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Geodesic Continued Fractions

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1. Introduction

Every rational number can be expressed uniquely in the form p/q, where p and q are coprime integers and q is positive; we describe such rationals as *reduced*. Two reduced rationals p/q and r/s are *Farey neighbors* if $|ps - qr| = 1$. As usual, we adjoin the point ∞ to the set $\mathbb{Q}$ of rationals to form $\mathbb{Q}_\infty$. We then define $1/0$ to be the reduced form of ∞, and p/q to be a Farey neighbor of ∞ if and only if $|p.0 - q.1| = 1$ (i.e., if and only if p/q is an integer). The *Farey graph* $\mathcal{F}$ is the graph whose set of vertices is $\mathbb{Q}_\infty$ and whose edges join each pair of Farey neighbors (and only these). We denote the path in $\mathcal{F}$ that passes through the vertices $v_1, \dots, v_n$ in this order by $\langle v_1, \dots, v_n \rangle$. A concrete realization of $\mathcal{F}$ is obtained by joining each pair of Farey neighbors by a hyperbolic line in the upper half-plane model $\mathbb{H}$ of the hyperbolic plane. It is well known that any two such hyperbolic lines have at most an endpoint in common, and this set of hyperbolic lines induces the *Farey tessellation* of $\mathbb{H}$ into mutually disjoint, nonoverlapping, ideal hyperbolic triangles (see e.g. [7; 8; 15]). Henceforth $\mathcal{F}$ refers to this model of the Farey graph, which is illustrated in Figure 1.

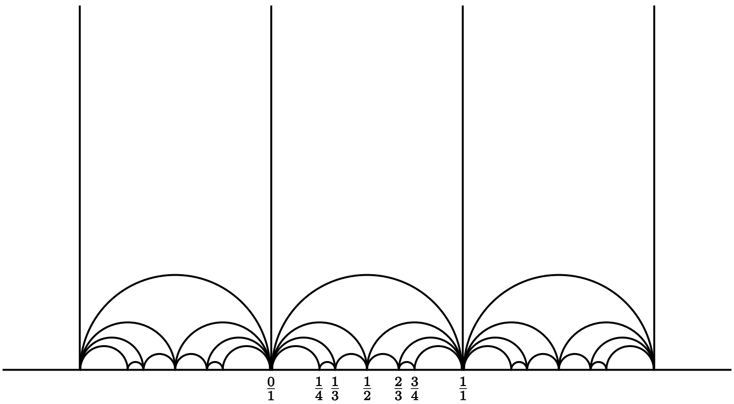

Figure 1 The Farey graph

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Let

$$[b_1, \dots, b_n] = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots + \cfrac{1}{b_n}}}}. \tag{1.1}$$

An *integer continued fraction expansion* (or, for brevity, an ICF expansion) of a rational number x is an expansion $x = [b_1, \dots, b_n]$ in which all b_i are integers (but not necessarily positive). The *convergents* of an ICF-expansion of x, namely $[b_1, \dots, b_i]$ for $i = 1, \dots, n$, form a finite sequence $C_1, \dots, C_n$ of vertices of $\mathcal{F}$, where C_1 is an integer and $C_n = x$. We shall see that if we express C_i as an irreducible rational A_i/B_i then $|A_i B_{i+1} - B_i A_{i+1}| = 1$, so that C_i and C_{i+1} are Farey neighbors, and this implies that $\langle \infty, C_1, \dots, C_n \rangle$ is a path from ∞ to x in $\mathcal{F}$. The converse is also true; that is, any path in $\mathcal{F}$ from ∞ to x is the sequence of convergents of some ICF expansion of x (see Theorem 3.1). Since the usual continued fraction expansion (with $b_2, b_3, \dots$ positive) of a rational x provides a path from ∞ to x, we see that $\mathcal{F}$ is connected. Thus there is a "natural" distance ρ on $\mathbb{Q}_\infty$, where $\rho(u, v)$ is the least number of edges that must be traversed to move from one vertex u to another vertex v. In particular, the shortest ICF expansion of x has length $\rho(\infty, x)$. These remarks allow us to discuss ICF expansions of x as paths in $\mathcal{F}$ from ∞ to x and also the shortest ICF expansions of x as geodesic paths in $\mathcal{F}$ from ∞ to x; we shall call these shortest expansions the *geodesic expansions* of x.

Our first result gives an explicit algorithm for constructing a geodesic path in $\mathcal{F}$ from a rational x to ∞. To describe this, we construct the *first parent map* $\alpha: \mathbb{Q} \to \mathbb{Q}_\infty$, where (with a few exceptions) $\alpha(x)$ is the Farey neighbor of x with the smallest denominator. This denominator is less than the denominator of x, so if we iterate α then we obtain a path, which we call the *ancestral path,* in $\mathcal{F}$ that joins x to ∞.

Theorem 1.1. *The ancestral path from x to ∞ is a geodesic path in $\mathcal{F}$.*

In terms of continued fractions, this gives an algorithm for providing a geodesic expansion of a given rational x. However, in general there is more than one geodesic path from ∞ to x; for example, $\langle \infty, 0, \frac{1}{2} \rangle$ and $\langle \infty, 1, \frac{1}{2} \rangle$ are different geodesic paths joining ∞ to $\frac{1}{2}$ (and these correspond to the continued fractions $[0, 2]$ and $[1, -2]$, respectively). Our next result bounds the number of geodesic paths from ∞ to x (equivalently, the number of geodesic expansions of x). As usual, F_n denotes the nth Fibonacci number; thus $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, and so on.

Theorem 1.2. *Suppose that x is rational and that $\rho(\infty, x) = n$. Then there are at most F_n geodesics from ∞ to x and, for each n, this bound is best possible.*

Finally, we characterize those ICF expansions that are geodesic expansions, and those rationals that have a unique geodesic expansion, and for this we need some terminology. A finite sequence is said to be *inefficient* if it is of the form $2\varepsilon_1, 3\varepsilon_2, 3\varepsilon_3, \dots, 3\varepsilon_{m-1}, 2\varepsilon_m$, where $m \geq 2$ and $\varepsilon_1, \varepsilon_2, \dots$ is an alternating sequence in $\{-1, 1\}$ (i.e., $\varepsilon_1 = \pm 1$ and $\varepsilon_{i+1} = -\varepsilon_i$). For example, the sequences $2, -2$ and $2, -3, 2$ and $-2, 3, -3, 3, -3, 2$ are inefficient; later we give a geometric interpretation of inefficient sequences.

THEOREM 1.3. *The ICF expansion* $[b_1, \dots, b_n]$ *is a geodesic expansion if and only if*:

(i) $|b_i| \geq 2$ *for* $i = 2, \dots, n$ *and*

(ii) $b_2, \dots, b_n$ *does not contain an inefficient subsequence of consecutive* b_i.

Furthermore, a geodesic expansion $[b_1, \dots, b_n]$ *of a rational* x *is the only geodesic expansion of* x *if and only if* $|b_i| \geq 3$ *for* $i = 2, \dots, n$.

We discuss infinite ICF expansions in Section 10.

In [4], Ford constructed a bijection between $\mathbb{Q}_\infty$ and the collection of what are now known as *Ford circles.* In a certain sense, the Farey graph $\mathcal{F}$ is the dual of the collection of Ford circles. Ford showed that there is a bijective correspondence between the ICF expansions of a rational x and particular finite chains of Ford circles, where each circle in the chain is tangent to its neighbors. This is equivalent to our Theorem 3.1, but it is easier to prove from the Farey graph than from Ford circles. Some of the other ideas we use also occur in the literature but not, as far as we know, to study geodesic expansions; see, for example, [6; 8; 9; 10; 11; 15]. In [10; 11] the authors construct a *Farey tree,* but we do not discuss this tree here because paths in it do not generally correspond to geodesic expansions. From a more algebraic perspective, the usual continued fraction expansion of x is obtained from the division algorithm. We can also obtain ICF expansions from the nearest integer algorithm, and some authors study semi-regular continued fractions (in which the numerators are ± 1 and the coefficients are subject to certain inequalities); see, for example, [5; 13; 16]. In particular, Srinivasan studies shortest semi-regular continued fractions in [16] and uses algebraic means to obtain a result similar to the first part of Theorem 1.3. Our main objective is to give a coherent account of all ICF expansions, especially the geodesic expansions, from the perspective of graph theory and without imposing any of the constraints that are sometimes found in the literature.

