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Boundary Behavior of the Kobayashi–Royden Metric in Smooth Pseudoconvex Domains

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1. Introduction and Main Results

In this paper we discuss the problem of the boundary behavior of the Kobayashi–Royden metric (mainly) in the normal direction in smooth bounded pseudoconvex domains. We show two main results. One of the results states in particular that the Kobayashi–Royden metric in the normal direction in some class of smooth bounded pseudoconvex domains is estimated from below by such expressions as $1/d_D^{7/8}(z)$ (where $d_D(z)$ denotes the distance of z from the boundary of D). This improves the result of [Fu], where the author obtained the lower estimate with the exponent 5/6. On the other hand, we demonstrate that a careful study of an example in [FL] shows that the optimal exponent in the lower estimate of the Kobayashi–Royden metric in the normal direction is smaller than 1 (for C^k-smoothness, $k < \infty$). We also specify some obstacles for the rate of the increase in the C^∞ case.

Recall that the Kobayashi–Royden metric has a localization property (see e.g. [G; R]); therefore, we lose no generality in concentrating on domains that are defined globally. Recall also that one of reasons to study the boundary behavior of the Kobayashi–Royden metric is the problem of deciding whether any bounded smooth pseudoconvex domain is Kobayashi complete (see e.g. [JP]). The hope was that the Kobayashi–Royden metric in the normal direction would explode near the boundary as $1/d_D(z)$; it was one of the ideas that were used to show that smooth bounded pseudoconvex domains are Kobayashi complete. However, after many years of uncertainty, an example of Fornæss and Lee [FL] showed that such a lower bound is not valid. More precisely, the example is the following.

THEOREM 1 (see [FL]). *For any given increasing sequence $(a_\nu)_\nu$, $a_\nu \to \infty$, of positive numbers, there exist a bounded smooth pseudoconvex domain $D \subset \mathbb{C}^3$ and a decreasing sequence $(\delta_\nu)_\nu$ with $\delta_\nu \to 0$ such that*

$$\kappa_D(P_{\delta_\nu}; n) \le \frac{1}{(a_\nu \delta_\nu)},$$

where P is a suitable point from ∂D, $P_{\delta_\nu} = P - \delta_\nu n$, and n is the unit outward normal vector to ∂D at P.

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In the other direction, Fu showed that smooth bounded pseudoconvex domains have the following lower estimate (our formulation is weaker than the one in the original paper).

THEOREM 2 (see [Fu]). *Let D be a bounded C^3-smooth pseudoconvex domain given by the formula $D = \{r < 0\}$, where r is a C^3-smooth defining function (meaning that its Levi form is semipositive definite on the complex tangent space of any boundary point). Then there is a constant $c > 0$ such that*

$$\kappa_D(z; X) \geq c\frac{|\langle \partial r(z), X\rangle|}{|r(z)|^{2/3}}, \quad z \in D,\ X \in \mathbb{C}^n.$$

Moreover, it follows from [Fu] that if we make some additional assumption on the vector X and on the points z (e.g., that X is the unit outward normal vector to some boundary point P and that z lies on the line passing through P in the direction X), then in the preceding estimate we may replace the exponent $2/3$ with 5/6. We shall see in Theorem 4 that in many cases the exponent 5/6 may be replaced by 7/8.

Fu also conjectured that, in the class of smooth domains, the lower estimate of the Kobayashi–Royden metric as in Theorem 2 may be taken to be of the form $1/d_D^{1-\varepsilon}(z)$ with $\varepsilon > 0$ arbitrarily small. Note that the example of Fornæss and Lee shows that the exponent cannot be taken to be equal to 1 (equivalently, ε cannot be equal to 0).

However, a careful study of the Fornæss–Lee example shows that, in the case of C^k-smooth domains, an estimate as conjectured by Fu does not hold. In particular, we have the following result.

