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Rigidification of Holomorphic Germs with Noninvertible Differential

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0. Introduction

Our aim in this paper is to study the structure of noninvertible holomorphic germs $f:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$. We shall consider only dominant holomorphic germs, as defined next.

Definition 0.1. Let $f:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ be a holomorphic germ. Then f is *dominant* if $\det(df_p)$ is not identically zero.

In this paper we are particularly interested in the following classes of holomorphic germs.

Definition 0.2. Let $f:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ be a holomorphic germ, and denote by $\mathrm{Spec}(df_0)=\{\lambda_1,\lambda_2\}$ the set of eigenvalues of df_0. Then f is said to be:

- *attracting* if $|\lambda_i|<1$ for $i=1,2$;
- *superattracting* if $df_0=0$;
- *nilpotent* if df_0 is nilpotent (i.e., if $df_0^2=0$; in particular, superattracting germs are nilpotent germs);
- *semi-superattracting* if $\mathrm{Spec}(df_0)=\{0,\lambda\}$ with $\lambda\neq0$;
- *of type* $(0,D)$ if $\mathrm{Spec}(df_0)=\{0,\lambda\}$ and $\lambda\in D$, where $D\subset\mathbb{C}$ is a subset of the complex plane.

In particular, the semi-superattracting germs are the ones of type $(0,\mathbb{C}^*)$.

We shall denote by $\mathbb{D}$ the open disk of radius 1 centered at 0.

A typical problem one would like to solve is to find a classification up to local (holomorphic, formal or topological) conjugacy; this problem is mostly solved in dimension 1, and there are classifications of germs in dimension 2 in only a few cases. One of these cases is the formal and holomorphic classification of attracting rigid germs proved by Favre [F].

Definition 0.3. Let $f:(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ be a (dominant) holomorphic germ. We denote by $\mathcal{C}(f)=\{z\mid\det(df_z)=0\}$ the *critical set* of f and by $\mathcal{C}^\infty(f)=\bigcup_{n\in\mathbb{N}}f^{-n}\mathcal{C}(f)$ the *generalized critical set* of f. Then a (dominant) holomorphic germ f is *rigid* if:

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(i) $\mathcal{C}^\infty(f)$ (is empty or) has normal crossings at the origin; and
(ii) $\mathcal{C}^\infty(f)$ is forward f-invariant.

REMARK 0.4. In [F] the condition (ii) is not explicitly stated in the definition of a rigid germ, but it is implicitly used. The second property does not follow from the first one: if, for example, we consider the map $f(z,w) = (\lambda z^p, z(1+w^2))$ with $p \geq 1$ and $\lambda \in \mathbb{C}^*$, then the generalized critical set is $\{zw = 0\}$ but $f(z,0) = (\lambda z^p, z)$ and hence $\mathcal{C}^\infty(f)$ is not forward f-invariant.

One way to study the local dynamics of a generic holomorphic germ in $(\mathbb{C}^2,0)$ and to find some invariants up to conjugacy is suggested by continuous local dynamics (see [IY, Chaps. 1 and 2] for main techniques in continuous local dynamics and [Sei] for Seidenberg's theorem): we can blow up the fixed point (the origin), replacing the ambient space by a more complicated space but simplifying the map, and study the lift $\hat{f}$ of f. But a single blow-up is often not enough, and one is led to consider a composition of point blow-ups $\pi\colon X \to (\mathbb{C}^2,0)$ over the origin (called *modification*).

A clever way to study all modifications at the same time was introduced by Favre and Jonsson [FJ1]. Take the set of all modifications $\mathfrak{B}$; for every $\pi \in \mathfrak{B}$ we can consider a simplicial graph Γ_π^* whose vertices are the irreducible components of the exceptional divisor of π (we shall call these vertices *exceptional components*). Taking the direct limit of these simplicial graphs, we obtain the ($\mathbb{Q}$-)*universal dual graph* Γ^*, which has a natural $\mathbb{Q}$-tree structure. Since it is easier to work with $\mathbb{R}$-trees, we can take the completion Γ of Γ^*, called the *universal dual graph*.

Favre and Jonsson also showed that the universal dual graph is strictly related to the set $\mathcal{V}$ of all centered and normalized valuations on the ring of formal power series in two coordinates: $\mathcal{V}$ admits an $\mathbb{R}$-tree structure and is isomorphic (in the strong sense) to Γ.

It is this isomorphism, which relates the geometry of exceptional components to the algebra of valuations, that allows us to define the action $f_\bullet\colon \mathcal{V} \to \mathcal{V}$ on the valuative tree $\mathcal{V}$ induced by a holomorphic germ $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ (see [FJ2]).

Favre and Jonsson [FJ2] studied the dynamical behavior of $f_\bullet$ when $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ is superattracting; in particular, they proved that one can find a modification $\pi\colon X \to (\mathbb{C}^2,0)$ and a point $p \in \pi^{-1}(0)$ such that the lift $\hat{f}\colon (X,p) \to (X,p)$ defined as a birational map by $\hat{f} = \pi^{-1} \circ f \circ \pi$ is actually holomorphic in p and rigid. This can be done by finding a fixed point $\nu_\star$ for $f_\bullet$ (this is called *eigenvaluation*) and then studying the basin of attraction around this eigenvaluation. We shall call this process *rigidification*.

DEFINITION 0.5. Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ be a (dominant) holomorphic germ. Let $\pi\colon X \to (\mathbb{C}^2,0)$ be a modification and $p \in \pi^{-1}(0)$ a point in the exceptional divisor of π. Then we shall call the triple $(\pi, p, \hat{f})$ a *rigidification* of f if the lift $\hat{f} = \pi^{-1} \circ f \circ \pi$ is a holomorphic rigid germ in p.

We shall follow the Favre–Jonsson strategy for finding eigenvaluations and rigidifications, extending their result to all (dominant) holomorphic germs. We remark that the rigidification process is trivial if df_0 is invertible (because the map f is itself rigid). Our main result can be stated as follows.

THEOREM 0.6. *Every (dominant) holomorphic germ* $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ *admits a rigidification.*

The nilpotent case is much the same as the superattracting case dealt with in [FJ1] (see Remark 4.2). Hence we shall focus on the semi-superattracting case, proving a sort of uniqueness of the rigidification process that can be stated as follows.

THEOREM 0.7. *Let f be a (dominant) semi-superattracting holomorphic germ. Then f admits a unique eigenvaluation $\nu_\star$, which must be a (possibly formal) curve valuation with multiplicity $m(\nu_\star) = 1$. Let us denote $\nu_\star = \nu_C$, with $m(C) = 1$. Then one (and only one) of the following statements holds.*

(i) *The set of valuations fixed by $f_\bullet$ consists only of the eigenvaluation $\nu_\star$; there exists only one contracted critical curve valuation ν_D, and in this case it must be $m(D) = 1$.*
(ii) *The set of valuations fixed by $f_\bullet$ consists of two valuations, the eigenvaluation $\nu_\star$ and a curve valuation ν_D; here D is a (possibly formal) curve with $m(D) = 1$.*

In both cases, C and D have transverse intersection; that is, their intersection number is $C \cdot D = 1$.

We shall prove the formal classification of semi-superattracting rigid germs (the first case of Theorem 0.8 actually follows from the holomorphic classification of such germs given in [F]).

THEOREM 0.8. *Let* $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ *be a (holomorphic) semi-superattracting rigid germ. Let $\lambda \in \mathbb{C}^*$ be the nonzero eigenvalue of df_0.*

(i) *If $|\lambda| < 1$ or $\lambda = e^{2\pi i\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then f is formally conjugated to the map*
$$(z,w) \mapsto (\lambda z, z^c w^d).$$
(ii) *If $|\lambda| > 1$, then f is formally conjugated to the map*
$$(z,w) \mapsto (\lambda z, z^c w^d(1 + \varepsilon z^l)),$$
where $\varepsilon \in \{0,1\}$ if $\lambda^l = d$ (the resonant *case) and $\varepsilon = 0$ otherwise.*
(iii) *If there exists an $r \in \mathbb{N}^*$ such that $\lambda^r = 1$, then f is formally conjugated to the map*
$$(z,w) \mapsto \big(\lambda z(1 + z^s + \beta z^{2s}), z^c w^d(1 + \varepsilon(z^r))\big);$$
here $r|s$, $\beta \in \mathbb{C}$, ε is a formal power series in z^r, and $\varepsilon \equiv 0$ if $d \geq 2$.

In all cases we have $c \geq 0$, $d \geq 1$, and $c + d \geq 2$.

REMARK 0.9. In the resonant case of part (ii), it seems difficult to understand which of the two possible normal forms (with $\varepsilon = 0$ or 1) is the normal form of a given germ f—whether we consider the dynamics of f or the action of $f_\bullet$ (see also Remark 3.8).

We shall also present two counterexamples (see Counterexamples 3.10 and 3.12) that show how the holomorphic classification of rigid germs of type $(0, \mathbb{C} \setminus \mathbb{D})$ is not trivial, meaning that it does not coincide with the formal classification.

Finally, we shall use the holomorphic classification of attracting rigid germs given in [F] to give holomorphic normal forms for a rigidification for type $(0, \mathbb{D}^*)$ (see Proposition 4.3), and we also use Theorem 0.8 to give formal normal forms for a rigidification for type $(0, \mathbb{C} \setminus \mathbb{D})$ (see Proposition 4.4 and Proposition 4.5).

Using a different language, Theorem 0.6 states that one can suppose a germ to be rigid up to birational conjugacy. Then the normal forms of a rigidification give us normal forms for the birational classification of these germs. In the semi-superattracting case we prove Theorem 0.7, a type of uniqueness of this process, that leads (see Example 4.9) to a type of uniqueness for these normal forms. The dynamics of these rigidifications $\hat{f}$, which are easier to study than the initial germ f itself, give us information on the dynamics of f (by projection), and the birational classification gives us information on the holomorphic classification in a very consistent way. In fact (see Remark 2.1), the action of $\hat{f}_\bullet$ is related to the action of $f_\bullet$ in a suitable basin of attraction in the valuative tree.

This paper is divided into four sections. In Section 1 we recall the construction of the valuative tree $\mathcal{V}$ and its isomorphic equivalent, the universal dual graph Γ, as in [FJ1]; the action $f_\bullet$ induced by a (dominant) holomorphic germ f; and the existence of an eigenvaluation and of a basin of attraction, as in [FJ2], adapted to deal with the general case. In Section 2 we prove Theorem 0.6 and Theorem 0.7, and in Section 3 we deal with the classification of semi-superattracting rigid germs. In Section 4, we compute normal forms for a rigidification in every case and then conclude with some remarks on the rigidification process.