2. Basic Ideas

Because the coefficients b_i in (1.1) can be negative, it is not always possible to evaluate $[b_1, \dots, b_n]$ via the usual rules of arithmetic (since division by zero may be required). For this reason, we redefine $[b_1, \dots, b_n]$ by

$$[b_1, \dots, b_n] = S_{b_1} \cdots S_{b_n}(\infty),$$

where S_w is the Möbius transformation defined by $S_w(z) = w + 1/z$ for each complex number w. This definition is consistent with (1.1), and it is used throughout the theory of complex continued fractions (see [1; 12]).

The *convergents* of $[b_1, \dots, b_n]$ are $C_1, \dots, C_n$, where $C_k = [b_1, \dots, b_k]$; since

$$C_k = S_{b_1} \cdots S_{b_k}(\infty) = S_{b_1} \cdots S_{b_{k-1}}(b_k),$$

they determine the b_i inductively by $b_1 = C_1$ and $b_k = (S_{b_1} \cdots S_{b_{k-1}})^{-1}(C_k)$. The usual numerical evaluation of $[b_1, \dots, b_n]$ requires us first to evaluate $z_1 = b_n$, then $z_2 = b_{n-1} + 1/z_1$, then $z_3 = b_{n-2} + 1/z_2$, and so on. In other words, we calculate $S_{b_n}(\infty)$, then $S_{b_{n-1}} S_{b_n}(\infty)$, then $S_{b_{n-2}} S_{b_{n-1}} S_{b_n}(\infty)$, and so forth, and numerical evaluation will fail (because of division by zero) precisely when for some k we have $S_{b_k} \cdots S_{b_n}(\infty) = \infty$ or, equivalently, $S_{b_1} \cdots S_{b_n}(\infty) = S_{b_1} \cdots S_{b_{k-1}}(\infty)$. Thus numerical evaluation fails if and only if some intermediate convergent is the value of the continued fraction or, equivalently, if the corresponding path in $\mathcal{F}$ passes through the final vertex before the path ends.

Given an ICF expansion $[b_1, \dots, b_n]$, we define sequences $A_0, A_1, \dots, A_n$ and $B_0, B_1, \dots, B_n$ of integers by $A_0 = 1$, $B_0 = 0$, and

$$\begin{pmatrix} A_i & A_{i-1} \\ B_i & B_{i-1} \end{pmatrix} = \begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_i & 1 \\ 1 & 0 \end{pmatrix}, \quad i = 1, \dots, n.$$

These matrices correspond to the Möbius maps S_{b_i}, so we see that $S_{b_1} \cdots S_{b_i}(\infty) = A_i/B_i$. Moreover, taking determinants of both sides of this equation yields $|A_i B_{i-1} - A_{i-1} B_i| = 1$. This means that A_i and B_i are coprime and that A_i/B_i and A_{i-1}/B_{i-1} are Farey neighbors.

Finally, we shall write $u \sim v$ if and only if vertices u and v of $\mathcal{F}$ are Farey neighbors. One can easily check that if $u \sim v$ then $S_b(u) \sim S_b(v)$ and $S_b^{-1}(u) \sim S_b^{-1}(v)$. The underlying fact here is that the S_b, $b \in \mathbb{Z}$, lie in (and, in fact, generate) the extended Modular group, which is the automorphism group of $\mathcal{F}$.

3. Paths and Continued Fractions

We shall now establish the promised correspondence between paths from ∞ and continued fractions. (This result is proved in [4, Sec. 6] using Ford circles.)

THEOREM 3.1. *Let x be any rational number. Then $C_1, \dots, C_n$, with $C_n = x$, are the consecutive convergents of some ICF expansion of x if and only if $\langle \infty, C_1, \dots, C_n \rangle$ is a path in $\mathcal{F}$ from ∞ to x.*

Proof. Suppose first that $C_1, \dots, C_n$ are the consecutive convergents of the ICF expansion $[b_1, \dots, b_n]$ of x; thus $C_k = S_{b_1} \cdots S_{b_k}(\infty)$ and $C_n = x$. Now $C_1 = S_{b_1}(\infty) = b_1$ (an integer), so $\langle \infty, C_1 \rangle$ is an edge in $\mathcal{F}$. Next,

$$C_k = S_{b_1} \cdots S_{b_k}(\infty) \quad \text{and} \quad C_{k+1} = S_{b_1} \cdots S_{b_k} S_{b_{k+1}}(\infty) = S_{b_1} \cdots S_{b_k}(b_{k+1}),$$

and since ∞ and b_{k+1} are Farey neighbors, so are C_k and C_{k+1}. Hence there is an edge in $\mathcal{F}$ from C_k to C_{k+1}, which means that $\langle \infty, C_1, \dots, C_n \rangle$ is a path from ∞ to x.

Now suppose that $\langle \infty, v_1, \dots, v_n \rangle$ is a path in $\mathcal{F}$ from ∞ to x (so that $v_n = x$). We need to construct a sequence $b_1, \dots, b_n$ of integers such that $v_k = S_{b_1} \cdots S_{b_k}(\infty)$ for $k = 1, \dots, n$, for then the v_k are the convergents of $[b_1, \dots, b_n]$, which equals x.

First, as $v_1 \sim \infty$, we see that v_1 is an integer. Thus we put $b_1 = v_1$, so that $v_1 = S_{b_1}(\infty)$. Now suppose that integers $b_1, \dots, b_k$ have been defined such that $v_k = S_{b_1} \cdots S_{b_k}(\infty)$. Define the real number b_{k+1} by $b_{k+1} = S_{b_k}^{-1} \cdots S_{b_1}^{-1}(v_{k+1})$. This certainly guarantees that $v_{k+1} = S_{b_1} \cdots S_{b_{k+1}}(\infty)$, so it only remains to show that b_{k+1} is an integer. Now $v_k \sim v_{k+1}$, so that $S_{b_1} \cdots S_{b_k}(\infty) \sim S_{b_1} \cdots S_{b_k}(b_{k+1})$. Since each $S_{b_i}^{-1}$ preserves the relation $\sim$, we see that $\infty \sim b_{k+1}$ and so b_{k+1} is an integer. This induction argument completes the proof. $\square$

4. Farey Parents

To facilitate our study of geodesics in $\mathcal{F}$, we introduce a lexicographic order $\prec$ on $\mathbb{Q}_\infty$ that orders the reduced rationals first by their nonnegative denominator and then by their numerator. Formally, we define $\prec$ on $\mathbb{Q}_\infty$ (regarded as the set of reduced rationals) by $p/q \prec r/s$ if either (i) $q < s$ or (ii) $q = s$ and $p < r$. For x in $\mathbb{Q}_\infty$, let $\mathcal{N}(x)$ be the set of Farey neighbors of x. By definition, $\mathcal{N}(\infty) = \mathbb{Z}$. Each finite rational x has at most two reduced rational Farey neighbors of a given denominator; therefore the set $\mathcal{N}(x)$ is well-ordered by $\prec$ in the sense that

$$\mathcal{N}(x) = \{y_1, y_2, y_3, \dots\}, \quad \text{where } \; y_1 \prec y_2 \prec y_3 \prec \cdots . \tag{4.1}$$

We call y_1 the *first parent,* and y_2 the *second parent,* of x (these are the *Farey parents* of x), and we define maps α and β by $\alpha(x) = y_1$ and $\beta(x) = y_2$. Note that ∞ has no parents and that $\alpha(x) = \infty$ if and only if $x \in \mathbb{Z}$. In the more geometric setting of [4], x is replaced by its Ford circle $\mathcal{C}_x$, and y is a Farey parent of x if and only if $\mathcal{C}_y$ is tangent to, and has larger radius than, $\mathcal{C}_x$. The idea of parents also appears in [11], where the phrases *old parent* and *young parent* are used instead of *first parent* and *second parent.*

In order to state our main result about Farey parents, we recall the *Farey sum* of two reduced rationals a/c and b/d:

$$\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}.$$

Observe that $a/c \oplus b/d$ lies between a/c and b/d and that the Farey sum of two reduced Farey neighbors is also reduced. Our next theorem is the key result about Farey parents (recall that $u \sim v$ means that u and v are Farey neighbors).

