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On the Representation of Holomorphic Functions on Polyhedra

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1. Introduction

1.1. Oka's Theorem

The following beautiful theorem of Oka, which gives a representation for holomorphic functions defined on p-polyhedra in $\mathbb{C}^d$, has played a significant role in the development of several complex variables.

THEOREM 1.1 (Oka [26], as presented in [7]). *Let $\delta_1, \dots, \delta_m$ be a collection of polynomials in d variables normalized such that the p-polyhedron K_δ defined by*

$$K_\delta = \{\lambda \in \mathbb{C}^d \mid |\delta_l(\lambda)| \le 1 \textit{ for } l = 1, \dots, m\}$$

lies in $\mathbb{D}^d$. If ϕ is holomorphic on a neighborhood of K_δ, then there exists a function Φ, holomorphic on a neighborhood of $(\mathbb{D}^-)^{d+m}$, such that

$$\phi(\lambda) = \Phi(\lambda, \delta(\lambda))$$

for all $\lambda \in K_\delta$.

Introduced originally in 1936 to give an elegant new proof of the Oka–Weil approximation theorem [26; 33], Oka's theorem was a stem theorem for the development of the theory of analytic sheaves—a powerful tool for applying function theory to domains of holomorphy and, more generally, Stein spaces [19; 21]. Basic to the understanding of polynomial convexity, Oka's theorem played an important role in the development of the theory of Banach algebras. Many operator theorists first learn of this theorem in the context of one of its many basic implications: the Arens–Calderon trick [10], which is fundamental to spectral theory and to the corresponding functional calculus for commuting tuples of operators [18; 28; 29].

1.2. Oka Mappings

In this paper we show how ideas that are currently evolving within the operator theory community can be adapted to obtain precise bounds for Oka's theorem.

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These bounds are defined using operator theory, and the problem of computing them—or, indeed, even estimating them in any meaningful fashion in terms of function theory—remains in large part unexplored.

In addition to these new bounds that we will obtain, there is a second contribution presented in this paper to our understanding of Oka's theorem. The idea is to drop Oka's normalization requirement and, more severely, to prevent the representing function Φ from "seeing" the coordinates λ. Specifically, we introduce the following definition.

DEFINITION 1.2. Let δ be an m-tuple of polynomials in d variables. We say that δ is an *Oka mapping* if, whenever ϕ is a function that is holomorphic on a neighborhood of K_δ, there exists a function Φ, holomorphic on a neighborhood of $(\mathbb{D}^-)^m$, such that $\phi = \Phi \circ \delta$ on K_δ.

With this language, Oka's theorem evidently becomes the assertion that if δ is an m-tuple of polynomials in d variables and if $K_\delta \subseteq \mathbb{D}^d$, then (λ, δ) is an Oka mapping. Of course, this leaves open the question of whether or not the map δ itself is an Oka map.

One approach to understanding Oka mappings is to use the Cartan extension theorem [16], which provides a purely geometric characterization of Oka mappings. We let

$$G_\delta = \{\lambda \in \mathbb{C}^d \mid |\delta_l(\lambda)| < 1 \text{ for } l = 1, \dots, m\}.$$

One always has G_δ equal to the interior of K_δ, which we shall denote by K_δ° (this is proved in Lemma 2.1); however, it need not be the case that $K_\delta = G_\delta^-$, the closure of G_δ.

THEOREM 1.3. *If δ is an m-tuple of polynomials in d variables, then δ is an Oka mapping if and only if there exists a $t < 1$ such that δ embeds $G_{t\delta}$ as an analytic submanifold in $\frac{1}{t}\mathbb{D}^m$ (i.e., such that δ is an injective, proper, and unramified mapping from $G_{t\delta}$ into $\frac{1}{t}\mathbb{D}^m$).*

The implication $\Leftarrow$ follows from [27, Thm. 7.1.5], and the converse follows from observing that, for some t and for each coordinate function λ^j, $1 \le j \le d$, there is a function Φ_j holomorphic on $\frac{1}{t}\mathbb{D}^m$ such that $\Phi_j(\delta(\lambda)) = \lambda^j$.

To return to operator theory, we consider an analogue of K_δ but with points in $\mathbb{C}^d$ replaced by d-tuples of pairwise commuting operators, $T = (T^1, T^2, \dots, T^d)$, acting on a complex Hilbert space. Thus we define

$$\mathcal{F}_\delta = \{T \mid \|\delta_l(T)\| \le 1,\ l = 1, \dots, m\}. \tag{1.4}$$

This one simple definition allows us to obtain a second condition for δ to be an Oka map, one that is stated in operator-theoretic terms.

PROPOSITION 1.5. *If δ is an m-tuple of polynomials in d variables, then δ is an Oka mapping if and only if there exists a $t < 1$ such that $\mathcal{F}_{t\delta}$ is bounded.*

We remark that the classical Oka theorem (Theorem 1.1) is an immediate corollary of both Theorem 1.3 and Proposition 1.5. This is because (i) for $t < 1$ but sufficiently close to 1, Oka's normalization condition $K_\delta \subseteq \mathbb{D}^d$ implies that $G_{t\delta} \subseteq \mathbb{D}^d$ and (ii) the map $(t\lambda, t\delta)$ is clearly an analytic embedding of $G_{t\delta}$ into $\frac{1}{t}\mathbb{D}^{m+d}$. Likewise, the family $\mathcal{F}_{(t\lambda,t\delta)}$ is bounded because (1.4) implies that, if $T \in \mathcal{F}_{(t\lambda,t\delta)}$, then $\|T^r\| \leq \frac{1}{t}$ for each $r = 1, \ldots, d$.

Not all δ are Oka mappings. For an m-tuple δ of polynomials in d variables that is not necessarily an Oka mapping, it is natural to ask the following question.