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1. The Valuative Tree

1.1. Modifications

The main objects that we wish to study are modifications (i.e., compositions of point blow-ups) and the lifts of maps over the exceptional divisor of a modification. We begin by fixing notation.

DEFINITION 1.1. Let X be a complex 2-manifold and $p \in X$ a point. We call a holomorphic map $\pi : Y \to (X, p)$ a *modification* over p if π is a composition of point blow-ups, with the first one being over p, and such that π is a biholomorphism outside $\pi^{-1}(p)$. We call $\pi^{-1}(p)$ the *exceptional divisor* of π, and we call every irreducible component of the exceptional divisor an *exceptional component*. We will denote by $\mathfrak{B}$ the set of all modifications over $0 \in \mathbb{C}^2$ and by Γ^*_π the set of all exceptional components of a modification π. We will call a point $p \in \pi^{-1}(0)$ on the exceptional divisor of a modification $\pi \in \mathfrak{B}$ an *infinitely near point* (we also consider $0 \in \mathbb{C}^2$ to be an infinitely near point).

1.2. Tree Structure

Here we fix notation for $\mathbb{R}$-trees. See [FJ1, Chap. 3] for definitions and proofs and [FJ2, Sec. 4] for properties of tree maps.

DEFINITION 1.2. Let $(\mathcal{T}, \leq)$ be an $\mathbb{R}$-tree. Maximal elements of $\mathcal{T}$ will be called *ends.*

Let $\tau_1, \tau_2 \in \mathcal{T}$ be two points. We shall denote by $[\tau_1, \tau_2]$ (resp., $[\tau_1, \tau_2)$ and (τ_1, τ_2)) the *closed* (resp., *semiopen* and *open*) *segment* between τ_1 and τ_2.

We shall denote by $T_\tau\mathcal{T}$ the *tangent space* of $\mathcal{T}$ over a point τ, and we denote by $\vec{v} = [\sigma] \in T_\tau\mathcal{T}$ a tangent vector over τ (represented by σ). Then the point τ is a *terminal point,* a *regular point,* or a *branch point* if $T_\tau\mathcal{T}$ has (respectively) one, two, or more than two tangent vectors.

Finally, let $\tau \in \mathcal{T}$ be a point in the tree and let $\vec{v} = T_\tau\mathcal{T}$ a tangent vector over it; we shall denote by

$$U_\tau(\vec{v}) := \{\sigma \in \mathcal{T} \mid \vec{v} = [\sigma]\}$$

the (*weakly*) *open set* associated to $\vec{v}$ in τ.

1.3. Universal Dual Graph

1.3.1. *Dual Graph of a Modification*

Given a modification $\pi \in \mathfrak{B}$, we can equip the set Γ^*_π of all exceptional components of π with a simplicial tree structure (i.e., an $\mathbb{N}$-tree structure; see [FJ1, pp. 51, 52]).

DEFINITION 1.3. We fix the set of vertices (Γ^*_π), and we say that two exceptional components are joined by an edge if and only if their intersection is nonempty. We will denote by $\leq_\pi$ the induced partial ordering (given by the correspondence between simplicial trees and $\mathbb{N}$ trees). Then $(\Gamma^*_\pi, \leq_\pi)$ will be called the *dual graph* of π.

DEFINITION 1.4. Let $\pi \in \mathfrak{B}$ be a modification. A point $p \in \pi^{-1}(0)$ in the exceptional divisor of π is a *free point* (resp., a *satellite point*) if π is a regular point (resp., a singular point) of $\pi^{-1}(0)$.

We remark that satellite points are also known as *corners* in literature. An equivalent definition will be that p is a free point if it belongs to only one exceptional component but is a satellite point if it belongs to exactly two exceptional components (which will have only one intersection point, with transverse intersection).

1.3.2. *Universal Dual Graph*

DEFINITION 1.5. The term *universal dual graph* will refer to the direct limit of dual graphs along all modifications in $\mathfrak{B}$:

$$(\Gamma^*, \leq) := \varinjlim_{\pi \in \mathfrak{B}} (\Gamma^*_\pi, \leq_\pi).$$

The universal dual graph is a way to see all exceptional components of all the possible modifications at the same time. The next result follows from this construction.

PROPOSITION 1.6 [FJ1, Props. 6.2 and 6.3]. *The universal dual graph Γ^* is a $\mathbb{Q}$-tree that is rooted at E_0, the exceptional component arising from the single blow-up of the origin $0 \in \mathbb{C}^2$. Moreover, all points are branch points for Γ^*. If we have an exceptional component $E \in \Gamma^*$ then the mapping $p \mapsto \vec{v}_p = [E_p]$, where $[E_p] \in T_E\Gamma^*$ is the tangent vector represented by the exceptional component arising from the blow-up of p, gives a bijection from E to $T_E\Gamma^*$.*

One can complete Γ^* to a complete $\mathbb{R}$-tree Γ, which will also be called the (complete) universal dual graph.

The (complete) universal dual graph is a powerful tool because of all the structure that arises from the completeness of $\mathbb{R}$. But we do not know how a holomorphic germ f acts on the universal dual graph. The answer to this question can be given thanks to the algebraic equivalent to the universal dual graph, the *valuative tree*.

1.4. Valuations

We shall denote by $R = \mathbb{C}[[x, y]]$ the ring or formal power series in two coordinates and by $K = \mathbb{C}((x, y))$ the quotient field of R (i.e., the field of Laurent series in two coordinates). Then R is a unique factorization domain (UFD) local ring with maximal ideal $\mathfrak{m} = \langle x, y \rangle$. Favre and Jonsson considered a slightly different concept of valuation; it takes values in $[0, +\infty]$, whereas classical Krull valuations take values in a (totally ordered) abelian group. Moreover, these authors focus their attention on *centered* valuations—in other words, valuations $\nu \colon R \to [0, +\infty]$ that take strictly positive values on $\mathfrak{m}$.

The set of all (centered) valuations can be endowed by a partial order as follows.

DEFINITION 1.7. Let ν_1 and ν_2 be two centered valuations. Then $\nu_1 \leq \nu_2$ if and only if $\nu_1(\phi) \leq \nu_2(\phi)$ for every $\phi \in R$.

The set of all (normalized) centered valuations with this partial order admits an $\mathbb{R}$-tree structure: we shall call $\mathcal{V}$ the *valuative tree*. Valuations are naturally embedded into Krull valuations, but the converse is not true (see the exceptional curve valuations [FJ1, p. 18] for details).

The next theorem is a classic result of algebraic geometry. For a modern exposition, see [ZS2, Part VI, Chap. 5] or [Ha].

THEOREM 1.8 [Ha, Thm. 4.7]. *Let ν be a Krull valuation on $K = \mathbb{C}((x, y)) \supset \mathbb{C}(x, y)$, let R_ν be the associated valuation ring, and let $\pi : X \to (\mathbb{C}^2, 0)$ be a modification. Then there exists a unique irreducible submanifold V of X such that R_ν dominates $\mathcal{O}_{X,V}$, the ring of regular functions in V. Moreover, if ν is centered then V is a point or an exceptional component in $\pi^{-1}(0)$.*

This V is called the center *of ν in X.*

The center of a valuation is the main concept allowing us to pass from valuations to exceptional components, and it gives an isomorphism between the valuative tree and the universal dual graph [FJ1, Thm. 6.22]. The center of a valuation also gives special open sets in the valuative tree (see [FJ1, Cor. 6.34] for some properties).

DEFINITION 1.9. Let $p \in \pi^{-1}(0)$ be an infinitely near point of a modification $\pi : X \to (\mathbb{C}^2, 0)$. We shall denote by $U(p) \subseteq \mathcal{V}$ the (weakly open) set of all valuations whose center in X is p.

For proofs and further details on valuations, see [ZS2, Part VI].

1.5. Classification of Valuations

We shall now describe the classification of valuations and their role in the valuative tree (cf. [FJ1, Chap. 1]).

1.5.1. *Divisorial Valuations*

Divisorial valuations are associated to an exceptional component E of a modification π. In particular, ν_E is defined by

$$\nu_E(\phi) := (1/b_E)\,\mathrm{div}_E(\pi^*\phi),$$

where div_E is the vanishing order along E, $\pi^*\phi = \phi \circ \pi$, and $1/b_E$ is necessary for a normalized valuation ($b_E \in \mathbb{N}^*$ is known as the *generic multiplicity* of ν_E [FJ1, p. 64] or the *second Faray weight* of E [FJ1, p. 122]). The set of all divisorial valuations is often denoted by $\mathcal{V}_{\mathrm{div}}$. The divisorial valuations are the branch points of the valuative tree, and in particular we have $T_{\nu_E}\mathcal{V} \cong E$.

The most important example is the *multiplicity valuation,* defined by

$$\nu_{\mathfrak{m}}(\phi) := m(\phi) = \max\{n \mid \phi \in \mathfrak{m}^n\};$$

it is associated to a single blow-up over the origin and plays the role of the root of $\mathcal{V}$. We will write $\nu_{\mathfrak{m}}$ if we want to consider the multiplicity as a valuation (or better, as a point on the valuative tree) and will write m if we want to consider only the multiplicity of an element of $R = \mathbb{C}[[x, y]]$.

1.5.2. *Irrational Valuations*

Irrational valuations are the regular points of the valuative tree. Divisorial and irrational valuations are called *quasi-monomial valuations,* and their set will be denoted by $\mathcal{V}_{\mathrm{qm}}$. For a geometric interpretation of quasi-monomial valuations, see [FJ1, pp. 16, 17].

Important examples of quasi-monomial valuations are *monomial valuations.* Fix local coordinates (x, y); then the monomial valuation of the weights (s, t) is defined by

$$\nu_{s,t}\biggl(\sum_{i,j} a_{i,j} x^i y^j\biggr) = \min\{si + tj \mid a_{i,j} \neq 0\}.$$

1.5.3. *Curve Valuations*
Curve valuations are ends of the valuative tree, and they are associated to a (formal) irreducible curve (germ) $C = \{\psi = 0\}$. In particular, ν_C is defined by

$$\nu_C(\phi) := \frac{C \cdot \{\phi = 0\}}{m(C)},$$

where by $C \cdot D$ we denote the standard intersection multiplicity between the curves C and D and where $m(C) = m(\psi)$ is the multiplicity of C (in 0). We will often use the notation ν_ψ instead of ν_C.

Analytic and nonanalytic curve valuations have the same algebraic behavior but play a different role as eigenvaluations, as we shall see in the proof of Theorem 0.6.

