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# On Invariants of Complete Intersections

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## Introduction and Background

When $R$ is a local hypersurface (that is, the quotient of a regular local ring by a regular element), Hochster [8, p. 98] defined the invariant $\theta^R$ for any pair of finitely generated $R$-modules $M, N$ such that the localization of $N$ at any prime other than the maximal ideal has finite projective dimension over the corresponding localized ring. This invariant is simply the difference in lengths of two consecutive Tor modules in high enough degree. The vanishing of this invariant is tied to the dimension inequality; to wit, when $R$ is an *admissible* hypersurface and the tensor product of $M$ and $N$ has finite length, $\theta^R(M,N) = 0$ if and only if $\dim M + \dim N \le \dim R$ [8, Thm. 1.4]. (A ring $R$ is *admissible* if a completion $\hat{R}$ of $R$ at a maximal ideal satisfies $\hat{R} \cong T/(f)$ and if the *dimension inequality, vanishing,* and *positivity* of Serre [14, V.5.1] hold for $T$; Serre showed that these conditions on $T$ hold when $T$ is a regular local ring containing a field.)

In this paper, we introduce a new invariant, denoted $\boldsymbol{\Theta}_c^R$, in the case that the ring is a standard graded complete intersection with isolated singularity, and we show that if the tensor product of a graded pair $M, N$ has finite length, then $\boldsymbol{\Theta}_c^R(M,N) = 0$ if and only if $\dim M + \dim N \le \dim R$. Moreover, $\dim M + \dim N \le \dim R + 1$ regardless of the value of $\boldsymbol{\Theta}_c^R(M,N)$. See Corollary 2.6.

This work continues in the spirit of recent research and takes its motivation from H. Dao, W. F. Moore et al., and Y. Kobayashi. To be specific, Dao [2; 3] provided an in-depth study of $\theta^R$, especially in the case that the ring has an isolated singularity at the maximal ideal, tying the invariant to questions of dimension and rigidity. Along with W. F. Moore, G. Piepmeyer, and M. E. Walker, the author continued this particular study of $\theta^R$ under the additional assumption that $R$ is graded and contains a field. In particular, we established the vanishing of $\theta^R$ for every pair of finitely generated modules when the dimension of $R$ is even [12, Thm. 3.2], proving, in the graded case, a conjecture posed by Dao [3, Conj. 3.15]. In addition, we showed that $\theta^R$ factors through cohomology and gave a formula for the pairing in odd dimension.

Let $R$ be a complete intersection (that is, the quotient of a regular ring $Q$ by a regular sequence of length $c$). If the $\operatorname{Tor}_j^R(M,N)$ eventually have finite length, then these lengths follow predictable patterns in high degree on even and odd indices.

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Specifically, as per [13, Prop. 2.1], there are polynomials of degree at most $c-1$ that determine the lengths of the even and odd Tor modules, respectively, for all $j \gg 0$. These are used by Moore et al. [13, Def. 2.2] to study an invariant first introduced by Dao [4, 4.2], namely the invariant $\eta_c^R(M, N)$. Moreover, when $R$ has an isolated singularity, we establish that $\eta_c^R(M, N)$ vanishes for all pairs of finitely generated modules when $c > 1$ [13, Thm. 4.5, Cor. 4.7].

Some key properties of $\eta_c^R$ differ fundamentally from those of $\theta^R$. Here we introduce and study a new invariant, $\boldsymbol{\Theta}_c^R(M, N)$, which shares many of the same properties as $\theta^R$, for example, how its vanishing relates to the dimension inequality, as noted previously. We establish results analogous to those in [12]. Additionally, we investigate the "expected dimension" of the intersection of two modules $M$ and $N$ versus its actual dimension, and we tie $\boldsymbol{\Theta}_c^R(M, N)$ to a generalized Bézout's theorem relating the degrees of the modules, their associated homology modules, and the ambient ring. See Theorem 2.4. Finally, we briefly return to $\eta_c^R$ and also tie it to a generalized Bézout's theorem. Some examples are calculated.

## 1. The $\boldsymbol{\Theta}_c^R$ Invariant

Let $R$ be the quotient of a Noetherian ring $Q$ by a regular sequence $f_1, \dots, f_c$ and let $M$ and $N$ be finitely generated $R$-modules. Suppose that, for all $j \gg 0$, the $Q$-modules $\operatorname{Tor}_j^Q(M, N)$ vanish and the $\operatorname{Tor}_j^R(M, N)$ have finite length.

Under these hypotheses, the authors [13, Prop. 2.1] show that there are polynomials of degree at most $c-1$ that determine the lengths of the even and odd Tor modules, respectively, for high indices. These polynomials make it possible to make the following definition.

DEFINITION 1.1 [13, Def. 2.2, Prop. 2.1]. With $R$, $M$, and $N$ as in the opening paragraph of this section, define

$$\eta_c^R(M, N) = \frac{(P_{\mathrm{ev}} - P_{\mathrm{odd}})^{(c-1)}}{2^c \cdot c!},$$

the $(c-1)$th iterated first difference of $P_{\mathrm{ev}} - P_{\mathrm{odd}}$, where the polynomials $P_{\mathrm{ev}}$ and $P_{\mathrm{odd}}$ depend on $M$ and $N$, have degree at most $c-1$, and satisfy

$$\operatorname{len}(\operatorname{Tor}_{2j}^R(M, N)) = P_{\mathrm{ev}}(j), \quad \operatorname{len}(\operatorname{Tor}_{2j+1}^R(M, N)) = P_{\mathrm{odd}}(j) \quad \text{for all } j \gg 0.$$

(The *first difference* of a polynomial $q(j)$ is the polynomial $q^{(1)}(j) = q(j) - q(j-1)$, and recursively one defines $q^{(i)} = (q^{(i-1)})^{(1)}$.)

When the ring is graded and $M$ and $N$ are finitely generated graded, the value for $\eta_c^R(M, N)$ can be realized via Hilbert series.

The Hilbert series of a finitely generated, nonnegatively graded $R$-module $M$ is $\sum_{i \geq 0} \dim_{\mathsf{k}}(M_i) t^i$, denoted $H_M(t)$. In fact, it is a rational function with a pole of order $\dim M$ at $t = 1$. To be specific,

$$H_M(t) = \frac{e_M(t)}{(1-t)^{\dim M}}, \tag{1.1}$$

where $e_M(t)$ is a polynomial in $\mathbb{Z}[t]$, sometimes called the *multiplicity polynomial* of $M$ [1, 1.1]. The value of $e_M(1)$ is always a positive integer, which is called the *multiplicity* or *degree* of $M$.

The modules $\mathrm{Tor}_j^R(M,N)$, which are graded when $M$ and $N$ are graded, will often be abbreviated to $T_j$, and their Hilbert series denoted by $H_{T_j}(t)$.

