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Properties of Meromorphic φ-normal Functions

Rauno Aulaskari & Jouni Rättyä

1. Introduction

Let $\mathcal{M}(\mathbb{D})$ denote the set of all meromorphic functions in the unit disc $\mathbb{D} := \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$, and let $\mathcal{T}$ stand for the set of all conformal self maps of $\mathbb{D}$. The class $\mathcal{N}$ of *normal functions* consists of those $f \in \mathcal{M}(\mathbb{D})$ for which the family $\{f \circ \tau : \tau \in \mathcal{T}\}$ is normal in $\mathbb{D}$ in the sense of Montel (i.e., ∞ is a permitted limit). By Marty's theorem, $f \in \mathcal{N}$ if and only if $\sup_{\tau\in\mathcal{T}}(f \circ \tau)^{\#}(z)$ is bounded on each compact subset of $\mathbb{D}$. Moreover, Lehto and Virtanen [27] showed that $f \in \mathcal{M}(\mathbb{D})$ is normal if and only if its spherical derivative $f^{\#}(z) := |f'(z)|/(1 + |f(z)|^2)$ satisfies $\sup_{z\in\mathbb{D}} f^{\#}(z)(1 - |z|^2) < \infty$.

There is a substantial body of literature on normal functions. Apart from the cited paper by Lehto and Virtanen [27], we mention the earlier work by Noshiro [30], the survey paper by Cambell and Wickes [9], and the papers by Anderson, Clunie, and Pommerenke [1], Lohwater and Pommerenke [28], and Zalcman [41] as well as the series of papers by Gavrilov [17; 18; 19], Lappan [23; 24; 25; 26], and Yamashita [38; 39; 40]. For more recent developments, see [5; 7; 11; 13; 20] and the references therein.

The purpose of this paper is to study subsets of $\mathcal{M}(\mathbb{D})$ that are defined by the condition $f^{\#}(z) = \mathcal{O}(\varphi(|z|))$, as $|z| \to 1^-$, where the function $\varphi(r)$ admits a sufficient regularity near 1 and exceeds $1/(1-r^2)$ in growth. These sets are larger than the class $\mathcal{N}$ of normal functions, and their members will be called φ-normal functions. These concepts are made precise in Definition 1. After that we give several examples of admissible functions φ. At the end of this section we illustrate what it means to change the growth restriction of spherical derivatives from $1/(1 - |z|^2)$ of normal functions to $\varphi(|z|)$ of φ-normal functions. Statements of the main results and their connections to existing literature are given in Section 2. Proofs are presented in Sections 3–9.

Definition 1. An increasing function $\varphi\colon [0, 1) \to (0, \infty)$ is called *smoothly increasing* if

$$\varphi(r)(1 - r) \to \infty \quad \text{as } r \to 1^- \tag{1.1}$$

and

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$$\mathcal{R}_a(z) := \frac{\varphi(|a + z/\varphi(|a|)|)}{\varphi(|a|)} \to 1 \quad \text{as } |a| \to 1^- \tag{1.2}$$

uniformly on compact subsets of $\mathbb{C}$. For a given such φ, a function $f \in \mathcal{M}(\mathbb{D})$ is called *φ-normal* if

$$\|f\|_{\mathcal{N}^\varphi} := \sup_{z\in\mathbb{D}} \frac{f^\#(z)}{\varphi(|z|)} < \infty. \tag{1.3}$$

The class of all φ-normal functions is denoted by $\mathcal{N}^\varphi$. Moreover, a function $f \in \mathcal{N}^\varphi$ is said to be *strongly φ-normal,* denoted by $f \in \mathcal{N}_0^\varphi$, if

$$f^\#(z) = o(\varphi(|z|)) \quad \text{as } |z| \to 1^-. \tag{1.4}$$

If φ is smoothly increasing then we will always further assume, without loss of generality, that $\varphi(r)(1-r) \geq 1$ for all $r \in [0,1)$. This because $\varphi^*(r) := \varphi(r) + (1-r)^{-1}$ satisfies $\mathcal{N}^{\varphi^*} = \mathcal{N}^\varphi$ and $\varphi^*(r)(1-r) \geq 1$ for all $r \in [0,1)$. Moreover, to shorten the notation, we set $\phi_a(z) := a + z/\varphi(|a|)$. Note that now $\mathcal{R}_a(z) = \varphi(|\phi_a(z)|)/\varphi(|a|)$ is well-defined for all $a, z \in \mathbb{D}$ since $\phi_a(z) \in \mathbb{D}$ as $\varphi(|a|)(1-|a|) \geq 1$.

We give two examples regarding smoothly increasing functions.

EXAMPLE 1. Assume $\varphi\colon [0,1) \to (0,\infty)$ is increasing such that (1.1) is satisfied. If $\psi := 1/\varphi$ is differentiable and convex on $[r_0, 1)$ for some $r_0 \in (0,1)$, then φ is smoothly increasing. To see this, let $K \subset \mathbb{C}$ be compact and choose $R > 0$ such that $K \subset \overline{D(0,R)}$. Then, by (1.1), there exists an $r_R \in (0,1)$ such that $\phi_a(z) \in \mathbb{D}$ for all $z \in \overline{D(0,R)}$ if $|a| \in (r_R, 1)$, and thus $\mathcal{R}_a(z)$ is well-defined in this case. Since ψ is decreasing, differentiable, and convex on $[r_0, 1)$ for some $r_0 \in (0,1)$, we have

$$\sup_{z\in K} \mathcal{R}_a(z) \leq \frac{\psi(|a|)}{\psi(|a| + R\psi(|a|))} \leq \frac{1}{1 + R\psi'(|a|)}$$

for all a such that $|a| > \max\{r_R, r_0\}$. Since (1.1) is satisfied, we also have that $\psi'(|a|) \to 0$ as $|a| \to 1^-$ by the convexity. Therefore,

$$\limsup_{|a|\to 1^-} \sup_{z\in K} \mathcal{R}_a(z) \leq 1.$$

In a similar manner we can show that

$$\liminf_{|a|\to 1^-} \sup_{z\in K} \mathcal{R}_a(z) \geq 1,$$

and hence (1.2) is satisfied.