1.5.4. *Infinitely Singular Valuations*
Infinitely singular valuations are the ones with $\operatorname{rk}\nu = \operatorname{ratrk}\nu = 1$ and $\operatorname{trdeg}\nu = 0$, and they share with curve valuations the role of ends of the valuative tree.

It is not so simple to give a geometric interpretation of infinitely singular valuations, but we can think of them as curve valuations associated to "curves" of infinite multiplicity. They can also be viewed as valuations with infinitely generated value groups.

1.6. Parameterizations

The valuative tree admits (at least) two natural parameterizations (skewness and thinness) and a concept of multiplicity, features that are useful for distinguishing the type of valuations. For definitions and properties we refer to [FJ1, Chap. 3]; all we need for this paper is the following result.

PROPOSITION 1.10 [FJ1, Thm. 3.46]. *The thinness $A\colon \mathcal{V} \to [2, \infty]$ is a parameterization for the valuative tree. Moreover*:

(i) *the multiplicity valuation is the only one with* $A(\nu_{\mathrm{m}}) = 2$;
(ii) *for divisorial valuations,* $A(\nu_E) \in \mathbb{Q}$;
(iii) *for irrational valuations,* $A(\nu) \in \mathbb{R} \setminus \mathbb{Q}$;
(iv) *for curve valuations,* $A(\nu_C) = \infty$; *and*
(v) *for infinitely singular valuations,* $A(\nu) \in (2, \infty]$.

1.7. Dynamics on the Valuative Tree

1.7.1. *Definition*
In this section we define the action $f_\bullet\colon \mathcal{V} \to \mathcal{V}$ induced by a holomorphic germ $f\colon (X, p) \to (Y, q)$, where X and Y are two complex 2-manifolds. We shall also assume that f is *dominant* (i.e., that $\operatorname{rk} df$ is not identically ≤ 1 near p).

A holomorphic germ $f\colon (X,p) \to (Y,q)$ naturally induces an action f^* on $R = \mathbb{C}[[x,y]]$ by composition: $\phi \mapsto f^*\phi = \phi \circ f$. The natural way to define an action on (centered) valuations seems to be the dual action $f_*\nu = \nu \circ f^*$; explicitly, we have $f_*\nu(\phi) = \nu(\phi \circ f)$. This definition works for Krull valuations but not for all valuations; if $\nu \in \mathcal{V}$, then clearly $f_*\nu$ is a valuation but it might not be proper. More precisely, $f_*\nu$ is not centered if and only if $\nu = \nu_C$ is a curve valuation, where $C = \{\phi = 0\}$ is an irreducible curve contracted to q by f (i.e., iff $f^*\mathfrak{m} \subseteq \langle\phi\rangle$). In this case, C must be a critical curve and $f_*\nu$ is not proper.

Definition 1.11. Let $f\colon (X,p) \to (Y,q)$ be a (dominant) holomorphic germ. We call *contracted critical curve valuations* for f the valuations ν_C with C a critical curve contracted to q by f. We denote by $\mathfrak{C}_f$ the set of all contracted critical curve valuations for f.

Remark 1.12. The set $\mathfrak{C}_f$ has a finite number of elements, all of which are ends for the valuative tree.

So if $\nu \in \mathcal{V} \setminus \mathfrak{C}_f$, then $f_*\nu$ is a centered valuation but is not normalized generally. The norm will be $f_*\nu(\mathfrak{m}) = \nu(f^*\mathfrak{m})$; we can renormalize this valuation and obtain an action $f_\bullet\colon \mathcal{V} \setminus \mathfrak{C}_f \to \mathcal{V}$.

Definition 1.13. Let $f\colon (X,p) \to (Y,q)$ be a (dominant) holomorphic germ. For every valuation $\nu \in \mathcal{V}$, we define $c(f,\nu) := \nu(f^*\mathfrak{m})$ as the *attraction rate of f along ν*; if $\nu = \nu_\mathfrak{m}$ is the multiplicity valuation, then we simply write $c(f) := c(f,\nu_\mathfrak{m})$ as the *attraction rate* of f. For every valuation $\nu \in \mathcal{V} \setminus \mathfrak{C}_f$, we define $f_\bullet\nu := f_*\nu / c(f,\nu) \in \mathcal{V}$. If $f\colon (X,p) \to (X,p)$, we will also define $c_\infty(f) := \lim_{n\to\infty} \sqrt[n]{c(f^n)}$ as the *asymptotic attraction rate* of f.

Up to fixed coordinates in p and q, we can consider a germ $f\colon (X,p) \to (Y,q)$ as a germ $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$. From now on we will state results in the latter case, but they can be easily extended to the general case.

In order to have an action on $\mathcal{V}$, we should extend $f_\bullet$ to contracted critical curve valuations.

Proposition 1.14 [FJ2, Prop. 2.7]. *Suppose C is an irreducible curve germ such that $f(C) = \{0\}$ (i.e., $\nu_C \in \mathfrak{C}_f$). Then $c(f,\nu_C) = \infty$. Furthermore, the limit of $f_\bullet\nu$ as ν increases to ν_C exists, and it is a divisorial valuation that we denote by $f_\bullet\nu_C$. This limit can be interpreted geometrically as follows. There exist modifications $\pi\colon X \to (\mathbb{C}^2,0)$ and $\pi'\colon X' \to (\mathbb{C}^2,0)$ such that f lifts to a holomorphic map $\hat{f}\colon X \to X'$ sending C to a curve germ included in an exceptional component $E' \in \Gamma^*_{\pi'}$ for which $f_\bullet\nu_C = \nu_{E'}$.*

Definition 1.15. Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ be a (dominant) holomorphic germ. For every $\nu \in \mathcal{V}$, we denote by $d(f_\bullet)_\nu\colon T_\nu\mathcal{V} \to T_{f_\bullet\nu}\mathcal{V}$ the *tangent map* induced by f at ν. We will often omit the point ν, and write $d(f_\bullet)_\nu = df_\bullet$, when we are considering the tangent map.

For other properties of the action $f_\bullet$, we refer to [FJ2, Secs. 2 and 3].

1.7.2. *Eigenvaluations and Basins of Attraction*
Given the regularity properties of $f_\bullet$ (see [FJ2, Thm. 3.1]) and a fixed point theorem for regular tree maps (see [FJ2, Thm. 4.5]), we can obtain eigenvaluations as follows.

THEOREM 1.16 [FJ2, Thm. 4.2]. *Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ be a (dominant) holomorphic germ. Then there exists a valuation $\nu_\star \in \mathcal{V}$ such that $f_\bullet\nu_\star = \nu_\star$ and $c(f,\nu_\star) = c_\infty(f) =: c_\infty$. Moreover, $\nu_\star$ cannot be a contracted critical curve valuation or a nonanalytic curve valuation if $c_\infty > 1$. If $\nu_\star$ is an end, then there exists a $\nu_0 < \nu_\star$ (arbitrarily close to $\nu_\star$) such that $c(f,\nu_0) = c_\infty$, $f_\bullet$ preserves the order on $\{\nu \geq \nu_0\}$, and $f_\bullet\nu > \nu$ for every $\nu \in [\nu_0,\nu_\star)$. Finally, we can find $0 < \delta \leq 1$ such that $\delta c_\infty^n \leq c(f^n) \leq c_\infty^n$ for every $n \geq 1$.*

DEFINITION 1.17. Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ be a (dominant) holomorphic germ. A valuation $\nu_\star \in \mathcal{V}$ is called a *fixed valuation* for f if $f_\bullet\nu_\star = \nu_\star$. It is called an *eigenvaluation* for f if it is either a quasi-monomial fixed valuation or a fixed valuation that is a strongly attracting end (see [FJ2, Sec. 4]).

REMARK 1.18. In the rest of this paper, we will consider quasi-monomial eigenvaluations whenever possible. Hence when we say that an eigenvaluation $\nu_\star$ "is an end" we are implicitly stating that quasi-monomial eigenvaluations do not exist.

COROLLARY 1.19. *Let f be a (dominant) holomorphic germ, and let $\nu_\star$ be an eigenvaluation for f. Then the following statements hold.*

(i) *If $c_\infty(f) > 1$, then $\nu_\star$ cannot be a nonanalytic curve valuation.*
(ii) *If $c_\infty(f) = 1$, then $\nu_\star$ cannot be a quasi-monomial valuation.*

Proof. The first assertion has already been stated in Theorem 1.16.

Let us suppose $c_\infty(f) = 1$. Then applying [FJ2, Lemma 7.7] to the eigenvaluation (and recalling that $c_\infty(f) = c(f,\nu_\star)$ by Theorem 1.16), we obtain

$$A(\nu_\star) = A(\nu_\star) + \nu_\star(Jf);$$

this equality is satisfied only if $A(\nu_\star) = \infty$. It follows that $\nu_\star$ cannot be a quasi-monomial valuation. $\square$

PROPOSITION 1.20 [FJ2, Prop. 5.2]. *Let f be a (dominant) holomorphic germ, and let $\nu_\star$ be an eigenvaluation for f.*

(i) *If $\nu_\star$ is an end for $\mathcal{V}$ then, for any $\nu_0 \in \mathcal{V}$ with $\nu_0 \leq \nu_\star$ and for ν_0 sufficiently close to $\nu_\star$, we have that $f_\bullet$ maps the segment $I = [\nu_0,\nu_\star]$ strictly into itself and is order preserving there. Moreover, if we set $U = U(\vec{v})$ for $\vec{v}$ the tangent vector at ν_0 represented by $\nu_\star$, then $f_\bullet$ also maps the open set U strictly into itself and $f_\bullet^n \to \nu_\star$ as $n \to \infty$ in U.*
(ii) *If $\nu_\star$ is divisorial, then there exists a tangent vector $\vec{w}$ at $\nu_\star$ such that, for any $\nu_0 \in \mathcal{V}$ representing $\vec{w}$ and sufficiently close to $\nu_\star$, we have that $f_\bullet$ maps the segment $I = [\nu_\star,\nu_0]$ into itself and is order preserving there. Moreover, if we set $U = U(\vec{v}) \cap U(\vec{w})$ then $f_\bullet(I) \subset\subset I$, $f_\bullet(U) \subset\subset U$, and $f_\bullet^n \to \nu_\star$ as $n \to \infty$ on U.*

(iii) *If $\nu_\star$ is irrational, then there exist $\nu_1, \nu_2 \in \mathcal{V}$, arbitrarily close to $\nu_\star$ and with $\nu_1 < \nu_\star < \nu_2$, such that $f_\bullet$ maps the segment $I = [\nu_1, \nu_2]$ into itself. Let $\vec{v}_i$ $(i = 1, 2)$ be the tangent vector at ν_i represented by $\nu_\star$, and set $U = U(\vec{v}_1) \cap U(\vec{v}_2)$. Then $f_\bullet(U) \subseteq U$. Furthermore, either $f_\bullet|_I^2 = \mathrm{id}_I$ or $f_\bullet^n \to \nu_\star$ as $n \to \infty$ on U.*

2. Rigidification

2.1. General Result

In this section we shall prove our main theorem (Theorem 0.6). We have five cases rather than the four of [FJ2, Thm. 5.1]; the new case is when we have a nonanalytic curve eigenvaluation, and it arises only when we deal with f having a nonnilpotent differential. For the other cases, we refer directly to [FJ2, Thm. 5.1].