LEMMA 1.2 [13, Lemma 3.6]. *Let* $\mathsf{k}$ *be a field and* $Q = \mathsf{k}[x_0,\dots,x_{n+c-1}]$, *where* $\deg x_l = 1$ *for all* $l$. *Set* $R = Q/(f_1,\dots,f_c)$, *where* $f_1,\dots,f_c$ *forms a regular* $Q$*-sequence and each* $f_i$ *is a homogeneous polynomial of degree* $d_i$ *for* $1 \le i \le c$. *Let* $M$ *and* $N$ *be finitely generated graded* $R$*-modules such that the* $\mathrm{Tor}_j^R(M,N)$ *eventually have finite length. Then, for* $E \gg 0$ *with* $E$ *an even integer, there is a unique polynomial* $\eta_{c,E}^R(M,N)(t)$ *in* $\mathbb{Q}[t]$ *such that*

$$\sum_{j\ge E}(-1)^j H_{T_j}(t) = \frac{\eta_{c,E}^R(M,N)(t)}{e_R(t)(1-t)^c} \quad \textit{and} \quad \eta_{c,E}^R(M,N)(1) = 2^c c!\cdot \eta_c^R(M,N),$$

*where* $e_R(t)$ *is the multiplicity polynomial of* $R$ *as in* (1.1). (*See also* (1.3) *in Lemma 1.7.*)

A more explicit, but equivalent, definition for $\eta_{c,E}^R(M,N)(t)$ under the assumptions of Lemma 1.2 is as follows.

DEFINITION 1.3. Let $s_1(t),\dots,s_c(t)$ denote the elementary symmetric functions on $t^{d_1},\dots,t^{d_c}$, and set $s_0(t)\equiv 1$. (To be specific $s_1(t) = t^{d_1}+\cdots+t^{d_c}$; $\sum_{1\le i<j\le c} t^{d_i+d_j},\dots$; and $s_c(t) = t^{d_1+\cdots+d_c}$.) Then for $E \gg 0$ and even, as in Lemma 1.2,

$$\eta_{c,E}^R(M,N)(t) = \sum_{j=0}^{c-1}\left(\sum_{i=0}^{c-(j+1)}(-1)^i s_i(t)(H_{T_{E+2j}}(t) - H_{T_{E+2j+1}}(t)\right).$$

It is easy to see that when $R$ is a hypersurface, this formula immediately reduces to the definition of $\theta^R(M,N)$, namely $\mathrm{len}(\mathrm{Tor}_E^R(M,N)) - \mathrm{len}(\mathrm{Tor}_{E+1}^R(M,N))$.

The following example uses a generic construction of complete intersections by Jorgensen [10, Exm. 2.8]. We will revisit this calculation in Section 3.

EXAMPLE 1.4. Let $R = \mathsf{k}[X,Y,Z_1,\dots,Z_8]/(XY, Z_1Z_2Z_3 - Z_4^3, Z_5Z_6 - Z_7Z_8)$, where $\mathsf{k}$ is a field, and set $M = R/(x,z_1,\dots,z_8)$ and $N = R/(y)$. Then $R$ is a complete intersection and $\mathrm{len}(\mathrm{Tor}_j^R(M,N))$ is 1 if $j \ge 0$ is even and 0 if $j$ is odd.

Therefore, $e_{T_{2j}}(t) = H_{T_{2j}}(t) = t^{2j}$ and, by Definition 1.3,

$$\begin{aligned}
&\eta_{3,E}^R(M,N)(t)\\
&\quad = (1-2t^2-t^3+t^4+2t^5)(H_{T_E}(t) - H_{T_{E+1}}(t))\\
&\qquad + (1-2t^2-t^3)(H_{T_{E+2}}(t) - H_{T_{E+3}}(t)) + (H_{T_{E+4}}(t) - H_{T_{E+5}}(t))
\end{aligned}$$

for $E \gg 0$. In particular, $\eta_{3,E}^R(M,N)(t) = t^E(1-t)^2(1+t)(1+t+t^2)$ and hence $\eta_3^R(M,N) = 0$.

If $R$ has an isolated singularity then, under the assumptions in the Lemma (where now the eventual finite length of the Tor modules is guaranteed), it is shown in [13] that $\eta_c^R(M, N)$ always vanishes when $c > 1$. We are thus motivated to define a new invariant by the fact that—unlike Hochster's original $\theta^R$ invariant, which is zero if and only if the dimension inequality is satisfied and the length of $M \otimes_R N$ is finite—the vanishing of $\eta_c^R(M, N)$ is independent of dimension and length considerations for $c > 1$. It is from this result that we define the new invariant $\mathbf{\Theta}_c^R(M, N)$.

*Throughout the remainder of this section and the next,* we make the following assumptions:

- $\mathsf{k}$ is a separably closed field;
- $R = \mathsf{k}[x_0, \dots, x_{n+c-1}]/(f_1, \dots, f_c)$, where $\deg x_l = 1$ for all $l$ and each $f_i$ is a homogeneous polynomial of degree $d_i$, $1 \le i \le c$;
- $f_1, \dots, f_c$ forms a regular sequence;
- $X = \operatorname{Proj} R$ is a smooth $\mathsf{k}$-variety, and $\mathfrak{m} = (x_0, \dots, x_{n+c-1})$ is the only nonregular prime of $R$;
- $M$ and $N$ are finitely generated graded $R$-modules;
- $E \gg 0$ is an even integer such that $\eta_{c,E}^R(M, N)(1) = 2^c c! \cdot \eta_c^R(M, N)$.

(1.2)

### *1.1. Definition and Preliminaries*

THEOREM 1.5 [13, Thm. 4.5]. *Under assumptions* (1.2), $\eta_{c,E}(M, N)(t)$ *has a zero at* $t = 1$ *of order at least* $c - 1$. *In particular,* $\eta_c^R(M, N) = 0$ *for* $c > 1$.

DEFINITION 1.6. Under assumptions (1.2), define $\mathbf{\Theta}_{c,E}^R(M, N)(t)$ to be the unique polynomial that satisfies

$$\eta_{c,E}^R(M, N)(t) = (1 - t)^{c-1} \mathbf{\Theta}_{c,E}^R(M, N)(t).$$

With this definition, the following relation is analogous to the one in Lemma 1.2.

