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Polarized Complexity-1 T-Varieties

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Introduction

It is well known that there is a correspondence between polarized toric varieties and lattice polytopes. The main result of this paper is to generalize this to the setting of normal varieties with effective complexity-1 torus action—that is, complexity-1 T-varieties. In order to do so, we introduce so-called *divisorial polytopes*. In short, a divisorial polytope on a smooth projective curve Y in a lattice M is a piecewise affine concave function

$$\Psi = \sum_{P \in Y} \Psi_P \cdot P \colon \square \to \operatorname{Div}_{\mathbb{Q}} Y$$

from some polytope in $M_{\mathbb{Q}}$ to the group of $\mathbb{Q}$-divisors on Y such that:

(i) $\deg \Psi(u) > 0$ for u from the interior of $\square$;
(ii) $\deg \Psi(u) > 0$ or $\Psi(u) \sim 0$ for u a vertex of $\square$; and
(iii) the graph of Ψ_P has integral vertices for every $P \in Y$.

We then show that, similarly to the toric case, there is a correspondence between polarized complexity-1 T-varieties and divisorial polytopes. We also describe how the smoothness, degree, and Hilbert polynomial of a polarized T-variety can be determined from the corresponding divisorial polytope.

There are two other logical approaches to describing a polarized complexity-1 T-variety. Indeed, T-invariant Cartier divisors on complexity-1 T-varieties were described in terms of *divisorial fans* and support functions in [PS], which also included a characterization of ampleness. On the other hand, a sufficiently high multiple of some polarizing line bundle gives a map to projective space such that the corresponding affine cone is a complexity-1 T-variety describable by a *polyhedral divisor* $\mathcal{D}$. We compare these two approaches with our divisorial polytopes and show how to pass from one description to another.

We also present two other results. First, we show how the complicated combinatorial data of a divisorial fan used to describe a general T-variety can be simplified to a so-called *marked fansy divisor* for complete complexity-1 T-varieties. Second, we address the problem of finding minimal generators for the multigraded $\mathbb{C}$-algebra corresponding to a polyhedral divisor $\mathcal{D}$ on a curve. This then gives us a method to determine whether projective embeddings of complexity-1 T-varieties are, in fact, projectively normal.

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We begin in Section 1 by recalling the construction of T-varieties from [AHS]. We specialize to the complexity-1 case and introduce marked fansy divisors. In Section 2, we then recall the description of T-invariant Cartier divisors. Section 3 is dedicated to divisorial polytopes. Here we prove the correspondence between divisorial polytopes and polarized complexity-1 T-varieties and also discuss properties of divisorial polytopes. In Section 4, we compare support functions and divisorial polytopes with polyhedral divisors corresponding to affine cones. Finally, in Section 5 we describe how to find minimal generators for affine complexity-1 T-varieties.

We remark that, even though this paper looks only at complexity-1 T-varieties, we believe that the correspondence between polarized T-varieties and divisorial polytopes should generalize to higher-complexity torus actions. To generalize the preceding definition of divisorial polytopes, we first replace Y by any normal projective variety; then the degree conditions in (i) and (ii) are replaced, respectively, by ampleness and semiampleness.

1. Polyhedral Divisors and T-Varieties

We recall several notions from [AHS] and then specialize these to the case of complexity-1 T-varieties. As usual, let N be a lattice with dual M and let $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$ be the associated $\mathbb{Q}$ vector spaces. For any polyhedron $\Delta \subset N_{\mathbb{Q}}$, let $\operatorname{tail}(\Delta)$ denote its tailcone—that is, the cone of unbounded directions in Δ. Thus, Δ can be written as the Minkowski sum of some bounded polyhedron and its tailcone. For any polyhedron $\Delta \subset N_{\mathbb{Q}}$ and vector u in the dual of its tailcone, let $\operatorname{face}(\Delta, u)$ be the set of Δ on which u attains its minimum. A *face* of Δ is then defined to be any subset of Δ of the form $\operatorname{face}(\Delta, u)$, or the empty set.

Let Y be a normal semiprojective variety over $\mathbb{C}$ and let $\sigma \subset N_{\mathbb{Q}}$ be a pointed polyhedral cone. By $\sigma^\vee$ we denote the dual cone of σ.

DEFINITION. A *polyhedral divisor* on Y with tailcone σ is a formal finite sum

$$\mathcal{D} = \sum_P \Delta_P \cdot P,$$

where P runs over all prime divisors on Y and Δ_P is a polyhedron with tailcone σ. By "finite" we mean that only finitely many coefficients differ from the tailcone. Note that the empty set is also allowed as a coefficient. If Y is a complete curve then we define the *degree* of a polyhedral divisor by

$$\deg \mathcal{D} := \sum_P \Delta_P,$$

where summation is via Minkowski addition. If Y is an affine curve, we define the degree as $\deg \mathcal{D} = \emptyset$.

We can evaluate a polyhedral divisor for every element $u \in \sigma^\vee \cap M$ via

$$\mathcal{D}(u) := \sum_P \min_{v \in \Delta_P} \langle v, u \rangle P$$

in order to obtain an ordinary divisor $\mathcal{D}(u)$ on the locus of $\mathcal{D}$, which is defined as $\operatorname{Loc}\mathcal{D} := Y \setminus \left(\bigcup_{\Delta_P=\emptyset} P\right)$.

DEFINITION. A polyhedral divisor $\mathcal{D}$ is called *proper* if $\mathcal{D}(u)$ is a semiample $\mathbb{Q}$-Cartier divisor for all $u \in \sigma^\vee$ and if $\mathcal{D}(u)$ is big for all u in the interior of $\sigma^\vee$. If Y is a curve, note that $\mathcal{D}$ is proper exactly when $\deg \mathcal{D} \subsetneq \sigma$, and for all $u \in \sigma^\vee$ with $\min_{v\in\deg\mathcal{D}}\langle v, u\rangle = 0$ it follows that a multiple of $\mathcal{D}(u)$ is principal.

To a proper polyhedral divisor we associate an M-graded $\mathbb{C}$-algebra and consequently an affine scheme admitting a $T^N = N \otimes \mathbb{C}^*$-action:

$$X(\mathcal{D}) := \operatorname{Spec} \bigoplus_{u\in\sigma^\vee\cap M} H^0(Y, \mathcal{D}(u)).$$

This construction gives a normal variety of dimension $\dim N_\mathbb{Q} + \dim Y$ together with an effective T^N-action.

REMARK. If $\mathcal{D}$ is a nonproper polyhedral divisor, we can still associate an M-graded $\mathbb{C}$-algebra as just described and consequently an affine scheme $X(\mathcal{D})$ with T^N-action. However, the resulting algebra need not be finitely generated; similarly, we can't say anything about the dimension of $X(\mathcal{D})$ or the effectiveness of the T^N-action.

In order to glue together the affine varieties with T^N-action, we require some further definitions.

DEFINITION. Let $\mathcal{D} = \sum_P \Delta_P \cdot P$ and $\mathcal{D}' = \sum_P \Delta'_P \cdot P$ be two polyhedral divisors on Y with respective tailcones σ and σ'.

- We define their *intersection* by

$$\mathcal{D} \cap \mathcal{D}' := \sum_P (\Delta_P \cap \Delta'_P) \cdot P.$$

- We say $\mathcal{D}' \subset \mathcal{D}$ if $\Delta'_P \subset \Delta_P$ for every point $P \in Y$.
- For $y \in Y$ a not necessarily closed point, we call $\mathcal{D}_y := \sum_{P\ni y} \Delta_P$ the *slice* of $\mathcal{D}$ at P and denote it by $\mathcal{D}_y$ as well.

If $\mathcal{D}' \subset \mathcal{D}$ and if both are proper, then we have the inclusion

$$\bigoplus_{u\in\sigma^\vee\cap M} H^0(Y, \mathcal{D}'(u)) \supset \bigoplus_{u\in\sigma^\vee\cap M} H^0(Y, \mathcal{D}(u)),$$

which corresponds to a dominant morphism $X(\mathcal{D}') \to X(\mathcal{D})$. We say that $\mathcal{D}'$ is a *face* of $\mathcal{D}$, written $\mathcal{D}' \prec \mathcal{D}$, if this morphism is an open embedding.