The functions $(1-r)^{-\alpha}$, $\alpha \in (1,\infty)$, and $\exp(1/(1-r))$ are smoothly increasing by Example 1. These functions are differentiable, but of course this is not necessary for (1.2) to be satisfied.

EXAMPLE 2. Assume $\varphi\colon [0,1) \to (0,\infty)$ is increasing such that (1.1) is satisfied. If $\psi = 1/\varphi$ satisfies the Lipschitz condition

$$\gamma_r := \sup_{r \le s < t < 1} \left| \frac{\psi(s) - \psi(t)}{s - t} \right| \le \gamma < \infty$$

for all $r \in (0,1)$ and if $\gamma_r \to 0^+$ as $r \to 1^-$, then (1.2) is satisfied. This follows by the inequality

$$\left| 1 - \frac{\varphi(|a|)}{\varphi(|\phi_a(z)|)} \right| \le \frac{\psi(|a|) - \psi(|a| + \psi(|a|)|z|)}{\psi(|a|)} \le \gamma_{|a|} R,$$

which is valid for all $z \in \overline{D(0,R)}$.

In view of Definition 1, every (strongly) normal function must be (strongly) φ-normal. Moreover, if $\varphi_\alpha(r) := (1 - r^2)^{-\alpha}$, $1 < \alpha < \infty$, then $\mathcal{N}^{\varphi_\alpha}$ coincides with the class $\mathcal{N}^\alpha$ of *α-normal functions*, and $\mathcal{N}_0^{\varphi_\alpha}$ equals to $\mathcal{N}_0^\alpha$, the class of *little* (or *strongly*) *α-normal functions*. For results on these classes, see [29; 33; 34; 35; 36; 37].

We will now compare (1.3) to the growth condition $\sup_{z \in \mathbb{D}} f^\#(z)(1 - |z|^2) < \infty$, which is satisfied by normal functions. To do this, we recall that the *chordal distance* between the points Z and W in the extended complex plane is

$$\chi(Z,W) := \begin{cases} \dfrac{|Z - W|}{\sqrt{1 + |Z|^2}\sqrt{1 + |W|^2}} & \text{if } Z, W \in \mathbb{C}, \\ \dfrac{1}{\sqrt{1 + |Z|^2}} & \text{if } Z \in \mathbb{C} \text{ and } W = \infty. \end{cases}$$

For any normal function f, a direct calculation gives

$$\chi(f(z), f(w)) \le \|f\|_{\mathcal{N}} \sup_{\zeta \in [z,w]} \frac{|z - w|}{1 - |\zeta|^2}, \quad z, w \in \mathbb{D}. \tag{1.5}$$

If now the *pseudohyperbolic distance* $\rho(z,w) := |z - w|/|1 - \bar{z}w|$ from z to w is less than or equal to a fixed $r \in (0,1)$, and so $w = (z - u)/(1 - \bar{z}u)$ for some $|u| < r$, then (1.5) yields

$$\chi(f(z), f(w)) \le C_1(r) \|f\|_{\mathcal{N}} \frac{|z - w|}{1 - |z|^2},$$

where $C_1(r) = (1 + r)/(1 - r^2)$. In a similar manner, if f is φ-normal then

$$\begin{aligned} \chi(f(z), f(w)) &\le \|f\|_{\mathcal{N}^\varphi} |z - w| \sup_{\zeta \in [z,w]} \varphi(|\zeta|) \\ &\le C_2(r) \|f\|_{\mathcal{N}^\varphi} |z - w| \varphi(|z|), \end{aligned} \tag{1.6}$$

where the second inequality follows by (1.2) and is valid if w belongs to the φ-disc $\Delta_\varphi(z,r) := \{\phi_z(u) : |u| < r\}$, where $r \in (0,1)$. This shows that the change of growth restriction of $f^\#(z)$ from $1/(1 - |z|^2)$ to $\varphi(|z|)$ affects in a natural manner the right-hand side of (1.5) and of (1.6). This "change of scale" will appear repeatedly in several places in the reminder of the paper, in statements of results and also in the proofs.

2. Results and Background

We begin with considering the zero distribution of φ-normal functions. One way to state the results is by means of the terminology of Nevanlinna theory. The *Nevanlinna counting function* of $f \in \mathcal{M}(\mathbb{D})$ is defined as $N(r, f) := \sum_{|z_n|<r} \log \frac{r}{|z_n|}$, where $\{z_n\}_{n=1}^{\infty}$ is the sequence of zeros of f listed according to multiplicities and ordered by increasing moduli. Lehto and Virtanen [27] (see also a related result by Nowak [31]) showed that if f is a normal function, then its counting function $N(r, f)$ is of logarithmic growth. The first result of this study establishes an analogue for φ-normal functions.