THEOREM 4.1. *Suppose that* $x \in \mathbb{Q}$. *Then*:

(i) $x = \alpha(x) \oplus \beta(x)$;
(ii) $\alpha(x) \sim \beta(x)$;
(iii) $\alpha(x) \prec \beta(x) \prec x \prec y$ *for every* y *in* $\mathcal{N}(x)$ *other than* $\alpha(x)$ *and* $\beta(x)$;
(iv) $\alpha(x)$ *is a parent of* $\beta(x)$.

We can illustrate Theorem 4.1 with the Fibonacci sequence F_n, where $F_{n+2} = F_{n+1} + F_n$, $F_1 = 1$, and $F_2 = 2$. The identity $F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}$ shows that F_n/F_{n+1} and F_{n+1}/F_{n+2} are reduced rationals and Farey neighbors. Since $(F_n/F_{n+1}) \oplus (F_{n+1}/F_{n+2}) = F_{n+2}/F_{n+3}$, we see that F_n/F_{n+1} and F_{n+1}/F_{n+2} are, respectively, the first and second parents of F_{n+2}/F_{n+3}.

Proof of Theorem 4.1. It is easy to check that (i)–(iv) hold when x is an integer, say $x = m$, since in that case $\alpha(x) = \infty$, $\beta(x) = m - 1$, and, in the notation of (4.1), $y_3 = m + 1$. Now suppose that x is the reduced rational a/c, where $c \geq 2$. Choose another reduced rational b/d such that $|ad - bc| = 1$, and let f be the element of the extended Modular group defined by $f(z) = (az + b)/(cz + d)$. Then, since f is an automorphism of $\mathcal{F}$,

$$\mathcal{N}\left(\frac{a}{c}\right) = \mathcal{N}(f(\infty)) = f(\mathcal{N}(\infty)) = f(\mathbb{Z}) = \left\{\frac{ak+b}{ck+d} : k \in \mathbb{Z}\right\}.$$

It is clear from this description of $\mathcal{N}(a/c)$ that we may now assume b/d to be chosen such that $0 < d < c$. Then

$$\{\alpha(x), \beta(x)\} = \left\{\frac{b}{d}, \frac{a-b}{c-d}\right\}, \quad \alpha(x) \neq \beta(x),$$

and it is a straightforward exercise to show that (i)–(iv) follow from this. □

5. Farey Parents and Geodesics

Our first result in this section connects Farey parents and geodesics.

THEOREM 5.1. *If $x \in \mathbb{Q}$, then each path from ∞ to x passes through a parent of x. If $\langle x_0, x_1, \dots, x_n \rangle$ (where $x_0 = \infty$) is a geodesic, then each x_{k-1} is a parent of x_k.*

Proof. The first assertion is trivially true if x is an integer, for then ∞ is a parent of x. So suppose that x is a nonintegral rational, and take a path $\langle x_0, x_1, \dots, x_n \rangle$ from ∞ to x; thus $n \geq 2$, $x_0 = \infty$, and $x_n = x$. Let u and v be the rational parents of x, where $u < v$. Then $x = u \oplus v$ and hence $x \in (u, v)$. Since $\infty \notin [u, v]$, there must be an integer i such that $x_i \notin (u, v)$ but $x_{i+1} \in (u, v)$. Suppose $x_i \neq u, v$. Then, in the concrete model of $\mathcal{F}$ in $\mathbb{H}$, the geodesic edge joining x_i to x_{i+1} intersects the geodesic edge joining u to v. This is impossible; therefore x_i is equal to either u or v, as required. The second assertion follows from the first because, for each vertex x_k, one of $x_1, \dots, x_{k-1}$, say x_t, is a parent of x_k and $\langle \infty, x_1, \dots, x_t, x_k, \dots, x_n \rangle$ is a path from ∞ to x. Because this cannot be shorter than the given geodesic path, we see that $t = k - 1$. □

We have created the *first parent map* $\alpha : \mathbb{Q} \to \mathbb{Q}_\infty$, which takes x to its first parent $\alpha(x)$. In particular, $\alpha(x) = \infty$ if and only if $x \in \mathbb{Z}$. If x is a nonintegral rational then, as $\alpha(x) \oplus \beta(x) = x$, and $\alpha(x)$ and $\beta(x)$ have strictly positive denominators, we see that $\alpha(x)$ has a strictly smaller denominator than x. Thus if we start with a nonintegral rational x and iterate the map α, we will eventually reach an integer and then reach ∞. Therefore, iteration of the map α creates a finite path from ∞ to x.

DEFINITION 5.2. Let x be a rational, and let m be the smallest integer with $\alpha^m(x) = \infty$. We call $\langle \alpha^m(x), \alpha^{m-1}(x), \dots, \alpha(x), x \rangle$ the *ancestral path* of x.

Theorem 1.1 asserts that an ancestral path is a geodesic, and we now prove this.

Proof of Theorem 1.1. For each rational x, let $m(x)$ be the smallest integer m such that $\alpha^m(x) = \infty$. Note that $m(\alpha(x)) = m(x) - 1$. Also, since the ancestral path joins x to ∞, we have $m(x) \geq \rho(\infty, x)$. Theorem 1.1 is clearly equivalent to the assertion that $m(x) = \rho(\infty, x)$, which we shall now prove by induction on $m(x)$.

First, $m(x) = 1$ if and only if $x \in \mathbb{Z}$ or, equivalently, if and only if $\rho(\infty, x) = 1$. Our induction hypothesis is that $\rho(\infty, y) = m(y)$ whenever $m(y) \leq k$, and we now consider any x with $m(x) = k + 1$. Let $\langle x_{r+1}, x_r, \ldots, x_1, x\rangle$, where $x_{r+1} = \infty$ and $r \geq 1$, be a geodesic path from ∞ to x. We need to show that $r = k$. By Theorem 5.1, x_1 is either $\alpha(x)$ or $\beta(x)$, and we consider each case in turn.

Suppose that $x_1 = \alpha(x)$. Because the given path is a geodesic, we see that $\rho(\infty, \alpha(x)) = r$. On the other hand, since $m(\alpha(x)) = m(x) - 1 = k$, the induction hypothesis implies that $\rho(\infty, \alpha(x)) = m(\alpha(x)) = k$ and so $r = k$ in this case. Now suppose that $x_1 = \beta(x)$. By Theorem 5.1, x_2 is a parent of $\beta(x)$ and $x_2 \neq \alpha(x)$ (else $\langle \infty, x_r, \ldots, x_2, x\rangle$ would be a shorter path from ∞ to x). Hence x_2 and $\alpha(x)$ are the two parents of x_1 and, by Theorem 4.1, $x_2 \sim \alpha(x)$. Thus $\langle \infty, x_r, \ldots, x_2, \alpha(x), x\rangle$ is also a geodesic path and so, by the $x_1 = \alpha(x)$ case, $r = k$. □

We prove one more result about paths and parents.