Proof of Theorem 0.6. Let $\nu_\star$ be an eigenvaluation for f (which exists by virtue of Theorem 1.16), and suppose that it is a nonanalytic curve valuation ν_C.

Pick ν_0 as in Proposition 1.20. By increasing ν_0, we can suppose ν_0 to be divisorial. Let $\pi \in \mathcal{B}$ be a modification such that $\nu_0 = \nu_{E_0}$. From [FJ2, Prop. 6.32] it follows that there exists a unique best approximation ν_E of $\nu_\star$ for π (it is unique because $\nu_\star$ is an end of $\mathcal{V}$). We have $\nu_0 \le \nu_E < \nu_C$, which can be chosen arbitrarily close to ν_C (by increasing ν_0). We now consider $U = U(p) = U_{\nu_E}([\nu_\star])$.

By Proposition 1.20 and [FJ2, Prop. 3.2], we have that $f_\bullet U \subset\subset U$, that the lift $\hat{f} = \pi^{-1} \circ f \circ \pi$ is holomorphic in p, and that $\hat{f}(p) = p$. By shrinking $U(p)$, we can avoid all critical curve valuations. Therefore, $\mathcal{C}^\infty(\hat{f}) = E$ has normal crossings. Moreover, E is contracted to p by $\hat{f}$ (because $f_\bullet \nu_E > \nu_E$), $\mathcal{C}^\infty(\hat{f})$ is forward $\hat{f}$-invariant, and $\hat{f}$ is rigid. □

REMARK 2.1. Studying the behavior of $\pi_\bullet$ for $\pi: (X, p) \to (\mathbb{C}^2, 0)$ a modification, we see that $\pi_\bullet$ is a bijection between $\mathcal{V}$ and $\overline{U(p)}$. Moreover, from the relation $\hat{f} = \pi^{-1} \circ f \circ \pi$ we see that $\pi_\bullet$ yields a conjugation between $\hat{f}_\bullet$ and $f_\bullet|_{\overline{U(p)}}$. So from the dynamics of $f_\bullet$ on $\overline{U(p)}$ we can obtain information on the rigidification $\hat{f}$. For example, if $f_\bullet^n \to \nu_\star$ then $\hat{f}$ will have a unique eigenvaluation $\pi_\bullet^{-1}\nu_\star$.

2.2. Semi-superattracting Case

In this section we deal with the semi-superattracting case and prove the uniqueness of the eigenvaluation in this case (see Theorem 0.7). We shall write

$$D_\lambda := \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}.$$

LEMMA 2.2. *Let f be a (dominant) semi-superattracting holomorphic germ such that $df_0 = D_\lambda$ with $\lambda \neq 0$. Let $\pi: X \to (\mathbb{C}^2, 0)$ be the single blow-up in $0 \in \mathbb{C}^2$, where $E := \pi^{-1}(0) \cong \mathbb{P}^1(\mathbb{C})$ is the exceptional divisor. Set $p = [1 : 0] \in E$, and let $\hat{f}: (X, p) \to (X, p)$ be the lift of f through π. Then $\hat{f}$ is a semi-superattracting holomorphic germ and $d\hat{f}_p \cong D_\lambda$.*

Proof. Since $df_0 = D_\lambda$, we have

$$f(z, w) = (\lambda z + f_1(z, w), f_2(z, w)) \tag{1}$$

with $f_1, f_2 \in \mathfrak{m}^2$. In the chart $\pi^{-1}(\{z \neq 0\})$ we can choose (u,t) coordinates in $p \in E$ such that

$$(z,w) = \pi(u,t) = (u, ut).$$

Hence for the lift $\hat{f} = \pi^{-1} \circ f \circ \pi$ we have

$$f \circ \pi(u,t) = (\lambda u + f_1(u,ut), f_2(u,ut)),$$

from which it follows that

$$\hat{f}(u,t) = \left(\lambda u + f_1(u,ut), \frac{f_2(u,ut)}{\lambda u + f_1(u,ut)}\right).$$

We have that u^2 divides $f_1(u,ut)$ and $f_2(u,ut)$; if we set $\hat{f} = (g_1, g_2)$, then

$$g_1(u,t) = \lambda u(1 + O(u)) \quad \text{and}$$

$$g_2(u,t) = \frac{u^2 O(1)}{\lambda u(1 + O(u))} = \alpha u + O(u^2)$$

for $\alpha = \lambda^{-1} a_{2,0}$, assuming that $f_2(z,w) = \sum_{i+j \geq 2} a_{i,j} z^i w^j$. Thus,

$$d\hat{f}_p = \begin{pmatrix} \lambda & 0 \\ \alpha & 0 \end{pmatrix} \cong D_\lambda.$$

Hence $\hat{f}$ is a holomorphic germ with $d\hat{f}_p \cong D_\lambda$. □

PROPOSITION 2.3. *Let f be a (dominant) semi-superattracting holomorphic germ such that $df_0 = D_\lambda$ with $\lambda \neq 0$, let $\nu_\star$ be an eigenvaluation for f, and let $(\pi, p, \hat{f})$ be a rigidification obtained from $\nu_\star$ as in Theorem 0.6. Then $d\hat{f}_p \cong D_\lambda$, and $\nu_\star = \nu_C$ is a (possibly formal) curve valuation with $m(C) = 1$.*

Proof. To prove this result, we follow the proof of [FJ2, Thm. 4.5] under the assumption $df_0 \cong D_\lambda$. Starting from any ν_0 (as in the proof of [FJ2, Thm. 4.5]), we take any end $\nu_0' > f_\bullet \nu_0$ and consider the induced tree map F_0 on $I_0 = [\nu_0, \nu_0']$. Let ν_1 be the (minimum) fixed point of F_0. Because $f_\bullet$ has no quasi-monomial eigenvaluations (see Corollary 1.19), $\nu_1 \geq f_\bullet \nu_0$. Up to choosing ν_0' such that $\nu_0' \notin d(f_\bullet)_{\nu_0}([f_\bullet \nu_0])$, we can suppose that $\nu_1 = f_\bullet \nu_0$.

Let us now apply this argument for $\nu_0 = \nu_{\mathfrak{m}}$. If f is as in (1), then $\nu_1 = f_\bullet \nu_0$ is a divisorial valuation associated to an exceptional component E_1 obtained from the exceptional component E_0 of a single blow-up of $0 \in \mathbb{C}^2$ by blowing up only free points (i.e., the generic multiplicity $b(\nu_{E_1})$ of ν_{E_1} is equal to 1); as a matter of fact, $f_* \nu_0(x) = 1$ while $f_* \nu_0(\phi) \in \mathbb{N}$ for every $\phi \in R$. Applying this argument recursively (as in the proof of [FJ2, Thm. 4.5]), we get the assertion on the type of eigenvaluation.

For the result on $d\hat{f}_p$ we need only observe that—in the proofs of Theorem 0.6 and [FJ2, Thm. 5.1] in the case of an analytic curve eigenvaluation—up to shrinking the basin of attraction we can choose the infinitely near point p such that ν_{E_p} has generic multiplicity $b(\nu_{E_p})$ equal to 1, where E_p denotes the exceptional component obtained by blowing up p. Then the modification π on the rigidification is the composition of blow-ups of free points, which allows us to apply (recursively) Lemma 2.2 and thereby obtain the thesis. □

LEMMA 2.4. *Let f be a (dominant) semi-superattracting holomorphic germ such that $df_0 = D_\lambda$ with $\lambda \neq 0$. Then, up to a (possibly formal) change of coordinates, we can suppose that*

$$f(z,w) = \big(\lambda z(1 + f_1(z,w)), w f_2(z,w)\big)$$

with $f_1, f_2 \in \mathfrak{m}$.

Proof. First of all, we can suppose that

$$f(z,w) = (\lambda z + g_1(z,w), g_2(z,w))$$

with $g_1, g_2 \in \mathfrak{m}^2$.

By Proposition 2.3 we know that there is an eigenvaluation $\nu_\star = \nu_C$ with $C = \{\phi = 0\}$ a (possibly formal) curve, where $\phi(z,w) = w - \theta(z)$ for a suitable θ. Up to the (possibly formal) change of coordinates $(z,w) \mapsto (z, w - \theta(z))$, we can suppose that $\phi = w$; in particular, since C is fixed by f, we can suppose that $w | g_2$. Then

$$f(z,w) = \big(\lambda z(1 + f_1(z,w)) + h(w), w f_2(z,w)\big)$$

with $f_1, f_2 \in \mathfrak{m}$ and $h \in \mathfrak{m}^2$. We put $g_2(z,w) = w f_2(z,w)$.

Now we need only show that, up to a (possibly formal) change of coordinates, $h \equiv 0$. We consider a change of coordinates of the form $\Phi(z,w) = (z + \eta(w), w)$ with $\eta \in \mathfrak{m}^2$, in which case $\Phi^{-1}(z,w) = (z - \eta(w), w)$. Therefore,

$$\begin{aligned}\Phi^{-1} \circ f \circ \Phi(z,w) = \big(&\lambda(z + \eta(w))(1 + f_1 \circ \Phi(z,w)) \\ &+ h(w) - \eta \circ g_2 \circ \Phi(z,w), w f_2 \circ \Phi(z,w)\big).\end{aligned} \tag{2}$$

We observe that the second coordinate of (2) is always divisible by w; we only have to show that there exists a suitable η such that the first coordinate of (2), valuated on $(0,w)$, is equal to 0. Hence we must solve

$$\lambda \eta(w)\big(1 + f_1(\eta(w), w)\big) + h(w) - \eta \circ g_2(\eta(w), w) = 0. \tag{3}$$

Set $\eta(w) = \sum_{n \geq 2} \eta_n w^n$, $h(w) = \sum_{n \geq 2} h_n w^n$, $1 + f_1(z,w) = \sum_{i+j \geq 0} f_{i,j} z^i w^j$, and $g_2(z,w) = \sum_{i+j \geq 2} g_{i,j} z^i w^j$ (with $g_{n,0} = 0$ for every n). Then

$$\lambda \sum_{i+j \geq 0} f_{i,j} \sum_{H \in \mathbb{N}^{i+1}} \eta_H w^{|H|+j} + \sum_{n \geq 2} h_n w^n = \sum_k \eta_k \bigg(\sum_{i+j \geq 2} g_{i,j} \eta(w)^i w^j\bigg)^k. \tag{4}$$

Comparing the coefficients of w^n on both sides yields

$$\lambda \eta_n + \text{l.o.t.} = \text{l.o.t.};$$

here l.o.t. (low-order term) denotes a suitable function that depends on η_h only for $h < n$. Thus, by (4), we have a recurrence relation for the coefficients η_n that is a solution of (3). □

Proof of Theorem 0.7. By Lemma 2.4, we can suppose (up to formal conjugacy) that

$$f(z,w) = \big(\lambda z(1 + g_1(z,w)), w g_2(z,w)\big),$$

where $g_1, g_2 \in \mathfrak{m}$. We shall denote $f_2(z,w) = w g_2(z,w)$.