DEFINITION. A *divisorial fan* is a finite set $\mathcal{S}$ of proper polyhedral divisors such that, for $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$, we have $\mathcal{D} \succ \mathcal{D}' \cap \mathcal{D} \prec \mathcal{D}'$ with $\mathcal{D}' \cap \mathcal{D}$ also in $\mathcal{S}$. The *tailfan* of $\mathcal{S}$ is the set of all $\operatorname{tail}(\mathcal{D})$ for $\mathcal{D} \in \mathcal{S}$. For a not necessarily closed point $y \in Y$, the polyhedral complex $\mathcal{S}_y$ defined by the polyhedra $\mathcal{D}_y$, $\mathcal{D} \in \mathcal{S}$, is called a *slice*

of $\mathcal{S}$. The set $\mathcal{S}$ is called *complete* if all slices $\mathcal{S}_y$ are complete subdivisions of $N_\mathbb{Q}$ and if Y is complete.

We may glue the affine varieties $X(\mathcal{D})$ via

$$X(\mathcal{D}) \leftarrow X(\mathcal{D} \cap \mathcal{D}') \rightarrow X(\mathcal{D}').$$

This construction yields a normal scheme $X(\mathcal{S})$ of dimension $\dim N_\mathbb{Q} + \dim Y$ with an effective torus action by T^N; furthermore, $X(\mathcal{S})$ is complete if and only if $\mathcal{S}$ is complete. Note that all normal varieties with effective torus action can be constructed in this manner.

For the rest of this section we will restrict to the case where Y is a curve; this is thus the case of complexity-1 T-varieties. As we have already seen, the criterion for properness of a polyhedral divisor simplifies nicely. This is true as well for the face relation. Let $\mathcal{D}, \mathcal{D}'$ be a polyhedral divisors on a curve Y with $\mathcal{D}$ proper. In this case, we say $\mathcal{D}' \blacktriangleleft \mathcal{D}$ if Δ'_P is a face of Δ_P for every point $P \in Y$ and if $\deg \mathcal{D} \cap \sigma' = \deg \mathcal{D}'$. We then have the following proposition.

PROPOSITION 1.1. *Let $\mathcal{D}, \mathcal{D}'$ be polyhedral divisors on a curve Y with $\mathcal{D}$ proper. Then $\mathcal{D}' \prec \mathcal{D}$ if and only if $\mathcal{D}' \blacktriangleleft \mathcal{D}$.*

We shall need several lemmas to prove this proposition.

LEMMA 1.2 (Refinement lemma). *Let $\mathcal{D}$ be a polyhedral divisor with affine locus Y, and let $\{U_i\}_{i \in I}$ be an affine covering of Y. The polyhedral divisors $\mathcal{D} + \emptyset \cdot (Y \setminus U_i) =: \mathcal{D}|_{U_i} \prec \mathcal{D}$ define open subsets $X(\mathcal{D}|_{U_i}) \hookrightarrow X(\mathcal{D})$, which cover $X := X(\mathcal{D})$.*

Proof. Every global section $f \in \Gamma(\mathcal{O}_Y)$ gives rise to a section $f \in \Gamma(X, \mathcal{O}_X)_0 = \Gamma(\mathcal{O}_Y)$. By [AHS, Prop. 3.1] we have $X_f = X(\mathcal{D} + \emptyset \cdot \operatorname{div}(f))$. Hence, for principal open subsets $U_i = Y_{f_i}$, the claim follows immediately.

Since Y is affine, by refining we can pass to a covering $\{U'_j\}_{j \in J}$ of principal open subsets and corresponding polyhedral divisors $\mathcal{D}|_{U'_j}$. Now the $X(\mathcal{D}|_{U'_j})$ define open subsets of $X(\mathcal{D}|_{U_i})$ and of X as well and also (by the preceding conclusion) cover them. Since the inclusions $X(\mathcal{D}|_{U'_j}) \hookrightarrow X$ factor through the $X(\mathcal{D}|_{U_i})$, the former already define an open covering of X. □

LEMMA 1.3 [AHS, Lemma 6.8]. *Assume that $\operatorname{Loc} \mathcal{D}' = \operatorname{Loc} \mathcal{D} \setminus Z$ and that $\mathcal{D}'_P = \operatorname{face}(\mathcal{D}_P, u)$ for some $u \in \sigma^\vee$ and all $P \in \operatorname{Loc} \mathcal{D}'$. Then $\mathcal{D}' \subset \mathcal{D}$ defines an open embedding if there is a semiample divisor E with support Z and $k \cdot \mathcal{D}(u) - E$ semiample for $k \gg 0$.*

LEMMA 1.4. *If $\mathcal{D}$ is proper and $\mathcal{D}' \blacktriangleleft \mathcal{D}$, then $\mathcal{D}'$ is proper, too, and the corresponding morphism $i\colon X(\mathcal{D}') \rightarrow X(\mathcal{D})$ is an open embedding.*

Proof. First we check the properness of $\mathcal{D}'$. For an affine locus there is nothing to prove, so we shall assume that $\mathcal{D}'$ and $\mathcal{D}$ have complete loci. If $\deg \mathcal{D}' = \deg \mathcal{D} \cap \sigma'$ then, by the properness of $\mathcal{D}$, we get $\deg \mathcal{D}' \subsetneq \sigma'$. Now for every

$u' \in (\sigma')^\vee$ there exists a decomposition $u' = u - u''$ such that $u \in \sigma^\vee$ and $u'' \in \sigma^\vee \cap (\sigma')^\perp$. First, note that $u \mapsto \mathcal{D}'(u)$ is by definition a concave map. Hence $\mathcal{D}'(u - u'') \geq \mathcal{D}'(u) + \mathcal{D}'(-u'')$ holds. The inclusion $\deg \mathcal{D}' \subset \sigma'$ implies the equality $\mathcal{D}'(-u'') = -\mathcal{D}'(u'')$. Moreover, the inclusion $\mathcal{D}' \subset \mathcal{D}$ allows us to conclude that $\mathcal{D}'(u) \geq \mathcal{D}(u)$ and $\mathcal{D}'(u'') \geq \mathcal{D}(u'')$. But since $\deg \mathcal{D}'(u'') = 0$, it follows that $\mathcal{D}(u'') = \mathcal{D}'(u'')$; in particular, $\deg \mathcal{D}(u'') = 0$. All together we get $\mathcal{D}(u') \geq \mathcal{D}(u) - \mathcal{D}(u'')$. Since $\mathcal{D}(u)$ and $-\mathcal{D}(u'')$ are semiample by the properness of $\mathcal{D}$, the same is true for $\mathcal{D}'(u')$.

We now check that i is an open embedding. Suppose first that $\mathcal{D}$ has affine locus. We shall assume additionally that $\Delta'_P = \operatorname{face}(\Delta_P, u)$ for some $u \in \sigma^\vee$ and all $P \in \operatorname{Loc} \mathcal{D}'$. Then i is indeed an open embedding by Lemma 1.3, since every divisor is semiample on an affine variety. Dropping the additional assumption, we may choose an open covering $\{U_j\}_{j \in J}$ of Y and refine $\mathcal{D}'$ by $\mathcal{D}'|_{U_j}$, as in Lemma 1.2, such that $(\mathcal{D}'|_{U_j})_P = \operatorname{face}(\mathcal{D}_P, u)$ for some u and all $P \in Y$. Now we infer that $X(\mathcal{D}'|_{U_j}) \to X(\mathcal{D})$ is an open embedding for every j and so, by the refinement lemma, we are done.

For $\mathcal{D}$ of complete locus and $\mathcal{D}'$ not, we again begin by assuming that $\Delta'_P = \operatorname{face}(\Delta_P, u)$ for some $u \in \sigma^\vee$ and all $P \in \operatorname{Loc} \mathcal{D}'$; in this case, too, we obtain our result by applying Lemma 1.3. We may choose any effective divisor with support $Y \setminus \operatorname{Loc} \mathcal{D}'$. The relation $\mathcal{D}' \prec \mathcal{D}$ implies that $\deg \mathcal{D}(u) > 0$. Hence, $\deg(k \cdot \mathcal{D}(u) - E) > 0$ for $k \gg 0$. For the general case we may once again refine $\mathcal{D}'$ as before to conclude that i is an open embedding.