QUESTION 1.6. *Given ϕ holomorphic on a neighborhood of K_δ, does there exist a Φ, holomorphic on a neighborhood of $(\mathbb{D}^-)^m$, such that $\phi = \Phi \circ \delta$ on K_δ?*

The approach to this question via the Cartan extension theorem would go something like this. First, we would hope that for $t < 1$ but sufficiently close to 1, $\delta(G_{t\delta})$ is an analytic variety in $\frac{1}{t}\mathbb{D}^m$. Then the condition for representing ϕ as in Oka's theorem would be that the function $\delta^\sim$, defined on $\delta(G_{t\delta})$ by the formula

$$\delta^\sim(\delta(\lambda)) = \phi(\lambda),$$

be a well-defined analytic function on $\delta(G_{t\delta})$ that could be extended via the Cartan theorem. Addressing the analyticity of $\delta^\sim$ would require the full strength of analytic sheaf theory and would proceed with great difficulty. A fundamental problem with this approach is that $\delta(G_{t\delta})$ need not be an analytic variety in $\frac{1}{t}\mathbb{D}^m$ for any $t \leq 1$. For example, if $d = m = 2$, $\delta_1 = \lambda^1$, and $\delta_2 = \lambda^1\lambda^2$, then

$$\delta(G_{t\delta}) = \left\{\lambda : |\lambda^1| < \tfrac{1}{t}, |\lambda^2| < \tfrac{1}{t}\right\} \setminus \left[\{0\} \times \tfrac{1}{t}(\mathbb{D}^d \setminus \{0\})\right].$$

To answer Question 1.6 in terms of operator theory, we return to $\mathcal{F}_\delta$ and note that, as a simple consequence of the spectral mapping theorem, we have $\sigma(T) \subseteq K_\delta$ whenever $T \in \mathcal{F}_\delta$. Thus, if ϕ is holomorphic on a neighborhood of K_δ then $\phi(T)$ can be defined by the Taylor functional calculus. Hence, for ϕ holomorphic on a neighborhood of K_δ, we may define

$$\|\phi\|_\delta^+ = \sup_{T \in \mathcal{F}_\delta} \|\phi(T)\|. \tag{1.7}$$

Question 1.6 can be answered in terms of the quantity defined in (1.7).

PROPOSITION 1.8. *If δ is an m-tuple of polynomials in d variables and if ϕ is holomorphic on a neighborhood of K_δ, then there exists a Φ, holomorphic on a neighborhood of $(\mathbb{D}^-)^m$, such that $\phi = \Phi \circ \delta$ on K_δ if and only if there exists a $t < 1$ such that $\|\phi\|_{t\delta}^+ < \infty$.*

1.3. Bounds for the Oka Representation

To describe our bounds for the Oka extension, we shall employ a norm on holomorphic functions essentially introduced by von Neumann in [32]. That paper, which has had a profound influence on the development of operator theory, was the first to demonstrate that norms defined with the aid of operators can be natural from the point of view of function theory.

THEOREM 1.9 (von Neumann's inequality [32]). *If C is a contraction acting on a complex Hilbert space and if Φ is a function holomorphic on a neighborhood of $\mathbb{D}^-$, then*

$$\|\Phi(C)\| \leq \max_{z\in\mathbb{D}^-}|\Phi(z)|. \tag{1.10}$$

One can reformulate von Neumann's inequality by stating that, if Φ is assumed to be holomorphic on a neighborhood of $\mathbb{D}^-$, then

$$\sup_{\|C\|\leq 1}\|\Phi(C)\| = \max_{z\in\mathbb{D}^-}|\Phi(z)|. \tag{1.11}$$

Twelve years after von Neumann published his inequality, Andô [9] proved a surprising and subtle generalization to two variables. If Φ is holomorphic on a neighborhood of $(\mathbb{D}^-)^2$ then, with $\mathcal{F}_2$ defined by

$$\mathcal{F}_2 = \{C = (C_1, C_2) \mid \|C_1\| \leq 1,\ \|C_2\| \leq 1,\ C_1C_2 = C_2C_1\}, \tag{1.12}$$

the following analogue of (1.11) obtains:

$$\sup_{C\in\mathcal{F}_2}\|\Phi(C)\| = \max_{z\in(\mathbb{D}^-)^2}|\Phi(z)|. \tag{1.13}$$

Unfortunately, when the operator theory community asked for the obvious analogue of (1.13) to hold in dimension 3, they were surprised to learn [17; 31] about examples of Φ being holomorphic on a neighborhood of $(\mathbb{D}^-)^3$ for which

$$\sup_{C\in\mathcal{F}_3}\|\Phi(C)\| > \max_{z\in(\mathbb{D}^-)^3}|\Phi(z)|. \tag{1.14}$$

Because it was the *right* side of (1.14)—defined as it is in terms of concrete function theory—that was thought to be the object of interest, the *left* side of (1.14) remained unexplored by operator theorists until the appearance in [2] of the following enshrinement of von Neumann's inequality as a definition. For $m \geq 1$, let

$$\mathcal{F}_m = \{C \mid C \text{ is an } m\text{-tuple of pairwise commuting contractions}\}.$$

This is the collection defined by (1.4) for $d = m$ and δ the identity map on $\mathbb{C}^m$.

DEFINITION 1.15. For $m \geq 1$ and $\Phi \in \mathrm{Hol}(\mathbb{D}^m)$, define $\|\Phi\|_m$ by

$$\|\Phi\|_m = \sup_{\substack{C\in\mathcal{F}_m \\ \sigma(C)\subseteq\mathbb{D}^m}}\|\Phi(C)\|.$$

The norm $\|\cdot\|_m$ occurs in many areas of multivariable function theory and operator theory—for example, in Nevanlinna–Pick interpolation [1; 3; 15], in realization theory [13; 14], in the theory of matrix monotone functions [6], in Carathéodory–Julia theorems on the polydisk [5], and so forth.