LEMMA 1.7. *The polynomial* $\mathbf{\Theta}_{c,E}^R(M, N)(t)$ *satisfies*

$$\sum_{j \ge E} (-1)^j H_{T_j}(t) = \frac{\mathbf{\Theta}_{c,E}^R(M, N)(t)}{e_R(t)(1 - t)}, \quad \textit{where } e_R(t) = \frac{\prod_{i=1}^c (1 - t^{d_i})}{(1 - t)^c}. \tag{1.3}$$

*Proof.* This follows easily from Lemma 1.2, Theorem 1.5, and Definition 1.6. □

PROPOSITION 1.8. *The value of* $\mathbf{\Theta}_{c,E}^R(M, N)(t)$ *at* $t = 1$ *is independent of* $E$. *Therefore,* $\mathbf{\Theta}_c^R(M, N) := \mathbf{\Theta}_{c,E}^R(M, N)(1)$ *is well-defined.*

*Proof.* For $E \gg 0$ and even, (1.3) holds. In particular, $\operatorname{len}(\operatorname{Tor}_j^R(M, N)) < \infty$ for $j \ge E$. Thus

$$\mathbf{\Theta}_{c,E}^R(M, N)(t) - \mathbf{\Theta}_{c,E+2}^R(M, N)(t) = e_R(t)(1 - t)(H_{T_E}(t) - H_{T_{E+1}}(t)),$$

which is an equation in $\mathbb{Q}[t]$. Evaluate at $t = 1$. □

PROPOSITION 1.9. $\mathbf{\Theta}_c^R(\cdot, \cdot)$ *is a symmetric pairing, and it is biadditive on short exact sequences of finitely generated graded* $R$*-modules.*

*Proof.* The first statement is obvious. For the biadditivity, let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated graded $R$-modules. Tensor with $N$ to obtain the following long exact sequence, where for $j$ large enough, the Tor modules have finite length:

$$\cdots \to \operatorname{Tor}_j^R(M', N) \to \operatorname{Tor}_j^R(M, N) \to \operatorname{Tor}_j^R(M'', N) \xrightarrow{\delta} \operatorname{Tor}_{j-1}^R(M', N) \to \cdots.$$

As $H_{T_j}(t)$ denotes the Hilbert series of $\operatorname{Tor}_j^R(M, N)$, let $H'_{T_j}(t)$ and $H''_{T_j}(t)$ denote the Hilbert series of $\operatorname{Tor}_j^R(M', N)$ and $\operatorname{Tor}_j^R(M'', N)$, respectively. Then for an even integer $E \gg 0$ such that all of the Tor modules have finite length for $j \geq E$ and Lemma 1.7 applies to all three pairs $(M, N), (M', N), (M'', N)$, we have

$$e_R(t)(1-t)\left(\sum_{j\geq E}(-1)^j H'_{T_j}(t) - \sum_{j\geq E}(-1)^j H_{T_j}(t) + \sum_{j\geq E}(-1)^j H''_{T_j}(t)\right)$$
$$= \mathbf{\Theta}^R_{c,E}(M', N)(t) - \mathbf{\Theta}^R_{c,E}(M, N)(t) + \mathbf{\Theta}^R_{c,E}(M'', N)(t). \quad (1.4)$$

Let $C_E$ be the cokernel of the map $\operatorname{Tor}_E^R(M, N) \to \operatorname{Tor}_E^R(M'', N)$. Since the Hilbert series is an additive function on the category of graded $R$-modules, the term in large parentheses in (1.4) can be replaced with the Hilbert series of $C_E$. Since $C_E$ has finite length, its Hilbert series is simply a polynomial, say $q(t)$. Thus, evaluating

$$e_R(t)(1-t)q(t) = \mathbf{\Theta}^R_{c,E}(M', N)(t) - \mathbf{\Theta}^R_{c,E}(M, N)(t) + \mathbf{\Theta}^R_{c,E}(M'', N)(t)$$

at $t = 1$ yields the result.

The proof of biadditivity in the second component follows from the fact that the pairing is symmetric. □

Corollary 1.10. *$\mathbf{\Theta}^R_c(\cdot, \cdot)$ defines a pairing on the Grothendieck group of finitely generated graded $R$-modules.*

### *1.2. Geometric Interpretation of* $\mathbf{\Theta}^R_c(M, N)$

Many of the properties satisfied by Hochster's original theta function are also satisfied by $\mathbf{\Theta}^R_c$. To establish this, we rely heavily upon our previous work and hence refer the reader to [12, Sec. 2] and [13, Sec. 4] for more explanation of the maps and groups used in this section. Good general references for this material are [7] and [6].

By $G(Z)$ and $K(Z)$ we denote the Grothendieck groups of coherent sheaves and locally free coherent sheaves, respectively, when $Z$ is a quasi-projective scheme over a field $\mathsf{k}$. Recall that $\operatorname{Proj} R = X \subseteq \mathbb{P}^{n+c-1}$, where the smooth variety $X$ has dimension $n-1$ and degree $d = d_1 \cdots d_c$. Since $\mathsf{k}$ is infinite, for an open subset $U$ containing $X$ there is a linear rational map $\mathbb{P}^{n+c-1} \dashrightarrow \mathbb{P}^{n-1}$ determining a regular function on $U$. This induces a finite, flat, regular map $\rho\colon X \to \mathbb{P}^{n-1}$ of degree $d$. Moreover,

$$\rho_*\colon K(X) \cong G(X) \to G(\mathbb{P}^{n-1}) \cong K(\mathbb{P}^{n-1})$$

and $\mathbb{Z}[t]/(1-t)^n \cong K(\mathbb{P}^{n-1})$ via $t \mapsto [\mathcal{O}(-1)]$ (see [7, Exr. III.5.4]). We identify $K(\mathbb{P}^{n-1}) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\mathbb{Q}[t]/(1-t)^n$.

LEMMA 1.11 (cf. [12, Lemma 4.2; 13, Lemma 4.3]). *Under assumptions* (1.2), *for any sufficiently large even integer $E$, the rational function*

$$(1-t)^{n-1}\frac{\boldsymbol{\Theta}_c^R(M,N)(t)}{(e_R(t))^2}$$

*does not have a pole at $t=1$. Its image in $\mathbb{Q}[t]_{(t)}/(1-t)^n = \mathbb{Q}[t]/(1-t)^n = K(\mathbb{P}^{n-1})_{\mathbb{Q}}$ satisfies the equation*

$$(1-t)^{n-1}\frac{\boldsymbol{\Theta}_{c,E}^R(M,N)(t)}{(e_R(t))^2} = \left(\frac{\rho_*([\tilde{M}])}{\rho_*(1)}\cdot\frac{\rho_*([\tilde{N}])}{\rho_*(1)} - \frac{\rho_*([\tilde{M}]\cdot[\tilde{N}])}{\rho_*(1)}\right).$$

*In particular, since* $\deg X = d$,

$$(1-t)^{n-1}\boldsymbol{\Theta}_c^R(M,N) = \left(\frac{d\cdot\rho_*([\tilde{M}])}{\rho_*(1)}\cdot\frac{d\cdot\rho_*([\tilde{N}])}{\rho_*(1)} - \frac{d^2\cdot\rho_*([\tilde{M}]\cdot[\tilde{N}])}{\rho_*(1)}\right).$$