Finally, if $\mathcal{D}$ and $\mathcal{D}'$ both have complete loci then $\deg \mathcal{D}' = \deg D \cap \sigma'$ implies that, for any u with $\sigma' = \operatorname{face}(\sigma, u)$, we have $\Delta'_P = \operatorname{face}(\Delta_P, u)$ for all $P \in Y$. Now we can again use Lemma 1.3 with $Z = \emptyset$ and $E = 0$, since $\mathcal{D}(u)$ is semiample by the properness condition. □

LEMMA 1.5. *Let $\mathcal{D}, \mathcal{D}'$ be two proper polyhedral divisors with $\mathcal{D}' \prec \mathcal{D}$. Then* $\deg \mathcal{D}' = \sigma' \cap \deg \mathcal{D}$.

Proof. If $\operatorname{Loc}(\mathcal{D})$ is affine, the claim is immediate. We can thus assume that $\operatorname{Loc} \mathcal{D}$ is complete for the rest of the proof. Recall from Proposition 3.4 and Definition 5.1 of [AHS] that $\mathcal{D}' \prec \mathcal{D}$ is equivalent to the following condition:

> For every $y \in Y$, there exists a $w_y \in \sigma^\vee \cap M$ and a D_y in the linear system $|\mathcal{D}(w_y)|$ such that $y \notin \operatorname{supp}(D_y)$, $\mathcal{D}'_y = \operatorname{face}(\mathcal{D}_y, w_y)$, and $\operatorname{face}(\mathcal{D}'_v, w_y) = \operatorname{face}(\mathcal{D}_v, w_y)$ for all $v \in Y \setminus \operatorname{supp}(D_y)$.

Now suppose that $\operatorname{Loc}(\mathcal{D}')$ is affine. Then we must show that

$$\sigma' \cap \deg \mathcal{D} = \emptyset. \tag{1}$$

For each w_y and D_y as before, the support of D_y cannot be empty because otherwise $\operatorname{Loc}(\mathcal{D}') = \operatorname{Loc}(\mathcal{D})$. In particular, $\deg(\mathcal{D}(w_y)) > 0$. Now, choosing y to be some general point gives us w_y such that $\sigma' = \operatorname{face}(\sigma, w_y)$ with $(\deg \mathcal{D})(w_y) > 0$. But this is equivalent to (1), since $\langle \sigma', w_y \rangle = 0$.

Suppose instead that $\operatorname{Loc}(\mathcal{D}')$ is complete. Given an element $v \in \deg \mathcal{D}'$, it follows from the properness of $\mathcal{D}'$ that $v \in \sigma'$ and from $\mathcal{D}' \subset \mathcal{D}$ that $v \in \deg \mathcal{D}$.

Therefore, $\deg \mathcal{D}' \subset \deg \mathcal{D} \cap \sigma'$. For the other inclusion we choose an element $v = \sum_y v_y \in \deg \mathcal{D} \cap \sigma'$ with $v_y \in \mathcal{D}_y$. Then we choose an element $u \in \sigma^\vee$ such that $\mathcal{D}'_z = \operatorname{face}(\mathcal{D}_z, u)$ for some $z \in Y$, and this implies that $\sigma' = \operatorname{face}(\sigma, u)$. Since $\langle v_y, u\rangle \geq \min\langle \mathcal{D}_y, u\rangle$ holds we get $0 = \sum_y \langle v_y, u\rangle \geq \sum_y \min\langle \mathcal{D}_y, u\rangle \geq 0$, where the first inequality follows from the fact that $v \in \sigma'$ and the last inequality from the properness of $\mathcal{D}$. Hence $\langle v_y, u\rangle = \min\langle \mathcal{D}_y, u\rangle$ holds for every $y \in Y$, and for $y = z$ we get $v_z \in \mathcal{D}'_z = \operatorname{face}(\mathcal{D}_z, u)$. Since this is true for every $z \in Y$, we conclude that $v = \sum_z v_z \in \sum_z \mathcal{D}'_Z = \deg \mathcal{D}'$. □

Proof of Proposition 1.1. The proposition follows directly from Lemmas 1.1–1.5. Indeed, Lemma 1.4 covers one direction. The other direction follows from Lemma 1.5 coupled with the fact that, if $\mathcal{D}' \prec \mathcal{D}$, then [AHS, Def. 5.1] ensures that Δ'_P is a face of Δ_P for every point $P \in Y$. □

Different divisorial fans $\mathcal{S}, \mathcal{S}'$ can in fact yield the same T-variety $X(\mathcal{S}) = X(\mathcal{S}')$. The differing divisorial fans simply correspond to different open affine coverings. On the other hand, divisorial fans with identical slices might yield differing T-varieties even in the complexity-1 case. However, for complete complexity-1 T-varieties, we can save the situation via the following definition.

DEFINITION. A *marked fansy divisor* on a curve Y is a formal sum $\Xi = \sum \Xi_P \cdot P$, together with a fan Σ and some subset $C \subset \Sigma$, such that the following statements hold.

(i) Ξ_P is a complete polyhedral subdivision of $N_\mathbb{Q}$, and $\operatorname{tail}(\Xi_P) = \Sigma$ for all $P \in Y$.
(ii) For full-dimensional $\sigma \in C$, the polyhedral divisor $\mathcal{D}^\sigma = \sum \Delta^\sigma_P \cdot P$ is proper; here Δ^σ_P is the unique element of Ξ_P with $\operatorname{tail}(\Delta^\sigma_P) = \sigma$.
(iii) For $\sigma \in C$ of full dimension and $\tau \prec \sigma$, we have $\tau \in C$ if and only if $\deg \mathcal{D}^\sigma \cap \tau \neq \emptyset$.
(iv) If $\tau \prec \sigma$ and $\tau \in C$, then $\sigma \in C$.

We say that the elements of C are *marked*. The support of a fansy divisor is the set of points $P \in Y$, where Ξ_P differs from the tailfan Σ.

Now, given any complete divisorial fan $\mathcal{S}$ on Y, we can associate a marked fansy divisor by setting $\Xi = \sum \mathcal{S}_P \cdot P$ and adding marks to the tailcones of all $\mathcal{D} \in \mathcal{S}$ with complete locus. We denote this marked fansy divisor $\Xi(\mathcal{S})$.

PROPOSITION 1.6. *For any marked fansy divisor Ξ, there exists a complete divisorial fan $\mathcal{S}$ with $\Xi = \Xi(\mathcal{S})$. If for two divisorial fans $\mathcal{S}, \mathcal{S}'$ we have that $\Xi(\mathcal{S}) = \Xi(\mathcal{S}')$, then $X(\mathcal{S}) = X(\mathcal{S}')$.*

Proof. Assume that Ξ is supported at $P_1, \ldots, P_r$. We construct a divisorial fan as follows. Consider the set

$$S = \{\mathcal{D}^\sigma \mid \sigma \in C\} \cup \{\Delta \cdot P_i + \textstyle\sum_{j \neq i} \emptyset \cdot P_j \mid \Delta \in \Xi^{(n)}_{P_i}, \operatorname{tail}(\Delta) \notin C\}.$$

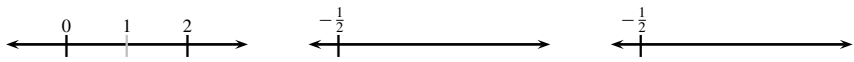
Now we obtain the divisorial fan $\mathcal{S}$ generated by S by adding all intersections of the polyhedral divisors in S. This is indeed a divisorial fan since part (ii) of the definition ensures that the polyhedral divisors with maximal tailcone are proper

while parts (iii) and (iv) ensure that the intersection of two polyhedral divisors is a face of both of them. Obviously we have $\Xi(\mathcal{S}) = \Xi$.

Now let $\mathcal{S}'$ be another divisorial fan with $\Xi(\mathcal{S}') = \Xi$. We obtain a common refinement $\mathcal{S}''$ of $\mathcal{S}$ and $\mathcal{S}'$ by considering all mutual intersections of divisors in $\mathcal{S}$ and $\mathcal{S}'$. To get the correct marks, the polyhedral divisors with complete locus in $\mathcal{S}'$ must be exactly the $\{\mathcal{D}^\sigma \mid \sigma \in C\}$. Hence, only polyhedral divisors with affine locus get refined. Now the claim follows by the refinement lemma. □

By Proposition 1.6, we can define $X(\Xi)$ to be $X(\mathcal{S})$ for any $\mathcal{S}$ with $\Xi = \Xi(\mathcal{S})$. Furthermore, every complete complexity-1 T-variety can be described via a marked fansy divisor. We thus can avoid divisorial fans and work instead with the somewhat more handy notion of marked fansy divisors.