We can now describe how to derive bounds for the Oka representation. Observe from (1.4) that if $s < t$ then $\mathcal{F}_{t\delta} \subseteq \mathcal{F}_{s\delta}$. Equally obvious from (1.7) is that if $\mathcal{F}_\delta \subseteq \mathcal{F}_\gamma$ then $\|\phi\|_\delta^+ \leq \|\phi\|_\gamma^+$. Together these implications show that $\|\phi\|_{t\delta}^+$ is a monotone decreasing function of t. We may therefore define

$$\rho(\phi) = \lim_{t\to 1^-}\|\phi\|_{t\delta}^+.$$

Theorem 1.8 can now be reformulated to assert that, if ϕ is holomorphic on a neighborhood of K_δ, then there exists a Φ (holomorphic on a neighborhood of $(\mathbb{D}^-)^m$) such that $\phi = \Phi \circ \delta$ on K_δ if and only if $\rho(\phi) < \infty$. The following theorem describes the bounds we have for the classical Oka setting.

THEOREM 1.16. *Let δ be an m-tuple of polynomials in d variables and let ϕ be holomorphic on a neighborhood of K_δ. If Φ is holomorphic on a neighborhood of $(\mathbb{D}^-)^m$ and if $\phi = \Phi \circ \delta$, then $\rho(\phi) \leq \|\Phi\|_m$. Furthermore, if $\varepsilon > 0$ then there exists a Φ holomorphic on a neighborhood of $(\mathbb{D}^-)^m$ such that $\phi = \Phi \circ \delta$ and $\|\Phi\|_m < \rho(\phi) + \varepsilon$.*

Note that Propositions 1.8 and 1.5 are immediate corollaries of Theorem 1.16 provided that, as asserted in Lemma 4.1, every function Φ that is holomorphic on a neighborhood of $(\mathbb{D}^-)^m$ has $\|\Phi\|_m$ finite.

1.4. H^∞_δ and $H^\infty_{\mathbf{m}}$

So far we have restricted ourselves to the classical Oka setting, in which one seeks to represent functions ϕ that are holomorphic on a neighborhood of K_δ. Sharper theorems are obtainable for functions defined only on G_δ. However, if ϕ is defined only on G_δ then (1.7) makes no sense because $\phi(T)$ need not be well-defined for all $T \in \mathcal{F}_\delta$. To accommodate this difficulty, we modify the definition (1.7) to sup only over those $T \in \mathcal{F}_\delta$ such that $\sigma(T) \subseteq G_\delta$. Thus, for ϕ a holomorphic function on G_δ, we define $\|\phi\|_\delta$ by the formula

$$\|\phi\|_\delta = \sup_{\substack{T \in \mathcal{F}_\delta \\ \sigma(T) \subseteq G_\delta}} \|\phi(T)\|. \tag{1.17}$$

Tautologically, we have $\|\phi\|_\delta \leq \|\phi\|^+_\delta \leq \|\phi\|_{t\delta}$ when ϕ is holomorphic on a neighborhood of K_δ and $t < 1$ is sufficiently close to 1.

Armed with this definition, we can define the space H^∞_δ as consisting of all functions ϕ that are holomorphic on G_δ and such that $\|\phi\|_\delta$ is finite. Let $\mathbf{m}$ denote the identity polynomial on $\mathbb{C}^m$. Then the norm $\|\Phi\|_m$ from Definition 1.15 is the same as the norm $\|\Phi\|_{\mathbf{m}}$, and we can define $H^\infty_{\mathbf{m}}$ as consisting of all functions Φ that are holomorphic on $\mathbb{D}^m$ and such that $\|\Phi\|_m$ is finite.

It turns out that H^∞_δ and $H^\infty_{\mathbf{m}}$ equipped with these norms are Banach spaces. The following theorem makes it clear that these spaces are natural ones in which to study Oka representations.

THEOREM 1.18. *Let δ be an m-tuple of polynomials in d variables, and assume that ϕ is a holomorphic function on G_δ. Then the following statements hold.*

(a) *There exists a $\Phi \in H^\infty_{\mathbf{m}}$ such that $\phi = \Phi \circ \delta$ if and only if $\phi \in H^\infty_\delta$.*
(b) *If $\Phi \in H^\infty_{\mathbf{m}}$ and $\phi = \Phi \circ \delta$, then $\|\phi\|_\delta \leq \|\Phi\|_m$.*
(c) *If $\phi \in H^\infty_\delta$, then there exists a $\Phi \in H^\infty_{\mathbf{m}}$ such that $\phi = \Phi \circ \delta$ and $\|\phi\|_\delta = \|\Phi\|_m$.*

1.5. Realization Formula

Our proofs rely on the existence of realizations.

DEFINITION 1.19. Let ϕ be a function on G_δ. We say that a 4-tuple (a, β, γ, D) is a *realization* for ϕ if $a \in \mathbb{C}$ and if there exists a decomposed Hilbert space, $\mathcal{M} = \bigoplus_{l=1}^{m} \mathcal{M}_l$, such that: the 2×2 matrix

$$V = \begin{bmatrix} a & 1 \otimes \beta \\ \gamma \otimes 1 & D \end{bmatrix}$$

acts isometrically on $\mathbb{C} \oplus \mathcal{M}$; $\delta(\lambda)$ acts on $\mathcal{M}$ via the formula

$$\delta(\lambda)\left(\bigoplus_{l=1}^{m} x_l\right) = \bigoplus_{l=1}^{m} \delta_l(\lambda) x_l;$$

and

$$\phi(\lambda) = a + \langle \delta(\lambda)(1 - D\delta(\lambda))^{-1}\gamma, \beta \rangle$$

for all $\lambda \in G_\delta$.

