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On Symplectic Automorphisms of Hyper-Kähler Fourfolds of $K3^{[2]}$ Type

Giovanni Mongardi

1. Introduction

An automorphism φ of a hyper-Kähler manifold X is *symplectic* if

$$\varphi^*(\sigma_X) = \sigma_X,$$

where σ_X is a holomorphic symplectic 2-form on X. Finite abelian groups of symplectic automorphisms of complex K3 surfaces have been classified by Nikulin in [11]. In particular, we know that a symplectic automorphism of finite order on a K3 surface over $\mathbb{C}$ has order at most 8.

This paper deals with symplectic automorphisms on hyper-Kähler fourfolds that are deformation equivalent to the Hilbert scheme of two points of a K3 surface; such fourfolds are known as manifolds of $K3^{[2]}$ type. Recall that manifolds of $K3^{[2]}$ type have $b_2 = 23$ and $H^2_{\mathbb{Z}} \cong U^3 \oplus E_8(-1)^2 \oplus (-2)$; here U is the hyperbolic plane, $E_8(-1)$ is the unique negative-definite even unimodular lattice of rank 8, and (-2) is the rank-1 lattice of discriminant -2.

Let Co_1 be Conway's sporadic simple group. The main result of this paper is the following theorem.

Theorem 1.1. *Let X be a hyper-Kähler manifold of $K3^{[2]}$ type and let G be a finite group of symplectic automorphisms of X. Then G is isomorphic to a subgroup of* Co_1.

We recall that Mukai [10] proved an analogous result for K3 surfaces (see also the proof of Kondo [8]). Namely, a finite group of symplectic automorphisms is a subgroup of Mathieu's group M_{23}.

A partial converse to Theorem 1.1 is provided by Proposition 2.12, which also gives a computational method of determining possible finite automorphism groups. We shall use Theorem 1.1 to prove the following result on symplectic automorphisms of order 11.

Proposition 1.2. *Let X be a fourfold of $K3^{[2]}$ type and let $\psi: X \to X$ be a symplectic automorphism of order 11. Then* $21 \geq h^{1,1}_{\mathbb{Z}}(X) \geq 20$. *Moreover,* $\mathrm{Bir}(X)$

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has a subgroup isomorphic to $\mathrm{PSL}_2(\mathbb{Z}_{/(11)})$. *Furthermore, if X is projective then* $h^{1,1}_{\mathbb{Z}}(X) = 21$.

Finally, we give an example of a fourfold of $\mathrm{K3}^{[2]}$ type with a symplectic automorphism of order 11. Our example is given by the Fano scheme of lines on a cubic fourfold (unique up to projectivities). Fano schemes of lines on cubic fourfolds were first studied in [3], where it is proved that they are of $\mathrm{K3}^{[2]}$ type.

The manifold $X_{\mathrm{Kl}} \subset \mathbb{P}^5_{\mathbb{C}}$, which is given with respect to homogeneous coordinates $(x_0 : \cdots : x_5)$ by

$$X_{\mathrm{Kl}} = V(x_0^3 + x_1^2 x_5 + x_2^2 x_4 + x_3^2 x_2 + x_4^2 x_1 + x_5^2 x_3), \tag{1}$$

is a $3:1$ cover of $\mathbb{P}^4_{\mathbb{C}}$ ramified along a cubic threefold (first studied by Klein [7]). We denote the Fano scheme of lines in X_{Kl} by F_{Kl}.

THEOREM 1.3. *The Fano scheme* F_{Kl} *is a hyper-Kähler fourfold of* $K3^{[2]}$ *type. It has a symplectic automorphism* φ *of order 11 that is induced by the element*

$$(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \to (x_0 : \omega x_1 : \omega^3 x_2 : \omega^4 x_3 : \omega^5 x_4 : \omega^9 x_5) \tag{2}$$

of $\mathrm{PGL}_6(\mathbb{C})$, *where* $\omega = e^{2\pi i/11}$.

We will describe other automorphisms of X_{Kl}. They are taken from the same work of Klein and were studied also by Adler [1].

The rest of the paper is organized as follows. In Section 2 we recall some results in lattice theory and then use them to prove Theorem 1.1 and Proposition 1.2. Section 3 briefly analyzes deformations of manifolds of $\mathrm{K3}^{[2]}$ type that have a symplectic automorphism of order 11. In that section we also compute $\mathrm{NS}(X)$ for one interesting polarization. In Section 4 we give the promised example of a manifold with an order-11 symplectic automorphism, and in Section 5 we describe more symplectic automorphisms of this example.

2. Lattice Theory

In this section we give a proof of Proposition 1.2 and a proof of Theorem 1.1 using several results on general lattice theory and on lattices defined by symplectic automorphisms. The interested reader can consult [12] for the main results concerning discriminant forms, [5] for a broader treatment of Sections 2.1 and 2.2, and [9] for proofs of the stated results on lattices defined by symplectic automorphisms.

Let L be a lattice—that is, a free $\mathbb{Z}$-module equipped with an integer-valued symmetric nondegenerate bilinear form $(\cdot,\cdot)_L$. We say that L is *even* if $(a,a)_L \in 2\mathbb{Z}$ for all $a \in L$ and that L is *unimodular* if $L^\vee = L$.