THEOREM 1. *Let* $\varphi\colon [0,1) \to (0,\infty)$ *be smoothly increasing and let* $f \in \mathcal{M}(\mathbb{D})$. *If* $f \in \mathcal{N}^{\varphi}$ *and* $f(0) \neq 0$, *then*

$$N(r, f) = \mathcal{O}\left(\int_0^r (\varphi(s))^2 \log \frac{r}{s}\, ds\right), \quad r \to 1^-. \tag{2.1}$$

Similarly, if $f \in \mathcal{N}_0^{\varphi}$ *and* $f(0) \neq 0$, *then*

$$N(r, f) = o\left(\int_0^r (\varphi(s))^2 \log \frac{r}{s}\, ds\right), \quad r \to 1^-. \tag{2.2}$$

The first statement in Theorem 1 remains true also when $\varphi\colon [0,1) \to (0,\infty)$ is arbitrary. This is also the case with the latter statement provided $\int_0^r (\varphi(s))^2 \log \frac{r}{s}\, ds \to \infty$ as $r \to 1^-$.

Theorem 1 is a consequence of the Ahlfors–Shimizu theorem, which says that $N(r, f) \leq T_0(r, f) + \log\bigl(\sqrt{1+|f(0)|^2}/|f(0)|\bigr)$, $f(0) \neq 0$. Here $T_0(r, f)$ is the *Ahlfors–Shimizu characteristic,* defined as

$$T_0(r, f) := \int_0^r \frac{A(t, f)}{t}\, dt, \quad 0 < r \leq 1, \tag{2.3}$$

where

$$A(t, f) := \frac{1}{\pi} \int_{D(0,t)} (f^{\#}(z))^2\, dA(z), \quad 0 < t < 1,$$

and $D(0,t) := \{z : |z| < t\}$.

The right-hand side of (2.1) and of (2.2) can be simplified when φ is given. For example, if

$$\varphi_{\alpha,\beta}(r) := \frac{1}{(1-r)^{\alpha}} \left(\log \frac{1}{1-r}\right)^{\beta}, \quad 0 \leq \alpha, \beta < \infty,$$

and if $f \in \mathcal{N}^{\varphi_{\alpha,\beta}}$, then as $r \to 1^-$ we have

$$N(r, f) = \begin{cases} \mathcal{O}(1) & \text{if } 0 \leq \alpha < 1, \\ \mathcal{O}\left(\left(\log \dfrac{1}{1-r}\right)^{1+\beta}\right) & \text{if } \alpha = 1, \\ \mathcal{O}\left(\dfrac{1}{(1-r)^{2\alpha-2}}\left(\log \dfrac{1}{1-r}\right)^{\beta}\right) & \text{if } \alpha > 1. \end{cases}$$

Recall that Yamashita [40] characterized normal functions by means of the Ahlfors–Shimizu characteristic. An analogue for φ-normal functions is given in the following result.

THEOREM 2. *Let* $\varphi\colon [0,1) \to (0,\infty)$ *be smoothly increasing and let* $f \in \mathcal{M}(\mathbb{D})$. *Then the following statements are equivalent*:

(1) $f \in \mathcal{N}^{\varphi}$;
(2) *for each* $\lambda \in (0,1)$ *we have* $\sup_{\lambda \le |a| < 1} T_0(1, f \circ \phi_a) < \infty$;
(3) *there exist* $\delta, \lambda \in (0,1)$ *such that* $\sup_{\lambda \le |a| < 1} T_0(\delta, f \circ \phi_a) < \infty$.

Moreover, the following statements are equivalent:

(1′) $f \in \mathcal{N}_0^{\varphi}$;
(2′) $\lim_{|a| \to 1^-} T_0(1, f \circ \phi_a) = 0$;
(3′) *there exists a* $\delta \in (0,1)$ *such that* $\lim_{|a| \to 1^-} T_0(\delta, f \circ \phi_a) = 0$.

Theorem 2 says that $f \in \mathcal{M}(\mathbb{D})$ is φ-normal if and only if it is of bounded characteristic "uniformly" in each φ-disc $\Delta_{\varphi}(a,r) = \{\phi_a(w) : |w| < r\}$, $0 < r \le 1$, when a is near the boundary. Another characterization of functions in $\mathcal{N}^{\varphi}$ involving these discs can be found in [2]; see also Lemma 13 in Section 4. We further note that the first part of Theorem 2 should be compared with a result by Wulan [35, Cor. 4.3.1].

The following characterization of φ-normal functions in terms of normal families appears to be useful in our study. The first assertion has been essentially proved in [3], but it is included here for the convenience of the reader.

THEOREM 3. *Let* $\varphi\colon [0,1) \to (0,\infty)$ *be smoothly increasing and let* $f \in \mathcal{M}(\mathbb{D})$. *Then* $f \in \mathcal{N}^{\varphi}$ *if and only if the family* $\{f \circ \phi_a : a \in \mathbb{D}\}$ *is normal in* $\mathbb{D}$. *Moreover,* $f \in \mathcal{N}_0^{\varphi}$ *if and only if* $(f \circ \phi_a)^{\#}$ *converges uniformly to zero on compact subsets of* $\mathbb{C}$ *as* $|a| \to 1^-$.

The proof of Theorem 3 shows that in the first assertion we may consider the normality also in the whole complex plane. In this case the assertion reads as follows: $f \in \mathcal{N}^{\varphi}$ if and only if the family $\{f \circ \phi_{a_n}\}$ is normal in $\mathbb{C}$ for any sequence $\{a_n\} \subset \mathbb{D}$ such that $|a_n| \to 1^-$ as $n \to \infty$.