THEOREM 3. (1) *For any positive integer k, there exist a C^k-smooth bounded pseudoconvex domain D in $\mathbb{C}^3$, a positive number ε, and a decreasing sequence $(\delta_\nu)_\nu$ with $\delta_\nu \to 0$ such that*

$$\kappa_D(P_{\delta_\nu}; n) \leq \frac{1}{\delta_\nu^{1-\varepsilon}}.$$

Here P is a suitable point from ∂D, $P_{\delta_\nu} = P - \delta_\nu n$, and n is the unit outward normal vector to ∂D at P.

(2) *For any $\alpha > 0$, there exist a C^∞-smooth bounded pseudoconvex domain D in $\mathbb{C}^3$ and a decreasing sequence $(\delta_\nu)_\nu$ with $\delta_\nu \to 0$ such that*

$$\kappa_D(P_{\delta_\nu}; n) \leq \frac{1}{\delta_\nu(-\log \delta_\nu)^\alpha};$$

here P is a suitable point from ∂D, $P_{\delta_\nu} = P - \delta_\nu n$, and n is the unit outward normal vector to ∂D at P.

Note that Theorem 3 shows that, even in the C^∞ case, the proof of the Kobayashi completeness of the smooth bounded pseudoconvex domain cannot proceed by showing that the Kobayashi–Royden metric (in the normal direction) behaves like a "regular" integrable function of $d_D(z)$. Therefore, when all smooth bounded pseudoconvex domains are Kobayashi complete, the proof would require a more subtle reasoning.

We can, however, also say something in the positive direction. Namely, we may slightly improve the estimate given in Theorem 2. Unfortunately, the better estimate holds for smooth domains defined as sublevel sets of smooth plurisubharmonic defining function. (Recall that not all smooth bounded pseudoconvex domains are locally sublevel sets of smooth plurisubharmonic functions; see [B; F].)

THEOREM 4. *Let $D = \{r < 0\}$ be a bounded domain in $\mathbb{C}^n$, where $r\colon U \mapsto \mathbb{R}$ is a C^4-smooth plurisubharmonic defining function for D. Then there is a constant $C > 0$ such that*

$$\kappa_D(z; X) \geq \frac{C|\langle n(z), X\rangle|}{d_D^{7/8}(z)}$$

as z tends to ∂D. The vectors X are taken so that $\|X\| = o(1/d_D(z))|\langle n(z), X\rangle|$, where $n(z)$ denotes the unit outward normal vector to ∂D at the point of ∂D of the smallest distance from z.

Before we start the proofs, recall the definition of the Kobayashi–Royden (pseudo)-metric of a domain $D \subset \mathbb{C}^n$ (for basic properties of the Kobayashi–Royden metric, see [JP]). For $z \in D$ and $X \in \mathbb{C}^n$,

$$\kappa_D(z; X) := \inf\{\alpha > 0 : \text{there is an } f \in \mathcal{O}(\mathbb{D}, D) \text{ with } f(0) = z,\ \alpha f'(0) = X\}.$$

2. Proofs

We start with some preliminary considerations.

Let $D = \{r < 0\}$ be a domain in $\mathbb{C}^n$, where $r\colon U \mapsto \mathbb{R}$ is a C^{k+1}-smooth plurisubharmonic defining function with $r(0) = 0$. Then (up to a linear isomorphism) the Taylor expansion at 0 of order k of r is of the form

$$r(z) = \operatorname{Re} z_n + \sum_{j=2}^{k} Q_j(z) + R_k(z),$$

where $Q_j(z) = \sum_{|\alpha|+|\beta|=j} a^j_{\alpha,\beta} z^\alpha \bar{z}^\beta$ (note that then $a^j_{\alpha,\beta} = \bar{a}^j_{\beta,\alpha}$). We may also write $Q_j(z) = \tilde{Q}_j(z) + \hat{Q}_j(z)$, where

$$\tilde{Q}_j(z) = \sum_{|\alpha|+|\beta|=j, |\alpha|,|\beta|>0} a^j_{\alpha,\beta} z^\alpha \bar{z}^\beta, \quad \hat{Q}_j(z) = 2\operatorname{Re} \sum_{|\alpha|=j} a^j_{\alpha,\alpha} z^\alpha =: 2\operatorname{Re} H_j(z).$$

It follows from Taylor's formula that $\mathcal{L}R_k(z)|_{S^{2n-1}} = O(\|z\|^{k-1})$, where $\mathcal{L}\tilde{r}(z)(X)$ is the Levi form of $\tilde{r}$ at the point z in the direction of X; here S^{2n-1} denotes the $(2n-1)$-dimensional sphere. Consequently, $\mathcal{L}R_k(z)(z) = O(\|z\|^{k+1})$.

