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Proximality and Pure Point Spectrum for Tiling Dynamical Systems

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1. Introduction

In order to understand the combinatorial properties of a single tiling T of Euclidean space $\mathbb{R}^n$, one may consider the collection Ω_T, called the *hull* of T, of all tilings of $\mathbb{R}^n$ that are locally indistinguishable from T. A tiling T' is in the hull of T if every finite collection of tiles in T' is exactly a translate of a finite collection of tiles of T. There is a natural topology on the hull, and an action of $\mathbb{R}^n$ on the hull by translation, such that the properties of the resulting topological dynamical system reflect combinatorial properties of the original tiling. A beautiful example of this correspondence between combinatorics and dynamics arises in diffraction theory. The diffraction spectrum of a point set in $\mathbb{R}^n$ (think of the points as atoms in a material and the diffraction spectrum as a picture of X-ray scattering) depends on the spatial recurrence properties of finite patterns of points in the point set. The point set determines a tiling of $\mathbb{R}^n$ by Veronoi cells and, provided the patches of the tiling are distributed in a sufficiently regular manner, the point set has a pure point diffraction spectrum (i.e., the material is a perfect quasicrystal) if and only if the $\mathbb{R}^n$-action on the hull of the tiling has a pure discrete dynamical spectrum [Dw; LMSo]. This claim means that, by definition, the Hilbert space $L^2(\Omega_T, \mu)$ is generated by the eigenfunctions of the action. Recall that eigenfunctions are (classes of) functions $f\colon \Omega_T \to \mathbb{C}$ satisfying $f(T' - v) = \exp(2\pi\imath\beta(v))f(T')$ for all $v \in \mathbb{R}^n$, almost all $T' \in \Omega_T$, and some linear functional $\beta\colon \mathbb{R}^n \to \mathbb{R}$ (the eigenvalue). Here μ is an invariant ergodic measure on Ω_T that is related to the diffraction via the construction of the autocorrelation measure. (We will consider strictly ergodic tiling dynamical systems, in which case this measure is unique.)

The study of continuous eigenfunctions is related to the study of equicontinuous factors of the dynamical system $(\Omega_T, \mathbb{R}^n)$. All continuous eigenfunctions together determine what is called the maximal equicontinuous factor $\pi_{\max}\colon (\Omega_T, \mathbb{R}^n) \to (X_{\max}, \mathbb{R}^n)$. One of the most common routes to determine whether the $\mathbb{R}^n$-action on the hull of the tiling has a pure discrete dynamical spectrum is therefore to examine whether $\pi_{\max}$ is almost everywhere one-to-one.

The equivalence relation whose equivalence classes are the fibers of $\pi_{\max}$ is called the *equicontinuous structure* relation, which resembles the proximal relation. The latter is not necessarily an equivalence relation, but a modification of it

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gives rise to a relation—known as regional proximality—that for minimal systems coincides with the equicontinuous structure relation. Our principal aim in this work is to use these notions of proximality to say something about the equicontinuous structure relation and, in particular, to shed light on the question just posed of whether (or not) the dynamical spectrum of a tiling system is a pure point.

The concept of proximality applies to rather general topological dynamical systems (X, G) and requires only that X be a compact uniform space and carry a continuous group G action α. Because the hull of a tiling is metrizable, we can work with a metric d; this approach involves an irrelevant choice of metric but is a little less abstract. We therefore consider a compact metric space (X, d) with a minimal continuous G-action. We require that G be a locally compact abelian group and, for some results, also that G be compactly generated.

Two points $x, y \in X$ are *proximal* if $\inf_{t\in G} d(\alpha_t(x), \alpha_t(y)) = 0$. A point $x \in X$ is *distal* if it is not proximal to any other point. We say that $x \in X_{\max}$ is *fiber distal* if $\pi_{\max}^{-1}(x)$ consists only of distal points. Let $\pi_{\max}\colon (X, G) \to (X_{\max}, G)$ be the maximal equicontinuous factor. We say that (X, G) has *finite minimal rank* if its minimal rank

$$\mathrm{mr} := \inf\{\#\pi_{\max}^{-1}(x) : x \in X_{\max}\}$$

is finite. We will establish the following result.

THEOREM (Theorem 2.15 in the main text). *Let (X, G) have finite minimal rank, suppose that the maximal equicontinuous factor is connected, and suppose that G is compactly generated. Then the proximal relation $\mathcal{P}$ coincides with the equicontinuous structure relation if and only if $\mathcal{P} \subset X \times X$ is closed (in the product topology).*

THEOREM (Lemmas 2.12 and 2.14 in the main text). *Let (X, G) have finite minimal rank, and suppose that its maximal equicontinuous factor is connected and admits at least one fiber distal point. Then the proximal relation $\mathcal{P}$ is closed if and only if the minimal rank is 1.*

Now let η be the (normalized) Haar measure on $X_{\max}$. We say that (X, G) is *almost everywhere fiber distal* if the set of fiber distal points has full measure in $X_{\max}$.

THEOREM (Theorem 2.25 in the main text). *Let (X, G) have finite minimal rank, have connected maximal equicontinuous factor, and be almost everywhere fiber distal. Let μ be an ergodic G-invariant Borel probability measure on X. Then $L^2(X, \mu)$ is generated by continuous eigenfunctions if and only if the proximal relation is closed.*

In particular, if all eigenfunctions are continuous then—under the hypothesis that (X, G) has finite minimal rank and the fiber distal points have full measure—the topological closedness of $\mathcal{P}$ is a neccessary and sufficient criterion for pure discrete spectrum. Thus it is of interest to investigate which type of tiling (or Delone set) systems satisfy the hypotheses. We shall identify those systems: regular model sets and Meyer substitution tilings.

The hull of a tiling has a laminated structure that comes from the group action. Unlike more general dynamical systems, this lamination admits canonical transversals. The extra transverse structure allows for the definition of a stronger notion of proximality. Namely, we call two tilings *strongly proximal* if, for each r, there is a ball of radius r on which they agree exactly. It is central to our analysis that the two notions of proximality coincide for the most important class of Meyer sets. A similar result applies to the regional proximal relation, of which there is a strong version that coincides with the usual one for Meyer sets. This fact, and the results of Section 5.3, are key to the efficient method for determining a pure discrete dynamical spectrum for Pisot family substitution tilings developed in [BaSW].

A large part of this paper is devoted to the study of two classes of Meyer systems, those defined by model sets and those defined by Meyer substitutions. Our findings can be summarized as follows.

- Model sets always have minimal rank 1 and so proximality is always closed. The set of fiber distal points has full measure if and only if the model set is regular.
- Meyer substitution tilings always have finite minimal rank, and the set of fiber distal points always has full measure.

The case of model sets seems a lot simpler; however, except for nice windows, we cannot control the maximal rank, $\sup\{\#\pi_{\max}^{-1}(x) : x \in X_{\max}\}$. Even so, we find that Meyer substitution tilings always have finite maximal rank (and hence finite minimal rank). This fact is extremely advantageous in other contexts as well. It is exploited in [BaO] to describe the branch locus in two-dimensional self-similar Pisot substitution tiling spaces and in [Ba] to characterize minimal directions in self-similar Pisot substitution tiling spaces of any dimension.

Finally, we consider syndetic proximality [C]. Two points x, y are *syndetically proximal* if, given any subset $A \subset G$ that contains a translate of each compact subset, we have $\inf_{t\in A} d(\alpha_t(x), \alpha_t(y)) = 0$. This is an equivalence relation, but it is not always closed. We will show that, for Meyer substitutions, syndetic proximality is indeed a closed equivalence relation. All three notions—syndetic proximality, proximality, and regional proximality—are the same for tilings of finite minimal rank if proximality is closed. Thus for Meyer substitutions we obtain (Corollary 6.7 in the main text) equivalence between the following statements:

(i) proximality is a closed relation;
(ii) proximality agrees with syndetic proximality;
(iii) the dynamical spectrum is purely discrete.

2. Maximal Equicontinuous Factors and Proximality

2.1. General Notions and Results

In this section we recall some aspects of the theory of topological dynamical systems relating to equicontinuity, proximality, and regional proximality. Most of this material can be found in Auslander's book [Au].

We consider a dynamical system (X, G), where X is a compact metrizable space and G is a locally compact abelian group acting continuously by α on X. We denote the action by $\alpha_t(x) = t \cdot x$ or, in the context of tilings, by $\alpha_v(T) = T - v$.

DEFINITION 2.1 (equicontinuity). A point $x \in X$ is *equicontinuous* if the family of homeomorphisms $\{\alpha_t\}_{t\in G}$ is equicontinuous at x. The dynamical system (X, G) is equicontinuous if all its points are equicontinuous.

Although the standard definition of equicontinuity uses a metric, it does not depend on the particular choice of metric as long as the metric is compatible with the topology. In fact, Auslander introduces this notion using the (unique) uniformity defined by the topology. Minimal equicontinuous systems have a simple structure: they are translations on compact abelian groups. This means that X has the structure of an abelian group and that the action is given by $\alpha_t(x) = x + \iota(t)$, where $\iota : G \to X$ is a group homomorphism.

THEOREM 2.2 (Ellis). *A minimal system (X, G) is equicontinuous if and only if it is conjugate to a minimal translation on a compact abelian group.*

If (X, G) is equicontinuous then the group structure on X arises as follows. Given any point $x_0 \in X$, the operation $t_1 \cdot x_0 + t_2 \cdot x_0 := (t_1 + t_2) \cdot x_0$ extends to an addition in X so that X becomes a group with x_0 as neutral element. Conversely, any translation on a compact abelian group is clearly equicontinuous.

We are interested in dynamical systems defined by aperiodic tilings of finite local complexity (FLC). Such systems are never equicontinuous [BaO], and it will prove fruitful to study the relation between these systems and their maximal equicontinuous factors.

DEFINITION 2.3 (maximal equicontinuous factor). An equicontinuous factor of (X, G) is *maximal* if any other equicontinuous factor of (X, G) factors through it. It is thus unique up to conjugacy and therefore referred to as *the* maximal equicontinuous factor. We denote it by $(X_{\max}, G)$ and the factor map by $\pi_{\max} : X \to X_{\max}$.

The maximal continuous factor always exists but may be trivial (i.e., a single point). The equivalence relation defined by $\pi_{\max}$—that is, $x \sim y$ if $\pi_{\max}(x) = \pi_{\max}(y)$—is called the *equicontinuous structure relation*.

The concept of proximality is central to the investigation of the equicontinuous structure relation.

DEFINITION 2.4 (proximality). Consider a compatible metric d on (X, G). Two points $x, y \in X$ are *proximal* if

$$\inf_{t\in G} d(t \cdot x, t \cdot y) = 0.$$

We denote by $\mathcal{P} \subset X \times X$ the proximal relation and write $x \sim_p y$ if $(x, y) \in \mathcal{P}$.

The proximal relation does not depend on the metric but only on the topology (it can also be formulated using the uniformity structure on X). It is easy to see that the proximal relation is trivial for equicontinuous systems, but the converse is not true. Systems for which the proximal relation is trivial are called *distal*.

Note that $\mathcal{P} = \bigcap_\varepsilon \mathcal{P}_\varepsilon$, where

$$\mathcal{P}_\varepsilon = \{(x, y) \in X \times X : \inf_{t \in G} d(t \cdot x, t \cdot y) < \varepsilon\}.$$

The proximal relation is not always closed. In other words, $\mathcal{P}$ need not be a closed subset of $X \times X$ and is not, in general, a transitive relation. However, we have the following statement.

THEOREM 2.5 [Au]. *If the proximal relation is closed then it is an equivalence relation.*

DEFINITION 2.6 (regional proximality). The regional proximal relation is $\mathcal{Q} := \bigcap_\varepsilon \overline{\mathcal{P}_\varepsilon}$. That is, two points $x, y \in X$ are *regionally proximal* if for all ε there exist $x' \in X$, $y' \in X$, and $t \in G$ such that $d(x, x') < \varepsilon$, $d(y, y') < \varepsilon$, and $d(t \cdot x', t \cdot y') < \varepsilon$.

THEOREM 2.7 [EGo]. *The equicontinuous structure relation is the smallest closed equivalence relation containing the regional proximal relation.*

The regional proximal relation is, in general, neither closed nor transitive. However, if the acting group is abelian then we have the following result.

THEOREM 2.8 [V]. *For minimal systems, the regional proximal relation is a closed equivalence relation and hence coincides with the equicontinuous structure relation.*

Even for minimal systems, the regional proximal relation is not necessarily the smallest closed equivalence relation containing the proximal relation. Indeed, $\mathcal{P}$ may be trivial while $\mathcal{Q}$ is not (there are minimal distal systems that are not equicontinuous).

The following definition is a generalization of a notion introduced by [BaKw] in the context of Pisot substitution tilings. For $\delta > 0$ and $x \in X_{\max}$, let $cr(x, \delta)$ be the maximal cardinality l of a collection $\{x_1, \ldots, x_l\} \subset \pi_{\max}^{-1}(x)$ with the property that $\inf_{t \in G} d(t \cdot x_i, t \cdot x_j) \geq \delta$ provided $i \neq j$.