Ambrozie and Timotin [8] proved that a function ϕ on G_δ has a realization if and only if $\|\phi\|_\delta^- \le 1$, where

$$\|\phi\|_\delta^- = \sup\{\|\phi(T)\| : \|\delta_l(T)\| < 1,\ 1 \le l \le m\}.$$

In Section 3 we develop the machinery showing that

$$\|\phi\|_\delta^- = \|\phi\|_\delta \quad \forall \phi \text{ holomorphic on } G_\delta$$

(Theorem 4.5). In fact, both norms agree with $\sup\{\|\phi(T)\|\}$ as T ranges over commuting d-tuples of diagonalizable matrices in $\mathcal{F}_\delta$ (Theorem 6.1).

2. H_δ^∞

Let $\delta = (\delta_1, \delta_2, \ldots, \delta_m)$ be an m-tuple of nonconstant polynomials with complex coefficients in d variables. We can view δ as a map from $\mathbb{C}^d$ into $\mathbb{C}^m$ and define two sets in $\mathbb{C}^d$ by $G_\delta = \delta^{-1}(\mathbb{D}^m)$ and $K_\delta = \delta^{-1}((\mathbb{D}^-)^m)$.

LEMMA 2.1. $G_\delta = K_\delta^\circ$.

Proof. Since $G_\delta \subseteq K_\delta$ and G_δ is open, it follows that $G_\delta \subseteq K_\delta^\circ$. If $\lambda \in K_\delta^\circ \setminus G_\delta$, then there exists an index l with $|\delta_l(\lambda)| = 1$. Since δ_l is assumed to be nonconstant, there exists a sequence $\lambda_n \to \lambda$ such that $|\delta_l(\lambda_n)| > 1$. In particular, $\lambda_n \notin K_\delta$ and so $\lambda \in \partial K_\delta$, a contradiction. This shows that $K_\delta^\circ \setminus G_\delta$ is empty. Therefore, $G_\delta = K_\delta^\circ$. □

Note that, even when $d = 1$, G_δ^- need not coincide with K_δ; and when $d > 2$, K_δ need not be compact and G_δ need not be bounded.

In what follows, T will always denote a d-tuple of pairwise commuting bounded operators acting on a Hilbert space. Let $\mathcal{F}_\delta$ be as in (1.4).

LEMMA 2.2. *If $T \in \mathcal{F}_\delta$ then $\sigma(T) \subseteq K_\delta$.*

Proof. Let $T \in \mathcal{F}_\delta$, and fix l. Since $\|\delta_l(T)\| \leq 1$, it follows that $\sigma(\delta_l(T)) \subseteq \mathbb{D}^-$. Hence, by the spectral mapping theorem, $\delta_l(\sigma(T)) \subseteq \mathbb{D}^-$. Thus, $\sigma(T) \subseteq \delta^{-1}((\mathbb{D}^-)^m) = K_\delta$. □

We now define an algebra of holomorphic functions on G_δ. For $f \in \mathrm{Hol}(G_\delta)$ and T with spectrum in G_δ, we can use the functional calculus to define $f(T)$. Let

$$\|f\|_\delta = \sup_{\substack{T \in \mathcal{F}_\delta \\ \sigma(T) \subseteq G_\delta}} \|f(T)\|, \tag{2.3}$$

and define H_δ^∞ as consisting of all $f \in \mathrm{Hol}(G_\delta)$ such that $\|f\|_\delta$ is finite.

PROPOSITION 2.4. *The space H_δ^∞ equipped with $\|f\|_\delta$ is a Banach algebra. Furthermore, if $H^\infty(G_\delta)$ denotes the space of bounded holomorphic functions on G_δ equipped with the sup norm $\|f\|_\infty$, then $H_\delta^\infty \subseteq H^\infty(G_\delta)$ and $\|f\|_\infty \leq \|f\|_\delta$ for all $f \in H_\delta^\infty$.*

Proof. That H_δ^∞ is a normed algebra is immediate from (2.3). If $\lambda \in G_\delta$, then λ can be viewed as an element of $\mathcal{F}_\delta$ and, moreover, $\sigma(\lambda) = \{\lambda\} \subseteq G_\delta$. Hence

$$\|f\|_\delta = \sup_{\substack{T \in \mathcal{F}_\delta \\ \sigma(T) \subseteq G_\delta}} \|f(T)\| \geq \sup_{\lambda \in G_\delta} |f(\lambda)| = \|f\|_\infty. \tag{2.5}$$

In particular, this implies that $H_\delta^\infty \subseteq H^\infty(G_\delta)$.

Now suppose $\{f_n\}$ is a Cauchy sequence in H_δ^∞. Since (2.5) implies that $\{f_n\}$ is also Cauchy in $H^\infty(G_\delta)$, there exists an $f \in H^\infty(G_\delta)$ such that $f_n \to f$ in $H^\infty(G_\delta)$. Therefore, by the continuity of the functional calculus,

$$\sigma(T) \subseteq G_\delta \implies f_n(T) \to f(T). \tag{2.6}$$

Now, since $\{f_n\}$ is Cauchy in H_δ^∞, there exists an M such that $\|f_n\| \leq M$ for all N. So if $T \in \mathcal{F}_\delta$ and $\sigma(T) \subseteq G$, then (2.6) implies that

$$\|f(T)\| = \lim_{n \to \infty} \|f_n(T)\| \leq M.$$

Now, because $\|f(T)\| \leq M$ whenever $T \in \mathcal{F}_\delta$ and $\sigma(T) \subseteq G_\delta$, equation (2.3) implies that $\|f\|_\delta \leq M$ and we see that $f \in H_\delta^\infty$.