An easy calculation then gives the following formula:

$$\begin{aligned}\mathcal{L}Q_j(z)(z) &= \sum_{\nu=0}^{j} \nu(j-\nu) \sum_{|\alpha|=\nu, |\beta|=j-\nu} a^j_{\alpha,\beta} z^\alpha \bar{z}^\beta \\ &= \sum_{\nu=1}^{j-1} \nu(j-\nu) \sum_{|\alpha|=\nu, |\beta|=j-\nu} a^j_{\alpha,\beta} z^\alpha \bar{z}^\beta.\end{aligned}$$

In particular, $\mathcal{L}Q_2(z)(z) = \tilde{Q}_2(z)$ and $\mathcal{L}Q_3(z)(z) = 2\tilde{Q}_3(z)$.

In the sequel we shall denote by C_j different constants that depend only on the domain D.

Proof of Theorem 4. We leave the notation as before and make use of the preceding considerations.

First we prove the desired estimate but with the exponent equal to 3/4 (instead of 7/8). The assumptions of the theorem imply that, for $z \in U$, we have the following estimate:

$$\tilde{Q}_2(z) + 2\tilde{Q}_3(z) + C_1\|z\|^4 \geq 0;$$

when combined with the property

$$\begin{aligned}\min\{\tilde{Q}_2(z) + \tilde{Q}_3(z), \tilde{Q}_2(-z) + \tilde{Q}_3(-z)\}& \\ \geq \min\{\tilde{Q}_2(z) + 2\tilde{Q}_3(z), \tilde{Q}_2(-z) + 2\tilde{Q}_3(-z)\},&\end{aligned}$$

this gives, for z close to 0, the inequality

$$\begin{aligned} r(z) &\geq \operatorname{Re} z_n + Q_2(z) + Q_3(z) - C_2\|z\|^4 \\ &\geq \operatorname{Re} z_n + 2\operatorname{Re}(H_2(z) + H_3(z)) - C_3\|z\|^4. \end{aligned}$$

Therefore, shrinking U if necessary, we have the inclusion

$$D \subset \{\operatorname{Re} z_n + 2\operatorname{Re}(H_2(z) + H_3(z)) - C_3\|z\|^4 < 0\}.$$

Now, for $\delta > 0$ small enough and for $X \in \mathbb{C}^n$ and $X \neq 0$, take $\varphi \in \mathcal{O}(\mathbb{D}, D)$ such that $\varphi(0) = (0, \dots, 0, -\delta)$ and $\kappa\varphi'(0) = X$ for $\kappa > 0$. Note that $\|\varphi(\lambda) - \varphi(0)\| \leq C_4|\lambda|$. For $r \in (0,1)$ define $\psi_r(\lambda) := \varphi(r\lambda)$. Then $\|\psi_r(\lambda)\| \leq \delta + C_4 r$, $\lambda \in \mathbb{D}$.

Define $\Psi(z) := z_n + 2H_2(z) + 2H_3(z)$, and put $\varphi_r := \Psi \circ \psi_r$. Then $\varphi_r(\mathbb{D}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < C_5(\delta + r)^4\} =: S_{r,\delta}$ for δ small enough. Therefore,

$$\begin{aligned} \frac{r}{\kappa}\left|X_n(1 + O(\delta)) + \sum_{j=1}^{n-1} O(\delta)X_j\right| &= \frac{r}{\kappa}|\Psi'(0, \dots, 0, -\delta)X| \\ &= |\varphi_r'(0)| \leq C_6(\delta + (\delta + r)^4), \end{aligned}$$

where the last inequality follows easily from the formula for the Kobayashi–Royden metric for $S_{r,\delta}$.