DEFINITION 2.9 (coincidence rank). The *coincidence rank* of a minimal system (X, G) is the number

$$\mathrm{cr} = \lim_{\delta \to 0^+} cr(x, \delta).$$

LEMMA 2.10. *The limit in Definition 2.9 does not depend on the choice of x. Moreover, if* cr *is finite for some x then there exists a $\delta_0 > 0$ such that, for all y,*

$$\mathrm{cr} = cr(y, \delta_0).$$

Proof. Clearly, $cr(x, \delta)$ is a decreasing integer-valued function of δ and $cr(x, \delta) = 1$ if δ is larger than the diameter of X. Furthermore, $cr(x, \delta) = cr(t \cdot x, \delta)$ for all $t \in G$. This implies that either $\lim_{\delta \to 0^+} cr(x, \delta) = +\infty$ or

$\lim_{\delta\to 0^+} cr(x,\delta) = cr(x_0,\delta_0)$ for some $\delta_0 > 0$ and all points x of the orbit of a point $x_0 \in X_{\max}$. We need to show that the result is the same for points $y \in X_{\max}$ in other orbits.

Let $l \in \mathbb{N}$ and let $\{x_1,\dots,x_l\} \subset \pi_{\max}^{-1}(x_0)$ with the property that

$$\inf_{t\in G} d(t\cdot x_i, t\cdot x_j) \geq \delta$$

provided $i \neq j$. Let $x_1' \in \pi_{\max}^{-1}(y)$. By transitivity, there exists a sequence $(t_n)_n$ such that $\lim_n t_n \cdot x_1 \to x_1'$. Taking subsequences, we may suppose that all other limits $\lim_n t_n \cdot x_i$ exist; let x_i' denote these limits. Since $\pi_{\max}$ is continuous and equivariant with respect to the action, it follows that $x_i' \in \pi_{\max}^{-1}(y)$ for all $i = 1,\dots,l$. Then

$$\begin{aligned} d(t'\cdot x_1', t'\cdot x_2') \geq d(t'\cdot(t_n\cdot x_1), t'\cdot(t_n\cdot x_2)) - d(t'\cdot x_1', t'\cdot(t_n\cdot x_1)) \\ - d(t'\cdot(t_n\cdot x_2), t'\cdot x_2). \end{aligned}$$

Keeping t' fixed we find, for all $\varepsilon > 0$, an n such that $d(t'\cdot x_1', t'\cdot(t_n\cdot x_1)) < \varepsilon$ and $d(t'\cdot(t_n\cdot x_2), t'\cdot x_2') < \varepsilon$. Hence $d(t'\cdot x_1', t'\cdot x_2') \geq \delta - 2\varepsilon$ and, since ε was arbitrary, we see that $cr(y,\delta) \geq cr(x,\delta)$. By a symmetric argument, $cr(x,\delta) \geq cr(y,\delta)$. □

DEFINITION 2.11 (minimal rank). The *minimal rank* of (X,G) is

$$\mathrm{mr} := \inf\{\#\pi_{\max}^{-1}(x) : x \in X_{\max}\}.$$

Any weakly mixing tiling system—for example, the $\mathbb{R}$-action on a substitution tiling space with non-Pisot inflation constant—has trivial maximal equicontinuous factor and hence infinite minimal rank.

LEMMA 2.12. *The coincidence rank* $\mathrm{cr} = 1$ *if and only if* $\mathcal{P} = \mathcal{Q}$. *Furthermore,* $\mathrm{cr} \leq \mathrm{mr}$.

Proof. Suppose that $\mathcal{P} = \mathcal{Q}$. Then, for all $x_1, x_2 \in \pi_{\max}^{-1}(x)$, we have

$$\inf_{t\in G} d(t\cdot x_1, t\cdot x_2) = 0$$

and hence $cr(x,\delta) = 1$.

Now suppose that $\mathrm{cr} = 1$. Since $cr(x,\delta)$ is a decreasing function of δ, it follows that $cr(x,\delta) = 1$ for all x and $\delta > 0$. In particular, two elements $x_1, x_2 \in \pi_{\max}^{-1}(x)$ cannot satisfy the following statement: there exists a $\delta > 0$ such that $\inf_{t\in G} d(t\cdot x_1, t\cdot x_2) \geq \delta$. Hence $(x_1,x_2) \notin \mathcal{P}_\delta^c$ for all $\delta > 0$; that is, $(x_1,x_2) \in \bigcap_\delta \mathcal{P}_\delta$.

The inequality is clear from the independence of $\lim_{\delta\to 0^+} cr(x,\delta)$ of x. □

DEFINITION 2.13 (distal, fiber distal). A point $x \in X$ is *distal* if, for all $y \in X - \{x\}$, $\inf_{t\in G} d(t\cdot x, t\cdot y) > 0$—in other words, if x is not proximal to any other point. A point $x \in X_{\max}$ is *fiber distal* if all points of the fiber $\pi_{\max}^{-1}(x)$ are distal. We denote by $X_{\max}^{\text{distal}} \subset X_{\max}$ the set of fiber distal points. The system (X,G) is called fiber distal if its maximal equicontinuous factor admits a fiber distal point.

Note that, since $\mathcal{P}$ is contained in the equicontinuous structure relation, a point x is distal if and only if it is not proximal to any other point in the fiber $\pi_{\max}^{-1}(\pi_{\max}(x))$.

LEMMA 2.14. *Let (X, G) be a minimal system with finite minimal rank. Then* $\mathrm{cr} = \#\pi_{\max}^{-1}(x)$ *whenever x is fiber distal. In particular,*

$$X_{\max}^{\mathrm{distal}} = \{x \in X_{\max} : \#\pi_{\max}^{-1}(x) = \mathrm{cr}\}$$

and $\mathrm{cr} = \mathrm{mr}$ *whenever $X_{\max}$ contains a fiber distal point.*

Proof. Let $x \in X_{\max}$ be a fiber distal point, and let $\{x_1, \ldots, x_k\} \subset \pi_{\max}^{-1}(x)$ for $k \geq \mathrm{mr}$. Then none of the points x_i is proximal to any other x_j. By definition, this means that $\mathrm{cr}(\pi_{\max}(x), \delta_0) \geq k$ for $0 < \delta_0 < \inf_{i \neq j \in \{1,\ldots,k\}} \inf_{t \in G} d(t \cdot x_i, t \cdot x_j)$. Hence $\mathrm{cr} \geq k \geq \mathrm{mr}$. We have already seen that $\mathrm{mr} \geq \mathrm{cr}$. This shows also that $\mathrm{cr} = \#\pi_{\max}^{-1}(x)$ if x is fiber distal.

Now suppose that $\mathrm{cr} = \mathrm{mr}$. Then there exists a $\xi \in X_{\max}$ such that $\pi_{\max}^{-1}(\xi) = \{x_1, \ldots, x_{\mathrm{mr}}\}$. Since $\mathrm{cr} = cr(\xi, \delta_0)$ for some δ_0, we must have

$$\inf_{t \in G} d(t \cdot x_i, t \cdot x_j) \geq \delta_0$$

for all $i \neq j$. Thus all x_i are distal.

Finally, if $\mathrm{cr} = \#\pi_{\max}^{-1}(x)$ then $\pi_{\max}^{-1}(x)$ cannot contain proximal points and so x must be fiber distal. □

Let $\tilde{\mathcal{P}}$ the smallest closed equivalence relation containing the proximal relation on a minimal compact metrizable dynamical system (X, G). We define X_p to be the quotient space $X_p := X/\tilde{\mathcal{P}}$ and denote its classes by $[x]_p$. Because $\tilde{\mathcal{P}}$ is closed, X_p is metrizable and the canonical projection is a closed continuous map [Ku]. Since the proximal relation is G-invariant, the action of G descends and so we have a factor system (X_p, G). Furthermore, $\pi_{\max}$ factors through the canonical projection just described; hence we get another factor map

$$\pi : X_p \to X_{\max}, \quad \pi([x]_p) = \pi_{\max}(x),$$

which is again closed.

THEOREM 2.15. *Let (X, G) be a minimal system with finite minimal rank and connected maximal equicontinuous factor, and suppose that G is compactly generated. Then $\mathcal{P} = \mathcal{Q}$ if and only if $\mathcal{P}$ is closed.*

The proof of this theorem is based on the following two lemmas. We know already that $\mathcal{P} = \mathcal{Q}$ is equivalent to $\mathrm{cr} = 1$. We must therefore show that $\mathcal{P}$ closed implies $\mathrm{cr} = 1$—assuming, of course, that cr is finite!

LEMMA 2.16. *Suppose that* cr *is finite. If $\mathcal{P}$ is closed, then π is a* cr*-to-1 map that is a local homeomorphism. In other words, any point in X_p admits a neighborhood on which π restricts to a homeomorphism onto its image.*

Proof. Note that if $\mathcal{P}$ is closed then, by Theorem 2.5, $\mathcal{P} = \tilde{\mathcal{P}}$. We first show that π is cr-to-1. For $x \in X_{\max}$, clearly $\#\pi^{-1}(x) \geq \mathrm{cr}$ because there exist elements $x_1, \ldots, x_{\mathrm{cr}} \in \pi_{\max}^{-1}(x)$ that belong to different $\mathcal{P}$-classes. Suppose now that

$x_1, \dots, x_l \in \pi_{\max}^{-1}(x)$ with $x_i \not\sim_p x_j$ for all $x_i \neq x_j$. Then there exists a $\delta > 0$ such that, for all $x_i \neq x_j$, $(x_i, x_j) \notin \mathcal{P}_\delta$. By definition, $l \leq cr(x, \delta) \leq \mathrm{cr}$.

Clearly, π is continuous and surjective. If π were not locally injective, then we could construct two sequences $(\xi_n)_n$ and $(\eta_n)_n$ in $X/\mathcal{P}$ such that

(i) $\xi_n \neq \eta_n$ for all n,
(ii) $\pi(\xi_n) = \pi(\eta_n)$ for all n, and
(iii) $\lim_n \xi_n = \lim_n \eta_n = [x]_p$ for some $x \in X$.

Suppose there exist such sequences $\xi_n = [x_n]_p$ and $\eta_n = [x_n']_p$. Without loss of generality, we may then suppose that $\lim_n x_n = x$ and $\lim x_n' = x'$ for some $x' \sim_p x$. It follows that $\inf_{t \in G} d(t \cdot x, t \cdot x') = 0$. For all $\varepsilon > 0$ there exists an $N_{\varepsilon,t}$ such that, for all $n \geq N_{\varepsilon,t}$, we have $d(t \cdot x_n, t \cdot x) < \varepsilon$ and $d(t \cdot x_n', t \cdot x') < \varepsilon$. So if t is such that $d(t \cdot x, t \cdot x') < \varepsilon$ and $n \geq N_{\varepsilon,t}$, then $d(t \cdot x_n, t \cdot x_n') < 3\varepsilon$. Now if $3\varepsilon < \delta_0$ (from Lemma 2.10) then $x_n \sim_p x_n'$ for all $n \geq N_{\varepsilon,t}$. This contradicts (i).

We have thus shown that π is a locally injective continuous surjection. Furthermore, π is a closed map and so the restriction of π to an open neighborhood (on which π is injective) yields a closed continuous invertible map onto the image of that neighborhood under π. The restriction is therefore a homeomorphism. □

The following lemma is Theorem 2 of [SaSe].

LEMMA 2.17 [SaSe]. *Let (Y, G) be a compact metrizable dynamical system, and let $\pi : (Y, G) \to (X, G)$ be a finite-to-one factor map that is a local homeomorphism. We suppose that X is connected and that G is a compactly generated abelian group; in other words, suppose there is a compact neighborhood K of the identity 0 in G such that $G = \bigcup_{n \in \mathbb{N}} nK$ for $nK := K + \cdots + K$. If (X, G) is equicontinuous, then (Y, G) is also equicontinuous.*

Proof of Theorem 2.15. Combining Lemmas 2.16 and 2.17, we find that (X_p, G) is equicontinuous and hence must already coincide with $(X_{\max}, G)$. Therefore, π is a conjugacy and $\mathrm{cr} = 1$. □

The preceding results will be sufficient for our analysis of tiling systems. For completeness, we add a corollary about distal systems and strengthen the last theorem. Recall that a dynamical system is called distal if all its points are distal.

COROLLARY 2.18. *For compact metrizable minimal distal systems with connected maximal equicontinuous factor, either* $\mathrm{cr} = 1$ *or* $\mathrm{cr} = +\infty$. *The first equality holds if and only if the system is equicontinuous.*

For compact Hausdorff Y, the space Y^Y of functions from Y to Y is a monoid under composition that is again compact in the product topology. If G acts on Y, then (a) each $t \in G$ defines a homeomorphism from Y to Y and (b) the Ellis monoid $E(Y)$ of the action (also called the enveloping semigroup) is the closure in the product topology of the set of all these homeomorphisms. The system (Y, G) is distal if and only if $E(Y)$ is a group, and $E(Y)$ is a group if and only if it does not have any proper minimal (left) ideals (see e.g. [Au, Chap. 5, Thm. 6]).