THEOREM 5.3. *Suppose that the continued fraction $[b_1, \ldots, b_n]$ corresponds to the path $\langle v_0, v_1, \ldots, v_n\rangle$ in $\mathcal{F}$, where $v_0 = \infty$.*

(i) *If $|b_i| \geq 2$ for $i = 2, \ldots, n$, then v_{i-1} is a parent of v_i for $i = 1, \ldots, n$.*
(ii) *If $|b_i| \geq 3$ for $i = 2, \ldots, n$, then $\langle v_0, v_1, \ldots, v_n\rangle$ is the ancestral path of v_n.*
(iii) *If $|b_i| \geq 3$ for some $i \geq 2$, then $\alpha(v_i) = v_{i-1}$ and $\beta(v_i) = [b_1, \ldots, b_{i-1}, b_i - b_i/|b_i|]$.*

Proof. Suppose first that $|b_i| \geq 2$ for $i = 2, \ldots, n$. Recall the sequences $A_0, A_1, \ldots$ and $B_0, B_1, \ldots$ from Section 2. We prove by induction that $|B_0| < |B_1| < |B_2| < \cdots < |B_n|$. First, $|B_0| = 0 < 1 = |B_1|$. Now suppose that $|B_{m-1}| < |B_m|$ for some $m \geq 1$. Since $B_{m+1} = B_m b_{m+1} + B_{m-1}$ we have

$$|B_{m+1}| \geq |B_m||b_{m+1}| - |B_{m-1}| \geq 2|B_m| - |B_{m-1}| > |B_m|,$$

which completes the inductive step. Since v_i is the irreducible rational A_i/B_i and since $|B_{i-1}| < |B_i|$, it follows that v_{i-1} is a parent of v_i for $i = 1, \ldots, n$; therefore (i) holds.

Clearly, (ii) follows from (iii), so we only have to prove (iii). Suppose then that $|b_i| \geq 3$ for some $i \geq 2$. Define $b'_i = b_i - b_i/|b_i|$, $A'_i = b'_i A_{i-1} + A_{i-2}$, $B'_i = b'_i B_{i-1} + B_{i-2}$, and $v'_i = A'_i/B'_i$. Since $|b'_i| \geq 2$, it follows from (i) that v_{i-1} is a parent of v'_i. Also, one can check that $v_{i-1} \oplus v'_i = v_i$. Hence, by Theorem 4.1, v_{i-1} and v'_i are, respectively, the first and second parents of v_i. □

6. Proof of Theorem 1.2

Theorem 1.2 asserts that (i) if $\rho(\infty, x) = n$, then there are at most F_n geodesics joining ∞ to x and (ii) for each n, there is some x with $\rho(\infty, x) = n$ and with exactly F_n geodesics from ∞ to x. We shall now prove this. For each rational x, let

$M(x)$ denote the number of geodesics from ∞ to x. We shall prove, by induction on n, that if $\rho(\infty, x) = n$ then $M(x) \leq F_n$.

If $n = 1$ then there is exactly one edge that joins x and ∞, so $M(x) = 1 = F_1$. Next, suppose that $n = 2$ and let $\langle\infty, u, x\rangle$ be a geodesic. By Theorem 5.1, u is either $\alpha(x)$ or $\beta(x)$; hence $M(x) \leq 2 = F_2$. Now suppose that for all vertices y with $\rho(\infty, y) = k$, where $k \leq n$, we have $M(y) \leq F_k$, and consider a vertex x with $\rho(\infty, x) = n + 1$ ($n \geq 2$). By Theorem 5.1, each geodesic from ∞ to x, say $\langle\infty, x_n, \dots, x_1, x\rangle$, must satisfy either $x_1 = \alpha(x)$ or $x_1 = \beta(x)$. Let $M_\alpha(x)$ be the number of geodesics from ∞ to x with $x_1 = \alpha(x)$ and let $M_\beta(x)$ be the number of geodesics from ∞ to x with $x_1 = \beta(x)$. Obviously, $M(x) = M_\alpha(x) + M_\beta(x)$.

Suppose there is a geodesic with $x_1 = \alpha(x)$; then $\rho(\infty, \alpha(x)) = n$. It follows from the induction hypothesis that $M(\alpha(x)) \leq F_n$. Since $M_\alpha(x) \leq M(\alpha(x))$, we see that $M_\alpha(x) \leq F_n$. Suppose next there is a geodesic $\langle\infty, x_n, \dots, x_2, x_1, x\rangle$ with $x_1 = \beta(x)$, and let $\gamma(x)$ denote the parent of $\beta(x)$ other than $\alpha(x)$. Because the geodesic does not pass through $\alpha(x)$, we must have $x_2 = \gamma(x)$. Since this is true of all geodesics with $x_1 = \beta(x)$ it follows that $M_\beta(x) \leq M(\gamma(x))$. Furthermore, since $\rho(\infty, \gamma(x)) = n - 1$, the induction hypothesis gives $M(\gamma(x)) \leq F_{n-1}$. Hence $M(x) = M_\alpha(x) + M_\beta(x) \leq F_n + F_{n-1} = F_{n+1}$, and the proof by induction is complete.

It remains to show that, for each n, there is a rational x_n with $\rho(\infty, x_n) = n$ and $M(x_n) = F_n$. Let $x_0 = \infty$ and $x_1 = 0$, and define rationals $y_1, x_2, y_2, x_3, \dots$ inductively, in this order, by

$$y_{n+1} = x_n \oplus x_{n+1}, \qquad x_{n+1} = x_n \oplus y_n.$$

It is clear that $x_n = \alpha(x_{n+1})$, $y_n = \beta(x_{n+1})$, and $x_n = \alpha(y_{n+1})$. This shows that $\langle x_0, \dots, x_{n-1}, x_n\rangle$ and $\langle x_0, \dots, x_{n-1}, y_n\rangle$ are both ancestral paths, so $\rho(\infty, x_n) = \rho(\infty, y_n) = n$. Let X_n and Y_n be the number of geodesics from ∞ to x_n and y_n, respectively. Since x_n and y_n are the parents of x_{n+1}, we see that $X_{n+1} = X_n + Y_n$. Next, since x_{n-1} is a parent of y_n, we have $Y_n \geq X_{n-1}$. This gives $X_{n+1} \geq X_n + X_{n-1}$, and since $X_1 = 1$ and $X_2 = 2$ it follows that $X_n \geq F_n$. However, we know that $X_n \leq F_n$, so $X_n = F_n$.

Next we show that two geodesics with the same endpoints are uniformly close.

THEOREM 6.1. *Suppose that* $\langle\infty, x_1, \dots, x_n, v\rangle$ *and* $\langle\infty, y_1, \dots, y_n, v\rangle$, *where* $n \geq 1$, *are geodesics. Then, for each* i, *either* $x_i = y_i$, *or* x_i *is parent of* y_i, *or* y_i *is a parent of* x_i.

Proof. Let $P(x, y)$ be the assertion "$x = y$ or x is a parent of y or y is a parent of x". The key observation is that if x and y are both parents of some rational z, then either $x = y$ or $\{x, y\} = \{\alpha(z), \beta(z)\}$; in the latter case, by Theorem 4.1, x is a parent of y or y is a parent of x. Thus, *if* x *and* y *are both parents of some* z, *then* $P(x, y)$ *is true*. Since x_n and y_n are parents of v, it follows that $P(x_n, y_n)$ is true. We shall now show, without any reference to v, that the truth of $P(x_n, y_n)$ implies the truth of $P(x_{n-1}, y_{n-1})$. This will then imply (by induction) that $P(x_i, y_i)$ is true for each i.

We assume, then, only that $P(x_n, y_n)$ is true. If $x_n = y_n$ then x_{n-1} and y_{n-1} are parents of w, where $w = x_n = y_n$, in which case $P(x_{n-1}, y_{n-1})$ is true. So from now on we may suppose that $x_n \neq y_n$, and then without loss of generality we may also suppose that x_n is a parent of y_n. Since $\rho(\infty, y_n) = n$ and since the ancestral path of y_n is a geodesic, it follows that $\rho(\infty, \alpha(y_n)) = n - 1$. Therefore, since $\rho(\infty, x_n) = n$, we have $x_n \neq \alpha(y_n)$ and so can deduce that $x_n = \beta(y_n)$. Next, since y_{n-1} is a parent of y_n and since $\rho(\infty, y_{n-1}) = n - 1$, we must have $y_{n-1} = \alpha(y_n)$. It now follows from Theorem 4.1(iv) that y_{n-1} is a parent of x_n. Finally, because x_{n-1} is also a parent of x_n, we see that $P(x_{n-1}, y_{n-1})$ is true in this case too. □

7. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3, which we break into five parts.