Given an even lattice L, the group $A_L = L^\vee/L$ is called the *discriminant group*. Let $l(A_L)$ be the minimal number of generators of A_L. On A_L there is a well-defined quadratic form q_{A_L} (induced by $(\cdot,\cdot)_L$) that takes values in $\mathbb{Q}/2\mathbb{Z}$ and is called the *discriminant form*. Let (l_+, l_-) denote the signature of the quadratic form induced by $(\cdot,\cdot)_L$ on $L \otimes \mathbb{R}$.

We abuse notation and define the signature $\mathrm{sign}(q)$ of a discriminant form q as $l_+ - l_-$ (modulo 8), where L is a lattice with discriminant form q. This notion is well-defined because two lattices M, M' such that $q_{A_M} = q_{A_{M'}}$ are stably equivalent; that is, there exist two unimodular lattices T, T' such that $M \oplus T \cong M' \oplus T'$.

Two lattices M and M' are said to have the same *genus* if $M \otimes \mathbb{Z}_p \cong M' \otimes \mathbb{Z}_p$ for all primes p. We remark that there could be several isometry classes of the same genus.

Let S be a nondegenerate sublattice of an unimodular lattice L, and let $M = S^{\perp}$. Then $A_M = A_S$ and $q_{A_M} = -q_{A_S}$. If G is a finite group of isometries of a lattice L, we let

$$T_G(L) = L^G$$

be the invariant lattice and let

$$S_G(L) = T_G(L)^{\perp}$$

be the co-invariant lattice.

The following lemmas are simplified versions of fundamental results on the existence of lattices and on the existence of primitive embeddings. Both general results were proven by Nikulin [12, Thm. 1.10.1, Thm. 1.12.2].

LEMMA 2.1. *Let A_T be a finite abelian group and let q_T be a quadratic form on A_T with values in $\mathbb{Q}/2\mathbb{Z}$. Suppose the following conditions are satisfied*:

- *there exists a lattice T' of rank $t_+ + t_-$ and discriminant form q_T over the group A_T*;
- $\operatorname{sign}(q_T) \equiv t_+ - t_-$ *modulo 8*;
- $t_+ \geq 0$, $t_- \geq 0$, *and* $t_+ + t_- \geq l(A_T)$.

Then there exists an even lattice T of signature (t_+, t_-), discriminant group A_T, and form q_{A_T}.

LEMMA 2.2. *Let S be an even lattice of signature (s_+, s_-). There exists a primitive embedding of S into some unimodular lattice L of signature (l_+, l_-) if and only if there exists a lattice M of signature (m_+, m_-) and discriminant form q_{A_M} such that*:

- $s_+ + m_+ = l_+$ *and* $s_- + m_- = l_-$;
- $A_M \cong A_S$ *and* $q_{A_M} = -q_{A_S}$.

LEMMA 2.3. *Let $L = U^3 \oplus E_8(-1)^2 \oplus (-2)$ and let G be a subgroup of $O(L)$. Then there exists a primitive embedding $L \to L' \cong U^4 \oplus E_8(-1)^2$ such that G extends to a group of isometries of L' and $S_G(L) = S_G(L')$.*

Proof. Let x be a vector of square 2 and $v \in L$ a vector of square -2 such that $(v, L) = 2\mathbb{Z}$. Let L' be the overlattice of $L \oplus \mathbb{Z}x$ generated by L and $\frac{x+v}{2}$, and extend the action of G to L' by letting G act as the identity on x. A direct computation shows that $S_G(L) = S_G(L')$. □

2.1. Niemeier Lattices and Leech Couples

In this section we recall Niemeier's list of negative-definite even unimodular lattices in dimension 24. We also introduce a class of lattices that will be of fundamental interest throughout Section 2. Detailed information about these lattices can be found in [5, Chap. 16] and in [12, Sec. 1.14].

Table 1 Niemeier Lattices

Name	Dynkin diagram	Maximal Leech-type group	Coxeter number	Generating glue code
N_1	D_{24}	1	46	[1]
N_2	$D_{16}E_8$	1	30	[10]
N_3	E_8^3	S_3	30	[000]
N_4	A_{24}	2	25	[5]
N_5	D_{12}^2	2	22	[12], [21]
N_6	$A_{17}E_7$	2	18	[31]
N_7	$D_{10}E_7^2$	2	18	[110], [301]
N_8	$A_{15}D_9$	2	16	[21]
N_9	D_8^3	S_3	14	[(122)]
N_{10}	A_{12}^2	4	13	[15]
N_{11}	$A_{11}D_7E_6$	2	12	[111]
N_{12}	E_6^4	$2.S_4$	12	[1(012)]
N_{13}	$A_9^2D_6$	2^2	10	[240], [501], [053]
N_{14}	D_6^4	S_4	10	[even perm. of $\{0,1,2,3\}$]
N_{15}	A_8^3	$S_3 \times 2$	9	[(114)]
N_{16}	$A_7^2D_5^2$	2^3	8	[1112], [1721]
N_{17}	A_6^4	$2.A_4$	7	[1(216)]
N_{18}	$A_5^4D_4$	$2.S_4$	6	[2(024)0], [33001], [30302], [30033]
N_{19}	D_4^6	$3 \times S_6$	6	[111111], [0(02332)]
N_{20}	A_4^6	$2.\mathrm{L}_2(5).2$	5	[1(01441)]
N_{21}	A_3^8	$2^3.\mathrm{L}_2(7).2$	4	[3(2001011)]
N_{22}	A_2^{12}	$2.M_{12}$	3	[2(11211122212)]
N_{23}	A_1^{24}	M_{24}	2	[1(00000101001100110101111)]
Λ	$\emptyset$	Co_0	0	$\emptyset$

DEFINITION 2.4. Let M be a lattice and let $G \subset O(M)$. Then (M, G) is Leech couple if the following conditions are satisfied:

- M is negative definite;
- M contains no vectors of square -2;
- G acts trivially on A_M;
- $S_G(M) = M$.