To see that $f_n \to f$ in H_δ^∞, fix $\varepsilon > 0$. Then choose N so that $m, n \geq N$ implies $\|f_n - f_m\|_\delta < \varepsilon$. If $T \in \mathcal{F}_\delta$ and $\sigma(T) \subseteq G_\delta$, we have

$$\|f_n(T) - f_m(T)\| < \varepsilon.$$

Thus, letting $m \to \infty$, we see that if $T \in \mathcal{F}_\delta$ and $\sigma(T) \subseteq G_\delta$ then

$$n \geq N \implies \|f_n(T) - f(T)\| \leq \varepsilon.$$

But then it follows from (2.3) that

$$n \geq N \implies \|f_n - f\|_\delta \leq \varepsilon,$$

so $f_n \to f$ in H_δ^∞. □

We close this section with the following proposition, which identifies (in terms of operator theory) when the space H^∞_δ contains the functions that are holomorphic on a neighborhood of K_δ. Define

$$\mathcal{F}^0_\delta = \{T \in \mathcal{F}_\delta : \sigma(T) \subset G_\delta\}.$$

PROPOSITION 2.7. *The following statements are equivalent*:

(a) $\phi \in H^\infty_\delta$ *whenever* ϕ *is holomorphic on a neighborhood of* K_δ;
(b) $\lambda^r \in H^\infty_\delta$ *for* $r = 1, \dots, d$;
(c) $\mathcal{F}^0_\delta$ *is bounded.*

Proof. Since λ^r is holomorphic on a neighborhood of K_δ, (a) implies (b). That (b) implies (c) follows immediately from (2.3). Suppose that (c) holds. If ϕ is holomorphic on a neighborhood of K_δ yet $\phi \notin H^\infty_\delta$, then there exist $T_n \in \mathcal{F}^0_\delta$ such that

$$\|\phi(T_n)\| \to \infty. \tag{2.8}$$

Since $\mathcal{F}^0_\delta$ is bounded, we have $T = \bigoplus T_n \in \mathcal{F}_\delta$. Therefore, by Lemma 2.2, $\sigma(T) \subseteq K_\delta$ and $\phi(T)$ is a well-defined operator. But then

$$\|\phi(T_n)\| \leq \left\|\bigoplus \phi(T_n)\right\| = \left\|\phi\left(\bigoplus T_n\right)\right\| = \|\phi(T)\|,$$

contradicting (2.8). □

3. Hereditary Calculus

For G an open set in $\mathbb{C}^d$, let $\mathcal{H}(G)$ denote the collection of functions $h = h(\lambda, \mu)$, defined for $(\lambda, \mu) \in G \times G$, such that h is holomorphic in λ on G for each fixed $\mu \in G$ and h is anti-holomorphic (i.e., $\bar{h}$ is holomorphic) in μ on G for each fixed $\lambda \in G$. If we equip $\mathcal{H}(G)$ with the topology of uniform convergence on compact subsets of $G \times G$, then $\mathcal{H}(G)$ is a locally convex topological vector space with a topology induced by a complete translation-invariant metric. Furthermore $\mathcal{H}(G)$ is isomorphic as a topological vector space with $\overline{\mathrm{Hol}(G)} \otimes \mathrm{Hol}(G)$, the completion of the projective tensor product, via the continuous linear extension to $\overline{\mathrm{Hol}(G)} \otimes \mathrm{Hol}(G)$ of the bilinear map defined by

$$\overline{\mathrm{Hol}(G)} \otimes \mathrm{Hol}(G) \ni \overline{g(\mu)} \otimes f(\lambda) \mapsto \overline{g(\mu)} f(\lambda) \in \mathcal{H}(G);$$

see [30, Thm. 51.6].

In particular, if B is a Banach space and if $u\colon \overline{\mathrm{Hol}(G)} \times \mathrm{Hol}(G) \to B$ is a jointly continuous B-valued bilinear map, then there exists a continuous linear map $\Gamma\colon \mathcal{H}(G) \to B$ such that

$$u(\overline{g(\mu)}, f(\lambda)) = \Gamma(\overline{g(\mu)} f(\lambda)) \tag{3.1}$$

for all $f, g \in \mathrm{Hol}(G)$ (see [22, p. 325]).

If $\mathcal{H}$ is a Hilbert space, we let $\mathcal{L}(\mathcal{H})$ denote the C^*-algebra of bounded operators on $\mathcal{H}$. If $T = (T^1, \dots, T^d)$ is a d-tuple of pairwise commuting elements of $\mathcal{L}(\mathcal{H})$ and if $\sigma(T) \subseteq G$, then (by the continuity of the functional calculus) u as defined by $u(\bar{g}, f) = g(T)^* f(T)$ is a jointly continuous $\mathcal{L}(\mathcal{H})$-valued bilinear map on $\overline{\mathrm{Hol}(G)} \times \mathrm{Hol}(G)$. Hence, if $\Gamma\colon \mathcal{H}(G) \to \mathcal{L}(\mathcal{H})$ is defined by (3.1), then we can define the *hereditary calculus for* T by setting $h(T) = \Gamma(h)$ for all $h \in \mathcal{H}$. Observe that from this definition it follows that

$$[\,\overline{g(\mu)}h(\lambda,\mu)f(\lambda)](T) = g(T)^* h(T) f(T) \tag{3.2}$$

for all $f, g \in \mathrm{Hol}(G)$ and all $h \in \mathcal{H}(G)$.