Substitute $r = \delta^{1/4}$ for δ small enough. Then we get the lower estimate

$$\kappa \geq C_7\delta^{1/4}\frac{|X_n|(1 + \alpha(\delta))}{\delta + (\delta + \delta^{1/4})^4},$$

where $\alpha(\delta) \to 0$ as $\delta \to 0$ (here we use that X was chosen such that $\|X\| = o(1/\delta)|X_n|$). Consequently, we get the lower estimate

$$\kappa_D((0, \dots, 0, -\delta); X) \geq C_6|X_n|/\delta^{3/4}$$

as δ tends 0, where the vectors are taken such that $o(1/\delta)|X_n| \geq \|X\|$.

Our aim is to show that we may replace the exponent 3/4 with 7/8. Keeping in mind the previous estimate and retaining our previous notation, we have the following inequality:

$$\left\| \varphi(\lambda) - \varphi(0) - \frac{\lambda}{\kappa} X \right\| \le C_7 |\lambda|^2.$$

Proceeding as before we obtain that, for $|\lambda| \le r$ and δ small enough,

$$\|\varphi(\lambda)\| \le \delta + \frac{rC_8}{\kappa}\|X\| + C_7 r^2$$

or $\|\psi_r(\lambda)\| \le \delta + \frac{rC_8}{\kappa}\|X\| + C_7 r^2$, $\lambda \in \mathbb{D}$.

We may assume without loss of generality that $\|X\|$ is bounded from above (or even equal to 1); hence, proceeding exactly as before, we get the inequality

$$\kappa \ge \frac{C_9 r |X_n|(1 + \alpha(\delta))}{\delta + (\delta + r/\kappa + r^2)^4}.$$

Because we already know that $\kappa \ge C_{10}/(\delta^{3/4})$ (here we need the first part of the proof), putting $r = \delta^{1/8}$ yields the following estimate:

$$\kappa_D((0, \dots, 0, -\delta); X) \ge \frac{C_{11}|X_n|}{\delta^{7/8}}$$

as δ tends to 0. Since X satisfies the inequality $\|X\| \le o(1/\delta)|X_n|$, this finishes the proof of the theorem. □

Remark 5. Consider r to be defined near 0 as $r(z) := \operatorname{Re} z_2 + p(z_1)$, where

$$\begin{aligned} p(z_1) &:= 2m^2 z_1^{m+l}\bar{z}_1^{m-l} + 4(m^2 - l^2)|z_1|^{2m} + 2m^2 z_1^{m-l}\bar{z}_1^{m+l} \\ &= |z_1|^{2m}(\operatorname{Re} 4m^2 e^{i2l\theta} + 4(m^2 - l^2)) \end{aligned}$$

for $m/2 \le l < m$ (here $z_1 = e^{i\theta}|z_1|$). Because p is a subharmonic function such that, for some values of θ, the last factor in the formula is negative (the example is taken from [La]), it is clear that we cannot hope to repeat the reasoning from the proof of Theorem 4 for general k—even the case $k = 2m = 4$ ($m = 2$, $l = 1$) encounters an obstacle. In other words, using that method does not give a better lower estimate. Nevertheless, we think that the lower estimate in the normal direction of the Kobayashi–Royden metric near the boundary of a C^{k+2}-smooth pseudoconvex domain may be of the form $1/d_D^{1-1/(2k)}(z)$; this would mean that, in the case of an infinitely smooth bounded pseudoconvex domain, the estimate with the exponent arbitrarily close to 1 (as suggested by Fu) may hold.

Remark 6. In the proof of Theorem 4, one needs to use the same reasoning twice. However, instead of repeating it one may use a result of Fu (to get the lower estimate of κ of the form $1/\delta^{2/3}$; in fact, it is sufficient to have the estimate of the form $1/\delta^{1/8}$). However, in its present form the proof is more self-contained.

Proof of Theorem 3. As mentioned previously, the domain that satisfies the properties claimed in the theorem was constructed in [FL]. Hence we recall the construction from there (keeping the notation from there, too). For the proof of the theorem we must add some estimates (mostly for derivatives) of the defined functions; also, for simplicity of calculations, at some places we make a special choice of certain sequences.