EXAMPLE. The subdivisions demarcated by the black vertical lines in Figure 1, together with marks for both $\mathbb{Q}_{\geq 0}$ and $\mathbb{Q}_{\leq 0}$, give a marked fansy divisor Ξ on $\mathbb{P}^1$ with $X(\Xi)$ equal to the unique log del Pezzo surface of degree 2 with one A_1 singularity and two A_3 singularities (see [S]). By further subdividing at the gray vertical line, we get a marked fansy divisor Ξ'. The corresponding T-variety comes together with a natural map $\varphi\colon X(\Xi') \to X(\Xi)$, which is a resolution of the A_1 singularity.



(a) Ξ_0 (b) Ξ_∞ (c) Ξ_1

Figure 1 The fansy divisor for a log del Pezzo surface

2. Invariant Cartier Divisors

Invariant Cartier divisors on complexity-1 T-varieties were described in [PS] in combinatorial terms. We recall this description here, specializing to complete T-varieties and replacing divisorial fans with marked fansy divisors. For any piecewise affine continuous function $f\colon N_\mathbb{Q} \to \mathbb{Q}$, set $f^0(v) = \lim_{k\to\infty} f(k \cdot v)/k$ for any $v \in N_\mathbb{Q}$. We call f^0 the linear part of f. Consider now some complete marked fansy divisor Ξ on a smooth projective curve Y with tailfan Σ.

DEFINITION. By $\mathrm{SF}(\Xi)$ we denote the set of all formal sums of the form

$$h = \sum_{P\in Y} h_P \otimes P,$$

where $h_P\colon N_\mathbb{Q} \to \mathbb{Q}$ are continuous functions such that:

(i) h_P is piecewise affine with respect to the subdivision Ξ_P;
(ii) h_P is integral—that is, if $k \cdot v$ is a lattice point for $k \in \mathbb{N}$ and $v \in N$, then $k \cdot h_P(v) \in \mathbb{Z}$;
(iii) h_P^0 does not depend on P, so we denote this so-called linear part of h by h^0;
(iv) $h_P \neq h^0$ for only finitely many P.

We call an element of $\mathrm{SF}(\Xi)$ a *support function*.

Consider σ a full-dimensional cone in Σ. We define

$$h|_\sigma(0) = \sum_P a_P \cdot P$$

where the a_P are determined by writing $h_P|_{\Delta_P^\sigma}(v) = \langle v, u\rangle + a_P$. We then define $\mathrm{CaSF}(\Xi)$ to consist of all $h \in \mathrm{SF}(\Xi)$ such that, for every marked $\sigma \in \Sigma$, $h|_\sigma(0)$ is a principal divisor on Y. Both $\mathrm{SF}(\Xi)$ and $\mathrm{CaSF}(\Xi)$ have a natural group structure. There is a group isomorphism from $\mathrm{CaSF}(\Xi)$ to the group $T\text{-}\mathrm{CaDiv}(X(\Xi))$ of T-invariant Cartier divisors on $X(\Xi)$; we denote the divisor associated to h by D_h. We call a support function h *ample* if D_h is ample.

PROPOSITION 2.1 [PS, 3.28]. *Consider $h \in \mathrm{CaSF}(\Xi)$. Then h is ample if and only if* (a) *h is strictly concave and* (b) *for all unmarked $\sigma \in \Sigma$ with σ full-dimensional, $-\deg h|_\sigma(0) > 0$.*

Given a support function $h \in \mathrm{CaSF}(\Xi)$, we define its weight polytope $\square_h \subset M_\mathbb{Q}$ by

$$\square_h = \{u \in M_\mathbb{Q} \mid h^0(v) \le \langle v, u\rangle \ \forall v \in N_\mathbb{Q}\}.$$

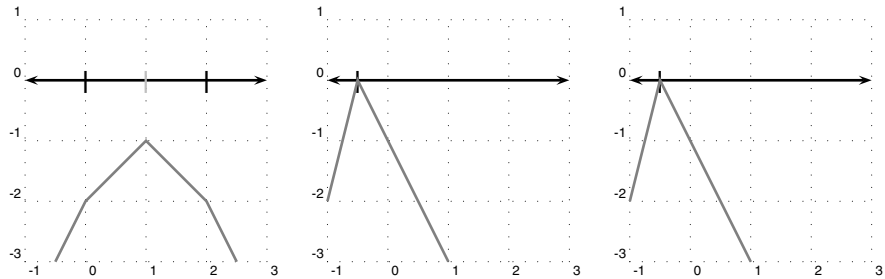
We then define the dual of h to be the piecewise affine concave function $h^*\colon \square_h \to \mathrm{Div}_\mathbb{Q} Y$ given by

$$h^* = \sum_{P \in Y} h_P^* \cdot P, \qquad h_P^*(u) = \min_{\substack{v \in \Xi_P \\ v \text{ vertex}}} \langle v, u\rangle - h_P(v).$$

PROPOSITION 2.2 [PS, 3.23]. *For $h \in \mathrm{CaSF}(\Xi)$ and $X = X(\Xi)$, we have*

$$H^0(X, D_h)_u = \begin{cases} H^0(Y, h^*(u)) & \text{if } u \in \square_h \cap M, \\ 0 & \text{if } u \notin \square_h \cap M. \end{cases}$$

EXAMPLE. Continuing the example from Section 1, the support function h pictured in Figure 2 corresponds to a divisor on $X(\Xi')$. In fact, using the formula for canonical divisors from [PS], one easily checks that $D_h = \phi^*(-2K) - E$, where K is a canonical divisor on $X(\Xi)$ and E is the exceptional divisor of φ. Using Proposition 2.1, we easily check that D_h is ample.



(a) h_0 (b) h_∞ (c) h_1

Figure 2 The support function for $\varphi^*(-2K) - E$

3. Divisorial Polytopes

DEFINITION. A divisorial polytope $(\Psi, \square)$ consists of a lattice polytope $\square \subset M_{\mathbb{Q}}$ and a piecewise affine concave function

$$\Psi = \sum \Psi_P \cdot P : \square \to \mathrm{Div}_{\mathbb{Q}} Y$$

such that:

(i) $\deg \Psi(u) > 0$ for u in the interior of $\square$;
(ii) $\deg \Psi(u) > 0$ or $\Psi(u) \sim 0$ for u a vertex of $\square$; and
(iii) for all $P \in Y$, the graph of Ψ_P is integral (i.e., has its vertices in $M \times \mathbb{Z}$).

We often will call the pair $(\Psi, \square)$ simply Ψ.

The set of divisorial polytopes for fixed lattice M and fixed curve Y actually form a natural semigroup. Indeed, for divisorial polytopes $(\Psi', \square')$ and $(\Psi'', \square'')$, we define $\Psi' + \Psi'' : (\square' + \square'') \to \mathrm{div}_{\mathbb{Q}} Y$ via

$$(\Psi' + \Psi'')(u) = \sum \max_{\substack{u'+u''=u \\ u' \in \square',\, u'' \in \square''}} \Psi'_P(u') + \Psi'_P(u'').$$

The neutral element is then obviously the constant function 0 on the 0 polytope. For any $k \in \mathbb{N}$ and divisorial polytope Ψ, we similarly define $k \cdot \Psi$ to be the k-fold sum of Ψ.

Before proceeding to associate a marked fansy divisor and support function to a divisorial polytope, we briefly recall the toric construction of a fan from a polytope. Consider a polytope $\square \subset M_{\mathbb{Q}}$. For every face F of $\square$, we consider the cone $\sigma_F \subset N_{\mathbb{Q}}$ consisting of all v such that $\langle v, \cdot \rangle$ obtains its minimum at F; these are exactly the inner normal vectors at F. The cones σ_F form a fan—the normal fan of $\square$, which can be seen as spanned by the regions where the piecewise linear function $\min_{u \in \square} \langle u, \cdot \rangle$ is linear. The corresponding face to a given cone σ of the normal fan we denote by F_σ. The described correspondence between faces of $\square$ and cones of the normal fan is inclusion reversing, and it maps faces of dimension r to cones of dimension $\dim N - r$. Moreover, we have $\langle u - u' \mid u, u' \in F \rangle = \sigma_F^{\perp}$.