Observe that (Λ, Co_0) as in Table 1 (last row) is a Leech couple.

Now we recall Niemeier's list of negative-definite even unimodular lattices of dimension 24. All of these lattices can be obtained by specifying a 0- or 24-dimensional negative definite Dynkin lattice such that every semisimple component has a fixed Coxeter number. In Table 1 we list the possible choices. We obtain a Niemeier lattice N from its Dynkin lattice A by adding a certain set of glue vectors that is a subset $G(N)$ of $A^\vee/A$. The precise definition of glue vectors is given in [5, Sec. 4], and we keep the notation from there. Note that the set of glue vectors forms an additive subgroup of $A^\vee/A$.

The maximal Leech-type group $\mathrm{Leech}(N)$ is defined to be the maximal subgroup G of $O(N)$ such that $(S_G(N), G)$ is a Leech couple. This group, first computed in [6], is isomorphic to $O(N)/W(N)$, where $W(N)$ is the Weyl group generated by reflections in -2 vectors.

The data are summarized in Table 1, where we have used the notation of [4] for groups and the notation of [5] for the glue code. In particular:

- n means a cyclic group of order n;
- p^n means an elementary p-group of order p^n;
- $G.H$ means any group F with a normal subgroup G such that $F/G = H$;
- $\mathrm{L}_m(n)$ means the group PSL_m over the finite field with n elements;
- S_n and A_n denote (respectively) permutation and alternating groups on n elements; and
- M_n and Co_n denote (respectively) the Mathieu and Conway groups.

For the glue code:

- $[abc]$ means a vector (x, y, z) with x, y, and z glue vectors of type a, b, and c, respectively;
- $[(abc)]$ means all glue vectors obtained from $[a, b, c]$ by cyclic permutations—thus, $[abc]$, $[bca]$, and $[cab]$.

It is well known (see [5, Chap. 26]) that all the Niemeier lattices can be defined as sublattices of $\Pi_{1,25} \cong U \oplus E_8(-1)^3$ by specifying a primitive isotropic vector v and setting $N = (v^\perp \cap \Pi_{1,25})/v$.

Example 2.5. Let $\Pi_{1,25} \subset \mathbb{R}^{26}$ (where the first coordinate of $\mathbb{R}^{26}$ is the positive-definite one) be as before. Let

$$v = (17, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, 5)$$

and

$$w = (70, 0, 1, 2, 3, 4, 5, \dots, 24)$$

be two isotropic vectors in the standard basis of $\mathbb{R}^{26}$. Then

$$\Lambda \cong (w^\perp \cap \Pi_{1,25})/w$$

and

$$N_{15} \cong (v^\perp \cap \Pi_{1,25})/v.$$

2.2. The "Holy" Construction and Automorphisms of the Leech Lattice

In this section we sketch the so-called holy construction (see [5, Chap. 24] for details) of the Leech lattice Λ from other Niemeier lattices. We shall use this construction later in proving Proposition 1.2.

Let $A_n(-1)$ be the negative definite Dynkin lattice defined by

$$\begin{aligned} A_n &= \left\{(a_1, \dots, a_{n+1}) \in \mathbb{Z}^{n+1},\ \textstyle\sum a_i = 0\right\}, \\ q_{A_n(-1)} &= -q_{A_n}. \end{aligned}$$

Let f_0 be the vector with -1 in the first coordinate and 1 in the second (and with 0 elsewhere). Let $g_0 = h^{-1}\left(-\frac{1}{2}n, -\frac{1}{2}n + 1, \dots, \frac{1}{2}n\right)$, where h is the Coxeter number of A_n. Let f_j and g_j, $j \in \{1, \dots, n\}$, be the respective images of f_0 and g_0 under cyclic permutations of coordinates. Notice that f_j, $j \neq 0$, are the simple roots of $A_n(-1)$.

Suppose now that $nm = 24$, so that $A_n(-1)^m$ is a 24-dimensional lattice contained in a Niemeier lattice N. Let $h_k = (g_{j_1}, \dots, g_{j_m})$, where $[j_1 j_2 \dots j_m]$ is a glue code obtained from Table 1 and $k \in G(N)$. Let $f_j^r = (0, \dots, 0, f_j, 0, \dots, 0)$, where f_j belongs to the rth copy of $A_n(-1)$. Let m_j^r and n_k be integers.

PROPOSITION 2.6 [5, Chap. 24]. *With notation as just described,*

$$\left\{ \sum_{r=1}^{m} \sum_{j=1}^{n} m_j^r f_j^r + \sum_{k \in G(N)} n_k h_k \;\Big|\; \sum_k n_k = 0 \right\} \tag{3}$$

is isometric to the negative definite Niemeier lattice N. Also,

$$\left\{ \sum_{r=1}^{m} \sum_{j=1}^{n} m_j^r f_j^r + \sum_{k \in G(N)} n_k h_k \;\Big|\; \sum_k n_k + \sum_{j,r} m_j^r = 0 \right\} \tag{4}$$

is isometric to the Leech lattice Λ and is known as the "holy" construction of Λ with hole N.