Colonna [14] used the Arzelà–Ascoli theorem to show that the preimages of two distinct points in the image set of a normal function are of bounded hyperbolic distance from each other. Recall that, for $z, w \in \mathbb{D}$, the *hyperbolic distance* from z to w is

$$d(z,w) := \frac{1}{2} \log \frac{1 + \rho(z,w)}{1 - \rho(z,w)}.$$

The cited result by Colonna can also be verified directly by using the inequality

$$\chi(f(z), f(w)) \le \|f\|_{\mathcal{N}} \log \frac{1}{1 - \rho(z,w)}, \quad z, w \in \mathbb{D},$$

which is slightly sharper than (1.5). In a similar manner one can show that, if $f \in \mathcal{N}_0$, then the hyperbolic distance of the preimages of two distinct points in the image set cannot remain bounded when the preimages approach the boundary.

The next result establishes an analogue for (strongly) φ-normal functions.

THEOREM 4. *Let $\varphi\colon [0,1) \to (0,\infty)$ be smoothly increasing and let $f \in \mathcal{M}(\mathbb{D})$. If $f \in \mathcal{N}^{\varphi}$, then there exists a $\delta > 0$ such that*

$$\rho(z,w)\varphi(|z|)(1-|z|) \geq \delta \tag{2.4}$$

for all $z, w \in \mathbb{D}$ such that $f(z) = Z \neq W = f(w)$.

If $f \in \mathcal{N}_0^{\varphi}$, then

$$\limsup_{z\in Z^*,\, |z|\to 1^-} \min_{w\in W^*} \rho(z,w)\varphi(|z|)(1-|z|) = \infty \tag{2.5}$$

for any distinct points $Z, W \in f(\mathbb{D})$ such that the sets $Z^* = \{z \in \mathbb{D} : f(z) = Z\}$ and $W^* = \{z \in \mathbb{D} : f(z) = W\}$ satisfy $\#Z^* = \infty = \#W^*$.

Theorem 4 says that the preimages of two distinct points in the image set of a φ-normal function are distributed according to the growth of $\varphi(r)$ as $r \to 1^-$. The faster the φ grows, the closer to each other the preimages of different points can be. For example, if all preimages of $Z \in f(\mathbb{D})$, $f \in \mathcal{N}^{\varphi}$, lie on the positive real axis and have an accumulation point in $z = 1$, then the preimages of $W \neq Z$ cannot be essentially closer than on the curves $\{r \pm i/\varphi(r) : r \in (0,1)\}$. We will see at the end of this section that the assertions in Theorem 4 are fairly sharp.

Lappan [23] (see also Campbell [8]) showed that the class of normal functions is closed neither under summation nor multiplication. Theorem 4 allows us to deduce that the same is true for $\mathcal{N}^{\varphi}$. Before stating this as Corollary 5, we set necessary definitions. For a given sequence $\{z_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ for which $\sum_{n=1}^{\infty}(1-|z_n|^2)$ converges (with the convention $|z_n|/z_n = 1$ for $z_n = 0$), the *Blaschke product* associated with the sequence $\{z_n\}_{n=1}^{\infty}$ is defined as

$$B(z) := \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}.$$

It is well known that such a product is analytic in $\mathbb{D}$, its modulus is bounded by 1, it is an inner function, and $\{z_n\}_{n=1}^{\infty}$ are precisely its zeros counting multiplicities [15].

COROLLARY 5. *Let $\varphi\colon [0,1) \to (0,\infty)$ be smoothly increasing, and let $f \in \mathcal{N}^{\varphi}$ with infinitely many poles. Then there exist a Blaschke product B and a φ-normal function g such that neither fB nor $f+g$ is φ-normal.*

Choose an infinite sequence $\{z_n\}_{n=1}^{\infty}$ of poles of f that satisfies the Blaschke condition $\sum_{n=1}^{\infty}(1-|z_n|^2) < \infty$. For each z_n, take $w_n \in \mathbb{D}$, not a pole of f, such that $\rho(z_n, w_n)\varphi(|z_n|)(1-|z_n|) \to 0$ as $n \to \infty$. Since φ is smoothly increasing, this implies $\rho(z_n, w_n) \to 0$ as $n \to \infty$, and so the sum $\sum_{n=1}^{\infty}(1-|w_n|^2)$ converges. Then the Blaschke product B associated with the sequence $\{w_n\}_{n=1}^{\infty}$ satisfies $fB \notin \mathcal{N}^{\varphi}$ by Theorem 4, since $\{z_n\}_{n=1}^{\infty}$ are poles and $\{w_n\}_{n=1}^{\infty}$ are zeros of fB. This also shows that the φ-normal function $g := f(B/2 - 1)$ satisfies $f + g \notin \mathcal{N}^{\varphi}$.

Theorem 6 contains Lohwater–Pommerenke [28] theorems for (strongly) φ-normal functions (see also a related result by Zalcman [41]). The first assertion was recently proved in [2], but the statement is included here for the sake of completeness. From the first assertion, one can easily obtain the second assertion by modifying the reasoning in [4], where an analogous result for little (or strongly) normal functions is established.