PROPOSITION 3.1. *Let Ξ be a marked fansy divisor, and let $g, h \in \mathrm{CaSF}(\Xi)$ be ample. Then*:

(i) *$(g^*, \square_g)$ and $(h^*, \square_h)$ are divisorial polytopes*;
(ii) *$(g + h)^* = g^* + h^*$; and*
(iii) *if $g^* = h^*$, then $g = h$.*

Proof. Every maximal cone $\sigma \in \mathrm{tail}\, \Xi$ corresponds to a vertex u_σ of $\square_g$. Moreover, the concaveness of g implies that $-g|_\sigma(0) = g^*(u_\sigma)$. Now the ampleness condition on g implies that $\deg g^*(u_\sigma) > 0$ for unmarked σ, and the Cartier condition implies that $g^*(u_\sigma) \sim 0$ for marked σ. Since g_P is integral, the same is true for the graph of g_P^* and the first claim follows. The remaining two claims are easily seen from the definitions of g^* and h^*. $\square$

We now show how to associate a marked fansy divisor and support function to a divisorial polytope $(\Psi, \square)$. We begin by setting $\Psi_P^*(v) = \min_{u\in\square}(\langle v, u\rangle - \Psi_P(u))$, which is a piecewise affine concave function on $N_\mathbb{Q}$. Now let Ξ_P be the polyhedral subdivision of $N_\mathbb{Q}$ induced by Ψ_P^* and take $\Xi = \sum \Xi_P \cdot P$. Furthermore, we add a mark to an element $\sigma \in \operatorname{tail}(\Xi)$ if $(\deg \circ \Psi)|_{F_\sigma} \equiv 0$, where $F_\sigma \prec \square$ is the face where $\langle \cdot, v\rangle$ takes its minimum for all $v \in \sigma$.

THEOREM 3.2. *With notation as in the construction just described, Ξ is a marked fansy divisor and $\Psi^* = \sum \Psi_P^* \cdot P \in \mathrm{CaSF}(\Xi)$ is a support function that satisfies the following two properties*:

(i) Ψ^* *is ample*;
(ii) $(\Psi^{**}, \square_{\Psi^*}) = (\Psi, \square)$.

Thus, the construction induces a correspondence between divisorial polytopes and pairs $(X, \mathcal{L})$ of complexity-1 varieties with an invariant ample line bundle.

Proof. The maximal polyhedra in Ξ_P consist of those v such that the minimum of $(\langle v, \cdot\rangle - \Psi_P(\cdot))$ is realized by the same vertex $u \in \square$. We will denote such a polytope by Δ_P^u. For $w \in \Delta_P^u$ and $v \in \sigma_u$ we obviously have $v + w \in \Delta_u$. Hence, the tailfan of Ξ_P is exactly the normal fan of $\square$.

Next we have to check that properties (ii)–(iv) for the markings of a fansy divisor are fulfilled. For condition (iv) we must check that, for any marked cone, all cones that contain it are also marked. By our setting of marks this corresponds to the fact that if $(\deg \circ \Psi)|_F \equiv 0$ holds then it is also true for all faces of F.

We now turn to conditions (ii) and (iii). Fix some vertex u of $\square$ with $\deg \Psi(u) = 0$, and let σ be the corresponding cone. We now consider some $v \notin \sigma$. This implies that $\langle v, \cdot\rangle$ does not become minimal at u. Since $\deg \Psi(u') \geq 0$, the minimum of $(\langle v, \cdot\rangle - \deg \Psi(\cdot))$ also cannot be realized at u and $v \notin \sum_P \Delta_P^u = \deg \mathcal{D}^\sigma$. Because $\deg \Psi(u') > 0$ for some u', we also infer that $0 \notin \deg \mathcal{D}^\sigma$. Hence we obtain $\deg \mathcal{D}^\sigma \subsetneq \sigma$.

We next assume that $\deg \mathcal{D}^\sigma \cap \tau \neq \emptyset$ for some face τ of σ. We choose some $v \in \deg \mathcal{D}^\sigma \cap \tau$. Since $v \in \deg \mathcal{D}^\sigma$ we know that $(\langle v, \cdot\rangle - \deg \Psi(\cdot))$ obtains its minimum at u. Hence $(\langle v, u'\rangle - \deg \Psi(u')) \geq (\langle v, u\rangle - \deg \Psi(u))$ for any element $u' \in \square$. For $u' \in F_\tau$ we get $\langle v, u\rangle = \langle v, u'\rangle$ since $u' - u \in \tau^\perp$. This implies that $\deg \Psi(u') = \deg \Psi(u) = 0$. Therefore, $(\deg \circ \Psi)|_{F_\tau} \equiv 0$. By construction of Ξ we thus have that τ is marked, too. For the other direction, assume that $(\deg \circ \Psi)|_{F_\tau} \equiv 0$ for some $\tau \prec \sigma \in C$. We choose any interior point $v \in \operatorname{relint} \tau$. We know that the elements of $\deg \mathcal{D}^\sigma$ are those v such that $(\langle v, \cdot\rangle - \Psi_P(\cdot))$ takes its minimum at $u = F_\sigma$. For any $u'' \notin F_\tau$ we then get $\langle v, u''\rangle > \langle v, u\rangle$ and hence $(\langle k \cdot v, u''\rangle - \deg \Psi(u')) > (\langle k \cdot v, u\rangle - \deg \Psi(u))$ for $k \gg 0$. Since $\deg \Psi(u') = \deg \Psi(u)$ holds for $u' \in F_\tau$, we conclude that $k \cdot v \in \deg \mathcal{D}^\sigma \cap \tau$. This proves (iii).

To finish the proof of (ii), assume that $\deg \mathcal{D}^\sigma(w) = 0$. We have to show that a multiple of $\mathcal{D}^\sigma(w)$ is principal. Without loss of generality, we may assume that $\tau = \operatorname{face}(\sigma, u')$ is a facet; thus $\tau = \sigma \cap \sigma'$ for another maximal cone σ' with corresponding vertex u'. Now $w = \lambda \cdot (u' - u)$ and $u' - u \in \tau^\perp$. By the last step we know that $\deg \mathcal{D}^{\sigma'} \cap \tau \neq \emptyset$ and hence, for every P, there is a $v_P \in \Delta_P^u \cap \Delta_P^{u'}$. This implies

that $(\langle v_P, u\rangle - \Psi_P(u)) = (\langle v_P, u'\rangle - \Psi_P(u'))$. Thus we obtain $\min\langle \Delta_P^u, u'-u\rangle = \langle v_P, u'-u\rangle = \Psi_P(u') - \Psi_P(u)$. Condition (ii) then follows from the fact that $\Psi(u)$ and $\Psi(u')$ are principal, since $\mathcal{D}^\sigma(\lambda\cdot(u'-u)) = \lambda\cdot(\Psi(u')-\Psi(u))$.

Now Ψ_P^* is strictly concave on Ξ_P by the construction of Ξ. Furthermore, for σ maximal we have $\Psi^*|_\sigma(0) = -\Psi(u_\sigma)$. Hence the ampleness follows from the condition $\deg\Psi(u) > 0$ for σ_u unmarked. Finally, a simple calculation shows that $(\Psi^{**}, \square_{\Psi^*}) = (\Psi, \square)$. □

REMARK. Two divisorial polytopes $(\Psi, \square)$ and $(\Psi', \square')$ give rise to isomorphic pairs $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ if and only if there exist isomorphisms $F\colon M' \to M$ and $\varphi\colon Y \to Y'$ as well as a linear map A from M' to the principal divisors on Y' such that

$$\square = F(\square') \quad \text{and} \quad \Psi' = \varphi^* F^* \Psi + A.$$

REMARK. Let $\Delta \subset M'_{\mathbb{Q}}$ be a polytope in some lattice M'. Consider an exact sequence

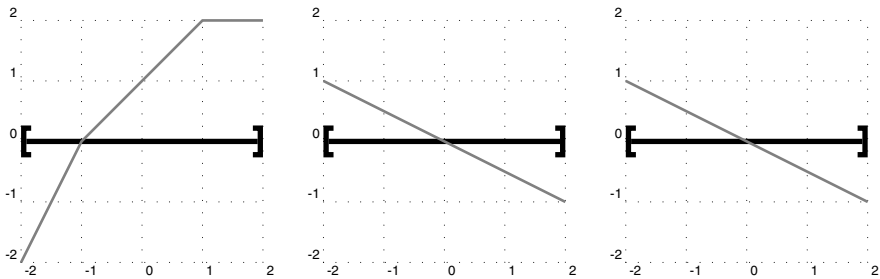
$$0 \to \mathbb{Z} \xrightarrow{F} M' \xrightarrow{G} M \to 0$$

corresponding to the torus inclusion $T_M \hookrightarrow T_{M'}$ of codimension 1. We choose a section $s\colon M \hookrightarrow M'$ and consider the map $\Psi_\Delta\colon G(\Delta) \to \mathrm{Div}(\mathbb{P}^1)$ given by

$$(\Psi_\Delta)_0(u) = \max\{a \in \mathbb{Q} \mid F_{\mathbb{Q}}(a) + s(u) \in \Delta \cap G_{\mathbb{Q}}^{-1}(u)\},$$
$$(\Psi_\Delta)_\infty(u) = -\min\{a \in \mathbb{Q} \mid F_{\mathbb{Q}}(a) + s(u) \in \Delta \cap G_{\mathbb{Q}}^{-1}(u)\}.$$

Then $(\Psi_\Delta, G(\Delta))$ is a divisorial polytope. Moreover, for Ψ_Δ the previous construction yields exactly the toric variety and the ample divisor corresponding to Δ but with the restricted torus action of T_M.