PART 1. *If $[b_1, \dots, b_n]$ is a geodesic expansion, then $|b_i| \geq 2$ for each $i \geq 2$.*

Proof. The result is trivial if $n = 1$, so we assume that $n \geq 2$. Let $x = [b_1, \dots, b_n]$. It is sufficient to show that if $|b_i| \leq 1$ for some $i \geq 2$, then there are integers $c_1, \dots, c_m$ with $x = [c_1, \dots, c_m]$ for $m < n$. We recall that $S_a(z) = a + 1/z$ and $[b_1, \dots, b_n] = S_{b_1} \cdots S_{b_n}(\infty)$. The proof is divided into four cases.

Case 1: $b_n = 0$. In this case, $n \geq 3$ (since $n = 2$ would imply that $x = [b_1, 0] = \infty \neq x$) and $x = [b_1, \dots, b_{n-1}, 0] = [b_1, \dots, b_{n-2}]$.

Case 2: $b_n = \pm 1$. In this case, $x = [b_1, \dots, b_{n-1} \pm 1]$.

Case 3: $b_k = 0$, where $2 \leq k \leq n - 1$ (so $n \geq 3$). Here we use the relation $S_a S_0 S_b = S_{a+b}$, which gives $x = [b_1, \dots, b_n] = [b_1, b_2, \dots, b_{k-2}, b_{k-1} + b_{k+1}, b_{k+2}, \dots, b_n]$.

Case 4: $b_k = \pm 1$, where $2 \leq k \leq n - 1$. Here we use the relation $S_a S_t S_b = S_{a+t} S_{-b-t} V$, where $V(z) = -z$, which holds when $t = \pm 1$. Note that $V S_a = S_{-a} V$ and $V(\infty) = \infty$. Thus, for example, $S_2 S_1 S_3 = S_3 S_{-4} V$ so that $[2, 1, 3] = [3, -4]$, and $S_2 S_1 S_3 S_5 = S_3 S_{-4} V S_5 = S_3 S_{-4} S_{-5} V$ so that $[2, 1, 3, 5] = [3, -4, -5]$. We leave the reader to handle the general case, and this completes the proof of Part 1. □

PART 2. *If $[b_1, \dots, b_n]$ is a geodesic expansion, then $b_2, \dots, b_n$ does not contain an inefficient subsequence of consecutive b_i.*

Proof. Again the proof uses relations among the S_a, and this time we need the following relations:

$$\begin{aligned} S_a S_2 (S_{-3} S_3)^m S_{-2} S_b &= S_{a+1} (S_{-3} S_3)^m S_{-3} S_{-b+1} V, \\ S_a S_2 (S_{-3} S_3)^m S_{-3} S_2 S_b &= S_{a+1} (S_{-3} S_3)^{m+1} S_{-b-1} V, \\ S_a S_{-2} (S_3 S_{-3})^m S_2 S_b &= S_{a-1} (S_3 S_{-3})^m S_3 S_{-b-1} V, \\ S_a S_{-2} (S_3 S_{-3})^m S_3 S_{-2} S_b &= S_{a-1} (S_3 S_{-3})^{m+1} S_{-b+1} V. \end{aligned} \tag{7.1}$$

The first of these relations shows that if $b_1, \ldots, b_n$ contains a subsequence of consecutive terms $a, 2, -3, 3, \ldots, -3, 3, -2, b$ then this can be replaced by the shorter sequence of coefficients $a+1, -3, 3, \ldots, -3, 3, -b+1$; therefore, $[b_1, \ldots, b_n]$ cannot be a geodesic expansion. The remaining cases are dealt with in a similar way using the other identities, and we omit the details. It only remains to prove the four identities, and to do this we let $U = S_{-3}S_3$. The first identity reduces to $fU^m = U^m f$, where $f(z) = -(2z+1)/(z+1)$, and this holds because $f^4 = U$. We leave the other proofs (which are similar) to the reader. □

Part 3. *If $|b_i| \geq 2$ for $i = 2, \ldots, n$ and if $b_2, \ldots, b_n$ does not contain an inefficient subsequence, then $[b_1, \ldots, b_n]$ is a geodesic expansion.*

Proof. Again assume that $n \geq 2$. Let $\mathcal{A}_x$ denote the class of those finite continued fraction expansions $[b_1, \ldots, b_n]$ of a rational x such that $|b_i| \geq 2$ for $i = 2, \ldots, n$ and $b_2, \ldots, b_n$ does not contain an inefficient subsequence of consecutive b_i. By Parts 1 and 2, $\mathcal{A}_x$ contains the continued fraction of the ancestral path. We shall define a map $\Pi\colon \mathcal{A}_x \to \mathcal{A}_x$ that fixes the continued fraction of the ancestral path and also preserves the length of elements of $\mathcal{A}_x$.

Let $\langle v_0, v_1, \ldots, v_n \rangle$, where $v_0 = \infty$ and $v_n = x$, be the path in the Farey graph corresponding to a member $[b_1, \ldots, b_n]$ of $\mathcal{A}_x$, and suppose that this path is not the ancestral path. By Theorem 5.3(i), v_{i-1} is a parent of v_i for each i. Let $m \geq 2$ be the largest integer such that v_{m-1} is the *second* parent of v_m. By Theorem 5.3(iii), this means that $|b_m| = 2$. Let $\varepsilon = b_m/|b_m|$. We define the image of $[b_1, \ldots, b_n]$ under Π to be $[b_1', \ldots, b_n']$, where $b_i' = b_i$ for all i except $b_{m-1}' = b_{m-1} + \varepsilon$, $b_m' = -b_m$, and, if $m < n$, $b_{m+1}' = b_{m+1} + \varepsilon$. We shall later prove that $[b_1', \ldots, b_n']$ is an element of $\mathcal{A}_x$.

Let $\langle v_0', v_1', \ldots, v_n' \rangle$ (where $v_0' = \infty$ and $v_n' = x$) be the path corresponding to $[b_1', \ldots, b_n']$ in $\mathcal{F}$. Observe that either this is the ancestral path or we can choose $m' \geq 2$ to be the largest integer such that $v_{m'-1}'$ is the *second* parent of $v_{m'}'$. We prove that $m' < m$. Thus, given c in $\mathcal{A}_x$, there is an integer $s \geq 0$ such that $\Pi^s(c)$ is the continued fraction corresponding to the ancestral path. Since Π preserves continued fraction length, we see that every member of $\mathcal{A}_x$ is a geodesic expansion. In graph-theoretic terms, the transformation Π corresponds to switching between the solid and dashed paths in Figure 2.

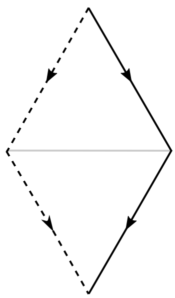

Figure 2 Two paths of equal length

It remains to prove that $m' < m$ and $[b'_1, \dots, b'_n] \in \mathcal{A}_x$. Recall that $|b_m| = 2$. For simplicity, suppose that $b_m = 2$; the case $b_m = -2$ is similar. Thus $b'_{m-1} = b_{m-1} + 1$, $b'_m = -2$, and, if $m < n$, $b'_{m+1} = b_{m+1} + 1$. For all other i we have $b'_i = b_i$.

First we show that $m' < m$. Using the identity $S_a S_2 S_b = S_{a+1} S_{-2} S_{b+1}$ and the equation $S_a S_2(\infty) = S_{a+1} S_{-2}(\infty)$, one can check that $v'_i = v_i$ for $i \neq m-1$ whereas $v'_{m-1} \neq v_{m-1}$. Since v'_{m-1} is a parent of v_m distinct from v_{m-1}, it must be the first parent of v_m. Hence $m' < m$.