THEOREM 6. *Let* $\varphi\colon [0,1) \to (0,\infty)$ *be smoothly increasing,* $-1 < \beta < 1$, *and let* $f \in \mathcal{M}(\mathbb{D})$. *Then* $f \notin \mathcal{N}^{\varphi}$ *if and only if there exist*

(1) *a sequence* $\{z_n\}$ *of points in* $\mathbb{D}$,
(2) *a sequence* $\{\rho_n\}$ *of positive real numbers,*
(3) *a sequence* $\{\sigma_n\}$ *of positive real numbers satisfying* $\sigma_n \to 0$ *as* $n \to \infty$, *and*
(4) *a constant* $c > 0$ *satisfying* $\varphi(|z_n|)\rho_n \le c\sigma_n$ *for all* $n \in \mathbb{N} := \{1, 2, \dots\}$

such that the sequence $\{\sigma_n^{-\beta} f(z_n + \rho_n \xi)\}$ *of functions converges spherically uniformly on each compact subset of* $\mathbb{C}$ *to a nonconstant meromorphic function.*

Moreover, $f \notin \mathcal{N}_0^{\varphi}$ *if and only if there exist*

(1′) *a positive constant* R,
(2′) *a sequence* $\{z_n\}$ *of points in* $\mathbb{D}$ *satisfying* $|z_n| \to 1^-$ *as* $n \to \infty$, *and*
(3′) *a sequence* $\{\rho_n\}$ *of positive real numbers satisfying* $\rho_n\varphi(|z_n|) < 1/R$

such that the sequence $\{f(z_n + \rho_n\xi)\}$ *of functions converges spherically uniformly on each compact subset of* $D(0, R)$ *to a nonconstant meromorphic function.*

Lappan [24] (see also the earlier result by Bagemihl and Seidel [6]) showed that $f \in \mathcal{M}(\mathbb{D})$ is normal if and only if $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} f(w_n)$ for all sequences $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ such that $\rho(z_n, w_n) \to 0$ as $n \to \infty$. Further, it is well known that $f \in \mathcal{M}(\mathbb{D})$ is strongly normal if and only if $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} f(w_n)$ for all sequences $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ such that $\lim_{n\to\infty} \rho(z_n, w_n) < 1$. These results have natural analogues for (strongly) φ-normal functions.

THEOREM 7. *Let* $\varphi\colon [0,1) \to (0,\infty)$ *be smoothly increasing and let* $f \in \mathcal{M}(\mathbb{D})$. *Then* $f \in \mathcal{N}^{\varphi}$ *if and only if* $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} f(w_n)$ *for all sequences* $\{z_n\}_{n=1}^{\infty}$ *and* $\{w_n\}_{n=1}^{\infty}$ *of points in* $\mathbb{D}$ *tending to the boundary such that*

$$\rho(z_n, w_n)\varphi(|z_n|)(1 - |z_n|) \to 0, \quad n \to \infty. \tag{2.6}$$

Moreover, $f \in \mathcal{N}_0^{\varphi}$ *if and only if* $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} f(w_n)$ *for all sequences* $\{z_n\}_{n=1}^{\infty}$ *and* $\{w_n\}_{n=1}^{\infty}$ *of points in* $\mathbb{D}$ *tending to the boundary such that*

$$\limsup_{n\to\infty} \rho(z_n, w_n)\varphi(|z_n|)(1 - |z_n|) < \infty. \tag{2.7}$$

It is worth noticing that if (2.6) is satisfied then $\rho(z_n, w_n) \to 0$ as $n \to \infty$. Therefore, $\rho(z_n, w_n)(1 - |z_n|)$ is comparable to $|w_n - z_n|$ for all sufficiently large n and thus (2.6) is equivalent to

$$|w_n - z_n|\varphi(|z_n|) \to 0, \quad n \to \infty.$$

In the sequel, we ignore this observation and adhere to the use of pseudohyperbolic distance in order to preserve the complete analogue with the classical case of normal functions.

If $f \in \mathcal{N}_0^{\varphi}$ and if $\lim_{r\to 1^-} f(r\zeta) = \alpha$ for $\zeta \in \partial\mathbb{D}$, then Theorem 7 implies $\lim_{z\to\zeta} f(z) = \alpha$ when z approaches ζ inside any *φ-angular domain* $\Omega(\zeta, c) := \{z : |\Im(z/\zeta)| \le c/\varphi(|z|)\}$, where $c > 0$.

In the case of unbounded analytic functions, there is a sense in which the first assertion in Theorem 7 can be improved. Toward this end, let H^∞ denote the space of bounded functions in the algebra $\mathcal{H}(\mathbb{D})$ of all analytic functions in $\mathbb{D}$.

THEOREM 8. *Let $\varphi\colon [0,1) \to (0,\infty)$ be smoothly increasing and let $f \in \mathcal{H}(\mathbb{D}) \setminus H^\infty$. Let $f \in \mathcal{N}^{\varphi}$, and let $\{z_n\}_{n=1}^\infty$ be a sequence of points in $\mathbb{D}$ such that $\lim_{n\to\infty} f(z_n) = \infty$. Then $\lim_{n\to\infty} f(w_n) = \infty$ for any sequence $\{w_n\}_{n=1}^\infty$ of points in $\mathbb{D}$ satisfying*

$$\sup_{n\in\mathbb{N}} \rho(z_n, w_n)\varphi(|z_n|)(1 - |z_n|) < \infty.$$

Theorem 8 can be used to show that the assertion in Corollary 5 remains valid when $f \in \mathcal{N}^{\varphi}$ is assumed to be an unbounded analytic function. The reasoning is similar to that yielding Corollary 5, and therefore we omit it.

Lappan's [25] five-point theorem says that if $\sup\{f^{\#}(z)(1-|z|^2) : z \in f^{-1}(E)\}$ is bounded for some five-point subset E of the image set $f(\mathbb{D})$, then f is a normal function. The next result is a version of this theorem for (strongly) φ-normal functions. Its proof uses Theorem 6 and imitates Lappan's original proof, so we omit the details in order to avoid unnecessary repetition.