For $a \in \mathcal{H}(G)$ we define $a^* \in \mathcal{H}(G)$ by $a^*(\lambda,\mu) = \overline{a(\mu,\lambda)}$. Given this notation, (3.2) takes on the more pleasing form

$$(g^* h f)(T) = g(T)^* h(T) f(T). \tag{3.3}$$

Define $\mathcal{R}(G) = \{a \in \mathcal{H}(G) \mid a = a^*\}$ and note that $\mathcal{R}(G)$ is a real locally convex space with the induced topology from $\mathcal{H}(G)$. Also, since $h^*(T) = h(T)^*$ whenever $\sigma(T) \subseteq G$ and $h \in \mathcal{H}(G)$, we see that if $\sigma(T) \subseteq G$ and $a \in \mathcal{R}(G)$ then $a(T)$ is self-adjoint. We say that $a \in \mathcal{H}(G)$ is *positive semidefinite,* and write $a \geq 0$, if

$$\sum_{i,j=1}^{n} a(\lambda_j,\lambda_i) c_j \overline{c_i} \geq 0 \tag{3.4}$$

whenever n is a positive integer, $\lambda_1, \dots, \lambda_n \in G$, and $c_1, \dots, c_n \in \mathbb{C}$. Set $\mathcal{P}(G) = \{a \in \mathcal{H}(G) \mid a \geq 0\}$. It then follows easily from (3.4) that $\mathcal{P}(G)$ is a closed cone in $\mathcal{R}(G)$.

PROPOSITION 3.5. *If $a \in \mathcal{H}(G)$, then $a \in \mathcal{P}(G)$ if and only if there exist a Hilbert space $\mathcal{M}$ and a holomorphic map $u\colon G \to \mathcal{M}$ such that*

$$a(\lambda,\mu) = \langle u(\lambda), u(\mu)\rangle_{\mathcal{M}} \tag{3.6}$$

for all $\lambda, \mu \in G$.

Proof. By Aronszajn's construction [4; 11] there is a Hilbert space $\mathcal{M}$ of functions on G, with reproducing kernel k, such that

$$a(\lambda,\mu) = \langle k_\lambda, k_\mu\rangle_{\mathcal{M}}, \quad \lambda,\mu \in G. \tag{3.7}$$

Since a is anti-holomorphic in μ for each fixed $\lambda \in G$, (3.7) implies that if $f \in \mathrm{span}\{k_\lambda \mid \lambda \in G\}$ then f is anti-holomorphic on G. Since $\mathrm{span}\{k_\lambda \mid \lambda \in G\}$ is dense in $\mathcal{M}$ and $\|k_\mu\|^2 = a(\mu,\mu)$ is bounded on compact subsets of G, it follows that f is actually anti-holomorphic on G for all $f \in \mathcal{M}$. So if we define $u(\lambda) = k_\lambda$, then $\langle u(\lambda), f\rangle = \overline{f(\lambda)}$ is holomorphic for all $f \in \mathcal{M}$ and we see that u is weakly holomorphic on G. And if u is weakly holomorphic, then u must be holomorphic. Thus (3.6) follows from (3.7). □

LEMMA 3.8. *If $a \in \mathcal{P}(G)$, then there exists a countable sequence $\{f_i\}$ in* $\mathrm{Hol}(G)$ *such that*

$$a(\lambda,\mu) = \sum_i \overline{f_i(\mu)} f_i(\lambda).$$

Proof. By Proposition 3.5, there exist a Hilbert space $\mathcal{M}$ and a holomorphic map $u\colon G \to \mathcal{M}$ such that (2.8) holds. Since u is holomorphic it follows that $\mathcal{M}_0$, the closed linear span of $\{u(\lambda) \mid \lambda \in G\}$ in $\mathcal{M}$, is separable. Let $\{e_i\}$ be a countable basis for $\mathcal{M}_0$, and define $f_i = \langle u(\lambda), e_i \rangle_{\mathcal{M}}$. $\square$

LEMMA 3.9. *Let $h \in \mathcal{R}(G)$ and $a \in \mathcal{P}(G)$, and assume that T is a d-tuple of pairwise commuting operators with $\sigma(T) \subseteq G$. If $h(T) \geq 0$, then $(ha)(T) \geq 0$.*

Proof. By Lemma 3.8 there is a sequence $\{f_i\}$ in $\mathrm{Hol}(G)$ such that $a = \sum_i f_i^* f_i$. Hence, if $h(T) \geq 0$, we have

$$(ha)(T) = \sum_i (f_i^* h f)(T) = \sum_i f_i(T)^* h(T) f(T) \geq 0. \qquad \square$$

DEFINITION 3.10. We say that $\mathcal{C} \subseteq \mathcal{R}(G)$ is a *hereditary cone* on G if $\mathcal{C}$ is a cone in $\mathcal{R}(G)$ with the property that $ha \in \mathcal{C}$ whenever $h \in \mathcal{C}$ and $a \in \mathcal{P}(G)$.

One way to construct hereditary cones is to let Ω be a subset of $\mathcal{R}(G)$ and then to define $\langle \Omega \rangle$ by

$$\langle \Omega \rangle = \left\{ \sum_{i=1}^n h_i a_i \;\middle|\; n \in \mathbb{N},\ h_1, \dots, h_n \in \Omega,\ a_1, \dots, a_n \in \mathcal{P}(G) \right\}.$$

Evidently, $\langle \Omega \rangle$ is the hereditary cone generated by Ω—in other words, the smallest hereditary cone $\mathcal{C} \subseteq \mathcal{R}(G)$ such that $\mathcal{C} \supseteq \Omega$.