It follows that the eigenvaluation $\nu_\star$ given by Proposition 2.3 is $\nu_\star = \nu_w$ and ν_z is either fixed by $f_\bullet$ or a contracted critical curve valuation. Hence we need only show that there are no other fixed valuations.

First of all, observe what happens—during the process used in the proof of Proposition 2.3—to tangent vectors at the valuation $\nu_0 = \nu_{\mathfrak{m}}$. Consider the family of valuations $\nu_{\theta,t}$, where $\theta \in \mathbb{P}^1(\mathbb{C})$ and $t \in [1,\infty]$, described as follows: if we put $\phi_\theta = w - \theta z$ for $\theta \in \mathbb{C}$ and if $\psi_\infty = z$, then $\nu_{\theta,t}$ is the valuation of skewness $\alpha(\nu_{\theta,t}) = t$ in the segment $[\nu_{\mathfrak{m}}, \nu_{\phi_\theta}]$ (i.e., the monomial valuation defined by $\nu_{\theta,t}(\phi_\theta) = t$ and $\nu_{\theta,t}(z) = 1$ if $\theta \in \mathbb{C}$); then $\nu_{\infty,t}(z) = t$ and $\nu_{\infty,t}(w) = 1$.

Thus we have that $\nu_1 = f_\bullet(\nu_{\mathfrak{m}}) = \nu_{0,m(f_2)}$, where m denotes the multiplicity function, and that $f_\bullet(\nu_{\theta,t}) \geq \nu_1$ for every $\theta \in \mathbb{C}$ and t. The latter statement follows because $f_\bullet(\nu_{\theta,t})(z) = 1 = \nu_1(z)$, $f_\bullet(\nu_{\theta,t})(w) \geq m(f_2) = \nu_1(w)$, and ν_1 is the minimum valuation that assumes these values on z and w.

We shall use $\vec{v}_\theta$ to denote the tangent vector in $\nu_{\mathfrak{m}}$ represented by $\nu_{\theta,\infty}$ and use $\vec{u}_\infty$ to denote the tangent vector in ν_1 represented by $\nu_{\mathfrak{m}}$. Then, by what we have shown so far, $df_\bullet(\vec{v}_\theta) \neq \vec{u}_\infty$ for every $\theta \neq \infty$ and hence there are no fixed valuations in $\overline{U_{\nu_{\mathfrak{m}}}(\vec{v}_\theta)}$ for every $\theta \neq 0, \infty$. Moreover, applying this argument recursively as in the proof of Proposition 2.3, we obtain that there are no other fixed valuations in $\overline{U_{\nu_{\mathfrak{m}}}(\vec{v}_0)}$ except for the eigenvaluation ν_w.

It remains to check for valuations in $\overline{U_{\nu_{\mathfrak{m}}}(\vec{v}_\infty)}$, and toward this end we consider $f_\bullet(\nu_{\infty,t})$. For simplicity we denote $\nu_{\infty,t} = \nu_{0,1/t}$ for every $t \in [0,1]$. Direct computation reveals that $f_*(\nu_{\infty,t})(z) = t$ and that

$$f_*(\nu_{\infty,t})(w) = \bigwedge_j (a_j t + b_j)$$

for suitable $a_j \in \mathbb{N}^*$ and $b_j \in \mathbb{N}$. It follows that (i) $f_\bullet(\nu_{\infty,t}) = \nu_{\infty,g(t)}$ for a suitable map $g(t)$ such that $g(t) < t$ and (ii) $d(f_\bullet)_{\nu_{\infty,t}}([\nu_w]) = [\nu_w]$ (where the latter tangent vector belongs to the proper tangent space). If we let t go to ∞ then the only fixed valuation in $U_{\nu_{\mathfrak{m}}}(\vec{v}_\infty)$ is ν_z, and we are done. $\square$

REMARK 2.5. Theorem 0.7 shows that every semi-superattracting germ f has two (formal) invariant curves: C, which is associated to the eigenvaluation and hence to the eigenvalue λ of df_0; and D, which is associated to the fixed or contracted critical curve valuation and hence to the eigenvalue 0 of df_0. If f is of type $(0, \mathbb{C} \setminus \bar{\mathbb{D}})$, then both these curves are actually holomorphic by the stable/unstable manifold theorem (see [A, Thms. 3.1.2 and 3.1.3]). In the general case of f of type $(0, \mathbb{C}^*)$, one can at least recover the manifold associated to the eigenvalue 0 of df_0 by using generalizations of the stable manifold theorem, such as the Hadamard–Perron theorem (see [A, Thm. 3.1.4]). In particular, the curve D is always holomorphic. However, C is not holomorphic in general (see e.g. Proposition 4.3).

3. Rigid Germs

In this section we introduce the classification of attracting rigid germs in $(\mathbb{C}^2, 0)$ up to holomorphic and formal conjugacy (for proofs, see [F]) as well as the classification of rigid germs of type $(0, \mathbb{C} \setminus \mathbb{D})$ in $(\mathbb{C}^2, 0)$ up to formal conjugacy. Stating our results will require three invariants as follows.

Table 1

Class	$\mathcal{C}^\infty(f)$	$\operatorname{tr} df_0$	$\det f$
1	0 (empty)		
2	1 (irreducible)	$\neq 0$	$= 1$
3			≥ 2
4		$= 0$	(≥ 2)
5	2 (reducible)	$\neq 0$	$(\neq 0)$
6		$= 0$	$\neq 0$
7			$= 0$

- *The generalized critical set*. If $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a rigid germ, then $C = \mathcal{C}^\infty(f)$ is a curve with normal crossings at the origin that can have either none, one, or two irreducible components. In other words, $\mathcal{C}^\infty(f)$ can be empty (if and only if f is a local biholomorphism in 0), an irreducible curve, or a reducible curve (with only two irreducible components); we will call f *regular, irreducible,* or *reducible,* respectively.
- *The trace*. If f is not regular then we have two cases: either $\operatorname{tr} df_0 \neq 0$, and df_0 has a zero eigenvalue and a nonzero eigenvalue; or $\operatorname{tr} df_0 = 0$, and df_0 is nilpotent.
- *The action on $\pi_1(\Delta^2 \setminus \mathcal{C}^\infty(f))$*. Because $\mathcal{C}^\infty(f)$ is backward invariant, f induces a map from $U = \Delta^2 \setminus \mathcal{C}^\infty(f)$ (here Δ^2 denotes a sufficiently small polydisc) to itself and hence an action f_* on the first fundamental group of U. If f is irreducible then $\pi_1(U) \cong \mathbb{Z}$, in which case f_* is completely described by $f_*(1) \in \mathbb{N}^*$ (f preserves orientation); if f is reducible, then $\pi_1(U) \cong \mathbb{Z} \oplus \mathbb{Z}$ and f_* is described by a 2×2 matrix with integer entries (in $\mathbb{N}$).

Definition 3.1. Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ a rigid germ. Then f belongs to:

class 1 if f is regular;
class 2 if f is irreducible, $\operatorname{tr} df_0 \neq 0$, and $f_*(1) = 1$;
class 3 if f is irreducible, $\operatorname{tr} df_0 \neq 0$, and $f_*(1) \geq 2$;
class 4 if f is irreducible and $\operatorname{tr} df_0 = 0$ (this implies $f_*(1) \geq 2$);
class 5 if f is reducible and $\operatorname{tr} df_0 \neq 0$ (this implies $\det f_* \neq 0$);
class 6 if f is reducible, $\operatorname{tr} df_0 = 0$, and $\det f_* \neq 0$;
class 7 if f is reducible, $\operatorname{tr} df_0 = 0$, and $\det f_* = 0$.

See Table 1.

Remark 3.2. If f is irreducible then, up to a change of coordinates, we can assume $\mathcal{C}^\infty(f) = \{z = 0\}$. Now, since $\{z = 0\}$ is backward invariant, we can write f in the form

$$f(z, w) = \big(\alpha z^p(1 + \phi(z, w)), f_2(z, w)\big)$$

with $\phi, f_2 \in \mathfrak{m}$. It can be easily seen that $f_* = p \geq 1$.

Analogously, if f is reducible then, up to a change of coordinates, we can assume $\mathcal{C}^\infty(f) = \{zw = 0\}$. Since $\{zw = 0\}$ is backward invariant, f can be written

$$f(z,w) = \big(\lambda_1 z^a w^b (1 + \phi_1(z,w)), \lambda_2 z^c w^d (1 + \phi_2(z,w))\big)$$

with $\phi_1, \phi_2 \in \mathfrak{m}$. In this case, f_* is represented by the 2×2 matrix

$$M(f) := \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5}$$

3.1. Attracting Rigid Germs

The classification up to holomorphic conjugacy of attracting rigid germs in $\mathbb{C}^2$ is given in [F, Chap. 1]. Here we need only the following remark.

REMARK 3.3. During the proof of [F, Step 1 on p. 491, first case on p. 498], the author starts from a germ of the form

$$f(z,w) = \big(\alpha z^p (1 + g(z,w)), f_2(z,w)\big)$$

with ϕ, $f_2 \in \mathfrak{m}$ and then uses the theorems of Kœnigs and Böttcher. (See, respectively, [F, Thms. 3.1 and 3.2] as well as [K] and [B] for the original papers and [M, Thms. 8.2 and 9.1] for a modern exposition of the proofs.) Favre [F] assumes, up to holomorphic conjugacy, that $g \equiv 0$ (and $\alpha = 1$ if $p \geq 2$). Yet that argument does not work. Denote by $\Phi(z,w) = (\phi_w(z), w)$ the conjugation given by those theorems, and let $\tilde{f} = \Phi \circ f \circ \Phi^{-1}$. We shall also put $f(z,w) = (f_w^{(1)}(z), f_w^{(2)}(z))$, and analogously for $\tilde{f}$.