*Proof.* Apply [13, Lemma 4.3] with $m=0$ and use the substitution

$$\eta_{c,E}^R(M,N)(t) = (1-t)^{c-1}\boldsymbol{\Theta}_{c,E}^R(M,N)(t). \qquad \square$$

The next two results use étale cohomology, and [5] serves as a good reference. The étale cohomology of a scheme $Z$ with coefficients in $\mathbb{Q}_\ell(i)$ will be denoted by $H^j_{\text{ét}}(Z,\mathbb{Q}_\ell(i))$, where $\ell$ is some fixed prime not equal to the characteristic of $\mathsf{k}$. Via the étale Chern character, there is a ring homomorphism

$$\mathrm{ch}_{\text{ét}}\colon K(Z)_{\mathbb{Q}} \to H^{2*}_{\text{ét}}(Z,\mathbb{Q}_\ell(*)),$$

where $H^{2*}_{\text{ét}}(Z,\mathbb{Q}_\ell(*)) = \bigoplus_i H^{2i}_{\text{ét}}(Z,\mathbb{Q}_\ell(i))$ is a commutative ring under cup product.

Let $Y$ be a smooth projective variety over $\mathsf{k}$. For the push-forward along the structure map $Y \to \operatorname{Spec}\mathsf{k}$, write

$$\int_Y\colon H^{2*}_{\text{ét}}(Y,\mathbb{Q}_\ell) \to \mathbb{Q}_\ell.$$

This mapping takes values in $H^{2*}_{\text{ét}}(\operatorname{Spec}\mathsf{k},\mathbb{Q}_\ell) = H^0_{\text{ét}}(\operatorname{Spec}\mathsf{k},\mathbb{Q}_\ell) = \mathbb{Q}_\ell$. In particular, if $\varsigma \in H^2_{\text{ét}}(\mathbb{P}^{n-1},\mathbb{Q}_\ell(1))$ is the class of a hyperplane, then $H^{2*}_{\text{ét}}(\mathbb{P}^{n-1},\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell[\varsigma]/\langle\varsigma^n\rangle$ and

$$\int_{\mathbb{P}^{n-1}}\varsigma^i = \begin{cases} 0 & \text{for } i = 0,\dots,n-2,\\ 1 & \text{for } i = n-1.\end{cases}$$

PROPOSITION 1.12. *Under assumptions* (1.2),

$$\boldsymbol{\Theta}_c^R(M,N) = \int_{\mathbb{P}^{n-1}}\bigl(\rho_*(\mathrm{ch}_{\text{ét}}[\tilde{M}])\cdot\rho_*(\mathrm{ch}_{\text{ét}}[\tilde{N}]) - d\cdot\rho_*(\mathrm{ch}_{\text{ét}}([\tilde{M}]\cdot[\tilde{N}]))\bigr).$$

*Proof.* The case $c = 1$ is simply [12, Prop. 3.1]. For larger values of $c$, the proof is similar but also uses results from [13]. To see this, let $c > 1$. Apply $\mathrm{ch}_{\text{ét}}$ to the second equation in Lemma 1.11 and simplify using the commutative diagram in [13, Lemma 4.4]; that is, $\rho_* \circ \mathrm{ch}_{\text{ét}} = (d/\rho_*(1)) \cdot \mathrm{ch}_{\text{ét}} \circ \rho_*$. This gives

$$\mathbf{\Theta}_c^R(M, N)\mathrm{ch}_{\text{ét}}(1-t)^{n-1} = \rho_*(\mathrm{ch}_{\text{ét}}[\tilde{M}]) \cdot \rho_*(\mathrm{ch}_{\text{ét}}[\tilde{N}]) - d \cdot \rho_*(\mathrm{ch}_{\text{ét}}([\tilde{M}] \cdot [\tilde{N}])).$$

The rest of the proof now follows exactly as in the proof of [12, Prop. 3.1]. □

The next result establishes the vanishing of $\mathbf{\Theta}_c^R$ in even dimension, a fact that has implications for our main theorem in Section 2; see Theorem 2.4.

THEOREM 1.13. *Under assumptions* (1.2), *if* $\dim R$ *is even then* $\mathbf{\Theta}_c^R$ *vanishes; that is, for every pair of finitely generated graded* $R$*-modules* $M$ *and* $N$, $\mathbf{\Theta}_c^R(M, N) = 0$.

*Proof.* The case $c = 1$ is simply [12, Thm. 3.2]. For larger values of $c$ the proof uses ideas similar to those in [13, Thm. 4.5, Cor. 4.7]. Let $c > 1$. Multiply the first equation in Lemma 1.11 by $d^2$. Next, apply $\mathrm{ch}_{\text{ét}}$ and simplify using the commutative diagram in [13, Lemma 4.4] (i.e., $\rho_* \circ \mathrm{ch}_{\text{ét}} = (d/\rho_*(1)) \cdot \mathrm{ch}_{\text{ét}} \circ \rho_*$) to obtain

$$\frac{d^2 \cdot \mathrm{ch}_{\text{ét}}((1-t)^{n-1} \cdot \mathbf{\Theta}_{c,E}^R(M, N)(t))}{\mathrm{ch}_{\text{ét}}((e_R(t))^2)} = \rho_* \, \mathrm{ch}_{\text{ét}}[\tilde{M}] \cdot \rho_* \, \mathrm{ch}_{\text{ét}}[\tilde{N}] - d \cdot \rho_*(\mathrm{ch}_{\text{ét}}[\tilde{M}] \cdot \mathrm{ch}_{\text{ét}}[\tilde{N}]). \quad (1.5)$$

Define a pairing $\Psi$ on $H_{\text{ét}}^{2*}(X, \mathbb{Q}_\ell(*))$ by the formula

$$\Psi(\alpha, \beta) = \rho_*(\alpha) \cdot \rho_*(\beta) - d \cdot \rho_*(\alpha \cdot \beta).$$

Then the right-hand side of (1.5) is $\Psi(\mathrm{ch}_{\text{ét}}[\tilde{M}], \mathrm{ch}_{\text{ét}}[\tilde{N}])$. We claim that $\Psi$ is zero when $\dim R$ is even.