PROPOSITION 2.19. *Let (X, G) be a compact Hausdorff dynamical system. Then (X_p, G) is distal.*

Proof. We must prove that the Ellis monoid $E(X_p)$ is a group. The canonical projection $\pi : X \to X_p$ induces a continuous semigroup homomorphism $\pi_* \colon E(X) \to E(X_p)$ determined by the equation

$$\pi_*(p)(y) = \pi(p(x)),$$

where x is any preimage of y. Suppose that $E(X_p)$ is not a group and therefore admits a proper minimal ideal I. Then I is closed and $\pi_*^{-1}(I)$ is a proper closed ideal of $E(X)$ (closed by continuity). Auslander [Au] shows that any closed ideal of the Ellis semigroup of a dynamical system contains an idempotent u. Any idempotent $u \in E(X)$ satisfies $u(x) \sim_p x$ for all $x \in X$. Hence $\pi(u(x)) = \pi(x)$ and so $\pi_*(u)(\pi(x)) = \pi(u(x)) = \pi(x)$ for all $x \in X$. Thus $\pi_*(u) = \text{id}$ and $\text{id} \in I$, contradicting the properness of I. □

COROLLARY 2.20. *Consider a minimal system (X, G) of finite minimal rank and with connected maximal equicontinuous factor. Then the equicontinuous structure relation is the smallest closed equivalence relation containing the proximal relation.*

Proof. Since the minimal rank of (X, G) is finite and since (X_p, G) is a factor sitting above the equicontinuous factor, it follows that the minimal rank of (X_p, G) must also be finite. By Proposition 2.19 and Corollary 2.18, this implies that (X_p, G) is equicontinuous and hence coincides with the maximal equicontinuous factor. □

2.2. The Maximal Equicontinuous Factor and the Dynamical Spectrum

So far we have seen that the maximal equicontinuous factor arises from dividing out the regional proximal relation (or, in the nonminimal case, by the closed equivalence relation generated by it). The maximal equicontinuous factor can also be described in another way, which is related to the topological dynamical spectrum of the system.

A *continuous eigenfunction* of a dynamical system (X, G) is a nonzero function $f \in C(X)$ for which there exists a (continuous) character $\chi \in \hat{G}$ such that

$$f(t \cdot x) = \chi(t) f(x). \tag{1}$$

This character χ is called the *eigenvalue* of f, and the set of all eigenvalues $\mathcal{E}$ forms a subgroup of the Pontryagin dual $\hat{G}$ of G. We say that the eigenfunction is *normalized* if its modulus is equal to 1.

Note that, by universality of the maximal equicontinuous factor, all continuous eigenfunctions on (X, G) factor through $\pi_{\max}$. Indeed, any normalized continuous eigenfunction $f : X \to \mathbb{T}^1$ gives rise to an equicontinuous factor of the form $(\mathbb{T}^1, G)$ that consequently sits below the maximal equicontinuous factor; thus,

$f = f' \circ \pi_{\max}$ for some $f' \in C(X_{\max})$. In particular, $\pi_{\max}(x) = \pi_{\max}(y)$ implies that all continuous eigenvalues take the same value on x as on y. Given that the maximal equicontinuous factor is a minimal translation on a compact abelian group, its eigenvalues are $\widehat{X_{\max}}$ and the continuous eigenfunctions separate the points of $X_{\max}$. Now if $\pi_{\max}(x) \neq \pi_{\max}(y)$ then there exists a continuous eigenfunction $\tilde{f}$ for $(X_{\max}, G)$ such that $\tilde{f}(\pi_{\max}(x)) \neq \tilde{f}(\pi_{\max}(y))$. Hence $\pi^*_{\max}(\tilde{f})$ is a continuous eigenfunction taking different values on x and y. Combining the two arguments shows that $\pi_{\max}(x) = \pi_{\max}(y)$ if and only if, for all continuous eigenfunctions f, one has $f(x) = f(y)$.

Let $\mathcal{F}$ be the norm closed subalgebra of $C(X)$ generated by the continuous eigenfunctions. We will argue that the maximal equicontinuous factor $X_{\max}$ can be identified with the spectrum $\hat{\mathcal{F}}$ of $\mathcal{F}$. (This spectrum is the space of nonzero *-algebra morphisms $\varphi\colon \mathcal{F} \to \mathbb{C}$ equipped with the subspace topology of the weak-* topology of the dual space $\mathcal{F}^*$.) The action of G on $C(X)$ via pull-back preserves the space of continuous eigenfunctions and so, by duality, gives rise to an action on $\hat{\mathcal{F}}$; for an eigenfunction f with eigenvalue χ this action is

$$(t \cdot \varphi)(f) := \chi(t)\varphi(f).$$

Therefore, the dual of the inclusion $\iota : \mathcal{F} \to C(X)$ yields a factor map $\hat{\iota} : \widehat{C(X)} \cong X \to \hat{\mathcal{F}}$. If we take into account the homeomorphism $x \mapsto \mathrm{ev}_x$ between X and the spectrum $\widehat{C(X)}$ of $C(X)$ (where the latter is given by the set of evaluation maps $\{\mathrm{ev}_x : x \in X\}$, $\mathrm{ev}_x(f) := f(x)$), then the map $\hat{\iota}(x)$ is simply the restriction of ev_x to the space of continuous eigenfunctions. By the previous remarks, $\pi_{\max}(x) = \pi_{\max}(y)$ if and only if $\hat{\iota}(x) = \hat{\iota}(y)$, from which it follows that $\pi_{\max}$ and $\hat{\iota}$ have the same fibers. Hence the factor $\hat{\iota}\colon X \to \hat{\mathcal{F}}$ is isomorphic to the maximal equicontinuous factor.

Finally, observe that $\hat{\mathcal{F}}$ can be identified with the Pontryagin dual $\hat{\mathcal{E}}$ of $\mathcal{E}$. Indeed, choose a point $x_0 \in X$ and normalize all continuous eigenfunctions to $f(x_0) = 1$. Then there is a bijection between normalized continuous eigenfunctions and eigenvalues (we assume minimality here, for otherwise one must choose several points). In particular, $\chi \leftrightarrow f_\chi$ because $f_\chi(t \cdot x_0) := \chi(t)$ extends by continuity to the unique normalized continuous eigenfunction to eigenvalue χ. Note that this bijection is also an abelian group isomorphism. So if we equip $\mathcal{E}$ with the discrete topology, then the continuous eigenfunctions form the group algebra $\mathbb{C}\mathcal{E}$ and hence $\hat{\mathcal{F}} = \hat{\mathcal{E}}$ (a compact abelian group) given that we have equipped $\mathcal{E}$ with the discrete topology. In sum, we have demonstrated the following result.

THEOREM 2.21. *Let (X, G) be a minimal dynamical system with abelian G, and let $\mathcal{E} \subset \hat{G}$ be the subgroup of (continuous) eigenvalues. Then the maximal equicontinuous factor is conjugate to $X \to \hat{\mathcal{E}}$ and given by $x \mapsto j_x$, where $j_x\colon \mathcal{E} \to \mathbb{T}^1$ is defined by $j_x(\chi) = f_\chi(x)$ and where the G-action on $\varphi \in \hat{\mathcal{E}}$ is given by $(t \cdot \varphi)(\chi) = \chi(t)\varphi(\chi)$.*

LEMMA 2.22. *Let H be a subgroup of $\hat{G}$. Consider the G-action $(t \cdot \varphi)(\chi) = \chi(t)\varphi(\chi)$ for $\varphi \in \hat{H}$ and $\chi \in H$. This action is locally free if and only if $\hat{G}/\bar{H}$ is* *compact for $\bar{H}$, the closure of H in $\hat{G}$. In particular, the action is free if and only if H is dense in $\hat{G}$.*

Proof. It should be clear that $t \in G$ acts freely if and only if $t \cdot \varphi \neq \varphi$ for all $\varphi \in \hat{H}$, which is the case whenever $\chi(t) \neq 1$ for at least one $\chi \in H$. By continuity, the latter condition can be rephrased as "whenever $\chi(t) \neq 1$ for at least one $\chi \in \bar{H}$". Consider the exact sequence of abelian groups

$$0 \to \widehat{\hat{G}/\bar{H}} \to G \xrightarrow{q} \hat{\bar{H}} \to 0,$$

which is the dual to the exact sequence $0 \to \bar{H} \to \hat{G} \to \hat{G}/\bar{H} \to 0$.

Suppose that $\hat{G}/\bar{H}$ is compact. This is the case whenever $\widehat{\hat{G}/\bar{H}}$ is discrete. Hence there exists $0 \in U \subset G$, an open neighborhood of the neutral element, such that $U \cap \widehat{\hat{G}/\bar{H}} = \{0\}$. It follows that $q(U) \cong U$. Since for any $0 \neq t \in \hat{\bar{H}}$ there exists a $\chi \in H$ such that $\chi(t) \neq 1$, we see that U acts freely on $\hat{H}$. Now if H is dense then q is an isomorphism and we can take $U = G$. The converse, which we will not use, is left as an exercise for the reader. □

We now consider the case where an additional homeomorphism $\Phi\colon X \to X$ is compatible with the G-action in the sense that

$$\Phi(t \cdot x) = \Lambda(t) \cdot \Phi(x)$$

for some group isomorphism $\Lambda\colon G \to G$. We assume also that Φ fixes the point $x_0 \in X$. This situation will be relevant later when we consider substitution tilings. If f is an eigenfunction with eigenvalue $\chi \in \hat{G}$, then

$$f(\Phi(t \cdot x)) = \chi(\Lambda(t)) f(\Phi(x));$$

this equality shows that $\Phi^* f$ is an eigenfunction with eigenvalue $\hat{\Lambda}\chi$. In particular: (a) Φ^* preserves $\mathcal{F}$ and hence induces an action $\Phi_{\max}$, on the maximal equicontinuous factor $X_{\max} = \hat{\mathcal{E}}$, that is equivariant with respect to the G-action; and (b) $\mathcal{E}$ is invariant under the dual isomorphism $\hat{\Lambda}\colon \hat{G} \to \hat{G}$. We now ask: When is $\Phi_{\max}$ ergodic with respect to the Haar measure η? This occurs precisely when the linear operator U_Φ defined on $L^2(X_{\max}, \eta)$ by $U_\Phi \psi := \psi \circ \Phi_{\max}$ has, up to normalization, only one eigenfunction with eigenvalue 1—that is, when $\psi = \psi \circ \Phi_{\max}$ implies that ψ is constant.

First note that if f is a continuous eigenfunction with eigenvalue χ, then $c_\chi := f(\Phi(x))/f(x)$ does not depend on x and so the above equation reads

$$\Phi^* f_\chi = c_{\hat{\Lambda}\chi} f_{\hat{\Lambda}\chi};$$

here f_χ is a normalized eigenfunction (i.e., $f_\chi(x_0) = 1$). Next observe that normalized eigenfunctions form an orthogonal base in $L^2(X_{\max}, \eta)$ and that $U_\Phi f_\chi = \Phi^* f_\chi$. We can therefore solve the eigenvalue equation $U_\Phi \psi = \psi$ in the preceding basis and so obtain, for $\psi = \sum_{\chi \in \mathcal{E}} a_\chi f_\chi$,

$$a_{\hat{\Lambda}^{-1}\chi} = c_\chi a_\chi \quad \forall \chi \in \mathcal{E}.$$

Since $\sum_{\chi\in\mathcal{E}}|a_\chi|^2$ must be finite and since $|c_\chi| = 1$, it follows that a solution other than $\psi = 1$ can exist only if $\hat{\Lambda}$ admits a nontrivial periodic orbit in $\mathcal{E}$. In particular we have our next lemma.

LEMMA 2.23. *If $G = \mathbb{R}^n$ and if Λ does not have any eigenvalues with root of unity, then $\Phi_{\max}$ is ergodic with respect to η.*

Proof. Upon identifying $\hat{\mathbb{R}}^n$ with $\mathbb{R}^{n*}$, we see that $\hat{\Lambda}$ becomes the transpose and therefore does not admit a nontrivial periodic orbit in $\hat{G}$ or hence in $\mathcal{E}$. □

We now consider conditions for the pure point dynamical spectrum. Let μ be a G-invariant Borel probability measure on X. An *L^2-eigenfunction* is an element $f \in L^2(X,\mu)$ satisfying the eigenvalue equation (1).

DEFINITION 2.24. The measure dynamical system (X, G, μ) has a *pure point dynamical spectrum* if the L^2-eigenfunctions span $L^2(X,\mu)$.

Since (X, G) is minimal, the maximal equicontinuous factor $(X_{\max}, G)$ is also minimal; in other words, $X_{\max}$ is a group containing $G/\mathrm{stab}(X)$ as a dense subgroup (where $\mathrm{stab}(X)$ is the stabilizer of X). Hence a G-invariant Borel probability measure on $X_{\max}$ is also $X_{\max}$-invariant. (Given that $X_{\max}$ is metrizable, the G-invariant Borel probability measures are regular and hence coincide via the Riesz representation theorem with G-invariant normed linear functionals on $C(X)$; by continuity of these functionals, G-invariance extends to $X_{\max}$-invariance.) Thus the only G-invariant Borel probability measure on $X_{\max}$ is the Haar measure η.