THEOREM 9. *Let $\varphi\colon [0,1) \to (0,\infty)$ be smoothly increasing and let $f \in \mathcal{M}(\mathbb{D})$. Then $f \in \mathcal{N}^{\varphi}$ if and only if there exists a set E of five distinct values in $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ such that*

$$\sup\{f^{\#}(z)/\varphi(|z|) : z \in \mathbb{D},\ f(z) \in E\} < \infty.$$

Moreover, $f \in \mathcal{N}_0^{\varphi}$ if and only if there exists a set E of five distinct values in $\hat{\mathbb{C}}$ with infinitely many preimages such that

$$\lim_{|z|\to 1^-,\ f(z)\in E} f^{\#}(z)/\varphi(|z|) = 0.$$

Let $f \in \mathcal{M}(\mathbb{D})$ and $R > 0$. If now $\sup\{|f'(z)|/\varphi(|z|) : |f(z)| < R\}$ is finite, then Theorem 9 implies $f \in \mathcal{N}^{\varphi}$. Conversely, if $f \in \mathcal{N}^{\varphi}$ then clearly

$$\sup_{|f(z)|<R} \frac{|f'(z)|}{\varphi(|z|)} = \sup_{|f(z)|<R} (1 + |f(z)|^2)\frac{f^{\#}(z)}{\varphi(|z|)} \le (1 + R^2)\|f\|_{\mathcal{N}^{\varphi}}.$$

A similar reasoning applies for $\mathcal{N}_0^{\varphi}$. Hence we obtain the following corollary, which (in view of Example 1) contains a result by Wulan [35, Thm. 4.5.1] as a special case. See also earlier results by Lappan [26].

COROLLARY 10. *Let $\varphi\colon [0,1) \to (0,\infty)$ be smoothly increasing and let $f \in \mathcal{M}(\mathbb{D})$. Then $f \in \mathcal{N}^{\varphi}$ if and only if there exist $R > 0$ and $M_R > 0$ such that*

$$\sup\{|f'(z)|/\varphi(|z|) : |f(z)| < R\} < M_R.$$

Moreover, $f \in \mathcal{N}_0^{\varphi}$ *if and only if there exist* $R > 0$ *such that*

$$\lim_{|z|\to 1^-,\, |f(z)|<R} |f'(z)|/\varphi(|z|) = 0.$$

As the last topic we consider φ-normal Blaschke quotients. In view of (2.4), the following definitions appear natural. An infinite sequence $\{z_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ is called *φ-separated* if there exists a $\delta > 0$ such that

$$\rho(z_k, z_n)\varphi(|z_n|)(1 - |z_n|) \geq \delta$$

for all distinct natural numbers k and n. Further, if there exists a $\delta > 0$ such that

$$\varphi(|z_n|)(1 - |z_n|) \prod_{k\neq n} \rho(z_k, z_n) \geq \delta$$

for all $n \in \mathbb{N}$, then $\{z_n\}_{n=1}^{\infty}$ is called *uniformly φ-separated.* Recall that $\{z_n\}_{n=1}^{\infty}$ is *separated* if $\inf_{k\neq n} \rho(z_k, z_n) > 0$ and *uniformly separated* if

$$\inf_n \prod_{k\neq n} \rho(z_k, z_n) > 0.$$

Thus every (uniformly) separated sequence is (uniformly) φ-separated, but of course the converse is not true in general. A result by Carleson [10] states that $\{z_n\}_{n=1}^{\infty}$ is an interpolating sequence for H^{∞} if and only if it is uniformly separated. Therefore Blaschke products associated with uniformly separated sequences are often called *interpolating Blaschke products.*

An infinite sequence $\{z_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ is called *strongly φ-separated* if

$$\limsup_{n\to\infty} \min_{k\in\mathbb{N}} \rho(z_k, z_n)\varphi(|z_n|)(1 - |z_n|) = \infty.$$

Further, if

$$\limsup_{n\to\infty} \varphi(|z_n|)(1 - |z_n|) \prod_{k\neq n} \rho(z_k, z_n) = \infty,$$

then $\{z_n\}_{n=1}^{\infty}$ is called *strongly uniformly φ-separated.*

Cima and Colwell [12] (see also a related result by Colonna [14]) showed that the quotient of two interpolating Blaschke products with disjoint zeros is normal if and only if its zeros and poles form a uniformly separated sequence. Theorem 11 generalizes this result for (strongly) φ-normal functions via (uniformly) φ-separated sequences. It is worth noticing that if $B_1/B_2 \in \mathcal{N}^{\varphi}$ then $\{z_n\}_{n=1}^{\infty} \cup \{w_n\}_{n=1}^{\infty}$ is φ-separated by Theorem 4, so Theorem 11 shows that the assertions in Theorems 4 and 7 are fairly sharp.