Next we check that $|b'_i| \geq 2$ for $i = 2, \dots, n$. This inequality certainly holds for $i \neq m-1$ and $i \neq m+1$. For $i = m-1$, observe that $b'_{m-1} \in \{-1, 0, 1\}$ if and only if $b_{m-1} \in \{-2, -1, 0\}$. However, $|b_{m-1}| \geq 2$ by assumption and, since $-2, 2$ is an inefficient sequence, we cannot have $b_{m-1} = -2$. Hence $|b'_{m-1}| \geq 2$. Reasoning similarly allows us to deduce also that $|b'_{m+1}| \geq 2$.

Finally, in order to reach a contradiction, suppose that $b'_p, b'_{p+1}, \dots, b'_q$ is an inefficient sequence of consecutive b'_i (where $2 \leq p < q \leq n$). This sequence must contain at least one of the terms b'_{m-1}, b'_m, and b'_{m+1} (because $b'_i = b_i$ for $i \neq m-1, m, m+1$). Since $b'_m = 2$, only four possible cases arise: (a) $2 \leq p < q = m-1$; (b) $2 \leq p < q = m$; (c) $m = p < q \leq n$; and (d) $m+1 = p < q \leq n$. Each of these cases leads to a contradiction, and the necessary argument in each case is similar, so we only provide the details for (a). Suppose then that $q = m-1$. Hence $|b'_{m-1}| = 2$. If $b'_{m-1} = 2$ then $b_{m-1} = 1$, which is a contradiction. If $b'_{m-1} = -2$ then one can check that $b_p, b_{p+1}, \dots, b_{m-1}, b_m$ is an inefficient sequence of consecutive b_i, which is also a contradiction.

We conclude that $b'_2, \dots, b'_n$ does not contain an inefficient subsequence of consecutive b'_i. Since $v'_n = v_n$ we see that $[b'_1, \dots, b'_n] \in \mathcal{A}_x$, so the proof is complete. □

We have now proved the first assertion of Theorem 1.3.

PART 4. *If $[b_1, \dots, b_n]$ is the unique geodesic expansion of x, then $|b_i| \geq 3$ for $i = 2, \dots, n$.*

Proof. Of course, it is sufficient to show that, if $|b_k| = 2$ for some k with $2 \leq k \leq n$, then x has an alternative expansion of the same length. When $b_n = 2$ we have the alternative expansion $[b_1, \dots, b_{n-2}, b_{n-1} + 1, -2]$, and when $b_n = -2$ we have the alternative $[b_1, \dots, b_{n-2}, b_{n-1} - 1, 2]$. If $b_k = \pm 2$ (where $2 \leq k \leq n-1$) then the identity $S_a S_2 S_b = S_{a+1} S_{-2} S_{b+1}$ shows that, again, x has an alternative expansion of the same length. □

PART 5. *If $x = [b_1, \dots, b_n]$, and $|b_i| \geq 3$ for $i = 2, \dots, n$, then this is the unique geodesic expansion of x.*

Proof. We therefore suppose that $|b_i| \geq 3$ for each $i \geq 2$ and let $\langle v_0, v_1, \dots, v_n \rangle$, where $v_0 = \infty$ and $v_n = x$, be the path in $\mathcal{F}$ corresponding to $[b_1, \dots, b_n]$. Recall that $M(y)$ denotes the number of geodesics from ∞ to y. According to Theorem 5.3(iii), $\alpha(v_i) = v_{i-1}$ and $\beta(v_i) = [b_1, \dots, b_{i-1}, b_i - b_i/|b_i|]$ for $i \geq 2$. Since $|b_i - b_i/|b_i|| \geq 2$, it follows from Part 3 that $[b_1, \dots, b_{i-1}, b_i - b_i/|b_i|]$ is a geodesic expansion. Hence $\rho(\infty, \beta(v_i)) = i$. Since also $\rho(\infty, v_i) = i$, no geodesic path from ∞ to v_i can pass through $\beta(v_i)$, which means that every geodesic path from ∞ to v_i must pass through $\alpha(v_i) = v_{i-1}$. It follows that $M(v_i) = M(v_{i-1})$. Therefore $M(v_n) = M(v_1) = 1$, so there is only one geodesic from ∞ to v_n. $\square$

The graph-theoretic significance of conditions (i) and (ii) in Theorem 1.3 is exhibited in Figure 3. A coefficient $b_i = 0$ if and only if the corresponding path in $\mathcal{F}$ retraces itself along an edge. A coefficient $b_i = \pm 1$ if and only if the path in $\mathcal{F}$ traverses two sides of a triangle. Likewise, an inefficient subsequence of consecutive b_i corresponds to a nongeodesic path in $\mathcal{F}$.

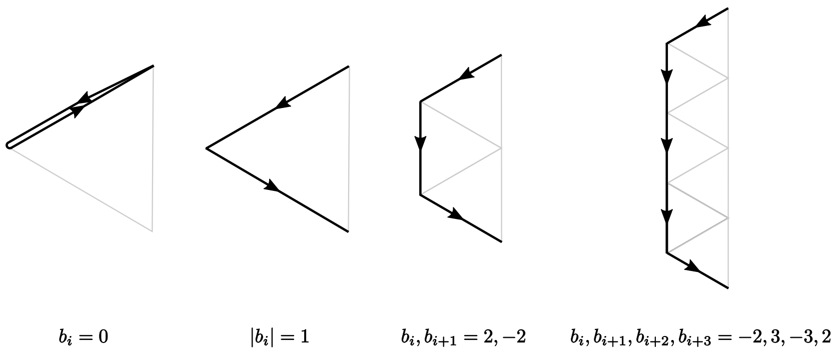

Figure 3 Paths that are not geodesics

8. Farey Subdivision

In this section we illustrate our earlier work by studying a single example in detail. First, the repeated bisection of an interval is commonplace in real analysis, but as an alternative we can subdivide a rational interval by the Farey sum (instead of the arithmetic mean) of the endpoints. We shall call this process the *Farey subdivision* of an interval. We take a positive rational x, start with the interval $[0, \infty]$, and then repeatedly perform the Farey subdivision where, at each stage, we choose the interval that contains x. The process terminates when x appears as the point of subdivision (see [5; 14]). If x is negative then we begin the procedure with $[-\infty, 0]$, where $-\infty$ is the same vertex as ∞ in $\mathcal{F}$ but has the reduced form $-1/0$ rather than $1/0$. Suppose now that $[p/q, r/s]$ is one interval in the Farey subdivision process. Then p/q and r/s are Farey neighbors and are also the Farey parents of the next division point $p/q \oplus r/s$. Thus the two parents of a division point are also division points, and given Theorem 5.1 this implies that *all geodesic paths from* ∞ *to* x are contained within the subgraph of $\mathcal{F}$ whose vertices are the endpoints arising from the Farey subdivision of $[0, \infty]$ that ends at x. We illustrate this process by an example.

The Farey subdivision of $[0, \infty]$ that leads to $22/39$ yields the decreasing sequence

$$[0,\infty], [0,1], \left[\tfrac{1}{2},1\right], \left[\tfrac{1}{2},\tfrac{2}{3}\right], \left[\tfrac{1}{2},\tfrac{3}{5}\right], \left[\tfrac{1}{2},\tfrac{4}{7}\right], \left[\tfrac{5}{9},\tfrac{4}{7}\right], \left[\tfrac{9}{16},\tfrac{4}{7}\right], \left[\tfrac{9}{16},\tfrac{13}{23}\right] \qquad (8.1)$$

of nested intervals, and the next division point is $22/39$. Our preceding remarks show that the corresponding subgraph of $\mathcal{F}$ (illustrated in Figure 4) contains all geodesics from ∞ to $22/39$, and it is easy to see from this that there are exactly six such geodesic paths. One of them is the ancestral path $\langle \infty, 0, \frac{1}{2}, \frac{4}{7}, \frac{9}{16}, \frac{22}{39} \rangle$; another is the path $\langle \infty, 1, \frac{1}{2}, \frac{4}{7}, \frac{9}{16}, \frac{22}{39} \rangle$ resulting from an application of the nearest integer division algorithm. The usual expansion of $22/39$ (with positive integer coefficients) corresponds to the path $\langle \infty, 0, 1, \frac{1}{2}, \frac{4}{7}, \frac{9}{16}, \frac{22}{39} \rangle$, but this is not a geodesic.