THEOREM 2.25. *Let (X, G, μ) be a minimal dynamical system with ergodic Borel probability measure μ. Assume that (X, G, μ) has finite minimal rank, that the maximal equicontinuous factor is connected, and that $X_{\max}^{\mathrm{distal}}$ has full Haar measure. Then the following statements are equivalent.*

(i) *The continuous eigenfunctions generate $L^2(X,\mu)$.*
(ii) *The minimal rank is 1.*
(iii) *Proximality is a closed relation.*

Proof. Equivalence of assertions (ii) and (iii) has already been shown. The push-forward of the measure μ under $\pi_{\max}$ is a G-invariant Borel probability measure and hence equals η.

Suppose that $\mathrm{mr} = 1$. By Lemma 2.14, $\mathrm{cr} = 1$. Thus the hypothesis that $X_{\max}^{\mathrm{distal}}$ has full measure implies that $\pi_{\max}$ is a measure isomorphism and that $\pi_{\max}^*(L^2(X_{\max},\eta)) = L^2(X,\mu)$. Since the linear span of the continuous eigenfunctions is dense in $L^2(X_{\max},\eta)$ and since $\pi_{\max}$ is continuous, the linear span of the continuous eigenfunctions is dense in $L^2(X,\mu)$ as well.

Now let $\mathrm{mr} = \mathrm{cr}$ be greater than 1 (but finite). We will see that $\pi_{\max}\colon X \to X_{\max}$ has the nature of a cr-to-1 "measurable covering projection"; this permits the construction of an L^2 function that, on a set of positive measure, distinguishes between $\pi_{\max}$-preimages of points. As pull-backs by $\pi_{\max}$, eigenfunctions cannot make such distinctions and so the constructed function cannot be approximated by a linear combination of them.

Let $X' := \pi_{\max}^{-1}(X_{\max}^{\text{distal}})$ and let π' be the restriction of $\pi_{\max}$ to X'. Then (a) $\pi'\colon X' \to X_{\max}^{\text{distal}}$ is a closed map ($\pi_{\max}$ is closed) that is exactly cr-to-1 everywhere and (b) for each $x \in X_{\max}^{\text{distal}}$ we have $\pi_{\max}^{-1}(x) = \{x_1, \ldots, x_{\text{cr}}\}$ with $d(x_i, x_j) \geq \delta_0$ for $i \neq j$ and $\delta_0 > 0$ as in Lemma 2.10. It follows that π' must be injective on $\delta_0/2$ balls and so $\pi'\colon X' \to X_{\max}^{\text{distal}}$ must be a cr-to-1 local homeomorphism. In particular, we can find an open set $U' \subset X_{\max}^{\text{distal}}$ such that $\pi'^{-1}(U') \subset \bigcup_{i=1}^{\text{cr}} B_{\delta_0/2}(x_i)$. Since U' is open it naturally follows that there exists an open set $U \subset X_{\max}$ such that $U' = U \cap X_{\max}^{\text{distal}}$. Let $U_i := B_{\delta_0/2}(x_i) \cap \pi_{\max}^{-1}(U)$. Because $X_{\max}^{\text{distal}}$ has full measure, X' also has full measure and so $\pi_{\max}|_{U_i}\colon U_i \to U$ is a function that is almost everywhere bijective and bi-measurable.

For each U, U_i as just described, let η_U^i be the push-forward of $\eta|_U$ by $(\pi_{\max}|_{U_i})^{-1}$. That is, $\eta_U^i(A) := \eta(\pi_{\max}(A))$ for Borel $A \subset U_i$. Then $\mu|_{U_i} \ll \eta_U^i$. Let $J_{U_i}\colon U_i \to \mathbb{R}$ be the Radon–Nikodym derivative of $\mu|_{U_i}$ with respect to η_U^i. By the uniqueness of that derivative, the J_{U_i} paste together to yield a Borel-measurable function $J\colon X \to \mathbb{R}$. Then, by the G-invariance of μ and η, the function J is also G-invariant. The ergodicity of μ implies that J must be μ-a.e. constant, and this constant equals $1/\text{cr}$.

Next we let $h := \mathbf{1}_{U_1} - \mathbf{1}_{U_2}$ be the difference of indicator functions for some U, U_1, U_2 as before and suppose that $f\colon X \to \mathbb{C}$ is a continuous eigenfunction. Then there is an $f'\colon X_{\max} \to \mathbb{C}$ such that $f|_{U_i} = f' \circ \pi_{\max}|_{U_i}$ for $i = 1, 2$. The scalar product of f and h is

$$\langle f, h\rangle = \int_{U_1} f\, d\mu - \int_{U_2} f\, d\mu = \int_U f' \frac{1}{\text{cr}}\, d\eta - \int_U f' \frac{1}{\text{cr}}\, d\eta = 0.$$

In other words, h is orthogonal to all continuous eigenfunctions. Since $\mu(U_1) = \frac{1}{\text{cr}}\eta(U)$ and since $\eta(U) > 0$ (given that U is open), h is not the zero function; thus we see that the linear span of the continuous eigenfunctions is not dense in $L^2(X, \mu)$. □

COROLLARY 2.26. *Consider a minimal dynamical system with ergodic G-invariant Borel probability measure* (X, G, μ), *of finite minimal rank, all of whose dynamical eigenfunctions are continuous. Assume that* $X_{\max}^{\text{distal}}$ *has full Haar measure and that the maximal equicontinuous factor is connected. Then the dynamical spectrum of* (X, G, μ) *is pure point if and only if the proximality relation is closed.*

3. Proximality for Tilings and Delone Sets

3.1. Preliminaries

By a *tile* τ in $\mathbb{R}^n$ we mean a compact subset of $\mathbb{R}^n$ that is the closure of its interior. It is sometimes useful to label tiles with marks. In that case one should rather speak of a tile as an ordered pair $\tau = (\text{spt}(\tau), m)$; here $\text{spt}(\tau)$, the *support* of τ, is compact and the closure of its interior, and m is a *mark* taken from some finite set of marks. The *interior* of a tile is then simply the interior of its support:

$\mathring{\tau} := \operatorname{int}(\operatorname{spt}(\tau))$. Two tiles $\tau = (\operatorname{spt}(\tau), m)$ and σ are *translationally equivalent* if there is a $v \in \mathbb{R}^n$ with $\tau + v := ((\operatorname{spt}(\tau) + v, m) = \sigma$.

A *patch* is a collection of tiles with pairwise disjoint interiors. The *support* of a patch P, $\operatorname{spt}(P)$, is the union of the supports of its constituent tiles; the *diameter* of P, $\operatorname{diam}(P)$, is the diameter of its support; and a *tiling* of $\mathbb{R}^n$ is a patch with support $\mathbb{R}^n$. We denote the translation action on patches (and tilings) also by $P \mapsto P - v$, $v \in \mathbb{R}^n$.

A collection Ω of tilings of $\mathbb{R}^n$ has *translationally finite local complexity* if, for each R, there exist only finitely many translational equivalence classes of patches $P \subset T \in \Omega$ with $\operatorname{diam}(P) \leq R$. It is useful to consider a metric topology on sets of tilings, which is expressed with the help of R-patches. Given a tiling T and $R \geq 0$, the patch $B_R[T] := \{\tau \in T : \bar{B}_R(0) \cap \operatorname{spt}(\tau) \neq \emptyset\}$ is called the *R-patch* of T at 0. If Ω is a collection of tilings of $\mathbb{R}^n$ with FLC, then the following metric d can be used:

$$d(T, T') := \inf\left\{\frac{\varepsilon}{\varepsilon + 1} : \exists \|v\|, \|v'\| \leq \frac{\varepsilon}{2} \text{ s.t. } B_{1/\varepsilon}[T - v] = B_{1/\varepsilon}[T' - v']\right\}. \quad (2)$$

In other words, in this metric two tilings are close if a small translate of one agrees with the other in a large neighborhood of the origin. Removing the possibility of translation by a small vector yields another metric,

$$d_0(T, T') := \inf\left\{\frac{\varepsilon}{\varepsilon + 1} : B_{1/\varepsilon}[T] = B_{1/\varepsilon}[T']\right\}, \quad (3)$$

which does not induce the same topology but is also useful.

We will call a collection Ω of tilings of $\mathbb{R}^n$ an *n-dimensional tiling space* if Ω has FLC, is closed under translation (i.e., $T \in \Omega$ and $v \in \mathbb{R}^n$ together imply $T - v \in \Omega$), and is compact in the metric d. (All tiling spaces in this paper are assumed to have finite local complexity, but we will occasionally include the FLC hypothesis for emphasis.) For example, if T is an FLC tiling of $\mathbb{R}^n$, then

$$\Omega_T = \{T' : T' \text{ is a tiling of } \mathbb{R}^n \text{ and every patch of } T' \text{ is a translate of a patch of } T\}$$

is an n-dimensional tiling space known as the *hull* of T [AP]. All tiling spaces we speak of are hulls; hence they and their maximal equicontinuous factors are connected.

A n-dimensional tiling space Ω is *repetitive* if, for each patch P with compact support that occurs in some tiling in Ω, there exists an R such that, for all $T \in \Omega$, a translate of P occurs as a subpatch in $B_R[T]$. If Ω is repetitive, then the action of $\mathbb{R}^n$ on Ω by translation is minimal.

For a tiling T, let $p \colon T \to \mathbb{R}^n$ satisfy the following conditions: $p(\tau) \in \operatorname{spt}(\tau)$; if $\tau \in T$ and $\tau + v \in T$, then $p(\tau + v) = p(\tau) + v$. Such an assignment p will be called a *puncture map.* If the tiling has FLC then the set of its punctures $p(T) = \{p(\tau) : \tau \in T\}$ is a Delone set—in other words, a subset of $\mathbb{R}^n$ that is uniformly discrete and relatively dense. ("Uniformly discrete" means there is an $r > 0$ such that $\# B_r(x) \cap \mathcal{L} \leq 1$ for all $x \in \mathbb{R}^n$; and "relatively dense" means there is an R such that $B_R(v) \cap \mathcal{L} \neq \emptyset$ for each $x \in \mathbb{R}^n$.) Note that if p and p' are two choices of puncture maps for an FLC tiling, then there is a finite set F such that $p(T) - p(T) \subset p'(T) - p'(T) + F$.

A puncture map on a tiling T extends naturally (by translation) to all tilings in the hull Ω_T and thus defines a transversal in the hull—namely, the set of $T' \in \Omega_T$ such that $0 \in p(T')$. When the metrics d and d_0 defined previously are restricted to this transversal, they become equivalent.

The definitions we have given for tilings all have analogues for Delone sets, and whether we deal with tilings or Delone sets is mainly a matter of convenience. One could, for instance, represent a tiling T by the Delone set of its punctures. (Strictly speaking, for that one might need to consider the marked Delone set, or Delone multiset, $\{(p(\tau), m) : \tau \in T$ and $m = m(\tau)$, the mark of $\tau\}$. Everything we do hereafter with Delone sets could likewise be done with marked Delone sets; we trust the reader can make the necessary adjustments.) But there are other possibilities. If the tiling is both polyhedral and FLC, then one could represent it also by the Delone set of its vertices. In the other direction, when dealing with Delone sets we can carry over the notions defined for tilings if we consider the Dirichlet tiling associated with the Delone set (the tiling defined by the dual of the Voronoi complex), which is a polyhedral tiling that has the points of the Delone set as vertices. With this in mind, we may define the R-patches of a Delone set as the R-patches of its associated Dirichlet tiling and thereby obtain a metric on collections of Delone sets as well. Alternatively, one could use the more standard definition of the R-patch at x of the Delone set $\mathcal{L}$ as the set $\{y \in \mathcal{L} : \|x - y\| \leq R\}$ to obtain a metric. The two metrics are different but define the same topology; all our results are independent of the choice of metric.

3.2. Strong Proximality of Tilings and the Meyer Property

We consider dynamical systems $(\Omega, \mathbb{R}^n)$, where Ω is an n-dimensional tiling space and $\mathbb{R}^n$ acts by translation. When we speak of the proximal or the regional proximal relation for tilings of Ω, we mean proximality or regional proximality for the metric d that defines the compact topology of Ω. The following two definitions and subsequent results are mostly formulated for tilings but have obvious counterparts for Delone sets.

Definition 3.1 (strong proximality). Two tilings $T_1, T_2 \in \Omega$ are *strongly proximal* if, for all r, there exists a $v \in \mathbb{R}^n$ such that $B_r[T_1 - v] = B_r[T_2 - v]$. Strong proximality is thus proximality for the metric d_0.

Definition 3.2 (strong regional proximality). Two tilings $T_1, T_2 \in \Omega$ are *strongly regionally proximal* if, for all r, there exist $S_1, S_2 \in \Omega_T$ and $v \in \mathbb{R}^n$ such that

$$B_r[T_1] = B_r[S_1], \quad B_r[T_2] = B_r[S_2], \quad B_r[S_1 - v] = B_r[S_2 - v].$$

Strong regional proximality is thus regional proximality for the metric d_0.