Let $\alpha$ be in the image of the ring map $\rho^* \colon H_{\text{ét}}^{2*}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell(*)) \to H_{\text{ét}}^{2*}(X, \mathbb{Q}_\ell(*))$, say $\alpha = \rho^*(\alpha')$. Recall that $\rho_*(1) = d = d_1 \cdots d_c$ and employ the projection formula $\rho_*(\rho^*(\alpha') \cdot \omega) = \alpha' \cdot \rho_*(\omega)$, where $\omega$ is taken to be 1 or $\beta$. Then

$$\rho_*(\rho^*(\alpha')) \cdot \rho_*(\beta) - d \cdot \rho_*(\rho^*(\alpha') \cdot \beta) = \alpha' \rho_*(1) \cdot \rho_*(\beta) - d \cdot \alpha' \rho_*(\beta).$$

In other words, $\Psi(\alpha, \beta) = 0$ in this case. The same holds true if $\beta = \rho^*(\beta')$ for some $\beta' \in H_{\text{ét}}^{2*}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell(*))$. Now the map

$$\rho^* \colon H_{\text{ét}}^{2j}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell(j)) \to H_{\text{ét}}^{2j}(X, \mathbb{Q}_\ell(j))$$

is an isomorphism in every degree except for possibly degree $2j = n - 1$ because $X$ is a complete intersection in projective space (see [15, XI.1.6]). Therefore, when the dimension of $R$ is even, one of these two cases must hold, and hence the claim is verified. Clearly, since $\Psi$ is zero when $\dim R$ is even, so too is the left-hand side of (1.5). Thus it follows that $\mathrm{ch}_{\text{ét}}((1-t)^{n-1}\mathbf{\Theta}_{c,E}^R(M, N)(t)) = 0$ as well.

The remainder of the proof now follows along the lines of [13, Thm. 4.5]. To wit, since the Chern character map with coefficients from $\mathbb{Q}_\ell$ induces an isomorphism on projective space, namely,

$$\mathrm{ch}_{\text{ét}} : K(\mathbb{P}^{n-1}) \otimes \mathbb{Q}_\ell \xrightarrow{\cong} H^{2*}_{\text{ét}}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell(*)),$$

it follows that $(1-t)^{n-1} \cdot \mathbf{\Theta}^R_{c,E}(M,N)(t) = 0$ in $\mathbb{Q}[t]/(1-t)^n$. In other words, $(1-t)^n$ divides $(1-t)^{n-1} \cdot \mathbf{\Theta}^R_{c,E}(M,N)(t)$ in $\mathbb{Q}[t]$, and hence $\mathbf{\Theta}^R_{c,E}(M,N)(t)/(1-t)$ is a polynomial. Consequently, $\mathbf{\Theta}^R_c(M,N) := \mathbf{\Theta}^R_{c,E}(M,N)(1) = 0$. □

## 2. Expected Dimension versus Actual Dimension

In this section, *under the same assumptions as in* (1.2), our goal is to answer the question of when the actual dimension of $\operatorname{Supp} M \cap \operatorname{Supp} N$ is no less than $\dim M + \dim N - \dim R$, the "expected dimension" of the intersection, and to relate these values to the invariant $\mathbf{\Theta}^R_c(M,N)$. One motivation for this question is a paper by Kobayashi—in particular, [11, Thm. 8], which we reconsider in the language of the invariants $\mathbf{\Theta}^R_c(M,N)$ and $\eta^R_c(M,N)$. The latter is considered in Section 3, specifically Theorem 3.1.

Throughout this section, we set $p = \dim M + \dim N - \dim R$. We also recall a definition that we will use in Theorem 2.4.

DEFINITION 2.1 [11, p. 652]. Let $M$ be a finitely generated graded module over $R$. For each integer $j$ with $0 \le j \le \dim R$, define

$$e_j(M) = \begin{cases} 0 & \text{if } \dim M < j, \\ \deg M & \text{if } \dim M = j. \end{cases}$$

Recall that the degree of a graded module $M$ is $e_M(1)$, as per (1.1). The notation is similar, but $e_j(M)$ equals $e_M(1)$ only in the case that $j = \dim M$.

REMARK 2.2. Kobayashi asserts the following. Let $M$ and $N$ be finitely generated graded $R$-modules for $R$ a graded algebra generated by homogeneous elements of degree 1 over a field. Set $p = \dim M + \dim N - \dim R$. Then:

(1) $\dim(M \otimes_R N) \ge p$;
(2) if $p \ge 1$ and the equality holds in (1) and if $R$ is nonsingular at every $\mathfrak{q} \in V(M) \cap V(N)$ such that $\dim R/\mathfrak{q} = p$, then $e_p(\operatorname{Tor}^R_j(M,N)) = 0$ for $j > \dim R - p$ and

$$\deg M \deg N = \deg R \cdot \sum_{j=0}^{\dim R - p} (-1)^j e_p(\operatorname{Tor}^R_j(M,N)). \qquad (*)$$

(Note that $V(M) = \{\mathfrak{q} \in \operatorname{Proj} R : \mathfrak{q} \supseteq \operatorname{ann}_R(M)\}$.)

However, Example 2.3 (cf. [8, Exm. 1.5]) provides a counterexample to (1) and (2) [11, Thm. 8], where $\mathbf{\Theta}^R_1$ is simply the original $\theta^R$ function.

EXAMPLE 2.3. Let $R = \mathbb{C}[X_1, X_2, Y_1, Y_2]/(X_1Y_1 - X_2Y_2)$ and set $M = R/(x_1, x_2)$. If $N = R/(y_1, y_2)$, then $\operatorname{len}(M \otimes_R N) < \infty$ and $p = 2 + 2 - 3 = 1$ provides a counterexample to (1). Next, reset $N = R/(x_1, y_2)$. Then $\dim(M \otimes_R N) = 1 = p$, and $\operatorname{len}(\operatorname{Tor}^R_j(M,N))$ is 0 if $j > 0$ is even and 1 if $j > 0$ is odd.

Therefore, $e_1(\mathrm{Tor}_0^R(M,N)) = 1$ and $e_1(\mathrm{Tor}_j^R(M,N)) = 0$ for $j = 1,2$. Also, $\deg R = 2$. Since $R$ is regular away from the irrelevant maximal ideal, the hypotheses in Remark 2.2(2) are satisfied, yet equation $(*)$ yields

$$(1)(1) = \deg M \deg N = \deg R \cdot \sum_{j=0}^{2} (-1)^j e_1(\mathrm{Tor}_j^R(M,N)) = 2(1).$$

In fact, $\mathbf{\Theta}_1^R(M,N) = -1$ in this example, and applying Theorem 2.4 via equation (2.2), where we can take $E$ as small as 2, yields

$$\deg M \deg N = \deg R \cdot \sum_{j=0}^{1} (-1)^j e_1(\mathrm{Tor}_j^R(M,N)) + \mathbf{\Theta}_1^R(M,N).$$

It is interesting to note that, under the assumptions in this section, Remark 2.2(1) is valid outside the lone case that $p = 1$ and $\dim R$ is odd. We demonstrate these ideas in our main result as follows.