By hypothesis, $\phi_w(z)$ is such that $\phi_w \circ f_w^{(1)} \circ \phi_w^{-1}(z) = \alpha z^p$ (with $\alpha = 1$ if $p \geq 2$). But

$$\tilde{f}_w^{(1)}(z) = \phi_{f_w^{(2)}(\phi_w^{-1}(z))} \circ f_w^{(1)} \circ \phi_w^{-1}(z),$$

which does not coincide with αz^p.

We also note that if $|\alpha| > 1$ then the Kœnigs theorem still applies; however, the result is false (see Counterexample 3.10). Nevertheless, one can obtain this result in the attracting case as follows.

We want to solve the conjugacy relation

$$\Phi \circ f = e \circ \Phi, \tag{6}$$

where e is a germ of the form

$$e(z,w) = (\alpha z^p, e_2(z,w))$$

with $e_2 \in \mathfrak{m}$. We look for a solution of the form

$$\Phi(z,w) = \big(z(1 + \phi(z,w)), w\big)$$

with $\phi \in \mathfrak{m}$. Then the conjugacy relation (6) (comparing the first coordinate) yields

$$(1 + g)(1 + \phi \circ f) = (1 + \phi)^p.$$

Now we can consider

$$1+\phi = \prod_{k=0}^{\infty}(1+g\circ f^k)^{1/p^{k+1}}, \tag{7}$$

which would work if that product converges. But since f is attracting, there exists $0<\varepsilon<1$ such that $\|f(z,w)\| \le \varepsilon\|(z,w)\|$, and since $g\in\mathfrak{m}$, there exists an $M>0$ such that $|g(z,w)|\le M\|(z,w)\|$. It follows that

$$\sum_{k=0}^{\infty} p^{-(k+1)}|g\circ f^k(z,w)| \le \sum_{k=0}^{\infty}\frac{M}{p}\left(\frac{\varepsilon}{p}\right)^k = \frac{M}{p-\varepsilon} < \infty,$$

so (7) defines a holomorphic germ ϕ and hence a holomorphic map Φ that satisfies the conjugacy relation (6) in the first coordinate.

To choose e_2 such that (6) holds also for the second coordinate, we must solve

$$f_2 = e_2\circ\Phi.$$

Yet because Φ is a holomorphic invertible map, we can just define $e_2 = f_2\circ\Phi$ and we are done.

Note that this approach would not work—not even formally—for rigid germs of type $(0,\mathbb{C}\setminus\mathbb{D})$.

3.2. Rigid Germs of Type $(0,\mathbb{C}\setminus\mathbb{D})$

In this section we study (formal) normal forms for rigid germs of type $(0,\mathbb{C}\setminus\mathbb{D})$. If $f(z,w)=\sum_{i,j} f_{i,j}z^iw^j$ is a formal power series and if $I=(i_1,\dots,i_k)$ and $J=(j_1,\dots,j_k)$ are two multi-indices, then we shall denote by $f_{I,J}$ the product

$$f_{I,J} = \prod_{l=1}^{k} f_{i_l,j_l}.$$

Moreover, when writing the dummy variables of a sum, we shall write the dimension of a multi-index after the multi-index itself. For example, $I(n)$ denotes a multi-index $I\in\mathbb{N}^n$. We group together the multi-indices with the same dimension, separating these groups by a semicolon, and omit the dimension when it is equal to 1. For example,

$$\sum_{n,m;\,I,J(n);\,K(m)}$$

is a sum over $n,m\in\mathbb{N}$, $I,J\in\mathbb{N}^n$, and $K\in\mathbb{N}^m$. As a convention, we view a multi-index of dimension 0 as an empty multi-index.

First of all, we need to recall the formal classification of (invertible) germs in one complex variable. See [M] for the proof and for the standard theory of dynamics in one complex variable.

Proposition 3.4 (Formal classification in $(\mathbb{C},0)$). *Let $f\colon(\mathbb{C},0)\to(\mathbb{C},0)$ be a holomorphic germ, and denote by $\lambda=f'(0)$ the multiplier.*

(i) *If* $\lambda = 0$, *then* f *is formally conjugated to* $z \mapsto z^p$ *for a suitable* $p \geq 2$.
(ii) *If* $\lambda \neq 0$ *and if* $\lambda^r \neq 1$ *for any* $r \in \mathbb{N}^*$, *then* f *is formally conjugated to* $z \mapsto \lambda z$.
(iii) *If* $\lambda^r = 1$, *then there exist* (*unique*) $s \in r\mathbb{N}^*$ *and* $\beta \in \mathbb{C}$ *such that* f *is formally conjugated to* $z \mapsto z(1 + z^s + \beta z^{2s})$.

REMARK 3.5. Suppose $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a semi-superattracting holomorphic germ. Then by Remark 2.5 we have two invariant curves—C and D, with transverse intersection and multiplicity equal to 1—that play the role of the unstable/stable manifold. In particular, the formal conjugacy classes of $f|_C$ and $f|_D$ are formal invariants.

Moreover, up to formal conjugacy, we can suppose that $C = \{w = 0\}$ and $D = \{z = 0\}$. Set $f = (f_1, f_2)$. Then, up to a formal change of coordinates, we can suppose that $f_1(z, 0)$ is equal to one of the formal normal forms given by Proposition 3.4.

Indeed, if $\phi \in \mathbb{C}[[z]]$ is the formal conjugation between $f_1(z, 0)$ and its formal conjugacy class $h(z)$, then the formal map $\Phi(z, w) = (\phi(z), w)$ is a conjugation between f and a map g with $g_1(\cdot, 0) = h(\cdot)$. We shall refer to the normal form h of a germ f as the *first* (*formal*) *action* of f.

LEMMA 3.6. *Let* $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *be a semi-superattracting holomorphic germ. Then, up to formal conjugacy, we can suppose that*

$$f(z, w) = (h(z), g(z, w)),$$

where h *is the first action of* f *and* $g \in \mathfrak{m}^2$.

Proof. We can suppose that f is of the form

$$f(z, w) = \big(\lambda z(1 + f_1(z, w)), g_2(z, w)\big),$$

where $f_1 \in \mathfrak{m}$ and $w | g_2 \in \mathfrak{m}^2$. We want to find a conjugation map of the form $\Phi(z, w) = (z(1 + \phi(z, w)), w)$ that conjugates f with

$$e(z, w) = \big(\lambda z(1 + e_1(z)), e_2(z, w)\big),$$

where $w | e_2 \in \mathfrak{m}^2$ and $\lambda z(1 + e_1(z)) = h(z)$.

Let us set $1 + f_1(z, w) = \sum_{i+j \geq 0} f_{i,j} z^i w^j$, $g_2(z, w) = \sum_{i+j \geq 2} g_{i,j} z^i w^j$, $1 + \phi(z, w) = \sum_{i+j \geq 0} \phi_{i,j} z^i w^j$, and $1 + e_1(z) = \sum_{i \geq 0} e_i z^i$. Then, for the first coordinate of the conjugacy equation $\Phi \circ f = e \circ \Phi$, we have:

$$\sum_{i+j \geq 0} \phi_{i,j} \lambda^{i+1} z^{i+1} \sum_{I, J \in \mathbb{N}^{i+1}} f_{I,J} z^{|I|} w^{|J|} \sum_{H, K \in \mathbb{N}^j} g_{H,K} z^{|H|} w^{|K|} \tag{8}$$

$$|| \tag{9}$$

$$\lambda \sum_h e_h z^{h+1} \sum_{N, M \in \mathbb{N}^{h+1}} \phi_{N,M} z^{|N|} w^{|M|}. \tag{10}$$

If we denote by $\mathbb{I}_{n,m}$ and $\mathbb{III}_{n,m}$ the coefficients of $z^n w^m$ in (8) and (10), respectively, then

$$\mathbb{I}_{n,m} = \sum_{\substack{i,j;I,J(i+1);H,K(j)\\ i+1+|I|+|H|=n\\ |J|+|K|=m}} \phi_{i,j}\lambda^{i+1} f_{I,J} g_{H,K} \quad \text{and} \quad \mathbb{III}_{n,m} = \sum_{\substack{h;N,M(h+1)\\ h+1+|N|=n\\ |M|=m}} \lambda e_h \phi_{N,M}.$$

If we denote by "l.o.t." all terms depending on $\phi_{i,j}$ for (i, j) lower than the ones that are compared in the equation (with respect to the lexicographic order), then

$$\delta_m^0 \phi_{n-1,0} \lambda^n f_{0,0}^n + \text{l.o.t.} = \mathbb{I}_{n,m} = \mathbb{III}_{n,m} = \lambda e_0 \phi_{n-1,m} + \text{l.o.t.}$$

In particular: for $n = 0$ we have $0 = \mathbb{I}_{0,m} = \mathbb{III}_{0,m} = 0$ for every $m \in \mathbb{N}$, and for every $m \geq 1$ we have $\mathbb{I}_{n,m} = \text{l.o.t.}$ for every $n \in \mathbb{N}^*$. Since $\lambda e_0 = \lambda \neq 0$, we can use (9) to define recursively $\phi_{n,m}$ for every $m \geq 1$ once we have defined the base step for $m = 0$.

But the case $m = 0$ is exactly the same as the one considered in the formal classification of $\tilde{f}(z) = \lambda z(1 + f_1(z, 0))$ as a map in one complex variable. So again recalling Remark 3.5 and combining our results, we can define a formal map Φ that solves the conjugacy relation $\Phi \circ f = e \circ \Phi$. □

Proof of Theorem 0.8. By Lemma 3.6 and some simple considerations on rigid germs (see Remark 3.2), we can suppose that

$$f(z, w) = \big(h(z), z^c w^d (1 + g(z, w))\big)$$

for a suitable $g \in \mathfrak{m}$, where $h(z) = \lambda z(1 + \delta(z))$ is the first action of f.