The glue code also provides several automorphisms of the Leech lattice, where the action of $t \in G(N)$ is given by sending h_w to h_{w+t}. The holy construction can be used to exhibit the action of certain elements of Co_1 on Λ explicitly, as in the following examples.

EXAMPLE 2.7. Let us apply this construction to the lattice A_{12}^2, where $G(N) = \mathbb{Z}_{/(13)}$. Let χ be an automorphism of Λ of order 13 generated by a nontrivial element g of $G(N)$ via this holy construction. Then χ cyclically permutes the simple roots of both copies of A_{12} and so has no fixed points in Λ.

EXAMPLE 2.8. Consider the lattice A_2^{12} and the cyclic permutation χ of the last 11 copies. This defines automorphisms, also denoted χ, of both N_{22} and Λ via the same action on glue vectors.

A direct computation shows that $T_\chi(N_{22})$ is spanned by

$$f_1^1,\ f_1^2,\ \sum_2^{12} f_1^i,\ \sum_2^{12} f_2^i,\ \sum_1^{11} h_j;$$

here h_j, $j \in \{1, \dots, 11\}$, are obtained from generators of the glue code as in Table 1. Moreover, $S_\chi(N_{22})$ has rank 20 and is spanned by

$$(f_1^k - \chi f_1^k),\ (f_2^k - \chi f_2^k),\ (g_j - \chi g_j), \tag{5}$$

where k runs from 2 to 12 and j is as before. These vectors lie in the set defined by (4), where g_j plays the role of h_k, $k \in G(N)$. Hence $S_\chi(N_{22})$ is contained in Λ and, since both $S_\chi(N_{22})$ and $S_\chi(\Lambda)$ are primitive, it follows that $S_\chi(N_{22}) = S_\chi(\Lambda)$.

EXAMPLE 2.9. A similar computation can be performed for A_1^{24}. We index the copies of A_1 by the set

$$\{\infty, 0, 1, \dots, 22\} = \mathbb{P}^1(\mathbb{Z}_{/(23)}).$$

The isometry χ of order 11 is defined by the permutation

$$(0)(15\,7\,14\,5\,10\,20\,17\,11\,22\,21\,19)(\infty)(3\,6\,12\,1\,2\,4\,8\,16\,9\,18\,13). \tag{6}$$

As before, this isometry preserves both N_{23} and Λ; the lattice $S_\chi(N_{23})$ is generated by the following vectors:

$$(f_1^k - \chi f_1^k),\ (f_1^l - \chi f_1^l),\ (g_j - \chi g_j). \tag{7}$$

Here k runs through the indexes contained in the first 11-cycle of (6), l runs through the second one, and j runs through the generators of the glue code contained in Table 1.

Once again, all of these generators also lie in Λ and so $S_\chi(N_{23}) = S_\chi(\Lambda)$. A direct computation shows that the lattice $S_{11} = S_\chi(N_{23})$ is given by the following quadratic form:

$$\begin{pmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{pmatrix},$$

where

$$\mathrm{A} = \begin{matrix}
-4 & 1 & -2 & -2 & -1 & 1 & -1 & 1 & -1 & -1 \\
1 & -4 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 2 \\
-2 & -1 & -4 & -2 & -1 & -1 & 0 & 1 & 0 & -1 \\
-2 & -1 & -2 & -4 & 0 & 0 & -2 & 0 & -1 & 0 \\
-1 & -1 & -1 & 0 & -4 & 1 & -1 & 2 & -2 & -1 \\
1 & -1 & -1 & 0 & 1 & -4 & 0 & -1 & 0 & 1 \\
-1 & -1 & 0 & -2 & -1 & 0 & -4 & 1 & -2 & 1 \\
1 & 1 & 1 & 0 & 2 & -1 & 1 & -4 & 0 & 0 \\
-1 & -1 & 0 & -1 & -2 & 0 & -2 & 0 & -4 & 0 \\
-1 & 2 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & -4
\end{matrix}$$

$$\mathrm{B} = \begin{matrix}
2 & 1 & -1 & 2 & -1 & -2 & -2 & 2 & 1 & -1 \\
-1 & -2 & 2 & 0 & -1 & 0 & 0 & -1 & -2 & 1 \\
1 & 0 & -1 & 2 & -2 & -1 & -1 & 0 & 0 & 1 \\
2 & 1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & -1 \\
1 & 0 & -1 & 0 & -2 & -2 & 0 & 1 & 1 & -1 \\
-2 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 1 \\
1 & 1 & 0 & -1 & 0 & -1 & 0 & 2 & 0 & -2 \\
-1 & 1 & 1 & 0 & 2 & 1 & 0 & -1 & 1 & 0 \\
0 & 1 & 1 & 0 & -1 & -2 & 0 & 2 & 0 & -2 \\
1 & 1 & -2 & 1 & 0 & 0 & 1 & 1 & 1 & 0
\end{matrix}$$

