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Brauer Groups of Singular del Pezzo Surfaces

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1. Introduction

The Brauer group of a variety X, which in this paper we take to mean the cohomology group $\operatorname{Br} X = \mathrm{H}^2(X, \mathbb{G}_\mathrm{m})$, was extensively studied by Grothendieck [8]. Brauer groups of singular varieties are not particularly well behaved: in particular, the Brauer group of a singular variety need not inject into the Brauer group of its function field. The purely local question, of understanding the Brauer group of the local ring of a singularity, has been well studied: see, for example, [6]. An interesting feature of the results discussed in this paper is that the calculation is a global one and often leads to elements of the Brauer group that are locally trivial in the Zariski topology. One individual example of such an element was given by Ojanguren [13], whose algebra is of order 3 and defined on a singular cubic surface with three A_2 singularities; it will be shown in what follows that this is the only type of singular cubic surface admitting a 3-torsion Brauer element. A more general framework for studying such examples, described by Grothendieck in [8], was developed by De Meyer and Ford [5] to give examples of toric surfaces admitting nontrivial, locally trivial Azumaya algebras.

In this paper we take a slightly different approach that, for varieties with rational singularities, shows how the calculation of the Brauer group can be made very explicit by using the intersection pairing. We then apply this to arguably the simplest interesting class of singular projective surfaces, namely the singular del Pezzo surfaces. These are easy to approach for two reasons: they have rational singularities; and they come with a natural desingularization that is a rational surface. In Proposition 1 we show how to combine the Leray spectral sequence for the desingularization with Lipman's detailed description of the local Picard groups above the singular points [9]. In particular, it follows that the Brauer group may be easily computed using the intersection form on the desingularization. For singular del Pezzo surfaces this is well understood, and in Section 3 we apply Proposition 1 to compute the Brauer groups of all singular del Pezzo surfaces over an algebraically closed field; the Brauer group depends only on the singularity type of the surface. The arguments, and hence the results, are valid in arbitrary characteristic.

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The principal motivation for this article is in applying the Brauer group to study rational points of del Pezzo surfaces, as first suggested by Manin [11]. For arithmetic questions, it is often more useful to work with a desingularization of the original variety, and so the Brauer group of the singular variety is not of obvious interest. However, there are some situations where one cannot avoid looking at the Brauer group of a singular variety; the situation we have in mind is that of a model of a del Pezzo surface over a local ring, where the Brauer group of the (possibly singular) special fibre must be taken into account.

2. The Brauer Group of a Surface with Rational Singularities

In this section we study the Brauer group of a surface Y having only isolated rational singularities over an algebraically closed base field. Following Lipman, by a *desingularization* $X \to Y$ we mean a proper birational morphism from a regular scheme X. If Y is a normal surface with finitely many rational singularities, then there is a unique minimal desingularization $X \to Y$ that may be constructed as a sequence of blow-ups at singular points.

We write $\mathrm{Br}(X/Y)$ to mean $\ker(\mathrm{Br}\, Y \to \mathrm{Br}\, X)$. If Y is integral with function field K, then there is a sequence of maps $\mathrm{Br}\, Y \to \mathrm{Br}\, X \to \mathrm{Br}\, K$. Since X is regular, $\mathrm{Br}\, X$ injects into $\mathrm{Br}\, K$; it follows that $\mathrm{Br}(K/Y) \cong \mathrm{Br}(X/Y)$.

Whenever A is an Abelian group, A^* denotes the group $\mathrm{Hom}(A, \mathbb{Z})$.