EXAMPLE. Consider the divisorial polytope Ψ on the interval $[-2, 2]$ pictured in Figure 3. One easily checks that the corresponding marked fansy divisor is exactly Ξ' from the example in Section 1 and that the corresponding support function is exactly the function h from the example in Section 2. Conversely, one easily checks that $h^* = \Psi$.



(a) Ψ_0 (b) Ψ_∞ (c) Ψ_1

Figure 3 A divisorial polytope on $\mathbb{P}^1$

We now describe how to read off simple geometric information about a projective T-variety from the corresponding divisorial polytope. For the following, we fix some divisorial polytope $(\Psi, \square)$ with corresponding projective variety X and ample divisor D. We first use our divisorial polytope to define some other polytopes.

DEFINITION. For a finite set of points $I \subset Y$, define

$$\Delta(\Psi, I) := \operatorname{Conv}\left(\left\{\left(u, \sum_{P \in I} \Psi_P(u)\right) \;\middle|\; u \in \square\right\} \cup \left\{\left(u, \sum_{P \notin I} -\Psi_P(u)\right) \;\middle|\; u \in \square\right\}\right) \subset M_{\mathbb{Q}} \times \mathbb{Q}.$$

For any point $P \in Y$, define

$$\tilde{\Delta}(\Psi, P) := \operatorname{Conv}\left(\{(u, \Psi_P(u)) \mid u \in \square\} \cup \square \times \min_{u \in \square} \Psi_P(u)\right) \subset M_{\mathbb{Q}} \times \mathbb{Q}.$$

Note that, although $\Delta(\Psi, I)$ need not have lattice vertices, $\tilde{\Delta}(\Psi, P)$ is always a lattice polytope.

PROPOSITION 3.3. *Let $m = \dim M_{\mathbb{Q}}$. Then*

$$D^{m+1} = (m+1)! \cdot \operatorname{vol} \Delta(\Psi, I)$$

for any set of points $I \subset Y$.

Proof. See [PS, Prop. 3.31]. □

For any polytope Δ with lattice vertex v, we say that Δ is *smooth at v* if the directions of Δ at v form a lattice basis. Now, for any $P \in Y$, consider some $v \in \square$ with $(v, \Psi_P(v))$ a vertex of the graph of Ψ_P.

DEFINITION. We say that Ψ is smooth at (P, v) if:

(i) for $\deg \Psi(v) > 0$, $\Delta(\Psi, P)$ is smooth at $(v, \Psi_P(v))$; or
(ii) for $\deg \Psi(v) = 0$, $Y = \mathbb{P}^1$ and there exist points $P_1, P_2 \in Y$ such that, for all points $P \neq P_1, P_2$, $(v, \Psi_P(v))$ is contained in only one full-dimensional polytope in Γ_{Ψ_P} (which additionally has integral slope) and the polytope $\Delta(\Psi, P_1)$ is smooth at $(v, \Psi_{P_1}(v))$.

PROPOSITION 3.4. *The T-variety X corresponding to $(\Psi, \square)$ is smooth if and only if, for every $P \in Y$ and every $v \in \square$ with $(v, \Psi_P(v))$ a vertex of Γ_{Ψ_P}, Ψ is smooth at (P, v).*

Proof. The vertices $(v, \Psi_P(v))$ of the graphs of Ψ_P correspond to affine invariant charts of the corresponding variety. If $\deg \Psi(v) > 0$ then the corresponding chart has affine locus, and one easily checks that criterion (i) corresponds to the hypothesis of Theorem 3.3 in [S]. On the other hand, if $\deg \Psi(v) > 0$ then the corresponding chart has complete locus and the criterion (ii) corresponds to the hypothesis of Proposition 3.1 in [S]. □

Finally, suppose that the divisor D is very ample and gives a projective embedding. We are interested in the Hilbert polynomial $\mathcal{H}_D$ of D. Recall that for natural

numbers k sufficiently large, $\mathcal{H}_D(k) = \dim H^0(X, k \cdot D)$. On the other hand, recall that for any lattice polytope Δ of dimension d there is a unique polynomial E_Δ of degree d, called the Ehrhart polynomial of Δ, such that $E_\Delta(k)$ is the number of lattice points in $k \cdot \Delta$ for any $k \in \mathbb{N}$.

DEFINITION. Let $\mathcal{P}$ be the set of all $P \in Y$ such that Ψ_P is not trivial. We then define the *Ehrhart polynomial* E_Ψ of the divisorial polytope Ψ by

$$E_\Psi(k) = E_\square(k) + \sum_{P \in \mathcal{P}} \Big(E_{\tilde{\Delta}(\Psi, P)}(k) - E_\square(k) \cdot \Big(1 - k \cdot \min_{u \in \square} \Psi_P(u)\Big)\Big).$$

REMARK. One easily checks that if Ψ only has nontrivial coefficients for two points P_1 and P_2, then $E_\Psi = E_{\Delta(\Psi, P_1)}$.

PROPOSITION 3.5. *We have*

$$E_\Psi \geq \mathcal{H}_D \geq E_\Psi - g(Y) \cdot E_\square.$$

Furthermore, if $\deg\lfloor \Psi(u) \rfloor \geq 2g(Y) - 1$ *for all* $u \in \square \cap M$, *then* $\mathcal{H}_D = E_\Psi - g(Y) \cdot E_\square$. *In particular, if* $Y = \mathbb{P}^1$ *then* $\mathcal{H}_D = E_\Psi$.

Proof. For any $k \in \mathbb{N}$, any $P \in \mathcal{P}$, and any $u \in k \cdot \square \cap M$, we have that $\lfloor (k \cdot \Psi)_P(u) \rfloor - k \cdot \min_{v \in \square} \Psi_P(v) + 1$ is equal to the number of lattice points in $k \cdot \tilde{\Delta}(\Psi, P)$ projecting to u. Summing over all $u \in k \cdot \square \cap M$ and $P \in \mathcal{P}$, we get

$$\sum_{u \in k \cdot \square \cap M} \deg\lfloor (k \cdot \Psi)(u) \rfloor = \sum_{P \in \mathcal{P}} \Big(E_{\tilde{\Delta}(\Psi, P)}(k) - E_\square(k) \cdot \Big(1 - k \cdot \min_{v \in \square} \Psi_P(v)\Big)\Big)$$

and thus

$$\sum_{u \in k \cdot \square \cap M} 1 + \deg\lfloor (k \cdot \Psi)(u) \rfloor = E_\Psi(k).$$

Now, for k large enough,

$$\mathcal{H}_D(k) = \sum_{u \in k \cdot \square \cap M} h^0(Y, (k \cdot \Psi)(u)).$$

Applying the Riemann–Roch theorem for curves, we have that

$$\deg\lfloor (k \cdot \Psi)(u) \rfloor + 1 - g(Y) \leq h^0(Y, (k \cdot \Psi)(u)) \leq \deg\lfloor (k \cdot \Psi)(u) \rfloor + 1$$

and the proposition follows. □

EXAMPLE. We apply Propositions 3.1–3.5 to the divisorial polytope Ψ from Figure 3. Regardless of the set of points $I \subset Y$, we always have $\operatorname{vol} \Delta(\Psi, I) = 3$ and thus that the corresponding divisor D has self-intersection number 6. We can also see that the corresponding projective surface is not smooth: Ψ is not smooth at $(P, \pm 2)$ for any point $P \in Y$. Finally, we will see in Section 5 that D is very ample, so we can calculate the Hilbert polynomial of D. Indeed, we have

$$\begin{aligned} E_\square(k) &= 4k + 1, \\ E_{\tilde{\Delta}(\Psi, 0)} &= 11k^2 + 6k + 1, \\ E_{\tilde{\Delta}(\Psi, \infty)} &= E_{\tilde{\Delta}(\Psi, 1)} = 4k^2 + 4k + 1 \end{aligned}$$

and thus

$$\mathcal{H}_D(k) = E_\Psi(k) = 3k^2 + 2k + 1.$$

4. Affine Cones

Let Ξ be a marked fansy divisor on a curve Y, and let $h \in \mathrm{CaSF}(\Xi)$ be such that D_h is globally generated. Then the sections of D_h determine a map $f\colon X(\Xi) \to \mathbb{P}^n$; we denote the image of f by X. Note that X also comes with a natural complexity-1 T-action, but in general X need not be normal. By $C(X)$ we denote the affine cone over X with respect to this embedding; let $\widetilde{C(X)}$ be the normalization of $C(X)$. The following proposition tells us how to describe $\widetilde{C(X)}$ in terms of a polyhedral divisor.