First let us make some comments. Note that the procedure works not only for $r_{n+1} = r_n^2/a_n$ but also under the assumption that $r_{n+1} \le r_n^2/a_n$; moreover, the choice of A_k may be done with the equality replaced by the inequality. Consequently, the series that we shall choose can be replaced by any subseries.

In what follows we shall write some inequalities for norms. With these inequalities, by (for instance) $l_n \le m_n$ we mean that $\limsup(l_n/m_n) < \infty$. The meaning of the equality is analogous (inequalities in both directions hold). The norms of functions are meant to be the supremum norms of functions (on some sets).

At the first stage we repeat the definition of a sequence of subharmonic functions, which is then adopted to the construction of a sequence of subharmonic functions of two variables defining a three-dimensional example. As mentioned earlier, the construction follows entirely from [FL].

At first we assume the existence of sequences $(a_n)_n$, $(r_n)_n$ such that the sequence $(a_n)_n$ is increasing to infinity and $r_{n+1} \le r_n^2/a_n$. We shall fix the sequences later. We define

$$u_n(z) := \frac{1}{8} - \operatorname{Re} z + \frac{\log|z|}{4\log a_n}, \quad z \in \mathbb{C},$$

and then define

$$R_n(z) := \begin{cases} \max\{u_n(z), 0\}, & \operatorname{Re} z \le b_n, \\ u_n(z), & \operatorname{Re} z > b_n; \end{cases}$$

here $0 < b_n \le 1$ is the smallest positive number such that

$$\frac{1}{8} - b_n + \frac{\log b_n}{4\log a_n} = 0.$$

At first we are interested in the norm of R_n on a closed disc of radius Ma_n/r_n (for some fixed $M > 1$). It is estimated from above by a_n/r_n.

We define

$$\tilde{R}_n(z) := \int_{\mathbb{C}} R_n(z - \varepsilon_n w)\chi(w)\, d\mu(w)$$

for some $0 < \varepsilon_n < r_n/2$, where $\mu = dx\, dy/m$. Here $m = \int_{\mathbb{C}} \chi(z)\, dx\, dy$ and $\chi\colon \mathbb{C} \mapsto [0,\infty)$ is a nonconstant C^∞ radial function such that $0 \le \chi \le 1$ and $\chi(z) = 0$ for $|z| \ge 1$. Then $\|\tilde{R}_n^{(k)}\|_{B(0,M)} \le (1/r_n)^k \cdot a_n/r_n$. Now we put $\rho_n(z) := \tilde{R}_n(a_n z/r_n)$. Therefore,

$$\|\rho_n^{(k)}\|_{B(0,M)} \le \left(\frac{a_n}{r_n^2}\right)^k \frac{a_n}{r_n}.$$

At this place we fix the sequences. We put $r_n = 1/a_n$. We also want to have $r_{n+1} := r_n^2/a_n = r_n^3$. In other words, we may choose $r_n := r_1^{3^n}$. Now fix for a while $\varepsilon \in (0, 1/3)$, and put $a_n = \delta_n^{-\varepsilon}$. So our choice of the numbers is $a_n = a^{\varepsilon 3^n}$ (where $a > 1$ is fixed), $r_n = (1/a)^{\varepsilon 3^n}$, and $\delta_n = (1/a)^{3^n}$. But the construction needs also an additional number $A_n = 1/2 + a_n/r_n + \log(1/r_n)/4\log a_n$ such that

$$\delta_n \le \frac{\delta_{n-1}}{A_n(1/2^n)}.$$

Given our choice of numbers, it follows that $A_n = a^{2\varepsilon 3^n}$ (in the asymptotic sense). The construction needs also that $\delta_n \le \delta_{n-1}/A_n 2^n$. This inequality must hold asymptotically (it follows from the reasoning) and so it is sufficient to see that, for large n

$$a^{-3^n} \leq \frac{1}{a^{3^{n-1}+2\varepsilon 3^n} 2^n};$$

this holds for ε as fixed previously.