PROPOSITION 1. *Let Y be a normal surface over an algebraically closed field k; suppose that Y has finitely many rational singularities, and let $f : X \to Y$ be the minimal desingularization. Let $\mathbf{E}$ denote the subgroup of* Pic X *generated by the classes of the exceptional curves of the resolution, and let θ :* Pic $X \to \mathbf{E}^*$ *be the homomorphism induced by the intersection pairing on* Pic X. *Then there is an exact sequence*

$$0 \to \mathrm{Pic}\, Y \xrightarrow{f^*} \mathrm{Pic}\, X \xrightarrow{\theta} \mathbf{E}^* \to \mathrm{Br}\, Y \xrightarrow{f^*} \mathrm{Br}\, X.$$

Proof. Since f is proper and birational, we have $f_*\mathbb{G}_\mathrm{m} = \mathbb{G}_\mathrm{m}$. It follows that, for any flat morphism of schemes $Y' \to Y$, if $f_{Y'} : X \times_Y Y' \to Y'$ denotes the base change of f then the following sequence is exact (see [1, Sec. 8.1, Prop. 4]):

$$0 \to \mathrm{Pic}\, Y' \xrightarrow{f_{Y'}^*} \mathrm{Pic}(X \times_Y Y') \to \mathcal{P}ic_{X/Y}(Y') \to \mathrm{Br}\, Y' \xrightarrow{f_{Y'}^*} \mathrm{Br}(X \times_Y Y'). \quad (1)$$

Taking $Y' = Y$ in (1) gives the exact sequence

$$0 \to \mathrm{Pic}\, Y \xrightarrow{f^*} \mathrm{Pic}\, X \to \mathcal{P}ic_{X/Y}(Y) \to \mathrm{Br}\, Y \xrightarrow{f^*} \mathrm{Br}\, X. \quad (2)$$

So it will be enough to exhibit an isomorphism $\alpha : \mathcal{P}ic_{X/Y}(Y) \to \mathbf{E}^*$ such that composing α with the natural homomorphism $\mathrm{Pic}\, X \to \mathcal{P}ic_{X/Y}(Y)$ gives the homomorphism θ described in the statement of the theorem. From now on, we work with $\mathcal{P}ic_{X/Y}$ as a sheaf only on the small étale site of Y in order to be able to talk about its stalks.

Step 1: Localization. Because the sheaf $\mathcal{P}ic_{X/Y}$ on $Y_{\text{ét}}$ is supported on the singular points, the natural map

$$\mathcal{P}ic_{X/Y}(Y) \to \prod_{P \text{ singular}} (\mathcal{P}ic_{X/Y})_P, \tag{3}$$

from the global sections of $\mathcal{P}ic_{X/Y}$ to the direct product of its stalks at the singular points, is an isomorphism.

At each singular point P, let $\tilde{Y}_P$ denote Spec $\mathcal{O}^{\text{sh}}_{Y,P}$, the spectrum of the Henselization of the local ring at P, and set $\tilde{X}_P = X \times_Y \tilde{Y}_P$. The stalk of $\mathcal{P}ic_{X/Y}$ at the geometric point P is naturally isomorphic to $\mathcal{P}ic_{X/Y}(\tilde{Y}_P)$, which is simply Pic $\tilde{X}_P$; this can be seen by taking $Y' = \tilde{Y}_P$ in (1) and using the facts that Pic $\tilde{Y}_P$ and Br $\tilde{Y}_P$ are trivial (for the latter, see [12, IV, Cor. 1.7]). Combining this with the isomorphism (3), we see that the natural map

$$\mathcal{P}ic_{X/Y}(Y) \to \prod_P \operatorname{Pic} \tilde{X}_P$$

is an isomorphism.

Step 2: Lipman's description of Pic $\tilde{X}_P$. For each singular point P of Y, denote by $\mathbf{E}_P$ the subgroup of Pic X generated by the exceptional curves lying over P. Let Y_P denote the spectrum of the Zariski local ring of Y at P, and let $X_P = X \times_Y Y_P$. We will use θ_P to denote the homomorphism Pic $X_P \to \mathbf{E}_P^*$ induced by the intersection pairing on X. (Lipman's definition of the map θ_P is slightly more general and involves dividing by the least degree of an invertible sheaf on each exceptional curve; because we are working over an algebraically closed field, all of our exceptional curves have a k-point and hence an invertible sheaf of degree 1.) Lipman [9, Part IV] studied the kernel and cokernel of θ_P in detail, defining an exact sequence