THEOREM 2.4. *Let $R$, $M$, and $N$ be the same as in assumptions* (1.2). *Set* $d_{T_j} = \dim \mathrm{Tor}_j^R(M,N)$ *and* $p = \dim M + \dim N - \dim R$. *Then the following statements hold.*

(1) *If $p \geq 2$, then $d_{T_0} \geq p$. Moreover, if $p = d_{T_0}$, then*

$$\deg M \deg N = \deg R \cdot \sum_{j=0}^{E-1} (-1)^j e_p(\mathrm{Tor}_j^R(M,N)). \tag{2.1}$$

(2) *If $p = 1$, then $d_{T_0} \geq p$ whenever* $\dim R$ *is even.*
*If $d_{T_0} = 0$, then* $\dim R$ *is odd and* $\deg M \deg N = \mathbf{\Theta}_c^R(M,N) \neq 0$; *that is, $\theta_{c,E}^R(M,N)(t)$ has a zero at $t = 1$ of order exactly $c - 1$.*
*If $d_{T_0} = 1$, then*

$$\deg M \deg N = \deg R \cdot \sum_{j=0}^{E-1} (-1)^j e_1(\mathrm{Tor}_j^R(M,N)) + \mathbf{\Theta}_c^R(M,N). \tag{2.2}$$

(3) *If $p \leq 0$, then $d_{T_0} \geq p$.*
*If $d_{T_0} = 0$, then $\mathbf{\Theta}_c^R(M,N) = 0$; that is, $\theta_{c,E}^R(M,N)(t)$ has a zero at $t = 1$ of order at least $c$.*
*If $p = 0 = d_{T_0}$, then*

$$\deg M \deg N = \deg R \cdot \sum_{j=0}^{E-1} (-1)^j e_0(\mathrm{Tor}_j^R(M,N)) - \mathbf{\Theta}'_{c,E}(M,N)(1), \tag{2.3}$$

*where $\mathbf{\Theta}'_{c,E}$ denotes the first derivative of $\mathbf{\Theta}_{c,E}^R(M,N)(t)$ with respect to $t$.*

REMARK 2.5. We note that under the assumption $d_{T_0} = p$, the finite sums can be written in terms of the degrees of the torsion modules, thus giving a Bézout-like result between those degrees and the degrees of $M$, $N$, and $R$. To be specific, the right-hand side of equation (2.1), (2.2), and (2.3), respectively, can be written as:

$$\deg R \cdot \sum_{\substack{0\le j\le E-1\\ d_{T_j}=p}} (-1)^j \deg(\operatorname{Tor}_j^R(M,N));$$

$$\deg M \deg N = \deg R \cdot \sum_{\substack{0\le j\le E-1\\ d_{T_j}=1}} (-1)^j \deg(\operatorname{Tor}_j^R(M,N)) + \boldsymbol{\Theta}_c^R(M,N);$$

$$\deg R \cdot \sum_{j=0}^{E-1} (-1)^j \deg(\operatorname{Tor}_j^R(M,N)) - \boldsymbol{\Theta}'_{c,E}(M,N)(1).$$

*Proof of Theorem 2.4.* We start with the equation from [1, Lemma 7]:

$$\frac{H_M(t)\cdot H_N(t)}{H_R(t)} = \sum_{j\ge 0}(-1)^j H_{T_j}(t),$$

which simplifies via (1.1) to

$$e_M(t)e_N(t) = e_R(t)\Bigl(\sum_{j\ge 0}(-1)^j H_{T_j}(t)\Bigr)(1-t)^p.$$

Splitting the sum at an even $E \gg 0$ and using equation (1.3), this can be reconfigured as

$$e_M(t)e_N(t) = \underbrace{e_R(t)\sum_{j=0}^{E-1}(-1)^j e_{T_j}(t)(1-t)^{p-d_{T_j}}}_{(\dagger)} + \boldsymbol{\Theta}_{c,E}^R(t)(1-t)^{p-1}. \tag{2.4}$$

The goal is to evaluate equation (2.4) at $t = 1$. Our argument relies on the fact that *the left-hand side has neither zero nor pole at* $t = 1$; to be specific, $e_M(1)e_N(1) = \deg M \deg N$. Thus, the argument involves simplifying the right-hand side and evaluating at $t = 1$. Essentially, this is an analysis of (†).

*Case 1: $p \ge 2$.* The rightmost term vanishes upon evaluation at $t = 1$. If $d_{T_0} < p$, then $d_{T_j} < p$ for all $j$ and so the entire right-hand side of equation (2.4) would be zero when evaluated at $t = 1$. Therefore, $d_{T_0}$ must be greater than or equal to $p$. Now assume $d_{T_0} = p$. Any of the individual terms in (†) with $d_{T_j} < p$ vanish upon evaluation at $t = 1$. Thus equation (2.1) follows easily.

*Case 2: $p = 1$.* If $d_{T_0} = 0$ then (†) is zero and, evaluating at $t = 1$, we have $\deg M \deg N = \boldsymbol{\Theta}_c^R(M,N)$. By Theorem 1.13, $\boldsymbol{\Theta}_c^R(M,N)$ vanishes when $\dim R$ is even; hence this equation is contradictory unless $\dim R$ is odd. Now assume $d_{T_0} = 1$. Any of the individual terms in (†) with $d_{T_j} = 0$ vanish upon evaluation at $t = 1$, yielding equation (2.2).

*Case 3: $p \le 0$.* The first sentence is obvious. Assume $d_{T_0} = 0$. Then $d_{T_j} = 0$ for all $j$. As a result, $e_{T_j}(1) = \deg T_j > 0$ for all $j$ such that $e_{T_j}(t)$ is nonzero. Now if $\boldsymbol{\Theta}_{c,E}^R(1) \ne 0$, then the rightmost term will have a pole at $t = 1$ of order $1 - p$, while any poles at $t = 1$ in (†), if they exist, will have order at most $-p < 1 - p$. Therefore, $\boldsymbol{\Theta}_{c,E}^R(1)$ must be zero. Write $\boldsymbol{\Theta}_{c,E}^R(t) = (1-t)q(t)$ for some $q(t) \in \mathbb{Q}[t]$.

Now assume that $p = 0 = d_{T_0}$. Thus, equation (2.4) becomes

$$e_M(t)e_N(t) = e_R(t)\sum_{j=0}^{E-1}(-1)^j e_{T_j}(t) + q(t).$$

Equation (2.3) follows by evaluating at $t = 1$. □

COROLLARY 2.6 (Finite-length case). *Let $R$, $M$, and $N$ be as in assumptions* (1.2). *In addition, assume that $M \otimes_R N$ has finite length. Then*:

(i) $\dim M + \dim N \le \dim R + 1$;
(ii) *if* $\dim R$ *is even, then* $\dim M + \dim N \le \dim R$;
(iii) $\dim M + \dim N \le \dim R$ *if and only if* $\mathbf{\Theta}_c^R(M, N) = 0$.