We want to find a conjugation Ψ of the form $\Psi(z, w) = \big(z, w(1 + \psi(z, w))\big)$ and where, for a suitable ε, Ψ is between f and

$$e(z, w) = \big(h(z), z^c w^d (1 + \varepsilon(z))\big).$$

Toward this end, we set $\delta(z) = \sum_{i\geq 0} \delta_i z^i$, $g(z, w) = \sum_{i+j\geq 2} g_{i,j} z^i w^j$, $1+\phi(z, w) = \sum_{i+j\geq 0} \phi_{i,j} z^i w^j$, and $1+\varepsilon(z) = \sum_{i\geq 0} \varepsilon_i z^i$. Then, for the second coordinate of the conjugacy equation $\Psi \circ f = e \circ \Psi$, we have:

$$z^c w^d \sum_{i+j\geq 0} \psi_{i,j} \lambda^i z^i \sum_{L\in\mathbb{N}^i} \delta_L z^{|L|} \sum_{I,J\in\mathbb{N}^{j+1}} g_{I,J} z^{|I|} w^{|J|} \tag{11}$$

$$|| \tag{12}$$

$$z^c w^d \sum_h \varepsilon_h z^h \sum_{H,K\in\mathbb{N}^d} \psi_{H,K} z^{|H|} w^{|K|}. \tag{13}$$

Denoting by $\mathbb{I}_{n,m}$ and $\mathbb{III}_{n,m}$ the coefficients of $z^{c+n} w^{d+m}$ in (11) and (13), respectively, yields

$$\mathbb{I}_{n,m} = \sum_{\substack{i,j;L(i);I,J(j+1)\\ i+cj+|L|+|I|=n\\ dj+|J|=m}} \psi_{i,j} \lambda^i \delta_L g_{I,J} \quad \text{and} \quad \mathbb{III}_{n,m} = \sum_{\substack{h;H,K(d)\\ h+|H|=n\\ |K|=m}} \lambda \varepsilon_h \psi_{H,K}.$$

Then, for $(n, m) \neq (0, 0)$, we have

$$\delta_m^0 \psi_{n,0} \lambda^n + \text{l.o.t.} = \mathbb{I}_{n,m} = \mathbb{III}_{n,m} = d\psi_{n,m} + \text{l.o.t.}$$

Therefore, if $m > 0$ then we can use (12) to define recursively $\psi_{n,m}$; with $m = 0$ there may be some resonancy problems when $\lambda^n = d$, which is exactly the condition expressed in parts (ii) and (iii) of the theorem. In these cases, the dependence of $\mathbb{III}_{n,0}$ on ε_h yields

$$\mathbb{III}_{n,0} = \varepsilon_n + \text{l.o.t.};$$

here l.o.t. denotes the dependence on lower-order terms ε_h with $h < n$. So for each n that gives us a resonance, there exists an ε_n that satisfies $\mathbb{I}_{n,0} = \mathbb{III}_{n,0}$. Putting this all together and (eventually) performing a conjugacy by a linear map, we obtain the thesis. □

REMARK 3.7. In the statement of Theorem 0.8, f belongs to class 2 if and only if $d = 1$, to class 3 if and only if $c = 0$, and to class 5 otherwise.

REMARK 3.8. The composition $\alpha \circ f_\bullet$, where α is either skewness or thinness, is not affected by slightly changing the nonnull coefficients of a germ f (provided we keep these coefficients nonnull); what changes is the action of the differential $df_\bullet$ in suitable tangent spaces. Therefore, the difference between normal forms in the resonant case of Theorem 0.8 lies in the action of $df_\bullet$, which is not invariant (by change of coordinates) and exhibits complicated behavior.

REMARK 3.9. Let $\phi(z, w) = \sum \phi_{n,m} z^n w^m$ be a formal power series. Then ϕ is holomorphic (as a germ in 0) if and only if there exists an M such that

$$|\phi_{n,m}| \leq M\alpha^n\beta^m.$$

In particular, if ϕ is holomorphic then $\limsup_n \sqrt[n]{|\phi_{n,m}|} < \infty$ for every $m \in \mathbb{N}$, and the same holds if we exchange the roles of m and n.

Next we show that, for rigid germs of type $(0, \mathbb{C} \setminus \bar{\mathbb{D}})$, one cannot generally perform the conjugacy of either Lemma 3.6 or Theorem 0.8 in a holomorphic way (this behavior is the opposite of the $(0, \mathbb{D})$ case).

COUNTEREXAMPLE 3.10. Here we demonstrate that the conjugation given by Lemma 3.6 cannot be always holomorphic. Let $f(z, w) = (\lambda z(1 + w), zw)$ with $|\lambda| > 1$ and $e(z, w) = (\lambda z, e_2)$, and let

$$\Phi(z, w) = \big(z(1 + \phi(z, w)), \psi(z, w)\big)$$

be the (formal) conjugation given by Lemma 3.6. Direct computation yields

$$\phi_{n,1} = \lambda^{n(n-1)/2},$$

and from Remark 3.9 it follows that ϕ is not holomorphic.

REMARK 3.11. Suppose we have a germ $f = (\lambda z, f_2)$, with $|\lambda| > 1$, that is formally conjugated to $(\lambda z, z^c w^d)$ (i.e., we are not in the resonance case). The proof of Theorem 0.8 shows also that the conjugation with the normal forms is unique when it is of the form $\Psi(z, w) = (z, w(1+\psi))$. But if we consider a general conjugation map $\Phi = (\phi_1, \phi_2)$ then—since we have two invariant curves $D = \{z = 0\}$

and $C = \{w = 0\}$—we have $z|\phi_1$ and $w|\phi_2$; because the first coordinate is λz, by direct computation we also have that $\phi_1(z, w) = z$. So Φ is unique up to a linear change of coordinates. Hence, in order to prove that two germs are formally but not holomorphically conjugated, we need only show that the conjugation found in the proof of Theorem 0.8 is not holomorphic.

COUNTEREXAMPLE 3.12. Here we show that also the conjugation given by Theorem 0.8 cannot be always holomorphic. Let $f(z, w) = (\lambda z, zw(1 + w))$ with $|\lambda| > 1$ and $e(z, w) = (\lambda z, zw)$, and let

$$\Psi(z, w) = \bigl(z, w(1 + \psi(z, w))\bigr)$$

be the (formal) conjugation given by Theorem 0.8. By direct computation we have

$$\psi_{n,1} = \lambda^{n(n-1)/2},$$

so again ψ is not holomorphic.

4. Normal Forms

4.1. Nilpotent Case

Favre and Jonsson studied the superattracting case (see [FJ2, Thm. 5.1]). The nilpotent case is almost the same; in fact, there is just one little difference between them. To explain this difference, we first prove the following lemma.

LEMMA 4.1. *Let f be a (dominant) holomorphic germ, where df_0 is noninvertible, $\nu_\star$ is an eigenvaluation for f, and $(\pi, p, \hat{f})$ is a rigidification obtained from $\nu_\star$ as in Theorem 0.6. Assume that $\nu_\star$ is not a divisorial valuation. Then $c_\infty(\hat{f}) = c_\infty(f)$.*

Proof. Directly from the definition of $\hat{f}$ as a lift of f, we have $\pi \circ \hat{f} = f \circ \pi$. Let $\mu_\star = \pi_\bullet^{-1}(\nu_\star)$ (in this case, $\mu_\star$ is an eigenvaluation for $\hat{f}$). Then

$$\begin{array}{c} c(\pi \circ \hat{f}, \mu_\star) = c(\hat{f}, \mu_\star) \cdot c(\pi, \hat{f}_\bullet \mu_\star) = c(\hat{f}, \mu_\star) \cdot c(\pi, \mu_\star) \\ \| \\ c(f \circ \pi, \mu_\star) = c(\pi, \mu_\star) \cdot c(f, \pi_\bullet \mu_\star) = c(\pi, \mu_\star) \cdot c(f, \nu_\star). \end{array}$$

From Theorem 1.16 we have $c(f, \nu_\star) = c_\infty(f)$ and $c(\hat{f}, \mu_\star) = c_\infty(\hat{f})$, so if $c(\pi, \mu_\star) < \infty$ then $c_\infty(f) = c_\infty(\hat{f})$. But $c(\pi, \mu_\star) = \infty$ if and only if $\mu_\star \in \partial U(p)$; following the proof of Theorem 0.6, this (always) happens if and only if $\nu_\star$ is a divisorial valuation. □

REMARK 4.2. The unique difference between the superattracting case and the nilpotent case is that $c_\infty(f) \geq 2$ in the former whereas $c_\infty(f) \geq \sqrt{2}$ in the latter. Moreover, from Lemma 4.1 it follows that, if the eigenvaluation $\nu_\star$ is not divisorial, then $c_\infty(f) = c_\infty(\hat{f})$ for the lift $\hat{f}$. Therefore, to obtain the result for the nilpotent case, we need only ignore the hypothesis $c_\infty(\hat{f}) \geq 2$ (when $\nu_\star$ is not divisorial).

4.1.1. *Germs of Type* $(0, \mathbb{D}^*)$

PROPOSITION 4.3. *Let f be a (dominant) holomorphic germ of type $(0, \mathbb{D}^*)$, let $\nu_\star$ be an eigenvaluation for f, and let $(\pi, p, \hat{f})$ be a rigidification obtained from $\nu_\star$ as in Theorem 0.6. Let $\lambda \in \mathbb{D}^*$ be the nonzero eigenvalue of df_0. Then $\nu_\star$ can be only a (formal) curve valuation, and the following statements hold.*

(i) *If $\nu_\star$ is a (noncontracted) analytic curve valuation, then $\hat{f} \cong (\lambda z, z^c w^d)$ with $c \geq 1$ and $d \geq 1$.*

(ii) *If $\nu_\star$ is a nonanalytic curve valuation, then $\hat{f} \cong (\lambda z, z^q w + P(z))$ with $q \geq 1$ and $P \in z\mathbb{C}[z]$ with $\deg P \leq q$ and $P \not\equiv 0$.*

Proof. The first assertion follows from Theorem 0.7.

(i) If $\nu_\star = \nu_C$ is a (noncontracted) analytic curve valuation, then directly from [FJ2, Thm. 5.1] we have that $\mathcal{C}^\infty(\hat{f}) = E \cup \tilde{C}$ or $\mathcal{C}^\infty(\hat{f}) = E$; in both cases, E is contracted and $\tilde{C}$ is fixed by $\hat{f}$. We also know from Proposition 2.3 that $\operatorname{tr} d\hat{f}_p = \lambda \neq 0$.

In the first case, $\mathcal{C}^\infty(\hat{f}) = E \cup \tilde{C}$ is reducible and so $\hat{f}$ is of class 5. Hence we can choose local coordinates (z, w) in p such that $E = \{z = 0\}$, $\tilde{C} = \{w = 0\}$, and

$$\hat{f}(z, w) = (\lambda z, z^c w^d)$$

with $c \geq 1$ and $d \geq 2$.