$$0 \to \operatorname{Pic}^0 X_P \to \operatorname{Pic} X_P \xrightarrow{\theta_P} \mathbf{E}_P^* \to G(Y_P) \to 0$$

attached to the resolution $X_P \to Y_P$, and showed that $\operatorname{Pic}^0 X_P = 0$ when Y_P has a rational singularity and that $G(Y_P) = 0$ when Y_P is Henselian. We thus obtain isomorphisms Pic $\tilde{X}_P \cong \mathbf{E}_P^*$ such that the composite homomorphism Pic $X \to \prod_P \operatorname{Pic} \tilde{X}_P \to \prod_P \mathbf{E}_P^*$ is θ_P.

Step 3: Globalization. Finally, note that two exceptional curves lying above distinct singularities of Y are disjoint, so in particular they have intersection number 0. Therefore the subgroups $\mathbf{E}_P \subseteq \operatorname{Pic} X$ are mutually orthogonal, and so $\mathbf{E} \cong \bigoplus_P \mathbf{E}_P$ and $\mathbf{E}^* \cong \prod_P \mathbf{E}_P^*$.

It is now easily verified that replacing $\mathcal{P}ic_{X/Y}(Y)$ in (2) with $\mathbf{E}^*$ does indeed lead to the desired exact sequence. □

COROLLARY 2. *If P is a singular point of Y, then* $\operatorname{Br}(X_P/Y_P)$ *is isomorphic to the cokernel of* θ_P : Pic $X \to \mathbf{E}_P^*$, *which is equal to Lipman's group* $G(Y_P)$.

Proof. Applying the proposition to Y_P shows that $\mathrm{Br}(X_P/Y_P)$ is isomorphic to $\mathrm{coker}(\mathrm{Pic}\,X_P \to \mathbf{E}_P^*)$, which by definition is equal to $G(Y_P)$. Since X is smooth, the restriction map $\mathrm{Pic}\,X \to \mathrm{Pic}\,X_P$ is surjective, and the statement follows. □

COROLLARY 3. *If Y has only one singularity P, then* $\mathrm{Br}(X/Y) \cong \mathrm{Br}(X_P/Y_P)$.

Proof. In this case $\mathbf{E} = \mathbf{E}_P$, so the statement follows immediately from Corollary 2. □

3. Singular del Pezzo Surfaces

In this section we apply Proposition 1 to compute the Brauer groups of singular del Pezzo surfaces. We refer to [2] and [4] for background details on singular del Pezzo surfaces.

Let X be a generalized del Pezzo surface over an algebraically closed field k and $f : X \to Y$ the morphism contracting the (-2)-curves (and nothing else), so that Y is the corresponding singular del Pezzo surface. The Picard group of X fits into a short exact sequence

$$0 \to Q \to \mathrm{Pic}\,X \xrightarrow{(\cdot, K_X)} \mathbb{Z} \to 0,$$

where Q is the subgroup orthogonal to the canonical class K_X under the intersection pairing. The exceptional curves of f are all contained in Q. Let $\mathbf{E}$ denote the subgroup of Q generated by all the exceptional curves of $X \to Y$ (equivalently, all the (-2)-curves on X).

PROPOSITION 4. $\mathrm{Br}\,Y$ *is isomorphic to* $(Q/\mathbf{E})_{\mathrm{tors}}$.

Proof. First, $\mathrm{Br}\,X$ is trivial because X is a rational surface. Hence, by Proposition 1, $\mathrm{Br}\,Y$ is isomorphic to the cokernel of the map $\theta : \mathrm{Pic}\,X \to \mathbf{E}^*$. Now θ factors as $\mathrm{Pic}\,X \to Q^* \to \mathbf{E}^*$, giving an exact sequence

$$\mathrm{coker}(\mathrm{Pic}\,X \to Q^*) \to \mathrm{Br}\,Y \to \mathrm{coker}(Q^* \to \mathbf{E}^*) \to 0.$$