The one-dimensional function ρ is now defined as

$$\rho(z) := \sum_{n=1}^{\infty} \delta_n \rho_n(z),$$

which defines a C^k-smooth function under the assumption

$$\sum_n \delta_n \|\rho_n^{(k)}\|_{B(0,M)} < \infty.$$

In other words, this gives the condition $\sum_n \delta_n (a_n^{k+1}/r_n^{2k+1}) < \infty$. But the last series is $\sum a^{((3k+2)\varepsilon-1)3^n}$, which is finite when $\varepsilon < 1/(3k+2)$.

Next we move to the construction of the proper function $\tilde{\rho}$. We define $V := \{(s,t) \in \mathbb{C}^2 : s^2 - t^3 = 0\}$. We want to have $\tilde{\rho}_n(s,t) = \rho_n(s/t) = \rho_n(\zeta)$ if $(s,t) = (\zeta^3, \zeta^2) \in V$.

Let $\tilde{r}_n := r_{n+1}^3$ and put $B_n := B(0, \tilde{r}_n) \subset \mathbb{C}^2$; we also put $B_n' := B(0, 3/4\tilde{r}_n)$. Then one may choose a small neighborhood U_n of V such that the projection $\pi : U_n \mapsto V$ is well-defined on $U_n \setminus B_n'$ (the formula is $\pi(s,t) := (s, s^{2/3})$ with a properly chosen branch of the power). We put $U_n := \{p \in \mathbb{C}^2 : \|p - \pi(p)\| < d_n^2\}$, where one may choose $d_n = r_{n+1}^3 = r_n^9$ (asymptotically in the aforementioned sense). Then

$$\|\pi^{(k)}\|_{U_n \setminus B_n'} = r_{n+1}^{-k} = \frac{1}{r_n^{3k}}.$$

We define $\tilde{\rho}_n := \rho_n \circ \pi$ on $U_n \setminus B_n$, and we may extend $\tilde{\rho}_n$ to a C^∞-smooth function on $B_n \cup U_n$ by letting it equal 0 on B_n. Now we note the next estimate,

$$\|\tilde{\rho}_n^{(k)}\|_{(U_n \setminus B_n) \cap B(0,M)} \leq \frac{1}{r_n^{6k+2}}.$$

Let $\chi : \mathbb{R} \mapsto [0,1]$ be a C^∞-smooth function that is equal to 1 on $[0, 1/2]$ and equal to 0 on $[1, \infty)$. Then we define another smooth extension of $\tilde{\rho}_n$ on $\mathbb{C}^2$ by the formula

$$p_n(z) := \begin{cases} 0, & z \in B_n, \\ \tilde{\rho}_n(z), & z \in U_n \setminus B_n,\ \|z - \pi(z)\| \leq \dfrac{d_n^2}{2} \\ \tilde{\rho}_n(z)\chi\left(\dfrac{\|z-\pi(z)\|^2}{d_n^2}\right), & z \in U_n \setminus B_n,\ \dfrac{d_n^2}{2} \leq \|z - \pi(z)\|^2 \leq d_n^2, \\ 0, & z \notin U_n \cup B_n. \end{cases}$$

Then one may verify that $\|p_n^{(k)}\|_{B(0,M)} \leq 1/r_n^{9k+2}$. Now take $C_n \geq 0$ such that $\mathcal{L}p_n(z)(X) \geq -C_n \|X\|^2$. It follows that we may take (asymptotically)

$$C_n = \frac{1}{r_n^{20}}$$

(use the estimate for the norm of p_n'').

