed point as in* (4.1). *Set* $a = b_{12}/a_{11} - 1$, $b = \gamma_3/a_{11}^2$, $c = b_{12}/\sqrt{2\gamma_3}$, $d = b_{22}/a_{11} - (\alpha_3 + \beta_3)/a_{11}^2 + \gamma_4/a_{11}^3$, $c_\pm = -c \pm \sqrt{c^2 + 4}$, *and* $e_\pm = -(a+3) \pm \sqrt{(a-1)^2 + 8b}$. *Then the following statements hold.*

(1) f *has no attracting domain in case either* $b_{11} \neq 0$ *or* $b_{11} = 0$, $a_{11} \neq 0$, $\gamma_3 \neq 0$, $b = a + 1$, *and* $b \neq -2$.
(2) f *has attracting domains in case* $b_{11} = a_{11} = \gamma_3 = 0$ *and* $a = 1$.
(3) f *is of degenerate type I in case either* $b_{11} = a_{11} = \gamma_3 = b_{12} = 0$ *or* $b_{11} = 0$, $a_{11} \neq 0$, $\gamma_3 \neq 0$, $a = -3$, *and* $b = -2$.

(4) *f is of degenerate type II in case either $b_{11} = a_{11} = \gamma_3 = 0$ and $b_{12} \neq 0$ or $b_{11} = \gamma_3 = b_{12} = 0$ and $a_{11} \neq 0$; in the latter case, f has attracting domains if* $\operatorname{Re} d > 0$.

(5) *f has attracting domains in case $b_{11} = a_{11} = 0$ and $\gamma_3 \neq 0$ if and only if* $\operatorname{Re}(cc_\pm - 4) > 0$ *or c is equal to $\pm 2i$ or $\pm\sqrt{3}i$.*

(6) *f has attracting domains in case $b_{11} = \gamma_3 = 0$, $a_{11} \neq 0$, and $a \neq \pm 1$ if and only if* $\operatorname{Re}(a-1) > 0$ *or* $\operatorname{Re}\left(\frac{2-2a}{a+1}\right) > 0$.

(7) *f has attracting domains in case $b_{11} = 0$, $a_{11} \neq 0$, $\gamma_3 \neq 0$, and $b \neq a+1$ if and only if* $\operatorname{Re}\left(-4 + \frac{(a+3)e_\pm}{2(b-a-1)}\right) > 0$ *or* $-4 + \frac{(a+3)e_\pm}{2(b-a-1)} = 0$.

5. Elliptic Jordan Fixed Point

Let f be a holomorphic map in $\mathbf{C}^2$ with an elliptic Jordan fixed point, where the eigenvalue is $\lambda = e^{i2\pi\theta}$, $\theta \in \mathbf{R}\backslash\mathbf{Q}$. In suitable local coordinates (z, w), f can be written as

$$\begin{aligned} z_1 &= \lambda(z + w + a_{11}z^2 + a_{12}zw + a_{22}w^2 + O(3)), \\ w_1 &= \lambda(w + b_{11}z^2 + b_{12}zw + b_{22}w^2 + O(3)). \end{aligned} \tag{5.1}$$

Under the blow-up $\{z = u,\ w = uv\}$, the blow-up map is given by

$$\begin{aligned} u_1 &= \lambda(u + uv + O(u^2)), \\ v_1 &= v - v^2 + b_{11}u + O(uv, u^2). \end{aligned} \tag{5.2}$$

Observe that (5.2) is a quasi-parabolic map in the direction $[u : v] = [0 : 1]$. The local dynamics of such maps has been studied in [7; 10; 11; 12]. Since the direction $[0 : 1]$ is not admissible and since $[1 : 0]$ is an elliptic direction, we need to consider the parabolic dynamics in a direction $[\alpha : 1]$ with $\alpha \neq 0$.

Consider an invertible linear transformation $\{x = v + au,\ y = bv + cu\}$ with $c - ab \neq 0$. Then (5.2) becomes

$$\begin{aligned} x_1 &= \tfrac{1}{c-ab}\big((c - b(a\lambda + b_{11}))x + (a(\lambda - 1) + b_{11})y + O(2)\big), \\ y_1 &= \tfrac{1}{c-ab}\big(b(c(1-\lambda) - bb_{11})x + (c\lambda - ab)y + O(2)\big). \end{aligned} \tag{5.3}$$

In order for the linear part of (5.3) to be diagonal (so that we can apply the quasi-parabolic theory or perform blow-ups), it is necessary that

$$\begin{aligned} a(\lambda - 1) + b_{11} &= 0, \\ c(1 - \lambda) - bb_{11} &= 0, \end{aligned}$$

from which we get $(c - ab)(\lambda - 1) = 0$. Since $\lambda \neq 1$, we must have $c - ab = 0$, a contradiction.

6. Hyperbolic Jordan Fixed Point

Let f be a holomorphic map in $\mathbf{C}^2$ with a hyperbolic Jordan fixed point, where the eigenvalue is λ ($|\lambda| \neq 1$); if $|\lambda| > 1$ then we can consider f^{-1} instead. So we will assume that $|\lambda| < 1$.

If $\lambda \neq 0$ then, in suitable local coordinates (z, w), f can be written as

$$\begin{aligned} z_1 &= \lambda(z + w + O(2)), \\ w_1 &= \lambda(w + O(2)). \end{aligned} \tag{6.1}$$

The nth iteration of f reads as

$$\begin{aligned} z_n &= \lambda^n(z + nw + O(2)), \\ w_n &= \lambda^n(w + O(2)). \end{aligned} \tag{6.2}$$

Since $|\lambda| < 1$, it follows that $n\lambda^n$ goes to zero as n goes to infinity; hence (z_n, w_n) goes to O as n goes to infinity for any (z, w) near O.

If $\lambda = 0$ then, in suitable local coordinates (z, w), f can be written as

$$\begin{aligned} z_1 &= w + O(2), \\ w_1 &= O(2). \end{aligned} \tag{6.3}$$

The nth iteration of f reads as

$$\begin{aligned} z_n &= O(2^{n-1}), \\ w_n &= O(2^{n-1}). \end{aligned} \tag{6.4}$$

Therefore, (z_n, w_n) goes to O as n goes to infinity for any (z, w) near O.

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