DEFINITION 3.11. For $\mathcal{F}$ a collection of pairwise commuting operator d-tuples and G an open set in $\mathbb{C}^d$, define $\mathcal{F}^{\perp}(G) \subseteq \mathcal{R}(G)$ by

$$\mathcal{F}^{\perp}(G) = \{h \in \mathcal{R}(G) \mid h(T) \geq 0 \text{ whenever } T \in \mathcal{F} \text{ and } \sigma(T) \subseteq G\}.$$

LEMMA 3.12. *If $\mathcal{F}$ is a collection of operators and G is an open set in $\mathbb{C}^d$, then $\mathcal{F}^{\perp}(G)$ is a hereditary cone on G.*

Proof. Let $h \in \mathcal{F}^{\perp}(G)$ and $a \in \mathcal{P}(G)$. If $T \in \mathcal{F}$ and $\sigma(T) \subseteq G$, then $h(T) \geq 0$. Hence, by Lemma 3.9, $(ha)(T) \geq 0$. Therefore, $ha \in \mathcal{F}^{\perp}(G)$. $\square$

4. The Realization Formula

We record the following two simple lemmas for future use. We choose to deduce them as corollaries of Proposition 2.7. Alternative direct and constructive proofs of them can be obtained using either the theory of power series or iterated Cauchy–Riesz–Dunford integrals.

LEMMA 4.1. *If Φ is holomorphic on a neighborhood of $(\mathbb{D}^-)^m$, then $\Phi \in H^\infty_{\mathbf{m}}$.*

Proof. The statement is an immediate consequence of (b) $\Rightarrow$ (a) in Proposition 2.7. $\square$

LEMMA 4.2. *If $s > 1$, then $H^\infty(s\mathbb{D}^m) \subseteq H^\infty_{\mathbf{m}}$. Furthermore, there exists a constant c, depending only on s and m such that*

$$\|\Phi\|_m \le c \sup_{\lambda\in\mathbb{D}^m} |\Phi(\lambda)|$$

for all $\Phi \in H^\infty(s\mathbb{D}^m)$.

For $\mathcal{B}$ a Banach space, we let ball$(\mathcal{B})$ denote the closed unit ball of $\mathcal{B}$.

DEFINITION 4.3. Let ϕ be a function on $\mathbb{D}^m$. Then we say that a 4-tuple (a, β, γ, D) is a *realization* for ϕ if $a \in \mathbb{C}$ and if there exists a decomposed Hilbert space, $\mathcal{M} = \bigoplus_{l=1}^m \mathcal{M}_l$, such that: the 2×2 matrix

$$V = \begin{bmatrix} a & 1 \otimes \beta \\ \gamma \otimes 1 & D \end{bmatrix}$$

acts isometrically on $\mathbb{C} \oplus \mathcal{M}$; z acts on $\mathcal{M}$ via the formula

$$z\left(\bigoplus_{l=1}^m x_l\right) = \bigoplus_{l=1}^m z_l x_l;$$

and

$$\phi(z) = a + \langle z(1 - Dz)^{-1}\gamma, \beta\rangle$$

for all $z \in \mathbb{D}^m$.

The following result was proved in [2].

THEOREM 4.4. *Let ϕ be a function defined on $\mathbb{D}^m$. Then the following statements are equivalent*:

(a) $\phi \in \text{ball}(H^\infty_{\mathbf{m}})$;
(b) $1 - \phi^*\phi \in \mathcal{F}_m^\perp$;
(c) ϕ *has a realization*.

Definition 4.3 and Theorem 4.4 can be extended to the H^∞_δ setting. We recall the following definition for the reader's convenience.

DEFINITION 1.19. Let ϕ be a function on G_δ. We say that a 4-tuple (a, β, γ, D) is a *realization* for ϕ if $a \in \mathbb{C}$ and if there exists a decomposed Hilbert space, $\mathcal{M} = \bigoplus_{l=1}^m \mathcal{M}_l$, such that: the 2×2 matrix

$$V = \begin{bmatrix} a & 1 \otimes \beta \\ \gamma \otimes 1 & D \end{bmatrix}$$

acts isometrically on $\mathbb{C} \oplus \mathcal{M}$; $\delta(\lambda)$ acts on $\mathcal{M}$ via the formula

$$\delta(\lambda)\left(\bigoplus_{l=1}^m x_l\right) = \bigoplus_{l=1}^m \delta_l(\lambda) x_l;$$

and

$$\phi(\lambda) = a + \langle \delta(\lambda)(1 - D\delta(\lambda))^{-1}\gamma, \beta\rangle$$

for all $\lambda \in G_\delta$.

We adopt the notation $\mathcal{C}_\delta$ for the hereditary cone in $\mathcal{R}(G_\delta)$ generated by the elements $1 - \delta_1^*\delta_1, \dots, 1 - \delta_m^*\delta_m$.

THEOREM 4.5. *Let ϕ be a function defined on G_δ. Then the following statements are equivalent*:

(a) $\phi \in \operatorname{ball}(H^\infty_\delta)$;
(b) $\|\phi\|^-_\delta \le 1$;
(c) $1 - \phi^*\phi \in \mathcal{C}_\delta$;
(d) ϕ *has a realization.*

Proof. The equivalence of (b), (c), and (d) is a special case of [8, Thm. 3]; see also [12].

By [8, Lemma 1], if

$$\|\delta_l(T)\| < 1, \quad 1 \le l \le m, \tag{4.6}$$

then $\sigma(T) \subset G_\delta$; hence tuples satisfying (4.6) lie in $\mathcal{F}_\delta$ and so $\|\phi\|^-_\delta \le \|\phi\|_\delta$. This means that $\mathcal{F}^\perp_\delta(G_\delta) \subseteq \mathcal{C}_\delta$. The other inclusion follows from Lemma 3.12. □

5. Oka Mappings

In this section we prove the theorems stated in the Introduction. Theorems 1.5, 1.8, and 1.16 will be deduced from Theorem 1.18.