An important question is: For what classes of tilings does proximality (resp., regional proximality) imply strong proximality (resp., strong regional proximality)? If the relations are the same then proximality (regional proximality) becomes a purely combinatorial property.

Recall that a *Meyer set* is a Delone set $\mathcal{L}$ such that $\mathcal{L} - \mathcal{L}$ is uniformly discrete. In particular, a Meyer set is always FLC. We will say that a tiling T is a *Meyer tiling,* or "has the Meyer property", if it has FLC and if the image $\mathcal{L} = p(T)$ of a puncture map is a Meyer set. Although the set $\mathcal{L}$ depends on the puncture map, the property of its being Meyer does not. If $\mathcal{L}$ is Meyer then so is $\mathcal{L} - \mathcal{L}$, per the following result.

PROPOSITION 3.3 (Meyer). *For a Meyer set $\mathcal{L}$, all finite combinations $\mathcal{L} \pm \mathcal{L} \pm \cdots \pm \mathcal{L}$ (with any choice of signs) are also Meyer.*

A proof, along with various different characterizations of Meyer sets, can be found in [M].

It is clear that if $\mathcal{L}'$ is any element in the hull $\Omega_{\mathcal{L}}$ of a Delone set $\mathcal{L}$ then $\mathcal{L}' - \mathcal{L}' \subset \mathcal{L} - \mathcal{L}$. Hence if $\mathcal{L}$ is Meyer then so is any such $\mathcal{L}'$. Furthermore, if $\mathcal{L}_1, \mathcal{L}_2 \in \Omega_{\mathcal{L}}$ and $v \in \mathcal{L}_1 \cap \mathcal{L}_2$, then $\mathcal{L}_1 - \mathcal{L}_2 = (\mathcal{L}_1 - v) - (\mathcal{L}_2 - v) \subset (\mathcal{L} - \mathcal{L}) - (\mathcal{L} - \mathcal{L})$; so if $\mathcal{L}$ is Meyer then $\bigcup_{\mathcal{L}_1, \mathcal{L}_2 \in \Omega_{\mathcal{L}} : \mathcal{L}_1 \cap \mathcal{L}_2 \neq \emptyset} \mathcal{L}_1 - \mathcal{L}_2$ is contained in a uniformly discrete set and thus cannot contain an accumulation point. For any $\mathcal{L}_1, \mathcal{L}_2 \in \Omega_{\mathcal{L}}$, we may pick w such that $\mathcal{L}_1 \cap (\mathcal{L}_2 - w) \neq \emptyset$. Then $\mathcal{L}_1 - \mathcal{L}_2 = \mathcal{L}_1 - (\mathcal{L}_2 - w) - w \subset (\mathcal{L} - \mathcal{L}) - (\mathcal{L} - \mathcal{L}) - w$. We see in this way that, if $\mathcal{L}$ is Meyer, then so is $\mathcal{L}_1 \pm \mathcal{L}_2 \pm \cdots \pm \mathcal{L}_k$ (for any choice of signs) for all $\mathcal{L}_1, \ldots, \mathcal{L}_k \in \Omega_{\mathcal{L}}$.

THEOREM 3.4. *Let Ω be the hull of a repetitive Meyer tiling. Then the proximal relation (resp., regional proximal relation) on Ω coincides with the strong proximal relation (resp., strong regional proximal relation).*

Proof. We give the proof for the regional proximality relation; the proof for strong proximality is similar and a bit simpler. Let $(T, T') \in \mathcal{Q}$. Since the orbit of T is dense, we may take x', y' in the definition of regional proximality to be of the form $x' = T - t$ and $y' = T - t'$. Hence for all $\varepsilon > 0$ there exist $t, t', v \in \mathbb{R}^n$ such that $d(T, T - t) < \varepsilon$, $d(T', T - t') < \varepsilon$, and $d(T - t - v, T - t' - v) < \varepsilon$. Moreover, we may adjust the t and the t' by a small amount (bounded by $\varepsilon/2$) so that $d(T, T - t) < \varepsilon$ implies $B_{1/\varepsilon}[T] = B_{1/\varepsilon}[T - t]$ and also $d(T - t - v, T - t' - v) < \varepsilon$ implies $B_{1/\varepsilon}[T - t - v] = B_{1/\varepsilon}[T - t' - v]$. Thus for all $r > 0$ there exist $t, t', v, z \in \mathbb{R}^n$, with $|z| \leq 1/r$, such that

(i) $B_r[T] = B_r[T - t]$,
(ii) $B_r[T' - z] = B_r[T - t']$, and
(iii) $B_r[T - t - v] = B_r[T - t' - v]$.

Our aim is to show that we can take $z = 0$. Let $\mathcal{L} = p(T)$ and $\mathcal{L}' = p(T')$, where p is a puncture map such that $0 \in \mathcal{L}$. The equality (i) implies that $t \in \mathcal{L}$, and the equality (iii) implies that $(t' + v) - (t + v) \in \mathcal{L} - \mathcal{L}$. Therefore, $t' \in \mathcal{L} - \mathcal{L} + \mathcal{L}$. From (ii) it follows that $(\mathcal{L}' - z) \cap (\mathcal{L} - t') \neq \emptyset$. Hence $(\mathcal{L}' - z) \cap (\mathcal{L} - \mathcal{L} + \mathcal{L} - \mathcal{L}) \neq \emptyset$ and so z is in the uniformly discrete set $\mathcal{L}' - \mathcal{L} + \mathcal{L} - \mathcal{L} + \mathcal{L}$. Since $|z| \leq 1/r$, there exists an r_0 such that $z = 0$ if $r \geq r_0$. Thus T and T' are strong regionally proximal. □

COROLLARY 3.5. *Consider the regional proximal relation $\mathcal{Q}$ on the hull of a repetitive Meyer tiling, and let $s > 0$. Up to translation, there are only finitely many pairs of patches of the form $(B_s[T], B_s[T'])$ with $(T, T') \in \mathcal{Q}$.*

Proof. We denote by $[(P, P')]$ the translational congruence class of a pair of patches (P, P'). Let $(T, T') \in \mathcal{Q}$ and $s > 0$. By FLC there is a finite list $\{P_1, \ldots, P_k\}$ of s-patches such that any s-patch is a translate of some P_i. Hence the set of translational congruence classes $[(B_s[S], B_s[S'])]$ with $(T, T') \in \mathcal{Q}$ is of the form $\{[(P_i, P_j - t_{i,j})] : (i, j, t_{i,j}) \in I\}$ for some subset I of $\{1, \ldots, k\}^2 \times B_{s'}(0)$ and some finite s'. We need to show that I is finite so assume the contrary. Let (i, j, t) be an accumulation point of I, and let $v_{i,j} \in p(P_i) - p(P_j)$ for p a puncture map. Then $t + v_{i,j}$ is an accumulation point of $\bigcup_{(i,j,t_{i,j}) \in I} p(P_i) - (p(P_j) - t_{i,j})$.

By Theorem 3.4 there exist $v \in \mathbb{R}^n$ and $S, S' \in \Omega$ such that $B_s[T] = B_s[S]$, $B_s[T'] = B_s[S']$, and $B_s[S - v] = B_s[S' - v]$. It follows that the set $\{[(B_s[T], B_s[T'])] : (T, T') \in \mathcal{Q}\}$ is contained in the set $\{[(B_s[S], B_s[S'])] : p(S) \cap p(S') \neq \emptyset\}$. The latter set thus contains $\{[(P_i, P_j - t_{i,j})] : (i, j, t_{i,j}) \in I\}$ and so $t + v_{i,j}$ must be an accumulation point of $\bigcup_{S, S' \in \Omega : p(S) \cap p(S') \neq \emptyset} p(S) - p(S')$. However, by the discussion preceding Theorem 3.4, that set does not contain any accumulation points. □

If $S = \{T_i\}_i$ is a set of tilings or Delone sets then we denote by $B_R[S]$ the corresponding collection of R-patches, $B_R[S] := \{B_R[T_i]\}_i$.

COROLLARY 3.6. *Consider the dynamical system of a repetitive Meyer tiling with finite coincidence rank. There is an R_0 such that, for all $y \in X_{\max}$ and $R \geq R_0$,*

$$\sup\{l : \# B_R[\pi_{\max}^{-1}(y) - v] \geq l \textit{ for all } v \in \mathbb{R}^n\} = \text{cr}.$$

Proof. By Lemma 2.10 there is $\delta_0 > 0$, and by Corollary 3.5 a corresponding R_0, so that for all y:

$$\begin{aligned} \text{cr} &= \sup\{l : \exists T_1, \ldots, T_l \in \pi_{\max}^{-1}(y) \text{ s.t. } \forall i \neq j \text{ and } \forall v,\ d(T_i - v, T_j - v) \geq \delta_0\} \\ &= \sup\{l : \exists T_1, \ldots, T_l \in \pi_{\max}^{-1}(y) \text{ s.t. } \forall i \neq j \text{ and } \forall v, \\ &\qquad\qquad B_{R_0}[T_i - v] \neq B_{R_0}[T_j - v]\}. \end{aligned}$$

By Lemma 2.10, the first equation holds if we replace δ_0 with any δ, $0 < \delta \leq \delta_0$; hence the second equation holds if we replace R_0 with any $R \geq R_0$. □

Let $R > 0$, and let

$$n^R(x) := \# B_R[\pi_{\max}^{-1}(x)] = \#\{B_R[T] : T \in \pi_{\max}^{-1}(x)\}.$$

For later use we establish the following result.

LEMMA 3.7. *For Meyer tilings with finite maximal rank, n^R is upper semicontinuous; that is, $\{x : n^R(x) \geq k\}$ is closed for all k. In particular, if R is sufficiently large (depending on the δ_0 of Lemma 2.10) and $m \in \mathbb{N}$, then $D^R(m) := \{x \in X_{\max} : n^R(x) \leq m\}$ is open.*

Proof. Recall that, for $T \in \Omega$, $B_R[T]$ is defined to be the collection of tiles in T that meet the closed (rather than open) ball of radius R at 0. Hence it follows that, for all R, there exists an $R' > R$ such that $B_{R'}[T] = B_R[T]$.

The fibers of π satisfy the following property: if $(x_n)_n$ is a sequence in $X_{\max}$ tending to x, then all accumulation points of sequences $(T_n)_n$, $T_n \in \pi_{\max}^{-1}(x_n)$, are contained in $\pi_{\max}^{-1}(x)$. Because $\pi_{\max}^{-1}$ is finite, it is also uniformly discrete; therefore, if n is large enough then we can define a map $f_n \colon \pi_{\max}^{-1}(x_n) \to \pi_{\max}^{-1}(x)$ by saying that $f_n(T)$ is the element of $\pi_{\max}^{-1}(x)$ that is closest to T.

Now let $B_R[f_n(T)] = B_R[f_n(T')]$. By the first remark there exists an $R' > R$ such that $B_{R'}[f_n(T)] = B_{R'}[f_n(T')]$. We may assume that n is large enough so that $d(f_n(T), T)$ is small enough to guarantee that $B_{R'}[T - v] = B_{R'}[f_n(T)]$ for some $|v| \leq R' - R$. Likewise, we may assume that n is large enough that $B_{R'}[T' - v'] = B_{R'}[f_n(T')]$ for some $|v'| \leq R' - R$. We may suppose that $R' - R$ is small, so the equality $B_{R'}[f_n(T)] = B_{R'}[f_n(T')]$ and the Meyer property together imply that $v = v'$ and hence $B_{R'}[T - v] = B_{R'}[T' - v]$. The latter implies $B_R[T] = B_R[T']$. This shows that $n^R(x) \geq n^R(x_n)$ and thus establishes the upper semicontinuity of n^R. □

4. Proximality in Model Sets

Model sets (a.k.a. "cut and project" patterns) are characterized by how they are constructed. We outline the construction here and refer the reader to [BLeM; FHK; M] for a thorough description.

The defining data of a model set consist of a lattice (co-compact subgroup) $\Gamma \subset \mathbb{R}^n \times H$ in the product of $\mathbb{R}^n$ with a locally compact abelian group H such that $\mathbb{R}^n$ is in irrational position w.r.t. Γ together with a window K (or acceptance domain, or atomic surface) that is a compact subset of H. We denote the boundary of K by ∂K and the quotient group $\mathbb{R}^n \times H/\Gamma$ by $\mathbb{T}$. The latter is a compact abelian group that is often referred to as the LI-torus. Note that $\mathbb{R}^n$ acts on $\mathbb{T}$ by rotation: $v \cdot ((w,h) + \Gamma) = (w + v, h) + \Gamma$; hence $(\mathbb{T}, \mathbb{R}^n)$ is an equicontinuous dynamical system.

Let $\pi^{\parallel} \colon \mathbb{R}^n \times H \to \mathbb{R}^n$ be the projection onto the first factor and let $\pi^{\perp} \colon \mathbb{R}^n \times H \to H$ be the projection onto the second factor. We make the following standard assumptions.