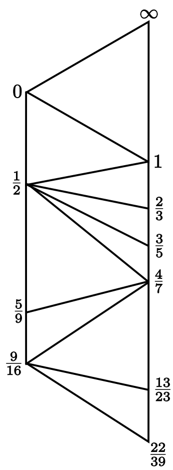

Figure 4 Farey subdivision of $[0, \infty]$ for the rational $22/39$

Sometimes one continued fraction expansion is obtained from another by (in the language of other authors) "removing the coefficient 1" or "singularisation", and this simply amounts to replacing the path on two sides of a triangle by the third side. In fact, this, and also the process of "insertion" (see [2]), is about the "homology" of $\mathcal{F}$, and these processes are valid precisely because of the existence of certain relations in the Modular group (of Möbius maps).

9. The Nearest Integer Division Algorithm

There are algorithms other than the ancestral path algorithm for constructing geodesics between ∞ and a rational x. For example, the nearest integer division algorithm yields a geodesic ICF expansion of x and, as we saw in Section 8, this is not always the same as the geodesic expansion derived from the ancestral path (see [13, p. 168], where Perron proves that the nearest integer division algorithm generates one of the shortest possible semi-regular continued fractions of a rational). However, both the ancestral path and nearest integer division algorithms are special cases of a more general algorithm for constructing geodesics, which we now explain (without proof).

Given a vertex q of $\mathcal{F}$, we define a map $\alpha_q : \mathcal{F} \setminus \{q\} \to \mathcal{F}$ as follows. Let $x \in \mathcal{F} \setminus \{q\}$. If x and q are adjacent then we define $\alpha_q(x) = q$. Otherwise, there is a unique pair of Farey neighbors u and v that are both adjacent to x and that separate x from q in the sense that any path from x to q must pass through one of u or v. The vertices x, u, and v form a Farey triangle. (If q is ∞ then u and v are the Farey parents of x.) There is also a uniquely defined vertex w that is distinct from x but is also adjacent to both u and v. On the ideal boundary $\mathbb{R}_\infty$ of $\mathbb{H}$ (which is topologically equivalent to a circle), the vertex q lies between u and w, or it lies between v and w, or it is equal to w. In the first case we define $\alpha_q(x) = u$, in the second case we define $\alpha_q(x) = v$, and in the third case we define $\alpha_q(x)$ to be either one of u or v. The first case is illustrated in Figure 5, where we use the unit disc model of the hyperbolic plane.

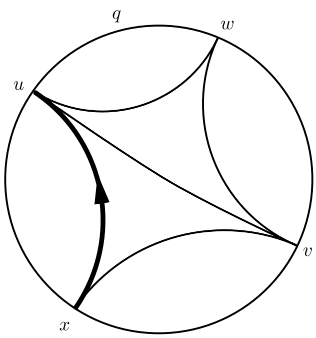

Figure 5 The vertex q lies between u and w, so $\alpha_q(x) = u$

The first parent map α coincides with α_∞ (provided that α_∞ is defined appropriately on $\mathbb{Z} + \frac{1}{2}$; the vertices in $\mathbb{Z} + \frac{1}{2}$ correspond to the points just discussed for the third case, in which an arbitrary choice was made). Moreover, if h is an element of the extended Modular group that satisfies $h(\infty) = q$, then $\alpha_q = h\alpha h^{-1}$. It follows that the iterates $x, \alpha_q(x), \alpha_q^2(x), \dots, q$ form a geodesic path from x to q, which is mapped by h^{-1} to the ancestral path from $h^{-1}(x)$ to ∞.

Next, observe that $\alpha_x(\infty)$ is the nearest integer to the rational x. Applying α_x repeatedly shows that the geodesic $\langle \infty, \alpha_x(\infty), \alpha_x^2(\infty), \dots, x \rangle$ is the path obtained by applying the nearest integer division algorithm to x. (Again, we must make a judicious choice of the nearest integer when working with elements of $\mathbb{Z} + \frac{1}{2}$.) In short, the ancestral path iterates $x, \alpha_\infty(x), \alpha_\infty^2(x), \dots$ form a geodesic from x to ∞, and the nearest integer division algorithm iterates $\infty, \alpha_x(\infty), \alpha_x^2(\infty), \dots$ form a geodesic from ∞ to x.

10. Infinite Continued Fractions

In this final section we briefly discuss the theory of infinite geodesic continued fractions. To keep the discussion short, we omit some of the straightforward details of our arguments.

An *infinite integer continued fraction* is a formal expression

$$[b_1, b_2, \dots] = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}, \tag{10.1}$$

where each coefficient b_i is an integer. The *convergents* of $[b_1, b_2, \dots]$ are the vertices $S_{b_1} \cdots S_{b_k}(\infty)$ of $\mathcal{F}$, and they form an infinite path in $\mathcal{F}$ with one fixed end at ∞. Every infinite path with one fixed end at ∞ corresponds, in this fashion, to a unique infinite integer continued fraction. The sequence of convergents may or may not converge in $\mathbb{R}_\infty$. We say that an element x of $\mathbb{R}_\infty$ has an *integer continued fraction expansion* $[b_1, b_2, \dots]$ if $S_{b_1} \cdots S_{b_k}(\infty) \to x$ as $k \to \infty$. It is well known that each irrational has a unique continued fraction expansion $[b_1, b_2, \dots]$ with $b_2, b_3, \dots$ positive integers. In general, however, both irrational and rational numbers have infinitely many infinite integer continued fraction expansions.

There is an alternative and ultimately equivalent way of assigning values to convergent infinite integer continued fractions that employs the theory of ends of infinite graphs (see [3]). We avoid this theory by working with the concrete model of the Farey graph in the hyperbolic plane with ideal boundary $\mathbb{R}_\infty$.

An infinite path $\langle \infty, v_1, v_2, \dots \rangle$ in $\mathcal{F}$ is a *geodesic* provided that $\langle \infty, v_1, \dots, v_n \rangle$ is a geodesic for each n. A real number is said to have an *infinite geodesic expansion* if it has an infinite continued fraction expansion that corresponds to a geodesic. Notice that, by Theorem 1.3, a continued fraction $[b_1, b_2, \dots]$ that corresponds to an infinite geodesic must satisfy $|b_i| \geq 2$ for $i = 2, 3, \dots$. It then follows from the Śleszyński–Pringsheim theorem (see [12, Thm. I.1]) that $[b_1, b_2, \dots]$ converges to a finite real number.

We finish with two theorems about infinite geodesics. In proving these theorems we use the following fundamental lemma, which is a stronger version of the first part of Theorem 5.1. It is a simple consequence of the fact that Farey geodesics in the concrete model $\mathcal{F}$ of the Farey graph do not intersect in $\mathbb{H}$.

LEMMA 10.1. *Let $[a, b]$ be a real Farey interval, and suppose that $\langle v_0, v_1, v_2, \dots \rangle$ is a path in $\mathcal{F}$ such that v_0 lies outside $[a, b]$ but v_n, for some integer n, lies inside (a, b). Then there is an integer i such that either $v_i = a$ or $v_i = b$.*

In other words, removing from $\mathcal{F}$ a pair of neighboring Farey vertices (and all edges connected to these vertices) disconnects $\mathcal{F}$ into two components. Consider, for example, removing 0 and ∞ from $\mathcal{F}$.