It follows from the description of Q in [4, II.4] that $\mathrm{Pic}\,X \xrightarrow{\theta} Q^*$ is surjective. Indeed, one easily checks that the basis of Q given by the simple roots α_i described there can be extended (for example, by adjoining one exceptional class E_1) to a basis of $\mathrm{Pic}\,X$. So we are left with an isomorphism between $\mathrm{Br}\,Y$ and $\mathrm{coker}(Q^* \to \mathbf{E}^*)$. To compute the latter group, we take the short exact sequence

$$0 \to \mathbf{E} \to Q \to (Q/\mathbf{E}) \to 0$$

and apply $\mathrm{Hom}(\cdot, \mathbb{Z})$ to obtain the longer exact sequence

$$0 \to (Q/\mathbf{E})^* \to Q^* \to \mathbf{E}^* \to \mathrm{Ext}^1(Q/\mathbf{E}, \mathbb{Z}) \to \mathrm{Ext}^1(Q, \mathbb{Z}).$$

Since Q is a free Abelian group, it follows that $\mathrm{Ext}^1(Q, \mathbb{Z}) = 0$ and so $\mathrm{Br}\,Y$ is isomorphic to $\mathrm{Ext}^1(Q/\mathbf{E}, \mathbb{Z})$, which by a standard calculation is isomorphic to $(Q/\mathbf{E})_{\mathrm{tors}}$. □

We note the following interesting corollary.

COROLLARY 5. *Let Y be a singular del Pezzo surface over an algebraically closed field, and denote by Y^{ns} the nonsingular locus of Y. Then there is an isomorphism of abstract groups* $\operatorname{Br} Y \cong \operatorname{Pic}(Y^{\mathrm{ns}})_{\mathrm{tors}}$.

Proof. Since Y^{ns} is isomorphic to the complement of the exceptional curves in X, we have $\operatorname{Pic} Y^{\mathrm{ns}} \cong (\operatorname{Pic} X)/\mathbf{E}$ and so $\operatorname{Pic}(Y^{\mathrm{ns}})_{\mathrm{tors}} \cong (Q/\mathbf{E})_{\mathrm{tors}}$. □

It remains to enumerate the possible singularity types of del Pezzo surfaces and to compute $Q/\mathbf{E}$ in each case. The algorithm for listing the possible configurations of (-2)-curves is well known, as is the list of possible configurations, so we only summarize the algorithm very briefly. The free Abelian group Q, together with the negative definite intersection pairing, is isomorphic to the root lattice of a particular root system depending only on the degree of the surface. Within this root lattice, the exceptional divisors of the desingularization $X \to Y$ form a set of simple roots in some subroot system and, indeed, form a Π-system in the sense of Dynkin [7, Sec. 5]. To list the Π-systems contained in Q, we use the following two theorems from [7].

- Theorem 5.2: Every Π-system is contained in a Π-system that is of maximal rank—in other words, that spans Q as a vector space.
- Theorem 5.3: The Π-systems of maximal rank may be all be obtained from some set of simple roots in Q by iterating the following procedure, called an *elementary transformation*. Starting with a set of simple roots, choose one connected component of the associated Dynkin diagram; adjoin the most negative root of that component and discard one of the original simple roots of that component.

So, starting from any choice of simple roots in Q, we can obtain all Π-systems up to the action of the Weyl group. Not quite all of these can actually be achieved as configurations of (-2)-curves; see [14], though it is not immediately clear that the methods there also apply in positive characteristic.

Let us remark that, given a root system R, the primes dividing $\#(\mathbb{Z}R/\mathbb{Z}R')_{\mathrm{tors}}$ for R' a closed subsystem of R are called *bad primes* (see e.g. [10, Apx. B]). A corollary of Proposition 4 is that the primes that can divide the order of the Brauer group of a singular del Pezzo surface of degree d are the bad primes of the associated root system. It turns out that the bad primes are simply those occurring as coefficients when a maximal root is expressed in terms of simple roots, so they are easily listed. There are no bad primes for A_n; 2 is the only bad prime for D_n ($n \geq 4$); 2 and 3 are the bad primes for E_6 and E_7; and E_8 has bad primes 2, 3, and 5.