PROPOSITION 4.1. *With h as just described, set*

$$\mathcal{D} = \sum_P \mathrm{Conv}(\Gamma_{-h_P}) \cdot P,$$

where Γ_{-h_P} is the graph of $-h_P$. If the map f is birational, then $\widetilde{C(X)} = X(\mathcal{D})$ and $\mathcal{D}$ is a proper polyhedral divisor on Y.

Proof. The homogeneous coordinate ring of X with respect to the given embedding is $A = \bigoplus_{k\geq 0} S^k(H^0(X(\Xi), D_h))$, where S^k is the kth symmetric product. Thus, $C(X) = \operatorname{Spec} A$. Now the integral closure of A is $\tilde{A} = \bigoplus_{k\geq 0} H^0(X(\Xi), k \cdot D_h)$; this follows from [Ha, Exer. II.5.14(a)] and the fact that f is birational. Thus $\widetilde{C(X)} = \operatorname{Spec} \tilde{A}$.

On the other hand, we claim that

$$\bigoplus_{k\geq 0} H^0(X, k \cdot D_h) = \bigoplus_{(u,k)\in \mathrm{tail}(\mathcal{D})\cap(M\times\mathbb{Z})} H^0(Y, \mathcal{D}((u,k))).$$

Indeed, $H^0(X, k \cdot D_h) = \bigoplus_{u\in\square_{k\cdot h}} H^0(Y, (k\cdot h)^*(u))$ by Proposition 2.2. Furthermore, $\square_{k\cdot h} = \{u \in M_\mathbb{Q} \mid (u,k) \in \mathrm{tail}\,\mathcal{D}\}$. The claim then follows from the fact that, for $P \in Y$,

$$\mathcal{D}((u,k))_P = \min\langle \mathrm{Conv}(\Gamma_{-h_P}), (u,k)\rangle = \min_{v\in N_\mathbb{Q}} \langle v, u\rangle - k \cdot h_P(v) = (k\cdot h)^*(u).$$

We thus have

$$\widetilde{C(X)} = \operatorname{Spec} \bigoplus_{(u,k)\in \mathrm{tail}(\mathcal{D})\cap(M\times\mathbb{Z})} H^0(Y, \mathcal{D}((u,k))) = X(\mathcal{D}).$$

Since f is birational, it follows that $\deg h^*(u) > 0$ for u in the interior of $\square_h$; therefore, $\mathcal{D}$ is proper. □

REMARK. A sufficient criterion for f to be birational is that $h^*(u)$ be very ample for some $u \in \square_h \cap M$ and that the set

$$\{u \in \square_h \cap M \mid \dim H^0(Y, h^*(u)) > 0\}$$

generate the lattice M.

REMARK. Let X be the image in projective space of some complexity-1 T-variety $\tilde{X}$ via a birational map corresponding to an invariant, globally generated, ample divisor D. Suppose now that the normalized affine cone over X is given by $\widetilde{C(X)} = X(\mathcal{D})$, where $\mathcal{D}$ is a polyhedral divisor on some smooth projective curve Y with corresponding lattice N'. Choose some isomorphism $N' \cong N \oplus \mathbb{Z}$, where the second term in the direct sum corresponds to the natural $\mathbb{C}^*$-action on the cone $\widetilde{C(X)}$. Reversing Proposition 4.1, we can easily recover a marked fansy divisor Ξ and a support function $h = \sum h_P \cdot P$ such that $\tilde{X} = X(\Xi)$ and $D = D_h$. Indeed, let $h\colon N_{\mathbb{Q}} \to \mathbb{Q}$ be defined by

$$-h(v)_P = \min \pi_2(\pi_1^{-1}(v) \cap \mathcal{D}_P),$$

where π_i is the projection of $N_{\mathbb{Q}} \oplus \mathbb{Q}$ onto the ith factor. Let Ξ be the polyhedral subdivision of N induced by the piecewise affine function h. We add marks to a top-dimensional cone σ in the tailfan of Ξ if $h|_\sigma(0)$ is principal, and we add marks to lower-dimensional cones τ if, for some full-dimensional marked σ, $\tau \prec \sigma$ and $\deg \mathcal{D}^\sigma \cap \tau \neq \emptyset$. Then one easily checks that Ξ is a marked fansy divisor, $h \in \mathrm{CaSF}(\Xi)$, $X = X(\Xi)$, and the embedding $X \hookrightarrow \mathbb{P}^n$ is given by the linear system D_h. Note that this procedure for determining Ξ from $\mathcal{D}$ coincides with a special case of the procedure in [AHe, Sec. 5], although we now also retain information on the linear system D_h of the embedding. The description of the corresponding divisorial polytope is even more simple: one easily checks that $\square_h$ is the projection of $(M_{\mathbb{Q}} \times \{1\}) \cap \mathrm{tail}\,\mathcal{D}^\vee$ onto $M_{\mathbb{Q}}$ and that $h^*(u) = \mathcal{D}((u, 1))$.

EXAMPLE. It is not difficult to check that the divisor D_h coming from the support function h on Ξ' of Figure 2 is globally generated. In fact, it follows from the proof of [PS, 3.27] that any semiample divisor D_h is globally generated if $h^*(u)$ is globally generated for all $u \in \square_h$. For $Y = \mathbb{P}^1$, this is always the case; thus, D_h in our example is globally generated and defines a morphism to projective space with some T-invariant image X. Using the first remark following the proof of Proposition 4.1, it is straightforward to check that this map is birational. By the proposition, we then know that $\widetilde{C(X)} = X(\mathcal{D})$, where $\mathrm{tail}(\mathcal{D})$ is generated by $(-1, 2)$, $(1, 2)$, $\mathcal{D}_0$ has vertices $(-1, 2)$, $(0, 1)$, $(1, 2)$, and $\mathcal{D}_\infty$ and $\mathcal{D}_1$ have sole vertex $(-1/2, 0)$.

5. Finding Generators

Recall that, for an affine toric variety coming from some pointed cone σ, a unique set of minimal generators of the corresponding multigraded algebra can be determined by calculating a Hilbert basis of $\sigma^\vee$. The goal of this section is to present a similar result for complexity-1 T-varieties. We can then use this result to determine when a projective embedding is projectively normal.

Let $\mathcal{D}$ be a proper polyhedral divisor with tailcone σ on a smooth projective curve Y. For $u \in \sigma^\vee \cap M$ we define $\mathcal{A}_u := H^0(Y, \lfloor \mathcal{D}(u) \rfloor)$ and

$$\mathcal{A} = \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{A}_u.$$

Thus, our goal is to find generators of the $\mathbb{C}$-algebra $\mathcal{A}$.