Put

$$A_n := \left\{ z \in U_n \setminus B_n : \frac{d_n^2}{2} \le \|z - \pi(z)\|^2 \le d_n^2 \right\}$$

and

$$q(s,t) := e^{\|(s,t)\|^2} |s^2 - t^3|^2, \quad (s,t) \in \mathbb{C}^2.$$

Note that $\mathcal{L}q(s,t)(X) \ge |s^2 - t^3|^2 \|X\|^2$. Therefore, if we take (asymptotically) $c_n = d_n^2 = r_n^{18}$, then

$$\mathcal{L}q(z)(X) \ge c_n \|X\|^2.$$

Put $K_n = 1/r_n^{38}$ (asymptotically). Then $-C_n + K_n c_n \ge 0$. Consequently, $\tilde{r}_n = p_n + K_n q$ is plurisubharmonic on $\mathbb{C}^2$. Certainly,

$$\|\tilde{r}_n^{(k)}\|_{B(0,M)} \le \max\left\{ \frac{1}{r_n^{9k+2}}, \frac{1}{r_n^{38}} \right\} =: \frac{1}{r_n^{m_k}}$$

(note that $m_k = 38$ for $k = 1, \dots, 4$ and that $m_k = 9k + 2$ for $k \ge 4$). Now the condition on C^k-smoothness of the example from [FL] follows from the C^k-smoothness of

$$\tilde{\rho} := \sum_n \delta_n \tilde{r}_n,$$

which is satisfied if

$$\infty > \sum_n \delta_n \frac{1}{r_n^{m_k}} = \sum_n a^{m_k \varepsilon - 1}. \tag{$*$}$$

The last inequality completes the proof for arbitrary $\varepsilon \in (0, 1/m_k)$.

To complete the construction, recall that Fornæss and Lee defined the domain as follows:

$$D := \{(s,t,w) \in \mathbb{C}^3 : \operatorname{Re} w + \tilde{\rho}(s,t) < 0\} \cap B(0,2).$$

Let us now move to the second part of the theorem.

We leave all the relations among the numbers a_n, δ_n, r_n with one exception. Namely, put $a_n = (-\log \delta_n)^\alpha$ for $\alpha > 0$. Explicitly, we have $\delta_n = (1/a)^{3^n}$ and $r_n = 1/a_n$. Then the convergence of the final sequence ($*$) (with the δ_n and r_n just introduced) is easily satisfied. And even though the relation $r_{n+1} \le r_n^2/a_n$ is not satisfied now, it is easy to see that, instead of taking the whole sequence, while defining the function $\tilde{\rho}$ we may also choose an arbitrary subsequence which easily guarantees that the desired inequality is satisfied. One may then prove that, after we choose these relations, the number A_n satisfies the desired inequality as well.

The considerations so far lead us to conclude that the following relation is sufficient for the construction of a C^∞-smooth domain with the boundary behavior of the Kobayashi–Royden metric in the normal direction equal to $1/(\delta_n(-\log \delta_n)^\alpha)$:

$$\sum \delta_n a_n^k < \infty \quad \text{for any positive integer } k.$$

The last condition is, as one may verify, satisfied. $\square$

References

[B] M. Behrens, *Plurisubharmonic defining functions of weakly pseudoconvex domains in* $\mathbb{C}^2$, Math. Ann. 270 (1985), 285–296.

[F] J. E. Fornæss, *Plurisubharmonic defining functions,* Pacific J. Math. 80 (1979), 381–388.

[FL] J. E. Fornæss and L. Lee, *Asymptotic behavior of the Kobayashi metric in the normal direction,* Math. Z. 261 (2009), 399–408.

[Fu] S. Fu, *The Kobayashi metric in the normal direction and the mapping problem,* Complex Var. Elliptic Equ. 54 (2009), 303–316.

[G] I. Graham, *Boundary behavior of the Carathéodory and Kobayashi metrics on strongly pseudoconvex domains in* $\mathbb{C}^n$ *with smooth boundary,* Trans. Amer. Math. Soc. 207 (1975), 219–240.

[JP] M. Jarnicki and P. Pflug, *Invariant distances and metrics in complex analysis,* de Gruyter Exp. Math., 9, de Gruyter, Berlin, 1993.

[La] G. Laszlo, *Peak functions on finite type domains in* $\mathbb{C}^2$, Ph.D. thesis, Eötvös Loránd University, Budapest, 1988.

[R] H. Royden, *Remarks on the Kobayashi metric,* Several complex variables, II (College Park, 1970), Lecture Notes in Math., 185, pp. 125–137, Springer-Verlag, Berlin, 1971.

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