THEOREM 11. *Let* B_1 *and* B_2 *be interpolating Blaschke products associated with the disjoint sequences* $\{z_n\}_{n=1}^{\infty}$ *and* $\{w_n\}_{n=1}^{\infty}$*, and let* $\varphi\colon [0,1) \to (0,\infty)$ *be smoothly increasing. Then the following assertions are equivalent*:

(1) $B_1/B_2 \in \mathcal{N}^{\varphi}$;
(2) $\{z_n\}_{n=1}^{\infty} \cup \{w_n\}_{n=1}^{\infty}$ *is φ-separated*;
(3) $\{z_n\}_{n=1}^{\infty} \cup \{w_n\}_{n=1}^{\infty}$ *is uniformly φ-separated.*

Moreover, the following assertions are equivalent:

(1′) $B_1/B_2 \in \mathcal{N}_0^\varphi$;
(2′) $\{z_n\}_{n=1}^\infty \cup \{w_n\}_{n=1}^\infty$ *is strongly* φ*-separated*;
(3′) $\{z_n\}_{n=1}^\infty \cup \{w_n\}_{n=1}^\infty$ *is strongly uniformly* φ*-separated.*

The first assertion in Theorem 11 can be considered as a refinement of the fact that, for any increasing function $\varphi\colon [0,1) \to (0,\infty)$, there exist non-$\varphi$-normal Blaschke quotients [3; 39]. It also reveals that the union of two uniformly separated sequences is φ-separated if and only if it is uniformly φ-separated.

The proof of Theorem 11 combined with Theorem 4 shows two things. First, if B_1 and B_2 are Blaschke products associated with the disjoint sequences $\{z_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ such that $B_1/B_2 \notin \mathcal{N}^\varphi$, then there exists a sequence $\{a_k\}_{k=1}^\infty$ for which

$$|B_i(a_k)|\varphi(|a_k|)(1-|a_k|^2) \to 0, \quad k \to \infty, \tag{2.8}$$

for both $i = 1,2$ (note that (2.8) implies $B_i(a_k) \to 0$ as $k \to \infty$). Second, if B_1 and B_2 are Blaschke products associated with the disjoint interpolating sequences and if there exists a sequence $\{a_k\}_{k=1}^\infty$ for which (2.8) is satisfied for both $i = 1,2$, then $B_1/B_2 \notin \mathcal{N}^\varphi$. Since a similar reasoning can be applied for nonstrongly φ-normal Blaschke quotients, we obtain the following result.

COROLLARY 12. *Let B_1 and B_2 be interpolating Blaschke products associated with the disjoint sequences $\{z_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$. Let $\varphi\colon [0,1) \to (0,\infty)$ be smoothly increasing. Then $B_1/B_2 \notin \mathcal{N}^\varphi$ if and only if there exists a sequence $\{a_k\}_{k=1}^\infty$ for which* (2.8) *is satisfied for both $i = 1,2$. Moreover, $B_1/B_2 \notin \mathcal{N}_0^\varphi$ if and only if there exists a sequence $\{a_k\}_{k=1}^\infty$ for which*

$$\lim_{k\to\infty} |B_i(a_k)|\varphi(|a_k|)(1-|a_k|^2) < \infty$$

for $i = 1,2$.

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3. Proof of Theorem 1

Fubini's theorem shows that the Ahlfors–Shimizu characteristic of $f \in \mathcal{M}(\mathbb{D})$ can be represented as

$$T_0(r,f) = \frac{1}{\pi}\int_{D(0,r)} (f^\#(z))^2 \log\frac{r}{|z|}\,dA(z). \tag{3.1}$$

If $f \in \mathcal{N}^\varphi$, then

$$T_0(r,f) \le 2\|f\|_{\mathcal{N}^\varphi}^2 \int_0^r (\varphi(s))^2 \log\frac{r}{s}\,ds,$$

and since

$$N(r,f) \le T_0(r,f) + \log\frac{\sqrt{1+|f(0)|^2}}{|f(0)|} \tag{3.2}$$

by the Ahlfors–Shimizu theorem [21, Thm. 1.4, p. 12], the assertion in (2.1) follows.

If $f \in \mathcal{N}_0^{\varphi}$ then, for a given $\varepsilon > 0$, there exists an $r_\varepsilon \in (0,1)$ such that $f^{\#}(z) \leq \varepsilon\varphi(|z|)$ whenever $|z| \geq r_\varepsilon$. It follows that

$$T_0(r,f) \leq 2\|f\|_{\mathcal{N}^\varphi}^2 \int_0^{r_\varepsilon} (\varphi(s))^2 \log\frac{r}{s}\,ds + 2\varepsilon^2 \int_0^{r} (\varphi(s))^2 \log\frac{r}{s}\,ds.$$

Since $\int_0^1 (\varphi(s))^2 \log\frac{r}{s}\,ds$ diverges for any smoothly increasing function φ, (2.2) follows by (3.2).

4. Proof of Theorem 2

First assume that $f \in \mathcal{N}^{\varphi}$. Then (3.1) and (1.2) yield

$$\begin{aligned} T_0(1, f\circ\phi_a) &= \frac{1}{\pi}\int_{\mathbb{D}} ((f\circ\phi_a)^{\#}(z))^2 \log\frac{1}{|z|}\,dA(z) \\ &\leq \frac{\|f\|_{\mathcal{N}^\varphi}^2}{\pi}\int_{\mathbb{D}} (\mathcal{R}_a(z))^2 \log\frac{1}{|z|}\,dA(z) \leq C < \infty \end{aligned}$$

for all $a \in \mathbb{D}$, and therefore (2) follows. Since (2) clearly implies (3), it remains to show that (3) implies $f \in \mathcal{N}^{\varphi}$. To do this, we need the following lemma. Recall that $\Delta_\varphi(a,r) = \{\phi_a(w) : |w| < r\}$.