$$C = \begin{matrix} 2 & -1 & 1 & 2 & 1 & -2 & 1 & -1 & 0 & 1 \\ 1 & -2 & 0 & 1 & 0 & -1 & 1 & 1 & 1 & 1 \\ -1 & 2 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & -2 \\ 2 & 0 & 2 & 1 & 0 & -1 & -1 & 0 & 0 & 1 \\ -1 & -1 & -2 & 0 & -2 & -1 & 0 & 2 & -1 & 0 \\ -2 & 0 & -1 & 0 & -2 & 0 & -1 & 1 & -2 & 0 \\ -2 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 2 & -1 & 0 & 1 & 1 & 0 & 2 & -1 & 2 & 1 \\ 1 & -2 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & -2 & 0 & -2 & 0 \end{matrix}$$

$$D = \begin{matrix} -4 & -2 & 2 & -1 & 0 & 0 & 0 & -1 & -2 & 1 \\ -2 & -4 & 1 & 0 & -1 & 0 & -1 & -1 & -2 & 2 \\ 2 & 1 & -4 & 0 & -1 & 0 & 0 & 1 & 2 & 0 \\ -1 & 0 & 0 & -4 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 1 & -4 & -2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & -4 & -2 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 & -1 & -2 & -4 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 2 & 1 & -4 & 0 & 2 \\ -2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & -4 & 1 \\ 1 & 2 & 0 & -1 & 0 & -1 & 0 & 2 & 1 & -4 \end{matrix}$$

It is worth mentioning that S_{11} is in the same genus as $E_8(-1)^2 \oplus \left(\begin{smallmatrix} -2 & 1 \\ 1 & -6 \end{smallmatrix}\right)^2$ and $D_{16}^+(-1) \oplus \left(\begin{smallmatrix} -2 & 1 \\ 1 & -6 \end{smallmatrix}\right)^2$, where D_{16}^+ is a unimodular overlattice of the Dynkin lattice D_{16}.

2.3. *Finite Symplectic Automorphism Groups*

Let X be a hyper-Kähler manifold and let $G \subset \mathrm{Aut}(X)$. Then we put $S_G(X) = S_G(H^2(X,\mathbb{Z}))$ and $T_G(X) = T_G(H^2(X,\mathbb{Z}))$. The next two lemmas are contained in [9].

LEMMA 2.10. *If G is a finite group of symplectic automorphisms of a fourfold X of $K3^{[2]}$ type, then the following statements hold*:

(i) $S_G(X)$ *is nondegenerate and negative definite*;
(ii) $S_G(X)$ *contains no elements with square* -2;
(iii) $S_G(X) \subset \mathrm{Pic}(X)$;
(iv) G *acts trivially on* $A_{S_G(X)}$.

This lemma amounts to saying that $(S_G(X), G)$ is a Leech couple.

LEMMA 2.11. *Let $L = U^3 \oplus E_8(-1)^2 \oplus (-2)$, and let G be a finite subgroup of $O(L)$. Suppose that*:

(i) $S_G(L)$ *is nondegenerate and negative definite*; *and*
(ii) $S_G(L)$ *contains no element with square* (-2).

Then there exists a hyper-Kähler manifold X *of* $K3^{[2]}$ *type and a subgroup* $G' \subset \mathrm{Bir}(X)$ *such that* $G' \cong G$, $S_G(L) \cong S_{G'}(X)$, *and* $G'|_{H^{2,0}(X)} = \mathrm{Id}$.

PROPOSITION 2.12. *Let* (S, G) *be a Leech couple such that* S *is primitively contained in some Niemeier lattice* N, *and suppose there exists a primitive embedding* $S \to L$. *Then* G *extends to a group of bimeromorphisms on some hyper-Kähler manifold* X *of* $K3^{[2]}$ *type.*

Proof. This is an immediate consequence of Lemma 2.11: G acts trivially on A_S, so we can extend G to a group of isometries of L acting trivially on $S^{\perp_L}$. Thus we have $S_G(L) \cong S$. Then the conditions of Lemma 2.11 are satisfied because (S, G) is a Leech couple. □

We are now ready to prove the main result of this section.

Proof of Theorem 1.1. Let $b = \mathrm{rank}(S_G(X))$; by Lemma 2.10, $S_G(X)$ has signature $(0, b)$. By Remark 2.3 we have a lattice T' of signature $(4, 20 - b)$ such that $A_{T'} = A_{S_G(X)}$ and $q_{T'} = -q_{A_{S_G(X)}}$. We can therefore apply Lemma 2.1 to obtain a lattice T of signature $(0, 24 - b)$ and discriminant form $-q_{A_{S_G(X)}}$. Hence by Lemma 2.2 there exists a primitive embedding $S_G(X) \to N$, where N is one of the lattices contained in Table 1. Again by Lemma 2.10 we see that $(S_G(X), G)$ is a Leech couple, whence G lies inside $\mathrm{Leech}(N)$. Using the holy construction now yields $\mathrm{Leech}(N) \subset \mathrm{Leech}(\Lambda)$. A direct computation then shows that, for all $G \subset \mathrm{Leech}(N) \subset \mathrm{Leech}(\Lambda)$, we have $\mathrm{rank}(S_G(N)) = \mathrm{rank}(S_G(\Lambda))$ (after tensoring with $\mathbb{Q}$ they are both generated by elements of the form $v - g(v)$ with $v \in N$ and $g \in G$). The central involution of Co_0 clearly has a co-invariant lattice of rank 24, so we can restrict ourselves to Co_1. □

COROLLARY 2.13. *Let* ϕ *be a symplectic automorphism of prime order* p *on a hyper-Kähler fourfold* X *of* $K3^{[2]}$ *type. Then* $p \leq 11$.