THEOREM 6. *Let Y be a singular del Pezzo surface of degree d over an algebraically closed field. If $d \geq 5$, then* $\operatorname{Br} Y = 0$. *If $1 \leq d \leq 4$, then the Brauer group of Y is determined by its singularity type. The singularity types giving rise to nontrivial Brauer groups are listed in Tables 1–4. Each class in* $\operatorname{Br} Y$ *is represented by an Azumaya algebra. Except for the singularity types A_7 in degree 2 and*

Table 1 Brauer Groups of Singular del Pezzo Surfaces of Degree 4

Singularity type	Brauer group	Singularity type	Brauer group
$2A_1 + A_3$	$\mathbb{Z}/2\mathbb{Z}$	$4A_1$	$\mathbb{Z}/2\mathbb{Z}$

Table 2 Brauer Groups of Singular del Pezzo Surfaces of Degree 3

Singularity type	Brauer group	Singularity type	Brauer group
$A_1 + A_5$	$\mathbb{Z}/2\mathbb{Z}$	$2A_1 + A_3$	$\mathbb{Z}/2\mathbb{Z}$
$4A_1$	$\mathbb{Z}/2\mathbb{Z}$	$3A_2$	$\mathbb{Z}/3\mathbb{Z}$

Table 3 Brauer Groups of Singular del Pezzo Surfaces of Degree 2

Singularity type	Brauer group	Singularity type	Brauer group
$A_1 + 2A_3$	$\mathbb{Z}/4\mathbb{Z}$	$5A_1$	$\mathbb{Z}/2\mathbb{Z}$
$A_1 + A_5$	$\mathbb{Z}/2\mathbb{Z}$	$6A_1$	$(\mathbb{Z}/2\mathbb{Z})^2$
$A_1 + D_6$	$\mathbb{Z}/2\mathbb{Z}$	$7A_1$†	$(\mathbb{Z}/2\mathbb{Z})^3$
$2A_1 + A_3$	$\mathbb{Z}/2\mathbb{Z}$	$A_2 + A_5$	$\mathbb{Z}/3\mathbb{Z}$
$2A_1 + D_4$	$\mathbb{Z}/2\mathbb{Z}$	$3A_2$	$\mathbb{Z}/3\mathbb{Z}$
$3A_1 + A_3$	$\mathbb{Z}/2\mathbb{Z}$	$2A_3$	$\mathbb{Z}/2\mathbb{Z}$
$3A_1 + D_4$	$(\mathbb{Z}/2\mathbb{Z})^2$	A_7	$\mathbb{Z}/2\mathbb{Z}$
$4A_1$*	$\mathbb{Z}/2\mathbb{Z}$		

* There are (up to the action of the Weyl group) two different ways of embedding $4A_1$ into E_7, so there are two different singularity types of a degree-2 del Pezzo surface with root system $4A_1$. One of these has Brauer group $\mathbb{Z}/2\mathbb{Z}$; the other has trivial Brauer group.

† This subroot system does not arise from a del Pezzo surface [14].

for A_7, A_8, and D_8 in degree 1, the corresponding Azumaya algebras are locally trivial in the Zariski topology.

Proof. For $d \geq 5$, the relevant root system is of type A_n; hence there are no bad primes and the Brauer group is trivial. For $1 \leq d \leq 4$, the results of applying the algorithm described previously are listed in the tables. Since $\operatorname{Br} Y$ is torsion, it follows from a result proved by Gabber and independently by de Jong (see [3]) that every class is represented by an Azumaya algebra. It remains to prove the statement about Zariski-local triviality. If P is a singular point of a singular del Pezzo surface Y then Corollary 2 shows that, in the notation used there, $\operatorname{Br}(X_P/Y_P) \cong \operatorname{coker}(\operatorname{Pic} X \to \mathbf{E}_P^*)$; we need to show that $\operatorname{Br}(X_P/Y_P) = 0$. Replacing Y by a del Pezzo surface of the same degree, but with only one singularity of the same type as P, changes neither $\operatorname{Pic} X$, $\mathbf{E}_P^*$, nor the map between them, so we may assume