*Proof.* If $M \otimes_R N$ has finite length, then $p \le 1$ by Theorem 2.4(1). This establishes part (i). Next, if $\dim R$ is even then $p \ne 1$ by Theorem 2.4(2). Third, if the dimension inequality is satisfied, then Theorem 2.4(3) establishes the vanishing of $\mathbf{\Theta}_c^R(M, N)$. Conversely, if $\mathbf{\Theta}_c^R(M, N)$ vanishes then $\mathbf{\Theta}_{c,E}^R(M, N)(t)$ can be factored as $(1-t)q(t)$ for some $q(t) \in \mathbb{Q}[t]$. Putting this into equation (2.4), we have

$$e_M(t)e_N(t) = e_R(t)\sum_{j=0}^{E-1}(-1)^j e_{T_j}(t)(1-t)^p + q(t)(1-t)^p. \qquad (2.5)$$

Recall that the left-hand side of (2.5) has neither zero nor pole at $t = 1$. Likewise, none of $e_{T_j}(t)$ or $q(t)$ has a pole at $t = 1$ since all are polynomials. If $p > 0$, then the right-hand side of (2.5) is zero when evaluated at $t = 1$. Contradiction. □

Note that in the case where $c = 1$, we are simply dealing with Hochster's original $\theta^R$ function, and we can recover some of his results. See [8, Thm. 1.4].

## 3. The Invariant $\eta_c^R(M, N)$ Revisited

In this section, we consider the question of actual versus expected dimension from the perspective of the invariant $\eta_c^R(M, N)$, relaxing the isolated singularity hypothesis on $R$. Again, $p = \dim M + \dim N - \dim R$. Also, $\eta_{c,E}^R(M, N)(t)$ will be abbreviated to $\eta_{c,E}^R(t)$. We revisit Example 1.4, interpreting it via the following theorem, and also include an example in codimension 2.

THEOREM 3.1. *Let $\mathsf{k}$ be a field and $Q = \mathsf{k}[x_0, \dots, x_{n+c-1}]$, where $\deg x_l = 1$ for all $l$. Set $R = Q/(f_1, \dots, f_c)$, where $f_1, \dots, f_c$ forms a regular $Q$-sequence and each $f_i$ is a homogeneous polynomial of degree $d_i$ for $1 \le i \le c$. Let $M$ and $N$ be finitely generated graded $R$-modules such that the $\operatorname{Tor}_j^R(M, N)$ eventually have finite length, and let $E \gg 0$ be an even integer such that $\eta_{c,E}^R(M, N)(1) = 2^c c! \cdot \eta_c^R(M, N)$.*

(1) *If $p > c$, then $d_{T_0} \ge p$. Moreover, if $d_{T_0} = p$, then*

$$\deg M \deg N = \deg R \cdot \sum_{j=0}^{E-1}(-1)^j e_p(\operatorname{Tor}_j^R(M, N)). \qquad (3.1)$$

(2) *For $p = c$:*
  (a) *if $d_{T_0} < p$, then* $\deg M \deg N = \eta_c^R(M, N) \neq 0$;
  (b) *if $d_{T_0} = p$, then*

$$\deg M \deg N = \deg R \cdot \sum_{j=0}^{E-1} (-1)^j e_p(\operatorname{Tor}_j^R(M, N)) + 2^c c! \cdot \eta_c^R(M, N). \quad (3.2)$$

(3) *For $p < c$:*
  (a) *if $d_{T_0} < p$, then $\eta_{c,E}^R(t)$ has a zero at $t = 1$ of order exactly $c - p$ and*

$$\deg M \deg N = \frac{(-1)^{c-p}}{(c-p)!} \eta_{c,E}^{(c-p)}(1), \quad (3.3)$$

*where $\eta_{c,E}^{(c-p)}$ denotes the iterated derivative of $\eta_{c,E}^R(t)$ with respect to $t$;*
  (b) *if $d_{T_0} = p$, then $\eta_{c,E}^R(t)$ has a zero at $t = 1$ of order at least $c - p$ and*

$$\deg M \deg N = \deg R \cdot \sum_{j=0}^{E-1} (-1)^j e_p(\operatorname{Tor}_j^R(M, N)) + \frac{(-1)^{c-p}}{(c-p)!} \eta_{c,E}^{(c-p)}(1). \quad (3.4)$$

*Proof.* Via the method used in the proof of Theorem 2.4, we arrive at the equation

$$e_M(t) e_N(t) = \underbrace{e_R(t) \sum_{j=0}^{E-1} (-1)^j e_{T_j}(t)(1-t)^{p-d_{T_j}}}_{(\ddagger)} + \eta_{c,E}^R(t)(1-t)^{p-c}, \quad (3.5)$$

which we analyze as we did before—again relying heavily on the fact that *the left-hand side has neither zero nor pole at $t = 1$* (i.e., $e_M(1)e_N(1) = \deg M \deg N$).

*Case 1: $p > c$.* If $d_{T_0} < p$, then $d_{T_j} < p$ for all $j$ and so the entire right-hand side of equation (3.5) would be zero when evaluated at $t = 1$. Therefore, $d_{T_0}$ must be greater than or equal to $p$. When $d_{T_0} = p > c$, equation (3.1) follows easily.

*Case 2: $p = c$.* Then (3.5) becomes

$$e_M(t) e_N(t) = e_R(t) \sum_{j=0}^{E-1} (-1)^j e_{T_j}(t)(1-t)^{p-d_{T_j}} + \eta_{c,E}^R(t).$$

(a) If $d_{T_0} < p = c$, then (‡) vanishes when evaluated at $t = 1$ and so $\eta_{c,E}^R(t)$ can not; that is, $\deg M \deg N = \eta_c^R(M, N) \neq 0$.
(b) If $d_{T_0} = p = c$, then equation (3.2) follows easily.

*Case 3: $p < c$.*
(a) If $d_{T_0} < p$, then (‡) vanishes upon evaluation at $t = 1$. As a result, $\eta_{c,E}(t)$ must have a zero at $t = 1$ of order *exactly* $c - p$, else the right-hand side of equation (3.5) would either be zero or have a pole at $t = 1$. From this, equation (3.3) follows.