Proof. By Theorem 1.1, the order of a symplectic automorphism must divide the order of the group Co_1. That criterion excludes all primes except for 2, 3, 5, 7, 11, 13, and 23. An automorphism of order 23 has a co-invariant lattice that is negative definite and of rank 22, so it cannot embed into $H^2(X, \mathbb{Z})$. This result can be computed explicitly using an order-23 element of M_{24} and letting it act on Λ or on N_{23}. The only Niemeier lattice with an automorphism of order 13 is Λ, where all elements of order 13 are conjugate (see [4]). These automorphisms have no fixed points on Λ, as in Example 2.7. □

Proof of Proposition 1.2. By Theorem 1.1, $S_\psi(X)$ embeds in a Niemeier lattice N and ψ extends to an element of $\mathrm{Leech}(N)$. By Table 1, N can only be N_{22}, N_{23}, or Λ and—up to conjugacy—there is only one possible choice for $\psi \in O(N_{23})$ or $\psi \in O(\Lambda)$ and only two possible choices for $\psi \in O(N_{22})$.

We computed $S_\psi(N)$ in Examples 2.8 and 2.9, where we proved that it is isometric to S_{11}. We immediately have $20 \leq h^{1,1}_{\mathbb{Z}}(X) \leq 21$; hence clearly $h^{1,1}_{\mathbb{Z}}(X) = 21$ if X is projective. Now we wish to give an action of $\mathrm{L}_2(11) = \mathrm{PSL}_2(\mathbb{Z}_{/(11)})$ on S_{11}. We can suppose without loss of generality that $N = N_{23}$, in which case the action of $\mathrm{L}_2(11)$ is given by the following permutations of the standard coordinates of N_{23} (see [5, pp. 274, 280]):

$$\alpha = (15\,7\,14\,5\,10\,20\,17\,11\,22\,21\,19)(3\,6\,12\,1\,2\,4\,8\,16\,9\,18\,13);$$

$$\beta = (14\,17\,11\,19\,22)(20\,10\,7\,5\,21)(18\,4\,2\,6\,1)(8\,16\,13\,9\,12);$$

$$\gamma = (2\,4)(5\,10)(6\,18)(8\,12)(9\,16)(11\,17)(14\,19)(20\,21).$$

Now, by Proposition 2.12, this action of $\mathrm{L}_2(11)$ on S_{11} is induced by a group of birational transformations isomorphic to $\mathrm{L}_2(11)$. □

3. Deformation Behavior

In this section we analyze deformation classes of manifolds of $\mathrm{K3}^{[2]}$ type with a symplectic automorphism of order 11. We also look at their possible invariant polarizations.

DEFINITION 3.1. Let X be a hyper-Kähler manifold with Kähler class ω and symplectic form σ_X. Then there exists a family

$$\begin{array}{ccc} \mathrm{Tw}_\omega(X) & := & X \times \mathbb{P}^1 \\ & & \downarrow \\ S^2 & \cong & \mathbb{P}^1, \end{array} \tag{8}$$

called the twistor family, such that $\mathrm{Tw}_\omega(X)_{(a,b,c)} = X$ with complex structure given by the Kähler class $a\omega + b(\sigma_X + \bar{\sigma}_X) + c(\sigma_X - \bar{\sigma}_X)$.

LEMMA 3.2. *Let X be a hyper-Kähler manifold, and let $G \subset \mathrm{Aut}(X)$ be a finite group of symplectic automorphisms. Let ω be a G-invariant Kähler class. Then the action of G on X extends to a symplectic action of G on all the fibers of the twistor space associated to ω.*

Proof. Every fiber has a Kähler class that is a linear combination of σ_X, $\bar{\sigma}_X$, and ω. Because G is symplectic on X, these classes are all G-invariant; therefore, $G \subset \mathrm{Aut}(\mathrm{Tw}_\omega(X)_t)$ for all t. □

Let X be a manifold of $\mathrm{K3}^{[2]}$ type with a symplectic automorphism ψ of order 11, and let ω be a ψ-invariant Kähler class.

It follows from Proposition 1.2 that a nontrivial deformation of (X, ψ) has dimension at most 1. Moreover, the twistor family $\mathrm{Tw}_\omega(X)$ is naturally endowed with a symplectic automorphism of order 11 (as in Lemma 3.2). Hence $\mathrm{Tw}_\omega(X)$ is already a family of the maximal dimension for such pairs (X, ψ). We also have that the twistor family $\mathrm{Tw}_\omega(X)$ is actually a family over the base $(T_\psi(X)\otimes\mathbb{R})/\mathbb{C}^*$, where the $\mathbb{C}^*$ action is given by the identification $T_\psi(X)=\langle\omega,\sigma_X,\bar{\sigma}_X\rangle\cap H^2_{\mathbb{Z}}(X)$.

Thus what we really need to analyze are the possible lattices $T_\psi(X)$ up to isometry. We have already proved that there is only one isometry class of lattices $S_\psi(X)$—namely, that of S_{11}. Yet there might be several isometry classes of lattices $T_\psi(X)$. In fact, Theorem 1.1 and Proposition 2.12 can be used to compute the genus of $T_\psi(X)$ only.