Table 4 Brauer Groups of Singular del Pezzo Surfaces of Degree 1

Singularity type	Brauer group	Singularity type	Brauer group
$A_1 + A_2 + A_5$	$\mathbb{Z}/6\mathbb{Z}$	$4A_1 + A_3$	$(\mathbb{Z}/2\mathbb{Z})^2$
$A_1 + 3A_2$	$\mathbb{Z}/3\mathbb{Z}$	$4A_1 + D_4$ †	$(\mathbb{Z}/2\mathbb{Z})^3$
$A_1 + 2A_3$	$\mathbb{Z}/4\mathbb{Z}$	$5A_1$	$\mathbb{Z}/2\mathbb{Z}$
$A_1 + A_5$ *	$\mathbb{Z}/2\mathbb{Z}$	$6A_1$	$(\mathbb{Z}/2\mathbb{Z})^2$
$A_1 + A_7$	$\mathbb{Z}/4\mathbb{Z}$	$7A_1$ †	$(\mathbb{Z}/2\mathbb{Z})^3$
$A_1 + D_6$	$\mathbb{Z}/2\mathbb{Z}$	$8A_1$ †	$(\mathbb{Z}/2\mathbb{Z})^4$
$A_1 + E_7$	$\mathbb{Z}/2\mathbb{Z}$	$A_2 + A_5$	$\mathbb{Z}/3\mathbb{Z}$
$2A_1 + A_2 + A_3$	$\mathbb{Z}/2\mathbb{Z}$	$A_2 + E_6$	$\mathbb{Z}/3\mathbb{Z}$
$2A_1 + A_3$ *	$\mathbb{Z}/2\mathbb{Z}$	$3A_2$	$\mathbb{Z}/3\mathbb{Z}$
$2A_1 + 2A_3$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$4A_2$	$(\mathbb{Z}/3\mathbb{Z})^2$
$2A_1 + A_5$	$\mathbb{Z}/2\mathbb{Z}$	$A_3 + D_4$	$\mathbb{Z}/2\mathbb{Z}$
$2A_1 + D_4$	$\mathbb{Z}/2\mathbb{Z}$	$A_3 + D_5$	$\mathbb{Z}/4\mathbb{Z}$
$2A_1 + D_5$	$\mathbb{Z}/2\mathbb{Z}$	$2A_3$ *	$\mathbb{Z}/2\mathbb{Z}$
$2A_1 + D_6$	$(\mathbb{Z}/2\mathbb{Z})^2$	$2A_4$	$\mathbb{Z}/5\mathbb{Z}$
$3A_1 + A_3$	$\mathbb{Z}/2\mathbb{Z}$	A_7 *	$\mathbb{Z}/2\mathbb{Z}$
$3A_1 + D_4$	$(\mathbb{Z}/2\mathbb{Z})^2$	A_8	$\mathbb{Z}/3\mathbb{Z}$
$4A_1$ *	$\mathbb{Z}/2\mathbb{Z}$	$2D_4$	$(\mathbb{Z}/2\mathbb{Z})^2$
$4A_1 + A_2$	$\mathbb{Z}/2\mathbb{Z}$	D_8	$\mathbb{Z}/2\mathbb{Z}$

* Each of these five root systems may be embedded into E_8 in two distinct ways. In all five cases, one way results in a trivial Brauer group; the other results in the Brauer group shown in the table.

† These subroot systems do not arise from del Pezzo surfaces [14].

that P is the only singularity of Y. Then $\mathrm{Br}(X_P/Y_P) = \mathrm{Br}(X/Y) = \mathrm{Br}\,Y$ by Corollary 3. But the tables show that $\mathrm{Br}\,Y = 0$, except in the cases just listed. □

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