In the second case, $\mathcal{C}^\infty(f) = E$ is irreducible and so $\hat{f}$ is of class 2 or 3. However, since E is contracted to 0 by $\hat{f}$, it follows that $\hat{f}$ is of class 2. Hence we can choose local coordinates (z, w) such that $E = \{z = 0\}$, $\tilde{C} = \{w = 0\}$, and

$$\hat{f}(z, w) = (\lambda z, z^q w + P(z))$$

with $q \geq 1$. Since $\tilde{C}$ is fixed, we have $P \equiv 0$.

(ii) Suppose now that $\nu_\star = \nu_C$ is a nonanalytic curve valuation. We showed in the proof of Theorem 0.6 that $\mathcal{C}^\infty(\hat{f}) = E$ and that E is contracted to 0 by $\hat{f}$. We also know from Proposition 2.3 that $\operatorname{tr} d\hat{f}_p = \lambda \neq 0$, so $\hat{f}$ is of class 2 or 3. But only for maps in class 2 does $\hat{f}$ contract the component E in $\mathcal{C}^\infty(\hat{f})$. Hence we are in class 2, so we can choose local coordinates (z, w) at p such that $E = \{z = 0\}$ and such that

$$\hat{f}(z, w) = (\lambda z, z^q w + P(z))$$

with $q \geq 1$; here $P \in z\mathbb{C}[z]$ with $\deg P \leq q$. Since $f_\bullet^n \to \nu_\star$ in $U(p)$, no analytic curve valuation (besides ν_z) is fixed by $\hat{f}$ and thus $P \not\equiv 0$. □

4.1.2. *Germs of Type* $(0, \mathbb{C} \setminus \bar{\mathbb{D}})$

PROPOSITION 4.4. *Let f be a (dominant) holomorphic germ of type $(0, \mathbb{C} \setminus \bar{\mathbb{D}})$, let $\nu_\star$ be an eigenvaluation for f, and let $(\pi, p, \hat{f})$ be a rigidification obtained from $\nu_\star$ as in Theorem 0.6. Let $\lambda \in \mathbb{C} \setminus \bar{\mathbb{D}}$ be the nonzero eigenvalue of df_0. Then $\nu_\star$ can be only an analytic curve valuation and $\hat{f} \overset{\text{for}}{\cong} (\lambda z, z^c w^d(1 + \varepsilon z^l))$, where $\overset{\text{for}}{\cong}$ denotes "is formally conjugated to" and where $c \geq 1$, $d \geq 1$, $l \geq 1$, and $\varepsilon = 0$ if $\lambda^l \neq d$ or $\varepsilon \in \{0, 1\}$ if $\lambda^l = d$.*

Proof. According to Theorem 0.7, $\nu_\star$ must be a (formal) curve valuation.

Let us suppose $\nu_\star = \nu_C$ is a nonanalytic curve valuation. From the proofs of Theorem 0.6 and Proposition 1.20(i) we know that $f_\bullet^n \to \nu_C$ on a suitable open set $U = U(p)$ and hence $\hat{f}_\bullet^n \to \nu_{\tilde{C}}$ on $\mathcal{V} \setminus \nu_E$, where $\tilde{C}$ is the strict transform of C (and is also nonanalytic). Note that ν_E is an analytic curve valuation when considered on the valuative tree where $\hat{f}_\bullet$ acts. In particular, E is the only analytic curve fixed by $\hat{f}$—in contradiction with the stable/unstable manifold theorem [A, Thms. 3.1.2 and 3.1.3]—because we know from Proposition 2.3 that $\operatorname{Spec}(d\hat{f}_p) = \{0, \lambda\}$ and $|\lambda| > 1$. Hence $\nu_\star = \nu_C$ is a (noncontracted) analytic curve valuation.

The assertion on normal forms now follows from Theorem 0.8. □

4.1.3. *Germs of Type* $(0, \partial\mathbb{D})$

Proposition 4.5. *Let f be a (dominant) holomorphic germ of type $(0, \partial\mathbb{D})$, let $\nu_\star$ be an eigenvaluation for f, and let $(\pi, p, \hat{f})$ be a rigidification obtained from $\nu_\star$ as in Theorem 0.6. Let $\lambda \in \partial\mathbb{D}$ be the nonzero eigenvalue of df_0. Then $\nu_\star$ can be only a (formal) curve valuation, and the following statements hold.*

(i) *If λ is not a root of unity, then* $\hat{f} \overset{\text{for}}{\cong} (\lambda z, z^c w^d)$ *with* $c, d \geq 1$.

(ii) *If $\lambda^r = 1$ is a root of unity, then* $\hat{f} \overset{\text{for}}{\cong} (\lambda z(1 + z^s + \beta z^{2s}), z^c w^d(1 + \varepsilon(z^r)))$; *here $c, d \geq 1$, $r | s$, $\beta \in \mathbb{C}$, and ε is a formal power series in z^r or $\varepsilon \equiv 0$ if $d \geq 2$.*

Proof. The first assertion follows from Theorem 0.7, and the normal forms are given by Theorem 0.8. □

4.2. Some Remarks and Examples

Remark 4.6. The proof of Theorem 0.6 gives a general procedure for obtaining a rigid germ. But in specific instances we can choose an infinitely near point lower that the one indicated. In particular, if $\nu_\star$ is divisorial then $U = U(p)$ may be associated to a free point p and not to a satellite one. In this case we obtain an irreducible rigid germ of class 2 or 3, and it must be of class 3 because the generalized critical set E is fixed by $\hat{f}$. So, for example, if $\hat{f}$ is still attracting then $\hat{f} \cong (z^p, \alpha w)$, where $p \geq 2$ and $0 < |\alpha| < 1$ (with $\alpha = \lambda$ if $df_0 = D_\lambda$).

Example 4.7. We present an example of the phenomenon described in Remark 4.6. Set

$$f(z, w) = (z^n + w^n, w^n)$$

with $n \geq 2$ an integer. We easily see that $\nu_\mathfrak{m}$ is an eigenvaluation for f. We want to study the action of $\hat{f}$ on the exceptional component $E = E_0$ that arises from the single blow-up of the origin. We do this by checking the action of $f_\bullet$ on $\mathcal{E} := \{\nu_{y-\theta x} \mid \theta \in \mathbb{C}\} \cup \{\nu_x\}$, where we fix the correspondence $\theta \mapsto \nu_{y-\theta x}$ between $E \cong \mathbb{P}^1(\mathbb{C})$ and $\mathcal{E}$ (setting $\infty \mapsto \nu_x$). Direct computations show that

$$\hat{f}|_E : \theta \mapsto \frac{\theta^n}{1 + \theta^n}.$$

Now set $p = \theta \in E$ such that θ is a noncritical fixed point for $\hat{f}|_E$ (i.e., such that $\theta^n + 1 = \theta^{n-1}$), and lift f to a holomorphic germ $\hat{f}$ on the infinitely near point p. Using the same arguments as in the proof of Theorem 0.6, we can tell that $\hat{f}$ is a rigid germ.

We demonstrate this claim by direct computations. Let us make a blow-up in $0 \in \mathbb{C}^2$:

$$\begin{cases} z = u, \\ w = ut; \end{cases} \qquad \begin{cases} u = z, \\ t = w/z. \end{cases}$$

Then

$$\hat{f}(u,t) = \left(u^n(1+t^n), \frac{t^n}{1+t^n}\right).$$

Choosing local coordinates $(u, v := t - \theta)$, we obtain

$$\hat{f}(u,v) = (u^n(1+(v+\theta)^n), v\xi(v))$$

for a suitable invertible germ ξ. In particular, $\hat{f}$ is a rigid germ, belongs to class 3, and (by direct computation) is locally holomorphically conjugated to $(u,v) \mapsto (u^n, \alpha v)$ for a suitable $\alpha \neq 0$; in contrast, [FJ2, Thm. 5.1] would give us a germ that belongs to class 5. In this case we recover the result of [FJ2, Thm. 5.1] simply by taking the lift of $g = \hat{f}$ when we blow up the point $[0:1] \in E$; thus,

$$\hat{g}(x,y) = (x^n y^{n-1} \chi(y), v\xi(v))$$

for a suitable invertible germ χ, which is locally holomorphically conjugated to $(x^n y^{n-1}, \alpha y)$.

REMARK 4.8. We can apply [FJ2, Thm. 5.1] as well as Propositions 4.3–4.5 even when f is itself rigid, and this allows us to avoid some types of rigid germs. First of all, from the proof of [FJ2, Thm. 5.1] (and recalling Proposition 2.3), one can see that class 7 can always be avoided (hence class 7 is not "stable under blow-ups"). Moreover, from the proof of Theorem 0.6 we see that the germs obtained after lifting are such that $\hat{f}_\bullet$ always has only one fixed point $\mu_\star = \pi_\bullet^{-1}\nu_\star$ of the same type of $\nu_\star$, with two exceptions: either $\nu_\star$ is divisorial, and $\mu_\star$ turns out to be an analytic curve valuation (contracted by π); or $\nu_\star$ is an irrational eigenvaluation, in which case it may be that $\hat{f}_\bullet = \mathrm{id}$ on $[\nu_z, \nu_w]$.

In the first case, reapplying Propositions 4.3–4.5 yields the same type of germ. In the second case, up to local holomorphic conjugacy we have that $\hat{f}(z,w) = (z^n, w^n)$ with a suitable $n \geq 2$. Then all valuations on $[\nu_z, \nu_w]$ are eigenvaluations, and reapplying [FJ2, Thm. 5.1] gives a rigid germ that belongs to a different class. In particular, making a single blow-up on the origin and considering the germ at $[1:1]$, we obtain a germ of the form $(nz(1+h(z)), w^n)$—for a suitable holomorphic map h such that $h(0) = 0$—that is (by direct computation) holomorphically conjugated to (nz, w^n).

EXAMPLE 4.9. When reapplying [FJ2, Thm. 5.1], as in the previous remark, we usually obtain the same normal form type. But there are cases where the normal form can change (staying rigid).

Consider for instance the rigid germ $f(z,w) = (w^2, z^3)$. Then the only eigenvaluation $\nu_\star$ is the monomial valuation on (z,w) such that $\nu_\star(x) = 1$ and $\nu_\star(w) = \sqrt{3/2}$. Hence an infinitely near point p that works in [FJ2, Thm. 5.1] can be obtained after three blow-ups: the first at 0 (where we obtain E_0), the second at $[1:0] \in E_0$ (where we obtain E_1), and the third at $[0:1]$ (where we obtain E_2). We can choose $p = [0,1] \in E_2$, and the lift we then get is $\hat{f}(z,w) = (w^6, z)$.

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