A direct computation shows that there are two such lattices:

$$T^1_{11}=\begin{pmatrix}2&1&0\\1&6&0\\0&0&22\end{pmatrix}, \tag{9}$$

$$T^2_{11}=\begin{pmatrix}6&-2&-2\\-2&8&-3\\-2&-3&8\end{pmatrix}. \tag{10}$$

Hence there are two distinct families of hyper-Kähler manifolds endowed with a symplectic automorphism of order 11, whose existence follows from Lemma 2.11. We call these families $\mathrm{Tw}(X_1)$ and $\mathrm{Tw}(X_2)$.

Invariant Polarizations

In this subsection we look at possible invariant polarizations of small degree of $\mathrm{Tw}(X_1)$ and $\mathrm{Tw}(X_2)$—that is, at primitive vectors in T^1_{11} and T^2_{11}. We have computed several polarizations of degree up to 24.

PROPOSITION 3.3. *The minimal degree of an invariant polarization inside* $\mathrm{Tw}(X_1)$ *is 2, and there are no polarizations of degree 4, 12, 14, 16, or 20. Moreover, the least degree of a polarization f such that $(f,L)=2\mathbb{Z}$ (i.e., f has divisor 2) is 22.*

Proof. Let a,b,c be the basis of T^1_{11} in (9). A minimal polarization is given by the vector a, and a minimal polarization of degree 22 and divisor 2 is given by c. The rest is just a direct computation. □

PROPOSITION 3.4. *The minimal degree of an invariant polarization inside* $\mathrm{Tw}(X_2)$ *is 6, and there are no polarizations of degree 12, 14, 16, or 20. Moreover, the least degree of a polarization f such that $(f,L)=2\mathbb{Z}$ is 6.*

Proof. Let a,b,c be the basis of T^2_{11} in (10). A minimal polarization is given by the vector a, which also has divisor 2. The rest is just a direct computation. □

COROLLARY 3.5. *Let X be a manifold of $K3^{[2]}$ type with a symplectic automorphism ψ of order 11 and with an invariant polarization of degree 6 and divisor 2. Then*

$$\mathrm{NS}(X)\cong(6)\oplus E_8(-1)^2\oplus\begin{pmatrix}-2&1\\1&-6\end{pmatrix}^2 \tag{11}$$

and

$$T(X)=\langle\sigma_X,\bar{\sigma}_X\rangle\cap H^2(X,\mathbb{Z})\cong\begin{pmatrix}22 & 33\\ 33 & 66\end{pmatrix}. \tag{12}$$

Proof. We know that $\mathrm{NS}(X)$ is an overlattice of $S_{11}\oplus(6)$. Since there are no nontrivial overlattices (its discriminant group has no nontrivial isotropic elements) of $S_{11}\oplus(6)$, it follows that $\mathrm{NS}(X)=S_{11}\oplus(6)$. A direct computation shows that this lattice is isomorphic to $(6)\oplus E_8(-1)^2\oplus\begin{pmatrix}-2 & 1\\ 1 & -6\end{pmatrix}^2$. Finally, $T(X)$ is the orthogonal complement of the polarization in $T_\psi(X)$. By Propositions 3.3 and 3.4 we have $T_\psi(X)=T_{11}^2$, and a direct computation shows that $T(X)=\begin{pmatrix}22 & 33\\ 33 & 66\end{pmatrix}$. □

4. The Fano Scheme of Lines F_{Kl}

The main goal of this section is to prove Theorem 1.3. We start by giving some results, about Fano schemes of lines on cubic fourfolds, that are due to Beauville and Donagi [3].

THEOREM 4.1. *Let $X\subset\mathbb{P}^5$ be a smooth cubic fourfold, and let $F(X)$ be the scheme parameterizing lines contained in X. Then the following statements hold*:

- *$F(X)$ is a hyper-Kähler manifold*;
- *$F(X)$ is of $K3^{[2]}$ type*;
- *the Abel–Jacobi map*

$$\alpha\colon H^4(X,\mathbb{C})\to H^2(F(X),\mathbb{C}) \tag{13}$$

is an isomorphism of Hodge structures.

Proof of Theorem 1.3. Let ψ be the element of $\mathrm{PGL}_6(\mathbb{C})$ sending $(x_0:x_1:x_2:x_3:x_4:x_5)$ to $(x_0:\omega x_1:\omega^3x_2:\omega^4x_3:\omega^5x_4:\omega^9x_5)$, where $\omega=e^{2\pi i/11}$. A cubic polynomial is ψ-invariant if and only if it is in the linear span of

$$B=\{x_0^3,x_1^2x_5,x_2^2x_4,x_3^2x_2,x_4^2x_1,x_5^2x_3\}.$$

An easy computation shows that the differential of f has nontrivial zeroes (so that the cubic fourfold $V(f)$ is singular) if and only if f lies in the span of some proper subset of B. Yet this is not the case for

$$h=x_0^3+x_1^2x_5+x_2^2x_4+x_3^2x_2+x_4^2x_1+x_5^2x_3;$$

hence $X_{\mathrm{Kl}}=V(h)$, as defined in (1), is smooth and so we can apply Theorem 4.1. We obtain that F_{Kl} is a hyper-Kähler manifold that is deformation equivalent to $\mathrm{K3}^{[2]}$. We also obtain the Hodge isomorphism α given in (13).