(b) If $d_{T_0} = p < c$, then (‡) has no pole at $t = 1$. Thus, as in Case 3(a), $\eta_{c,E}(t)$ must have a zero at $t = 1$ of order at least $c - p$. Equation (3.4) follows. □

COROLLARY 3.2 (Finite-length case). *Let $R$, $M$, and $N$ be as in Theorem 3.1. In addition, assume that $M \otimes_R N$ has finite length. Then $\eta^R_{c,E}(M,N)(t)$ has a zero at $t = 1$ of order at least $c$ if and only if* $\dim M + \dim N \leq \dim R$.

*Proof.* If $M \otimes_R N$ has finite length, then $d_{T_j} = 0$ for all $j$; hence (3.5) becomes

$$e_M(t)e_N(t) = e_R(t)\sum_{j=0}^{E-1}(-1)^j e_{T_j}(t)(1-t)^p + \eta^R_{c,E}(t)(1-t)^{p-c}.$$

Assume $\dim M + \dim N \leq \dim R$. Now if $\eta^R_{c,E}(t)$ does not have a zero of order at least $c$ at $t = 1$, then the rightmost term will have a pole at $t = 1$ of order strictly greater than $-p$. However, the individual terms in (‡) either are zero or have poles at $t = 1$ of order exactly $-p$. Since all poles must cancel, $\eta^R_{c,E}(t)$ must have a zero of order at least $c$ at $t = 1$.

Conversely, if $\eta^R_{c,E}(t)$ has a zero of order at least $c$ at $t = 1$, write $\eta^R_{c,E}(t) = (1-t)^c q(t)$ for some polynomial $q(t) \in \mathbb{Q}[t]$. Then (3.5) becomes

$$e_M(t)e_N(t) = e_R(t)\sum_{j=0}^{E-1}(-1)^j e_{T_j}(t)(1-t)^p + q(t)(1-t)^p.$$

If $p > 0$, then the entire right-hand side vanishes upon evaluation at $t = 1$. Contradiction. Thus, the dimension inequality must be satisfied. □

To illustrate the preceding results, we provide some examples. The first is familiar from Example 1.4 and illustrates Case 3(a).

EXAMPLE 3.3. Let $R = \mathbb{C}[X, Y, Z_1, \dots, Z_8]/(XY, Z_1Z_2Z_3 - Z_4^3, Z_5Z_6 - Z_7Z_8)$, and set $M = R/(x, z_1, \dots, z_7)$ and $N = R/(y)$. Then $\dim R = 7$, $\dim M = 1$, and $\dim N = 7$, so $\dim M + \dim N = \dim R + 1$. Moreover, the length of $M \otimes_R N$ is finite; that is, $d_{T_0} = 0$, $p = 1$, and $c = 3$. The Hilbert series are

$$H_R(t) = \frac{(1+t)^2(1+t+t^2)}{(1-t)^7}, \quad H_M(t) = \frac{1}{1-t}, \quad H_N(t) = \frac{(1+t)(1+t+t^2)}{(1-t)^7},$$

where the numerators are $e_R(t)$, $e_M(t)$, and $e_N(t)$, respectively. We recall that $\eta^R_{3,E}(M,N)(t) = t^E(1-t)^2(1+t)(1+t+t^2)$. Then, by (3.3) in Theorem 3.1,

$$\deg M \deg N = \frac{(-1)^2}{2!}\eta''_{3,E}(1).$$

The left-hand side is $e_M(1)e_N(1) = (1)(6)$ while the right-hand side is $12/2$.

Our next example is illustrative of the most complicated cases involving $d_{T_0}$, $p$, and $c$ (namely, when $d_{T_0} > p$), which are not covered in Theorem 3.1. We detail in Theorem 3.5 what (little) we can say in these cases via our methods.

EXAMPLE 3.4 [9, Exm. 4.2]. Let $R = \mathbb{C}[X,Y,Z]/(XZ - Y^2, XY - Z^2)$, and set $M = R/(x, y)$ and $N = R/(x, z)$. Then $\dim R = 1$ and $\dim M = \dim N = 0$, so $\dim M + \dim N = \dim R - 1$. Moreover, the length of $M \otimes_R N$ is finite (i.e., $p = -1$, $d_{T_0} = 0$, and $c = 2$). The Hilbert series are

$$H_R(t) = \frac{(1+t)^2}{1-t}, \quad H_M(t) = 1 + t, \quad H_N(t) = 1 + t,$$

where $e_R(t)$ is the numerator of $H_R(t)$, $e_M(t) = H_M(t)$, and $e_N(t) = H_N(t)$.

Since $\operatorname{len}(\operatorname{Tor}_j^R(M, N))$ is 1 if $j = 0$ or 1 and is 0 if $j \geq 2$, it follows that $\eta^R_{2,E}(M, N)(t) \equiv 0$. Thus, via equation (3.5), we have

$$e_M(t)e_N(t) = e_R(t)\left(\frac{1}{1-t} - \frac{t}{1-t}\right) + 0.$$

Simplifying and evaluating at $t = 1$, we have $(2)(2) = \deg M \deg N = \deg R = (2)^2$.

THEOREM 3.5. *Let $R$, $M$, and $N$ be as in Theorem 3.1.*

(1) *If $d_{T_0} > p = c$, then*

$$\deg M \deg N = \deg R\left(Q(1) + \sum_{\substack{0\le j\le E-1\\ d_{T_j}=p}} (-1)^j e_p(\operatorname{Tor}_j^R(M, N))\right) + \eta^R_c(M, N),$$

*where $Q(t)$ is the polynomial $\sum_{0\le j\le E-1,\, d_{T_j}>p} (-1)^j e_{T_j}(t)(1-t)^{p-d_{T_j}}$.*

(2) *If $d_{T_0} > p \neq c$, then*

$$\deg M \deg N = \deg R\left(Q(1) + \sum_{\substack{0\le j\le E-1\\ d_{T_j}=p}} (-1)^j e_p(\operatorname{Tor}_j^R(M, N))\right),$$

*where*

$$Q(t) = \begin{cases} \text{the same polynomial as in part (1)} & \text{if } p > c,\\ \displaystyle\sum_{\substack{0\le j\le E-1\\ d_{T_j}>p}} (-1)^j e_{T_j}(t)(1-t)^{p-d_{T_j}} \\ \qquad\qquad + \eta^R_{c,E}(M, N)(t)(1-t)^{p-c} & \text{if } p < c.\end{cases}$$

*Proof.* The formulas again follow from equation (3.5). By hypotheses, some individual terms in (‡) have poles of order $d_{T_j} - p$ at $t = 1$. For (1), since the rightmost term, namely $\eta^R_{c,E}(t)$ (since $p = c$), does not and the left-hand side does not, these poles must cancel, leaving a polynomial $Q(t)$. The same holds true for (2), where the rightmost term either is zero when $p > c$ or is subsumed by the polynomial $Q(t)$ when $p < c$ because the poles at $t = 1$ on the right-hand side of (3.5) must cancel. □

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