- The restrictions of $\pi^{\parallel}$ and $\pi^{\perp}$ to Γ are one-to-one.
- The restrictions of $\pi^{\parallel}$ and $\pi^{\perp}$ to Γ have dense image.
- The window K is the closure of its interior.
- The stabilizer of K in H is trivial; that is, $h + K = K$ implies $h = 0$.

The data $(\mathbb{R}^n, H, \Gamma, K)$ determine a whole family of point patterns in $\mathbb{R}^n$. Indeed, for $x \in \mathbb{R}^n \times H$ we let

$$M_x := \{\pi^{\parallel}(\gamma + x) \in \mathbb{R}^n : \gamma \in \Gamma,\ \pi^{\perp}(\gamma + x) \in K\}$$

so that $M_x \times \{e\} = \mathbb{R}^n \times \{e\} \cap (\Gamma + x - \{0\} \times K)$, where $e \in H$ the neutral element. It is well known that M_x is a Delone set under these assumptions. A Delone set arising in this way is called a *model set*. Furthermore, $M_x = M_y$ if and only if $x - y \in \Gamma$. We define the set S of *singular points* by

$$S := \{x \in \mathbb{R}^n \times H : \pi^\perp(x) \in \partial K + \pi^\perp(\Gamma)\} = \mathbb{R}^n \times \{e\} + \Gamma + \{0\} \times \partial K$$

and denote its complement by S^c. Then M_x is repetitive if $x \in S^c$.

PROPOSITION 4.1. *The complement of S is a dense G_δ set. In particular, it is nonempty.*

Proof. By our assumptions, ∂K (and hence $R^n \times \{e\} + \{0\} \times \partial K$) has empty interior; therefore, the interior of S is also empty. This claim results from a simple application of the Baire category theorem; see [Sc] or [FHK] for the case $H = \mathbb{R}^k$. □

Suppose (for the sake of simplicity) that $0 \notin S$, and consider the hull Ω_M of $M = M_0$. It is well known that M_x and M_y are locally indistiguishable provided $x, y \in S^c$. Furthermore $M_x = M_y$ if and only if $x - y \in \Gamma$. Thus Ω_M is the completion of the set S^c/Γ with respect to the metric $\delta(x + \Gamma, y + \Gamma) = d(M_x, M_y)$. The metric δ does not extend continuously in the (quotient of the product) topology of $\mathbb{T}$, but the converse is the basis of one of the main structural theorems for model sets.

THEOREM 4.2. *The map $\{M_y \in \Omega_M : y \in S^c\} \ni M_x \mapsto x + \Gamma \in \mathbb{T}$ extends to a continuous surjection*

$$\mu : \Omega_M \to \mathbb{T}.$$

This surjection is equivariant with respect to the $\mathbb{R}^n$-action and is one-to-one precisely on S^c/Γ; in other words, it is precisely the nonsingular points that have a unique preimage.

Proof. For H a real vector space, a proof can be found in [FHK]. For the case of more general groups H, see [BLeM]. □

COROLLARY 4.3. *Repetitive model sets have minimal rank 1. In particular, $(\mathbb{T}, \mathbb{R}^n)$ is the maximal equicontinuous factor, $\mu = \pi_{\max}$, and two elements are proximal if and only if they are mapped to the same point by μ.*

Proof. The set of fiber distal points includes S^c/Γ, which is nonempty. Hence cr = mr = 1. By Lemma 2.14, the proximality relation coincides with the equicontinuous structure relation. □

A model set is called *regular* if ∂K has measure 0 (w.r.t. the Haar measure on H).

THEOREM 4.4. *For a repetitive regular model set, the set of fiber distal points $X_{\max}^{\text{distal}}$ has full Haar measure. Moreover, if the model set is not regular then $X_{\max}^{\text{distal}}$ has Haar measure 0.*

Proof. Since the proximality relation coincides with the equicontinuous structure relation, it follows that the set of distal points coincides with the nonsingular points and that $X_{\max}^{\text{distal}} = S^c/\Gamma$. Clearly, ∂K has strictly positive measure if and only if the complement of $X_{\max}^{\text{distal}} = S^c/\Gamma$ has strictly positive Haar measure. By ergodicity of the Haar measure, S/Γ must have full Haar measure if its measure is strictly positive. □

5. Proximality for Meyer Substitution Tilings

5.1. Basic Notions

Suppose that $\mathcal{A} = \{\rho_1, \dots, \rho_k\}$ is a set of translationally inequivalent tiles (called *prototiles*) in $\mathbb{R}^n$. Let Λ be an expanding linear isomorphism of $\mathbb{R}^n$; that is, let all eigenvalues of Λ have modulus strictly greater than 1. A *substitution* on $\mathcal{A}$ with expansion Λ is a function $\Phi\colon \mathcal{A} \to \{P : P \text{ is a patch in } \mathbb{R}^n\}$ with the properties that, for each $i \in \{1, \dots, k\}$, every tile in $\Phi(\rho_i)$ is a translate of an element of $\mathcal{A}$ and $\operatorname{spt}(\Phi(\rho_i)) = \Lambda(\operatorname{spt}(\rho_i))$. Such a substitution naturally extends to patches whose elements are translates of prototiles by $\Phi(\{\rho_{i(j)} + v_j : j \in J\}) := \bigcup_{j \in J}(\Phi(\rho_{i(j)}) + \Lambda v_j)$. A patch P is *allowed* for Φ if there is an $m \geq 1$, an $i \in \{1, \dots, k\}$, and a $v \in \mathbb{R}^n$ such that $P \subset \Phi^m(\rho_i) - v$. The *substitution tiling space* associated with Φ is the collection $\Omega_\Phi := \{T : T \text{ is a tiling of } \mathbb{R}^n \text{ and every finite patch in } T \text{ is allowed for } \Phi\}$. Clearly, translation preserves allowed patches, so $\mathbb{R}^n$ acts on Ω_Φ by translation.

The substitution Φ is *primitive* if, for each pair $\{\rho_i, \rho_j\}$ of prototiles, there is a $k \in \mathbb{N}$ such that a translate of ρ_i occurs in $\Phi^k(\rho_j)$. If Φ is primitive then Ω_Φ is repetitive.

If the translation action on Ω is free (i.e., if $T - v = T$ implies $v = 0$), then Ω is said to be *nonperiodic*. If Φ is primitive and if Ω_Φ is both FLC and nonperiodic, then Ω_Φ is compact in the metric just described, $\Phi\colon \Omega_\Phi \to \Omega_\Phi$ is a homeomorphism, and the translation action on Ω_Φ is minimal and uniquely ergodic [AP; So1; So2]. In particular, $\Omega_\Phi = \Omega_T$ for any $T \in \Omega_\Phi$. It will be with respect to the unique ergodic measure μ on Ω_Φ when we speak about the dynamical spectrum and L^2-eigenfunctions. Note that Φ preserves regional proximality, so there is an induced homeomorphism $\Phi_{\max}$ on the maximal equicontinuous factor $X_{\max}$ of Ω_Φ. We will assume that Φ fixes some tiling (otherwise, replace Φ by an appropriate power), which means that we can identify $\Phi_{\max}$ with the action of Φ on the Pontryagin dual $\hat{\mathcal{E}}$ of the group of eigenvalues (see Section 2.2).

All substitutions will be assumed to be primitive, aperiodic, and FLC.

THEOREM 5.1 [So3]. *All L^2-eigenfunctions of a substitution tiling space can be chosen to be continuous.*

We call a substitution a *Meyer substitution* if every tiling $T \in \Omega_\Phi$ has the Meyer property (i.e., if the set of punctures $p(T)$ is a Meyer set). This does not depend on the choice of punctures and therefore holds true also if punctures are control points in the sense of [LSo2]. We shall next consider primitive aperiodic Meyer substitutions. In this context, Ω_Φ is minimal and the Meyer property is satisfied for all $T \in \Omega_\Phi$ if it is satisfied for a single one.

DEFINITION 5.2 (Meyer substitution tiling). A *Meyer substitution tiling* is a tiling in the hull of a primitive aperiodic Meyer substitution.

5.2. Finite Rank and Fiber Distality of Meyer Substitutions

Recall that the maximal rank of a tiling is

$$\sup\{\#\pi_{\max}^{-1}(x) : x \in X_{\max}\},$$

which (of course) bounds the minimal rank.

For example, the so-called table substitution (see [BGGr]) is a Meyer substitution whose tiling system has minimal rank 4 and maximal rank 24.

THEOREM 5.3. *A Meyer substitution tiling system has finite maximal rank.*

Proof. By Corollary 3.5 there is N such that $\#\{B_1[T'] : \pi_{\max}(T') = \pi_{\max}(T)\} \leq N$ for all $T \in \Omega$. Suppose that $T_1, \dots, T_m$ are distinct tilings in Ω with $\pi_{\max}(T_i) = \pi_{\max}(T_j)$ for all i and j. Let R be large enough that $B_R[T_i] \neq B_R[T_j]$ for all $i \neq j$, and let k be large enough that $\Lambda^k(B_1(0)) \supset B_R(0)$. Then $B_1[\Phi^{-k}(T_i)] \neq B_1[\Phi^{-k}(T_j)]$ for $i \neq j$. Thus $m \leq N$ and $\pi_{\max}$ is at most N-to-one. □

Our next theorem extends the one-dimensional result of [BaKw].

THEOREM 5.4. *For a Meyer substitution tiling system, $X_{\max}^{\text{distal}}$ has full Haar measure.*

Proof. Since Λ is expanding, there is a $k \geq 1$ such that $B_R(0) \subset \Lambda^k(B_R(0))$. Replacing Φ by Φ^k, we may suppose that $k = 1$—that is, $B_R(0) \subset \Lambda(B_R(0))$—and hence $B_R[\Phi(T)] = B_R[\Phi(B_R[T])]$ for all $T \in \Omega$. It follows from this and the equality $\Phi_{\max} \circ \pi_{\max} = \pi_{\max} \circ \Phi$ that $n^R(\Phi_{\max}(x)) \leq n^R(x)$. Hence $D^R(m) := \{x \in X_{\max} : n^R(x) \leq m\}$ is invariant under $\Phi_{\max}$ for every $m \in \mathbb{N}$. By Lemma 3.7, $D^R(m)$ is open. By the ergodicity of $\Phi_{\max}$ with respect to the Haar measure (Lemma 2.23), $D^R(m)$ has full measure if it is nonempty. Since $X_{\max}^{\text{distal}} = \bigcap_{R \geq R_0} D^R(\text{cr})$, we are done once we show that $D^R(\text{cr}) \neq \emptyset$.

Consider a fiber with minimal rank; that is, choose $x \in X_{\max}$ such that $\pi_{\max}^{-1}(x) = \{T_1, \dots, T_{\text{mr}}\}$. Suppose that for all $r > 0$ there exists a $w \in \mathbb{R}^n$ such that, for all $t \in B_r(w)$, we have $n^R(x - t) \geq \text{mr}$; that is, all $B_R[T_i - t]$ $(1 \leq i \leq \text{mr})$ are distinct. Then we can find two sequences $(r_k)_k \to \infty$ and $(w_k)_k \in \mathbb{R}^n$ such that: the $(T_i - w_k)_k$ converge in Ω, let's say to S_i; $(x - w_k)_k$ converges in $X_{\max}$, say to y; and, for all $t \in B_{r_k}(0)$, all the $B_R[T_i - w_k - t]$ $(1 \leq i \leq \text{mr})$ are distinct. Taking $k \to \infty$, we conclude that all $B_R[S_i - t]$ with $1 \leq i \leq \text{mr}$ and $t \in \mathbb{R}^n$ are distinct. In particular, the S_i belong to the fiber of y and are pairwise nonproximal; hence $\text{cr} \geq \text{mr}$. This shows that $\text{cr} = \text{mr}$ and so $D^R(\text{cr})$ is not empty.