Let φ be the map induced on F_{Kl} by ψ. Using α, we have that φ is symplectic if and only if $\psi|_{H^{3,1}(X_{\mathrm{Kl}})}=\mathrm{Id}$.

By the formula in [13, Thm. 18.1] we have

$$H^{3,1}(X_{\mathrm{Kl}})=\left\langle\mathrm{Res}\left(\frac{\Omega}{h^2}\right)\right\rangle, \tag{14}$$

where $\Omega=\sum_i(-1)^ix_i\,dx_0\wedge\ldots\widehat{dx_i}\ldots\wedge dx_5$. Since ψ acts trivially on both Ω and h, it follows that $\varphi(\sigma_{F_{\mathrm{Kl}}})=\sigma_{F_{\mathrm{Kl}}}$ for any symplectic 2-form $\sigma_{F_{\mathrm{Kl}}}$ on F_{Kl}.

Finally, fixed points of ψ on X_{Kl} are the eigenvectors of ψ lying on X_{Kl}, which are the points $[e_1]$, $[e_2]$, $[e_3]$, $[e_4]$, and $[e_5]$; here $e_1 = (0,1,0,0,0,0) \in \mathbb{C}^6$ and so forth. A fixed line on X_{Kl} must contain two fixed points, so the fixed points on F_{Kl} are those parameterizing lines through those points:

$$\overline{[e_1][e_2]},\ \overline{[e_1][e_3]},\ \overline{[e_2][e_5]},\ \overline{[e_3][e_4]},\ \overline{[e_4][e_5]}.$$

This proves that φ is not the identity and hence it indeed has order 11. □

REMARK 4.2. Proposition 3.3 and Proposition 3.4 imply that $F_{\mathrm{Kl}} \subset \mathrm{Tw}(X_2)$, and Corollary 3.5 gives the Neron–Severi and transcendental lattices. Moreover, φ does not lift to any small projective nontrivial deformation of F_{Kl}, since $h^{1,1}_{\mathbb{Z}}(F_{\mathrm{Kl}}) = h^{1,1}_{\mathbb{C}}(F_{\mathrm{Kl}})$.

5. $\mathrm{L}_2(11)$ Acting on F_{Kl}

In this section we give explicitly the automorphisms of X_{Kl} that generate the action of $\mathrm{L}_2(11)$ from Proposition 1.2.

We remark that X_{Kl} is a 3 : 1 Galois cover of $\mathbb{P}^4$ ramified along the threefold

$$KA = V(x_1^2x_5 + x_2^2x_4 + x_3^2x_2 + x_4^2x_1 + x_5^2x_3), \tag{15}$$

where the covering map is simply the projection

$$[x_0, x_1, x_2, x_3, x_4, x_5] \to [x_1, x_2, x_3, x_4, x_5]$$

and the covering automorphism group is generated by

$$[x_0, x_1, x_2, x_3, x_4, x_5] \xrightarrow{\alpha} [\eta x_0, x_1, x_2, x_3, x_4, x_5], \quad \eta = e^{2\pi i/3}.$$

Note that α acts as multiplication by η on $H^{3,1}(X_{\mathrm{Kl}})$.

Clearly, any automorphism of $\mathbb{P}^4$ that preserves KA extends to an automorphism of X_{Kl}. By the results in [1] and [7], these automorphisms generate precisely the group $\mathrm{L}_2(11) = \mathrm{PSL}_2(\mathbb{Z}_{/(11)})$, which is a finite simple group of order 660. Hence the automorphism group of X_{Kl} contains $\mathbb{Z}_{/(3)} \times \mathrm{L}_2(11)$. Now we need only find generators of this group and then determine whether or not they act symplectically on F_{Kl}.

Permuting the coordinates on $\mathbb{P}^4$ by the cyclic permutation (1 4 2 3 5) preserves KA and thus induces an automorphism β of order 5 on F_{Kl}. From (14) we can check that β is a symplectic automorphism. Furthermore, a direct computation on the Jacobian ring $\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]/(\partial h/\partial x_0, \dots, \partial h/\partial x_5)$ of X_{Kl} shows that $\mathrm{rank}(S_\beta(F_{\mathrm{Kl}})) = 16$.

Let H be the kernel of the action of $\mathrm{L}_2(11)$ on $H^{2,0}(F_{\mathrm{Kl}})$. This kernel H is nontrivial because $\beta \in H$, so $H = \mathrm{L}_2(11)$. Therefore, $\mathrm{L}_2(11)$ acts symplectically on F_{Kl}. By [4], $\mathrm{L}_2(11)$ contains only elements of order 2, 3, 5, 6, and 11. A direct computation shows that

$$\mathrm{rank}(S_\alpha(F_{\mathrm{Kl}})) = \begin{cases} 8 & \text{if } \mathrm{ord}(\alpha) = 2, \\ 12 & \text{if } \mathrm{ord}(\alpha) = 3, \\ 16 & \text{if } \mathrm{ord}(\alpha) = 5 \text{ or } 6, \\ 20 & \text{if } \mathrm{ord}(\alpha) = 11. \end{cases}$$

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Department of Mathematics
Università degli studi RomaTRE
Largo San L. Murialdo 1
00146, Rome
Italy