THEOREM 10.2. *A real number x has an infinite geodesic expansion if and only if x is irrational.*

Proof. Suppose first that x is irrational. Let $\langle \infty, C_1, C_2, \dots \rangle$ be the path from ∞ to x generated by the nearest integer division algorithm. Given any positive integer n, choose a rational q sufficiently close to x that the nearest integer division algorithm applied to q yields a path that also begins with $\langle \infty, C_1, \dots, C_n \rangle$. As we saw in Section 9, this path is a geodesic. Hence $\langle \infty, C_1, C_2, \dots \rangle$ is a geodesic.

Suppose now that x is rational, and consider an infinite continued fraction expansion of x. After applying an element of the Modular group we can assume that $x = 0$, and our continued fraction expansion gives rise to an infinite path $\langle v_0, v_1, v_2, \dots \rangle$ in $\mathcal{F}$ that converges to 0. Choose a positive integer m such that $1/m < |v_0|$. The path converges to 0 and so, by Lemma 10.1, for every integer $n > m$ the path passes through one endpoint of one of the Farey intervals $[-1/n, 0]$ or $[0, 1/n]$. It follows that the path cannot be a geodesic because the distance in $\mathcal{F}$ between any two rationals of the form $\pm 1/n$ is at most 2. □

The next theorem is analogous to Theorem 1.3.

THEOREM 10.3. *The infinite integer continued fraction* $[b_1, b_2, \dots]$ *is an infinite geodesic expansion if and only if*

(i) $|b_i| \geq 2$ *for* $i = 2, 3, \dots$ *and*

(ii) $b_2, b_3, \dots$ *does not contain a (finite) inefficient subsequence of consecutive* b_i.

Furthermore, the infinite geodesic expansion $[b_1, b_2, \dots]$ *of an irrational* x *is the only geodesic expansion of* x *if and only if* (a) $|b_i| \geq 3$ *for* $i = 2, 3, \dots$ *and* (b) $b_1, b_2, \dots$ *does not eventually coincide with one of the periodic sequences* $3, -3, 3, -3, \dots$ *or* $-3, 3, -3, 3, \dots$.

Proof. The first assertion of equivalence follows immediately from the first assertion of equivalence in Theorem 1.3. We focus on the second assertion, which concerns the necessary and sufficient conditions for an infinite geodesic expansion to be unique. Suppose $|b_i| < 3$ for some integer i. It then follows from Theorem 1.3 that there is more than one infinite geodesic expansion of x. Next, suppose that $b_{n+1}, b_{n+2}, b_{n+3}, \dots = 3, -3, 3, \dots$ and that $|b_i| \geq 3$ for $i = 2, 3, \dots$. From (7.1) we see that $[a, -2, 3, -3, 3, -3, \dots] = [a - 1, 3, -3, 3, -3, \dots]$, so $[b_1, \dots, b_n + 1, -2, 3, -3, 3, -3, \dots] = [b_1, \dots, b_n, 3, -3, 3, -3, \dots]$. By the first part of the theorem, these are both infinite geodesic expansions, so again there is more than one infinite geodesic expansion of x.

Finally, suppose that $|b_i| \geq 3$ for $i = 2, 3, \dots$ and that $[b_1, b_2, \dots]$ and $[b_1', b_2', \dots]$ are distinct infinite geodesic expansions of x. We will prove that $b_1, b_2, \dots$ is eventually periodic with period $3, -3$.

After using an element of the extended Modular group to remove a finite number of terms from both continued fractions, we can assume that $b_1 \neq b_1'$ (and since both expansions are geodesic expansions, b_1 and b_1' differ by 1). Let m be the lower integer part of x, and let $I_1 = [m, m + 1]$. The Farey subdivision process applied to x yields an infinite nested sequence of Farey intervals $I_1 \supset I_2 \supset \cdots$, where $\bigcap_k I_k = \{x\}$ and where I_k and I_{k+1} share a common endpoint. Let $I_k = [p_k, q_k]$. Let $\gamma = \langle \infty, C_1, C_2, \dots \rangle$ be the infinite path in $\mathcal{F}$ corresponding to $[b_1, b_2, \dots]$, and let $\gamma = \langle \infty, C_1', C_2', \dots \rangle$ be the infinite path corresponding to $[b_1', b_2', \dots]$. Since x lies inside I_k and since ∞ lies outside I_k, it follows from Lemma 10.1 that both γ and γ' must pass through one of the endpoints of each interval I_k. Moreover, because $|b_i| \geq 3$, the uniqueness assertion of Theorem 1.3 implies that γ and γ' intersect only at ∞. A short induction argument now shows that one of

γ or γ' consists of all the points $\infty, p_1, p_2, \dots$ and that the other consists of all the points $\infty, q_1, q_2, \dots$ (after removing repetitions).

Suppose that $|b_n| \geq 4$ for some $n \geq 2$. By Theorem 5.3(iii), the parents of C_n are C_{n-1} and $w = [b_1, \dots, b_{n-1}, b_n - b_n/|b_n|]$. Now C_{n-1} and w are the endpoints of one of the Farey intervals I_k, and since C_{n-1} belongs to γ it follows that w belongs to γ'. However, $|b_n - b_n/|b_n|| \geq 3$ and so, by Theorem 1.3, there is a unique geodesic from ∞ to w, namely, the ancestral path. Using Theorem 5.3(iii) again reveals that the first parent of w is C_{n-1}, which yields a contradiction because C_{n-1} does not lie in γ'. Therefore, $|b_n| = 3$ for $n = 2, 3, \dots$.

Recall that the convergents $C_1, C_2, \dots$ are strictly monotonic; let us suppose that they are increasing. We will prove by induction that $b_n = (-1)^n 3$ for $n = 2, 3, \dots$. (Likewise, it can be shown that if $C_1, C_2, \dots$ is decreasing then $b_n = (-1)^{n-1} 3$ for $n = 2, 3, \dots$.) Since $b_1 + 1/b_2 = C_2 > C_1 = b_1$, we see that $b_2 = 3$. Suppose $b_n = (-1)^n 3$ for $n = 2, \dots, 2k+1$ (the even case can be dealt with similarly). When restricted to the topological circle $\mathbb{R}_\infty$, each map S_{b_i} reverses the usual orientation of $\mathbb{R}_\infty$. It follows that the composite map $f = S_{b_1} \cdots S_{b_{2k}}$ preserves the usual orientation of $\mathbb{R}_\infty$. Hence C_{2k}, C_{2k+1}, and C_{2k+2} occur in the same, counterclockwise order in $\mathbb{R}_\infty$ as $f^{-1}(C_{2k}) = \infty$, $f^{-1}(C_{2k+1}) = b_{2k+1}$, and $f^{-1}(C_{2k+2}) = b_{2k+1} + 1/b_{2k+2}$. Thus $b_{2k+1} + 1/b_{2k+2} > b_{2k+1}$, which means that $b_{2k+2} > 0$. Therefore $b_{2k+2} = 3$, and the inductive step is complete. $\square$

We finish by commenting on the relationship between paths in $\mathcal{F}$ and the cutting sequences studied in [15] (see also [6; 8; 9]). Cutting sequences involve the sequence of Farey triangles crossed consecutively by a curve in $\mathbb{H}$ that has one endpoint ζ in $\mathbb{H}$ and the other endpoint x in $\mathbb{R}$. This sequence of triangles can be identified with a path in the *dual graph* $\mathcal{F}'$ of $\mathcal{F}$. The vertices of $\mathcal{F}'$ are Farey triangles, and two vertices are joined by an edge if and only if the corresponding Farey triangles are adjacent. In graph-theoretic terms, $\mathcal{F}'$ is the unique connected tree in which each vertex has valency 3. Cutting sequences are often described by formal sequences of symbols L and R (L for "left" and R for "right"); these sequences may be considered to encode instructions for navigating a path in $\mathcal{F}'$. Let γ denote the hyperbolic line connecting ζ and x. The sequence of Farey triangles corresponding to the cutting sequence of γ coincides with the sequence of Farey triangles arising from an application of the Farey subdivision process to x. All geodesics from ∞ to x in $\mathcal{F}$ lie within the subgraph generated by this subdivision (we saw this for rational values of x in Section 8).

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