It remains to demonstrate that our assumption is satisfied. So let us suppose the contrary—namely, that there exists an $r > 0$ such that, for all $w \in \mathbb{R}^n$, there is a $t \in B_r(w)$ with $n^R(x - t) \leq \text{mr} - 1$. It follows that the lower density of points $t \in \mathbb{R}^n$ with $n^R(x - t) \leq \text{mr} - 1$ is strictly positive. Since for all t we have that

$n^R(x-t) \leq \mathrm{mr}$ (recall that x lies in a fiber of rank mr), the ergodic theorem implies that $\int_{X_{\max}} n^R(x)\,d\eta(x) < \mathrm{mr}$. Hence $D^R(mr-1)$ cannot have measure 0. Therefore, it must have measure 1. But then $\bigcap_{R\geq R_0} D^R(\mathrm{mr}-1)$ has measure 1 and so there must be a fiber of rank at most $\mathrm{mr}-1$, which contradicts the minimality of mr. In other words, the assumption of the preceding paragraph is correct; hence $D^R(\mathrm{cr})$ is nonempty and so $X_{\max}^{\mathrm{distal}}$ has full measure. □

COROLLARY 5.5. *The tiling flow on a Meyer substitution tiling space has pure discrete spectrum if and only if the proximal relation is closed.*

5.3. Pisot Family Substitutions

Geometry places rather strong conditions on the collection, $\mathrm{spec}(\Lambda)$, of eigenvalues of the expansion matrix Λ of a substitution. To begin with, all elements of $\mathrm{spec}(\Lambda)$ must have absolute value greater than 1 simply because Λ is an expansion. To say more, it is convenient to introduce the notion of a family. Let p be a monic and irreducible integer polynomial, and let $c > 0$ be a real number. A collection of complex numbers of the form

$$F_{p,c} := \{\lambda \in \mathbb{C} : p(\lambda) = 0,\ |\lambda| \geq c\}$$

is called a *family*. That is, a nonempty family is a collection of all the algebraic conjugates of some algebraic integer λ whose absolute values are no less than λ. In general, the elements of $\mathrm{spec}(\Lambda)$ must be algebraic integers [Ke1; LSo1], and if Λ is diagonalizable over $\mathbb{C}$ then $\mathrm{spec}(\Lambda)$ must be a union of families [KeSo]. In the special case that $\Lambda = \lambda I$, the substitution is called *self-similar* and the tiling space has a nontrivial equicontinuous factor (equivalently, the $\mathbb{R}^n$-action has eigenvalues) if and only if λ is a Pisot number: an algebraic integer, greater than 1, all of whose algebraic conjugates have absolute value less than 1. We shall refer to a family $F_{p,c}$ as a *Pisot family* if $c = 1$ and no element of $F_{p,c}$ has absolute value 1.

Recall that eigenvalues for a group action are continuous characters. When the group is $\mathbb{R}^n$, any such character takes the form $\chi(x) = e^{2\pi i\langle x,\beta\rangle}$ for some $\beta \in \mathbb{R}^n$. In this context, it is customary (as in the following theorem) to call β, rather than χ, an eigenvalue of the action.

THEOREM 5.6 [LSo2]. *Consider a primitive FLC substitution with diagonalizable expansion matrix* Λ, *and suppose that* $\mathrm{spec}(\Lambda)$ *consists of algebraic conjugates with the same multiplicity. Then the following statements are equivalent*:

(i) *the substitution is Meyer*;
(ii) $\mathrm{spec}(\Lambda)$ *is a Pisot family*;
(iii) *the eigenvalues are relatively dense in* $\mathbb{R}^n$;
(iv) *the maximal equicontinuous factor is nontrivial.*

To capture all these desirable qualities, we say that a substitution is a *Pisot family substitution* if it is primitive, aperiodic, and FLC and if its linear expansion is diagonalizable over $\mathbb{C}$ and has a Pisot family spectrum with all elements of the same multiplicity. The *degree* of such a substitution is the algebraic degree of the elements of the Pisot family, and its *multiplicity* is the common multiplicity of those elements as eigenvalues of the expansion.

Theorem 5.6 shows that, under the stated assumptions, either the maximal equicontinuous factor is a single point or the set of eigenvalues contains a subgroup that is relatively dense in $\hat{\mathbb{R}}^n$. We will improve this result by determining completely the form of the group of eigenvalues and the maximal equicontinuous factor for Pisot family substitutions. The key result is an extension of Corollary 3.5.

THEOREM 5.7. *Let Ω be the continuous hull of an aperiodic primitive FLC substitution Φ with expansion matrix Λ. Let $g\colon \Omega \to \mathbb{X}$ be a factor of the tiling flow such that the $\mathbb{R}^n$-action on $\mathbb{X}$ is locally free and*

(i) $g(T) = g(T')$ *implies* $g(\Phi(T)) = g(\Phi(T'))$.

Let p be a puncture map. Then the set

$$\mathcal{O} := \{x : \exists T, T' \text{ with } g(T) = g(T'),\ \tau \in T,\ \tau' \in T',\ \text{and } \mathring{\tau} \cap \mathring{\tau}' \neq \emptyset \quad \text{s.t. } p(\tau) - p(\tau') = x\}$$

is finite.

Proof. Suppose $\mathcal{O}$ is infinite. Then, for all n, there exist sequences $(T_n)_n, (T_n')_n \in \Omega$ with $\tau \in T_n$, $\tau_n' \in T_n'$, $g(T_n) = g(T_n')$, and $\mathring{\tau} \cap \mathring{\tau}_n' \neq \emptyset$ and with $p(\tau_m') \neq p(\tau_n')$ for $m \neq n$. The conclusion of the theorem is independent of the choice of puncture, so we may assume that the puncture of any tile is in the interior of the support of the tile. Translating, we may also assume $0 = p(\tau)$, and passing to a subsequence allows us to assume that

(ii) $T_n \to T$ and $T_n' \to T'$ for some T and T',
(iii) $p(\tau_n') \to p(\tau')$ for $\tau' \in T'$, and
(iv) $\Phi^n(T) \to \bar{T}$ and $\Phi^n(T') \to \bar{T}'$ for some $\bar{T}$ and $\bar{T}'$.

For $x_n := p(\tau_n') - p(\tau')$, there is a function $m\colon \mathbb{N} \to \mathbb{N}$ with $m(n) \to \infty$ as well as an $x \neq 0$ (but close enough to 0 that it acts freely) such that

(v) $\Lambda^{m(n)}(x_n) \to x$.

Note that, since $0 \in \mathring{\tau}$ and $\tau \in T_n$,

(vi) $\Phi^{m(n)}(T_n) \to \bar{T}$;

likewise, since (at least for large n) $T_n' - x_n$ and T' have exactly the same tiles at the origin, it follows that

(vii) $\Phi^{m(n)}(T_n' - x_n) \to \bar{T}'$.

By the continuity of g, $g(T_n) = g(T_n')$ implies $g(T) = g(T')$ and hence (i) and (iv) together imply $g(\bar{T}) = g(\bar{T}')$. Also, $g(T_n) = g(T_n')$ implies $g(\Phi^{m(n)}(T_n)) = g(\Phi^{m(n)}(T_n'))$, so

$$\lim g(\Phi^{m(n)}(T_n' - x_n)) \overset{\text{(vii)}}{=} g(\bar{T}') = g(\bar{T}) \overset{\text{(vi)}}{=} \lim g(\Phi^{m(n)}(T_n)) = \lim g(\Phi^{m(n)}(T_n')).$$

Since $\lim g(\Phi^{m(n)}(T_n' - x_n)) = \lim g(\Phi^{m(n)}(T_n')) - x$ by (v), we obtain a contradiction to the freeness of the action of x on $\mathbb{X}$. □

REMARK. Putnam proves in [P] that if g is any u-resolving factor map between Smale spaces (i.e., if g is injective on unstable sets), then g is finite-to-one. Theorem 5.7 is a corollary to that result under the assumption that g is a semiconjugacy with a suitably hyperbolic action on $\mathbb{X}$—as will be the case for applications of Theorem 5.7 in this paper.

The following statement is the required extension of Corollary 3.5.

COROLLARY 5.8. *Given the hypotheses of Theorem 5.7, up to translation there are only finitely many pairs of patches of the form* $(B_0[T], B_0[T'])$ *with* $g(T) = g(T')$.

COROLLARY 5.9. *Suppose that the hypotheses of Theorem 5.7 hold and that* $g(T) = g(T')$ *implies* $g(\Phi^{-1}(T)) = g(\Phi^{-1}(T'))$; *then* g *is boundedly finite-to-one*.

Proof. The proof of this result is the same as that for Theorem 5.3. □

We now identify $\hat{\mathbb{R}}^n$ with the dual vector space $\mathbb{R}^{n*}$ such that $b^* \in \mathbb{R}^{n*}$ corresponds to the character $t \mapsto e^{2\pi\imath b^*(t)}$. Then each endomorphism Λ on $\mathbb{R}^n$ has a dual endomorphism, which we denote Λ^*; with respect to an orthonormal basis, Λ^* is the transpose of Λ.

THEOREM 5.10. *Consider a Pisot family substitution of degree* d *and multiplicity* J *and with linear expansion* Λ. *There exists a lattice* Γ *of rank* dJ *that is relatively dense in* $\mathbb{R}^{n*}$ *and such that the group of eigenvalues of the* $\mathbb{R}^n$*-action is exactly*

$$\mathcal{E} = \varinjlim(\Gamma, \Lambda^*).$$

Proof. It was established in [LSo2] that, for a Pisot family substitution, there exist J vectors $b_1^*, \dots, b_J^* \in \mathbb{R}^{n*}$ such that $\{\Lambda^{*m} b_i^* : 1 \le i \le J, 0 \le m \le d-1\}$ is a collection of eigenvalues that is linearly independent over $\mathbb{Q}$ and spans $\mathbb{R}^{n*}$. (Linear independence over $\mathbb{Q}$ is not stated explicitly in [LSo2], but it can be easily derived from what is written there.) Let Γ' be the group generated by these eigenvalues; it is a lattice of rank dJ. Since $\mathcal{E}$ is invariant under Λ^*, it contains the group H generated by $\{\Lambda^{*m} b_i^* : 1 \le i \le J, m \in \mathbb{Z}\} = \varinjlim(\Gamma', \Lambda^*)$. The strategy is to show that $i : H \hookrightarrow \mathcal{E}$ is a finite index inclusion of H in $\mathcal{E}$, which is equivalent to the map $\hat{i} : \hat{\mathcal{E}} \to \hat{H}$ being finite-to-one. This can be seen as follows.

Let $g : X \to \hat{H}$ for $g = \hat{i} \circ \pi_{\max}$. This is a factor map because g is surjective and $\mathbb{R}^n$-equivariant, where the $\mathbb{R}^n$-action on $j \in \hat{H}$ is given by $(t \cdot j)(\Lambda^{*m} b_i^*) = e^{2\pi\imath \Lambda^{*m} b_i^*(t)} j(\Lambda^{*m} b_i^*)$. Since H is relatively dense in $\hat{G}$, Lemma 2.22 implies that this action is locally free. Furthermore, $\Lambda^{*m} \circ i = i \circ \Lambda^{*m}$ and so $\Lambda^{*m} \circ g = g \circ \Phi^m$. Hence g satisfies the assumptions of Corollary 5.9. It follows that $\hat{\imath}$ must be finite-to-one and that H is a finite index subgroup of $\mathcal{E}$.

Now it follows that there are finitely many vectors $w_1, \dots, w_k$ such that $\sum_i w_i + H = \mathcal{E}$. So for some N, $Nw_i \in H$ for all i (otherwise the index would be infinite). By definition of the direct limit, this means that $Nw_i \in \Lambda^{*-m}\Gamma'$ for some m and hence also that $w_i \in \frac{1}{N}\Lambda^{*-m}\Gamma'$. Let Γ be the group generated by $\Lambda^{*-m}\Gamma'$ and the w_i. Then we have the inclusions $\Gamma' \subset \Gamma \subset \frac{1}{N}\Gamma'$, all of finite index. Hence Γ has the same rank as Γ'. Note also that Γ is invariant under Λ^*, since both $\mathcal{E}$ and H are invariant. Thus we have $\Gamma \subset \mathcal{E}$ and $\mathcal{E} \subset \varinjlim(\Gamma, \Lambda^*)$. The invariance of $\mathcal{E}$ under Λ^* now implies that the last inclusion is an equality. □

COROLLARY 5.11. *The maximal equicontinuous factor is an inverse limit of dJ-tori,*

$$\hat{\mathcal{E}} = \varprojlim(\mathbb{T}^{dJ}, \widehat{\Lambda^*}),$$

where $\mathbb{T}^{dJ} = \hat{\Gamma}$. Its $\mathbb{R}^n$-action is free, and $\Phi_{\max}$ is ergodic with respect to Haar measure.

Proof. Only the last point requires comment. By definition, a linear expansion has no eigenvalues on the unit circle, so the result follows from Lemma 2.23. □

Recall that Φ is assumed to have a fixed point, so $\Phi_{\max} = \widehat{\Lambda^*}$. (Here $\widehat{\Lambda^*}$ is the dual map to Λ^*, which should not be confused with $\Lambda^{**} = \Lambda$ because dualization is w.r.t. the group Γ and not w.r.t. $\hat{\mathbb{R}}^n$.) In fact, by the Pisot family condition, $\widehat{\Lambda^*}$ can be written as $\mathrm{diag}(A, \dots, A)$ (with J copies) in some basis for some integer matrix A whose characteristic polynomial is the minimal monic polynomial having the eigenvalues of Λ as roots. Note that if $\det A = \pm 1$ then $\mathcal{E} = \Gamma$ and $\hat{\mathcal{E}} = \mathbb{T}^{dJ}$.

5.4. *Further Results*

The following lemma is a generalization of a result from [BaKw] that is based on the definition of the coincidence rank.

LEMMA 5.12. *Consider a Meyer substitution tiling system, $x \in X_{\max}$ and $T, T' \in \pi_{\max}^{-1}(x)$. If T and T' are not proximal, then they do not have a single tile in common*: $T \cap T' = \emptyset$.