Let g be the genus of Y and let c be the minimum of 0 and one less than the number of $P \in Y$ such that $\mathcal{D}_P$ is not a lattice polyhedron. Then, for any $u \in \sigma^\vee \cap M$,

$$\deg \lfloor \mathcal{D}(u) \rfloor \geq \deg \mathcal{D}(u) - c.$$

We now take Σ be the coarsest common refinement of the set of all normal fans of $\mathcal{D}_P$, where P is a point on Y. Note that $\mathcal{D}$ is linear on each cone of Σ. Each cone τ of Σ defines a subalgebra

$$\mathcal{A}_\tau := \bigoplus_{u \in \tau \cap M} \mathcal{A}_u.$$

Note that the union of all such subalgebras is again $\mathcal{A}$. For any cone $\tau \in \Sigma$, let τ' be a pointed cone and let $u_\tau \in M \cap \tau \cap -\tau$ be a weight such that $\tau = \tau' + \langle u_\tau \rangle$. Let $\mathrm{HB}(\tau')$ be the Hilbert basis of τ'; note that the semigroup $\tau \cap M$ is generated by $\mathrm{HB}(\tau') \cup \{u_\tau\}$. Furthermore, for $u \in \mathrm{HB}(\tau') \cup \{u_\tau\}$ we define $\alpha_u \in \mathbb{N}$ to be the smallest number such that:

(i) $\mathcal{D}(\alpha_u \cdot u)$ is principal and $\lfloor \mathcal{D}(\alpha_u \cdot u) \rfloor) = \mathcal{D}(\alpha_u \cdot u)$; or
(ii) $\alpha_u/2 \in \mathbb{N}$, $\deg \mathcal{D}(\alpha_u \cdot u) \geq 4g + 2 + 2c$, and $\lfloor \mathcal{D}((\alpha_u/2 \cdot u)) \rfloor) = \mathcal{D}((\alpha_u/2) \cdot u)$.

Observe that the properness of $\mathcal{D}$ guarantees that such an α_u exists. Also, some multiple of $\mathcal{D}(u_\tau)$ must be principal because $\deg \mathcal{D}(u_\tau) = 0$.

Finally, we set

$$\mathcal{G}_\tau := \left\{ \sum_{u \in \mathrm{HB}(\tau')} k_u \cdot u \;\middle|\; 0 \leq k_u \leq \alpha_u \right\} \cup \{\alpha_{u_\tau} \cdot u_\tau\}.$$

THEOREM 5.1. *For $\tau \in \Sigma$, the algebra $\mathcal{A}_\tau$ is generated in degrees $\mathcal{G}_\tau$. In particular, $\mathcal{A}$ is generated in degrees $\mathcal{G}_\mathcal{D} := \bigcup_{\tau \in \Sigma} \mathcal{G}_\tau$.*

We will need the following lemma.

LEMMA 5.2. *Let D_1, D_2 be divisors on a smooth curve Y_0. Then the natural map*

$$H^0(Y, D_1) \times H^0(Y, D_2) \to H^0(Y, D_1 + D_2)$$

is surjective if

(i) *D_1 is principal or*
(ii) *Y_0 is complete,* $\deg D_1 \geq 2g + 1$, *and* $\deg D_2 \geq 2g$.

Proof. The first case is immediate. The second case is due to Mumford [M]. □

Proof of Theorem 5.1. Fix some $\tau \in \Sigma$ and consider $u \in \tau' \cap M$ such that $u \notin \mathcal{G}_\tau$. Then there exist some $u' \in \mathrm{HB}(\tau')$ and $u'' \in \tau' \cap M$ such that $u = \alpha_{u'} u' + u''$. Suppose first that $\mathcal{D}(\alpha_{u'} u')$ is principal. Then

$$\lfloor \mathcal{D}(u) \rfloor = \lfloor \mathcal{D}(\alpha_{u'} u') \rfloor + \lfloor \mathcal{D}(u'') \rfloor$$

and it follows from Lemma 5.2(i) that $\mathcal{A}_u$ is generated by $\mathcal{A}_{\alpha_{u'} u'}$ and $\mathcal{A}_{u''}$. Now suppose instead that $\deg \alpha_{u'} \mathcal{D}(u') \geq 4g + 2 + c$. Then

$$\lfloor \mathcal{D}(u) \rfloor = \lfloor \mathcal{D}((\alpha_{u'}/2) u') \rfloor + \lfloor \mathcal{D}((\alpha_{u'}/2) u' + u'') \rfloor$$

and we have

$$\deg\lfloor \mathcal{D}((\alpha_{u'}/2)u')\rfloor \geq 2g+1+c,$$
$$\deg\lfloor \mathcal{D}((\alpha_{u'}/2)u'+u'')\rfloor \geq 2g+1+c-c \geq 2g+1.$$

Thus, it follows from Lemma 5.2(ii) that $\mathcal{A}_u$ is generated by $\mathcal{A}_{(\alpha_{u'}/2)u'}$ and $\mathcal{A}_{(\alpha_{u'}/2)u'+u''}$. Continuing this argument by induction, we can conclude that $\mathcal{A}_u$ is generated in degrees lying in $\mathcal{G}_\tau$ for any $u \in \tau' \cap M$.

Now consider any $u \in \tau \cap M$ such that $u \notin \mathcal{G}_\tau$. We can write $u = k\alpha_{u_\tau}u_\tau + u'$ for some $u' \in \tau' \cap M$ and $k \in \mathbb{N}$. Then, once again by Lemma 5.2(i), $\mathcal{A}_u$ is generated by $\mathcal{A}_{k\alpha_{u_\tau}u'_\tau}$ and $\mathcal{A}_{u'}$; but we have just shown that $\mathcal{A}_{u'}$ is generated by degrees in $\mathcal{G}_\tau$. Hence we may conclude that $\mathcal{A}_\tau$ is generated in degrees $\mathcal{G}_\tau$. The statement concerning $\mathcal{A}$ follows immediately. □

We can now use Theorem 5.1 to give a finite list of generators of $\mathcal{A}$. Note that we can consider $\mathcal{A}_0$ as a finitely generated $\mathbb{C}$-algebra—say, with generators $f_0^1, \ldots, f_0^{d_0}$. Now, for $u \in \mathcal{G}_\mathcal{D}$ and $u \neq 0$, let $f_u^1, \ldots, f_u^{d_u}$ generate $\mathcal{A}_u$ as an $\mathcal{A}_0$-module. Then the following corollary is immediate.

COROLLARY 5.3. *The algebra $\mathcal{A}$ is generated as a $\mathbb{C}$-algebra by*

$$\{f_u^i\}_{\substack{u\in\mathcal{G}_\mathcal{D}\\ 1\leq i\leq d_u}}.$$

The weight set $\mathcal{G}_\mathcal{D}$ and set of generators of $\mathcal{A}$ from Corollary 5.3 are in general not minimal. Yet it is immediately clear that, if the tail cone of $\mathcal{D}$ is full-dimensional, then

$$\mathcal{G}_\mathcal{D}^{\min} := \mathcal{G}_\mathcal{D} \setminus \left\{ u \in \mathcal{G}_\mathcal{D} \,\middle|\, \sum_{\substack{u'\in\mathcal{G}^\mathcal{D}\\ u-u'\in\sigma^\vee}} H^0(Y, \lfloor\mathcal{D}(u')\rfloor) \times H^0(Y, \lfloor\mathcal{D}(u-u')\rfloor) = H^0(Y, \lfloor\mathcal{D}(u)\rfloor) \right\}$$

is the unique minimal set of weights needed to generate $\mathcal{A}$. This set can be constructed by checking a *finite* number of conditions.

COROLLARY 5.4. *Let Ξ be a marked fansy divisor on a curve Y, and let $h \in \mathrm{CaSF}(\Xi)$ be such that D_h is very ample. Then the corresponding embedding is projectively normal if and only if all elements of $\mathcal{G}_\mathcal{D}^{\min}$ have last coordinate equal to 1, where $\mathcal{D}$ is defined as in Proposition 4.1.*

Proof. The embedding is projectively normal if and only if $\mathcal{A}$ is generated in degree 1 with respect to the relevant $\mathbb{Z}$-grading. □

EXAMPLE. Consider the polyhedral divisor $\mathcal{D}$ from the example in Section 4. One easily checks that $\mathcal{G}_\mathcal{D} = \{(-2,1),(-1,1),(0,1),(1,1),(2,1)\}$. Thus, the image X of $X(\Xi)$ under the linear system $|D_h|$ is projectively normal. It follows that D_h is very ample and that the corresponding map is actually an embedding. Indeed, on

the one hand, the quotient of $C(X)$ by $\mathbb{C}^*$ is clearly X; on the other hand, one can also check by calculation that the quotient of $C(X)$ by $\mathbb{C}^*$ is $X(\Xi)$.

REMARK. If Y is not a curve and instead we have $Y = \mathbb{P}^n$ or $Y = \mathbb{A}^n$, we can define a set $\mathcal{G}_{\mathcal{D}}$ similar to that in the preceding example and containing the weights generating $\mathcal{A}$. Indeed, in both cases we have statements similar to Lemma 5.2.

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