5.1. Proof of Theorem 1.18

First suppose that $\phi \in \operatorname{ball}(H^\infty_{\mathbf{m}})$. By Theorem 4.5, ϕ has a realization (a, β, γ, D) such that

$$\phi(\lambda) = a + \langle \delta(\lambda)(1 - D\delta(\lambda))^{-1}\gamma, \beta\rangle_{\mathcal{M}} \tag{5.1}$$

for all $\lambda \in G_\delta$. Therefore, if we define Φ on $\mathbb{D}^m$ by

$$\Phi(z) = a + \langle z(1 - Dz))^{-1}\gamma, \beta\rangle_{\mathcal{M}}, \tag{5.2}$$

then $\Phi \in \operatorname{ball}(H^\infty_{\mathbf{m}})$ (Theorem 4.4) and $\phi(\lambda) = \Phi(\delta(\lambda))$ for all $\lambda \in G_\delta$. This proves parts (a) and (c) of Theorem 1.18.

To prove part (b), assume that $\Phi \in H^\infty_{\mathbf{m}}$ and $\|\Phi\|_m = 1$. Define a function ϕ on G_δ by the formula $\phi(\lambda) = \Phi(\delta(\lambda))$. By Theorem 4.4, there exists a realization (a, β, γ, D) for Φ such that (5.2) holds for all $z \in \mathbb{D}^m$. Hence (5.1) holds for all $\lambda \in G_\delta$ and so, by Theorem 4.5, $\phi \in H^\infty_\delta$ and $\|\phi\|_\delta \le 1$. As a result, $\|\phi\|_\delta \le 1 = \|\Phi\|_m$. This proves (b) and completes the proof of Theorem 1.18.

5.2. Proof of Theorem 1.16

Let δ be an m-tuple of polynomials in d variables, and assume that ϕ is holomorphic on a neighborhood of K_δ.

First assume that Φ is holomorphic on a neighborhood of $(\mathbb{D}^-)^m$ and that $\phi = \Phi \circ \delta$. Fix $\varepsilon > 0$. Using Lemma 4.2, choose $t < 1$ but sufficiently close to 1 such that $\|\Phi(z/t)\|_m < \|\Phi\|_m + \varepsilon$. Define Ψ by setting $\Psi(z) = \Phi(z/t)$. Evidently, since $\Psi \in H^\infty_{\mathbf{m}}$ and $\phi(\lambda) = \Phi(\delta(\lambda)) = \Psi(t\delta(\lambda))$, it follows from Theorem 1.18(b) that $\|\phi\|_{t\delta} \le \|\Psi\|_m$. Therefore,

$$\rho(\phi) \le \|\phi\|_{t\delta} \le \|\Psi\|_m \le \|\Phi\|_m + \varepsilon.$$

Because ε is arbitrary, this proves the first assertion made in Theorem 1.16.

To prove the second assertion, fix $\varepsilon > 0$. Since

$$\lim_{t \to 1^-} \|\phi\|_{t\delta} = \rho(\phi),$$

there exists a $t < 1$ such that $\|\phi\|_{t\delta} < \rho(\phi) + \varepsilon$. By Theorem 1.18(c), there exists a $\Psi \in H^\infty_{\mathbf{m}}$ such that $\phi(\lambda) = \Psi(t\delta(\lambda))$ and $\|\Psi\|_m = \|\phi\|_{t\delta}$. Finally, note that if Φ is defined by $\Phi(z) = \Psi(tz)$ then, since $t < 1$, we have $\|\Phi\|_m < \|\Psi\|_m$. These constructions yield $\phi = \Phi \circ \delta$ and

$$\|\Phi\|_m \le \|\Psi\|_m = \|\phi\|_{t\delta} < \rho(\phi) + \varepsilon.$$

This completes the proof of Theorem 1.16.

6. Remarks

It is worth noting that the norm $\|\phi\|_\delta$ is always achieved by taking the supremum of $\phi(T)$ as T ranges over tuples of simultaneously diagonalizable matrices in $\mathcal{F}_\delta$. Indeed, in [23] it was asked whether (1.14) could hold for some generic C (where "generic" is understood to mean that all the eigenvalues are distinct); in [24] and [25] it was shown that (1.14) could hold in that case. The existence of such a C follows also from the nongeneric examples in [17] or [31] and the following theorem.

THEOREM 6.1. *Let ϕ be a function defined on G_δ. Then*

$$\|\phi\|_\delta = \sup\{\|\phi(T)\| : T \text{ is a } d\text{-tuple of generic matrices in } \mathcal{F}_\delta\}. \tag{6.2}$$

Proof. The inequality $\ge$ is obvious. Suppose the right-hand side of (6.2) is equal to 1. Since commuting *diagonalizable* matrices can be perturbed to commuting generic matrices (this need not be true for nondiagonizable matrices [20]), it follows that

$$\sup\{\|\phi(T)\| : T \text{ is a } d\text{-tuple of commuting diagonalizable matrices in } \mathcal{F}_\delta\}$$

is also equal to 1. If T is a commuting diagonizable d-tuple of $n \times n$ matrices, then we can choose common eigenvectors $v_1, \dots, v_n$ such that

$$T^r v_j = \lambda_j^r v_j, \quad 1 \le r \le d, \ 1 \le j \le n.$$

Let K be the Gram matrix $K_{ij} = \langle v_j, v_i \rangle$. Then the assertion $\|\delta_l(T)\| \le 1$ is equivalent to

$$[(1 - \overline{\delta_l(\lambda_i)}\delta_l(\lambda_j))K_{ij}] \ge 0. \tag{6.3}$$

Thus we have the following implication: if $\lambda_1, \dots, \lambda_n$ is a finite set in G_δ and if K is a positive-definite matrix such that (6.3) holds for $1 \le l \le m$, then

$$[(1 - \overline{\phi(\lambda_i)}\phi(\lambda_j))K_{ij}] \ge 0.$$

By the usual Hahn–Banach argument (see [4, Sec. 11.1]), this statement proves that $1 - \phi^*\phi$ is in $\mathcal{C}_\delta$ and hence that $\|\phi\|_\delta \le 1$. ☐

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