Proof. Recall that δ_0 is such that $\mathrm{cr} = cr(x, \delta_0)$. Therefore, if $T, T' \in \pi_{\max}^{-1}(x)$ and if T and T' are not proximal, then $\inf_v d(T - v, T' - v) \geq \delta_0$ and δ_0 does not depend on T, T'. By Theorem 5.6 and Corollary 3.5, we can reformulate this claim as follows:

$$\sup_v \{R : B_R[T - v] = B_R[T' - v])\} \leq R_0 < +\infty,$$

where R_0 does not depend on T, T'. Since the substitution Φ preserves fibers of $\pi_{\max}$ and respects the proximality relation, we also have $\sup_v\{R : B_R[\Phi(T) - v] = B_R[\Phi(T') - v])\} \leq R_0$. But $B_R[T - v] = B_R[T' - v]$ implies that $B_{\lambda R}[\Phi(T) - v] = B_{\lambda R}[\Phi(T') - v]$ for some $\lambda > 1$. This is possible only if $R_0 = 0$. □

The coincidence rank thus counts the maximal number of tilings in a fiber of $\pi_{\max}$ that are pairwise noncoincident in the following sense: they do not share a single tile.

6. Syndetic Proximality

6.1. General Results

In [C], Clay discusses proximality and regional proximality as well as the notion of syndetic proximality. We investigate this relation more closely for Meyer substitution tilings. Fairly general dynamical systems (X, G) are allowed in [C]; however, since we are interested in tilings of the Euclidean space, we restrict the discussion to $G = \mathbb{R}^n$ acting continuously on a compact metrizable space X.

For $\varepsilon > 0$ and $x, y \in X$, let $\Gamma_\varepsilon(x, y) = \{t \in \mathbb{R}^n : d(t \cdot x, t \cdot y) \leq \varepsilon\}$. Recall that a relatively dense subset $\Gamma \subset G$ of a topological group G is called *syndetic* if there exists a compact subset $K \subset G$ such that $K + \Gamma = G$. For $G = \mathbb{R}^n$, the notions of "syndetic" and "relatively dense" coincide.

DEFINITION 6.1 [C]. Two points $x, y \in X$ are *syndetically proximal,* written $x \sim_{syp} y$, if $\Gamma_\varepsilon(x, y)$ is syndetic for all $\varepsilon > 0$.

Note that a subset Γ is syndetic if and only if its complement does not contain a translate of every compact subset of $\mathbb{R}^n$. So if we denote by $\mathcal{A}$ the collection of all sets $A \subset \mathbb{R}^n$ that contain a translate of every compact subset of $\mathbb{R}^n$, then we can rephrase syndetic proximally as follows: $x, y \in X$ are proximal in A, written $x \sim_{A,p} y$, if $\inf_{t \in A} d(t \cdot x, t \cdot y) = 0$.

LEMMA 6.2. *$x \sim_{syp} y$ if and only if $x \sim_{A,p} y$ for all $A \in \mathcal{A}$.*

Proof. If $\Gamma_\varepsilon(x, y)$ is not syndetic then $\Gamma_\varepsilon(x, y)^c \in \mathcal{A}$. Hence for $A = \Gamma_\varepsilon(x, y)^c$ we cannot have $x \sim_{A,p} y$. If $\Gamma_\varepsilon(x, y)$ is syndetic then $\Gamma_\varepsilon(x, y)^c \notin \mathcal{A}$, in which case there exists a compact K such that $\Gamma_\varepsilon(x, y)^c$ does not contain any of its translates. Hence no $A \in \mathcal{A}$ is contained in $\Gamma_\varepsilon(x, y)^c$. But then all $A \cap \Gamma_\varepsilon(x, y) \neq \emptyset$. Since ε is arbitrary in that argument, the statement of the lemma follows. □

The following useful facts (and much more) are proved in [C].

THEOREM 6.3. (i) *Syndetic proximality is an equivalence relation* [C, Thm. 1].

(ii) *If proximality is closed, then it agrees with syndetic proximality* [C, Thm. 3(3)].

(iii) *Let $x \sim_{syp} y$, and let $\bar{x} = \lim t_n \cdot x$ and $\bar{y} = \lim t_n \cdot y$ for some sequence $(t_n)_n \subset G$. Then $\bar{x} \sim_{syp} \bar{y}$* [C, Lemma 4].

6.2. Syndetic Proximality for Meyer Substitutions

In this section, Φ will denote a substitution whose tiling space Ω has the Meyer property. We show that syndetic proximality is a closed relation on Ω. The group G is $\mathbb{R}^n$, and again we denote the action by $T - v$ instead of by $v \cdot T$.

We say that a pair of patches (P, P') occurs in the pair of tilings (T, T') if there is a $v \in \mathbb{R}^n$ such that $P - v \subset T$ and $P' - v \subset T'$. Let $\mathcal{R}$ be the relation on Ω defined as follows: $(S, S') \in \mathcal{R}$ if and only if, for each $v \in \mathbb{R}^n$ and each $r > 0$, there exist $T, T' \in \Omega$ with $T \sim_{syp} T'$ such that $(B_r[S - v], B_r[S' - v])$ occurs in (T, T').

LEMMA 6.4. *The relation $\mathcal{R}$ is closed, and $\mathcal{R} \subset \mathcal{Q}$.*

Proof. We show first that $\mathcal{R}$ is closed. Suppose $(S_n)_n$ and $(S'_n)_n$ are two converging sequences—toward S and S', respectively—such that $(S_n, S'_n) \in \mathcal{R}$. Let v and r be given. Then there is an N such that, for all $n \geq N$, there exist $\varepsilon_n, \varepsilon'_n$ such that

$$B_r[S - v - \varepsilon_n] = B_r[S_n - v], \qquad B_r[S' - v - \varepsilon'_n] = B_r[S'_n - v].$$

Furthermore, we may assume that the ε_n and ε'_n form sequences tending to 0. Since $(S_n, S'_n) \subset \mathcal{R}$, there exist T, T' such that $(B_r[S - v - \varepsilon_n], B_r[S' - v - \varepsilon'_n])$ occurs in $(T - \varepsilon_n, T' - \varepsilon'_n)$ and $T - \varepsilon_n \sim_{syp} T' - \varepsilon'_n$. By the Meyer property (Corollary 3.5) we must have $\varepsilon_n = \varepsilon'_n$ for large enough n; therefore, $T \sim_{syp} T'$. Moreover, $(B_r[S - v], B_r[S' - v])$ occurs in (T, T') and so $(S, S') \in \mathcal{R}$.

Let $(S, S') \in \mathcal{R}$. For each $k \in \mathbb{N}$ there are $T_k \sim_{syp} T'_k$ such that $(B_k[S], B_k[S'])$ occurs in (T_k, T'_k). Then $T_k \to S$ and $T'_k \to S'$. Since $T_k \sim_{syp} T'_k$ implies $(T_k, T'_k) \in \mathcal{Q}$ and since $\mathcal{Q}$ is closed, it follows that $(S, S') \in \mathcal{Q}$. □

LEMMA 6.5. *Suppose that $(T, T') \in \mathcal{Q}$ and that the pair of finite patches (P, P') occurs in (T, T'). Then there exist $l \in \mathbb{Z}$ and $L \in \mathbb{N}$ such that (P, P') occurs in $(\Phi^{-(kL+l)}(T), \Phi^{-(kL+l)}(T'))$ for all $k \in \mathbb{N}$.*

Proof. Let $r > 0$ be large enough that $P \subset B_r[T]$ and $P' \subset B_r[T']$. Since Φ^{-1} preserves regional proximality, we have $(\Phi^{-k}(T), \Phi^{-k}(T')) \in \mathcal{Q}$ for all k. Hence we may conclude from Corollary 3.5 that there are $k_i \to \infty$ such that

$$(Q, Q') := (B_r[\Phi^{-k_i}(T)], B_r[\Phi^{-k_i}(T')])$$

does not depend on i. Let i be large enough that

$$B_r[T] \subset \Phi^{k_i}(B_r[\Phi^{-k_i}(T)]), \qquad B_r[T'] \subset \Phi^{k_i}(B_r[\Phi^{-k_i}(T')]),$$

and let $j > i$ be large enough that

$$B_r[\Phi^{-k_i}(T)] \subset \Phi^{k_j - k_i}(B_r[\Phi^{-k_j}(T)]), \quad B_r[\Phi^{-k_i}(T')] \subset \Phi^{k_j - k_i}(B_r[\Phi^{-k_j}(T')]).$$

Let $L := k_j - k_i$. Then there is an $l' \in \{0, \dots, L - 1\}$ such that $k_s \equiv l'$ modulo L for infinitely many s. So (Q, Q') occurs in $(\Phi^{-(kL+l')}(T), \Phi^{-(kL+l')}(T'))$ for all $k \in \mathbb{N}$ and (P, P') occurs in $(\Phi^{-(kL+l)}(T), \Phi^{-(kL+l)}(T'))$ for all $k \in \mathbb{N}$, where $l := l' - k_i$. □

PROPOSITION 6.6. *Syndetic proximality is closed for Meyer substitution tiling spaces.*

Proof. We will prove that $\mathcal{R}$ is the same as the syndetic proximality relation. That $T \sim_{syp} T'$ implies $(T, T') \in \mathcal{R}$ is immediate, so suppose $(S, S') \in \mathcal{R}$. By Lemma 6.4, $(S, S') \in \mathcal{Q}$ and hence there is an increasing sequence of positive integers $k_i \to \infty$ such that

$$(P, P') := (B_1[\Phi^{-k_i}(S)], B_1[\Phi^{-k_i}(S')])$$

does not depend on i. By recognizability (i.e., since Φ is invertible), there is an $r > 0$ such that if

$$(B_r[T], B_r[T']) = (B_r[S], B_r[S'])$$

then

$$(B_1[\Phi^{-k_1}(T)], B_1[\Phi^{-k_1}(T')]) = (B_1[\Phi^{-k_1}(S)], B_1[\Phi^{-k_1}(S')]).$$

Since $(S, S') \in \mathcal{R}$, there are $T \sim_{syp} T'$ with $(B_r[T], B_r[T']) = (B_r[S], B_r[S'])$. We apply Lemma 6.5 to $(\Phi^{-k_1}(T), \Phi^{-k_1}(T'))$ and so obtain $l \in \mathbb{Z}$ and $L \in \mathbb{N}$ such that (P, P') occurs in

$$(\Phi^{-(kL+l)}(\Phi^{-k_1}(T)), \Phi^{-(kL+l)}(\Phi^{-k_1}(T')))$$

for all $k \in \mathbb{N}$. Let $l' \in \{0, \dots, L-1\}$ be such that $k_i \equiv l' \bmod L$ for infinitely many i, say $k_{i_j} = m_j L + l'$ with $m_j \in \mathbb{N}$ and $m_j \to \infty$. Let $\bar{T} := \Phi^{l+k_1-l'}(T)$ and $\bar{T}' := \Phi^{l+k_1-l'}(T')$. Then $\bar{T} \sim_{syp} \bar{T}'$ and (P, P') occurs in $(\Phi^{-k_{i_j}}(\bar{T}), \Phi^{-k_{i_j}}(\bar{T}'))$ for all j. Suppose $B_1[\Phi^{-k_{i_j}}(S)] - v_j \in \Phi^{-k_{i_j}}(\bar{T})$ and $B_1[\Phi^{-k_{i_j}}(S')] - v_j \in \Phi^{-k_{i_j}}(\bar{T}')$, and let $r_j \to \infty$ be such that $\Phi^{k_{i_j}}(B_1[\Phi^{-k_{i_j}}(S)]) \supset B_{r_j}[S]$ and $\Phi^{k_{i_j}}(B_1[\Phi^{-k_{i_j}}(S)]) \supset B_{r_j}[S]$. Then $\bar{T} + \lambda^{k_{i_j}} v_j$ and $\bar{T}' + \lambda^{k_{i_j}} v_j$ agree with S and S' (respectively) on $B_{r_j}(0)$, so $\bar{T} + \lambda^{k_{i_j}} v_j \to S$ and $\bar{T}' + \lambda^{k_{i_j}} v_j \to S'$ as $j \to \infty$. It then follows from Lemma 3 that $S \sim_{syp} S'$. □

Corollary 6.7. *For Meyer substitution tiling spaces, proximality is closed if and only if it coincides with syndetic proximality.*

This result can be compared to [MoOp, Thm. 5.3], which states that—for constant-length substitutions with overall coincidence—the proximal relation coincides with the syndetically proximal relation. More precisely, the substitutions in question are symbolic substitutions and the proximal and syndetic proximal relation considered are for the $\mathbb{Z}$-action on the substitution sequence space. But given that the substitution has constant length, we can suspend the letters to intervals of equal length and then label them according to the letter type. Then the suspension flow corresponds to the translation action on the tiling space of a Meyer substitution tiling in the sense discussed in this paper. If we also require that the substitution be primitive, then overall coincidence implies pure point dynamical spectrum (for both the $\mathbb{Z}$-action on the substitution sequence space and the $\mathbb{R}$-action on the substitution tiling space Ω [De]) and thus, as we saw in Section 5.2, it implies closedness of the proximality relation. Hence [MoOp, Thm. 5.3] corresponds to a special case of one direction of Corollary 6.7.

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