Lemma 13. *Let $\varphi\colon [0,1) \to (0,\infty)$ be smoothly increasing and let $f \in \mathcal{M}(\mathbb{D})$. Then $f \in \mathcal{N}^{\varphi}$ if and only if there exist $r, \lambda \in (0,1)$ such that*

$$\sup_{\lambda \leq |a| < 1} \int_{\Delta_\varphi(a,r)} (f^{\#}(z))^2\,dA(z) < \pi. \tag{4.1}$$

Moreover, $f \in \mathcal{N}_0^{\varphi}$ if and only if

$$\lim_{|a|\to 1^-} \int_{\Delta_\varphi(a,r)} (f^{\#}(z))^2\,dA(z) = 0 \tag{4.2}$$

for all $r \in (0,1)$.

Proof. If $f \in \mathcal{N}^{\varphi}$, then (1.2) implies that there exist $C > 0$ and $\lambda \in (0,1)$ such that

$$\begin{aligned} \int_{\Delta_\varphi(a,r)} (f^{\#}(z))^2\,dA(z) &\leq \|f\|_{\mathcal{N}^\varphi}^2 \int_{\Delta_\varphi(a,r)} (\varphi(|z|))^2\,dA(z) \\ &\leq C\|f\|_{\mathcal{N}^\varphi}^2 \int_{\Delta_\varphi(a,r)} (\varphi(|a|))^2\,dA(z) \\ &= C\|f\|_{\mathcal{N}^\varphi}^2 \pi r^2 \end{aligned}$$

for all $|a| \geq \lambda$. The assertion (4.1) follows by choosing r sufficiently small.

If (4.1) is satisfied, then

$$\sup_{\lambda \leq |a| < 1} \frac{1}{\pi}\int_{D(0,r)} ((f\circ\phi_a)^{\#}(z))^2\,dA(z) < 1$$

and hence

$$\sup_{\lambda \le |a| < 1} (f \circ \phi_a)^{\#}(0) = \sup_{\lambda \le |a| < 1} f^{\#}(a)/\varphi(|a|) < \infty$$

by Dufresnoy's theorem [16] (or [32, p. 83]). It follows that $f \in \mathcal{N}^{\varphi}$.

If $f \in \mathcal{N}_0^{\varphi}$ then, for a given $\varepsilon > 0$, there exists a $\lambda_\varepsilon \in (0,1)$ such that $f^{\#}(z) \le \varepsilon\varphi(|z|)$ for all $z \in \Delta_\varphi(a,1)$ whenever $\lambda_\varepsilon \le |a| < 1$. It follows that

$$\int_{\Delta_\varphi(a,r)} (f^{\#}(z))^2\, dA(z) \le \varepsilon^2 \int_{\Delta_\varphi(a,1)} (\varphi(|z|))^2\, dA(z) \le C\varepsilon^2$$

for all $r \in (0,1)$ and for all a sufficiently close to the boundary. Thus (4.2) is satisfied. The converse follows easily by Dufresnoy's theorem. □

To complete the proof of Theorem 2, assume that there exist $\delta, \lambda \in (0,1)$ such that

$$C := \sup_{\lambda \le |a| < 1} T_0(\delta, f \circ \phi_a) < \infty.$$

Then, for all $|a| \in (\lambda, 1)$ and $\gamma \in (0,1)$,

$$\begin{aligned} C &\ge \frac{1}{\pi} \int_{\Delta_\varphi(a,\gamma\delta)} (f^{\#}(z))^2 \log \frac{\delta}{\varphi(|a|)|z-a|}\, dA(z) \\ &\ge \frac{1}{\pi} \log \frac{1}{\gamma} \int_{\Delta_\varphi(a,\gamma\delta)} (f^{\#}(z))^2\, dA(z). \end{aligned}$$

By choosing γ sufficiently small, we obtain

$$\sup_{\lambda \le |a| < 1} \int_{\Delta_\varphi(a,\gamma\delta)} (f^{\#}(z))^2\, dA(z) < \pi,$$

and therefore $f \in \mathcal{N}^{\varphi}$ by Lemma 13. The second assertion concerning (1′), (2′), and (3′) can be proved by an argument similar to that used here.

5. Proof of Theorem 3

To prove the first assertion, let first $f \in \mathcal{N}^{\varphi}$ and let $z \in D(0,r)$, where $r \in (0,1)$ is fixed. Then

$$(f \circ \phi_a)^{\#}(z) = \frac{f^{\#}(\phi_a(z))}{\varphi(|a|)} \le \|f\|_{\mathcal{N}^{\varphi}} \mathcal{R}_a(z)$$

for all $a \in \mathbb{D}$ and $z \in D(0,r)$. It follows that $(f \circ \phi_a)^{\#}(z)$ is uniformly bounded in $D(0,r)$ for all $a \in \mathbb{D}$. Therefore, Marty's theorem implies that the family $\{f \circ \phi_a : a \in \mathbb{D}\}$ is normal in $\mathbb{D}$.

Conversely, let the family $\{f \circ \phi_a : a \in \mathbb{D}\}$ be normal in $\mathbb{D}$, and assume to the contrary of the assertion that $f \notin \mathcal{N}^{\varphi}$. Then there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ such that $\lim_{n\to\infty}|z_n| = 1$ and $f^{\#}(z_n)/\varphi(|z_n|) \to \infty$ as $n \to \infty$. However, Marty's theorem implies that there exists a $C > 0$ such that

$$\frac{f^{\#}(z_n)}{\varphi(|z_n|)} = (f \circ \phi_{z_n})^{\#}(0) \le C$$

for all $n \in \mathbb{N}$. This establishes a contradiction, and so $f \in \mathcal{N}^{\varphi}$ as desired.

To prove the second assertion, let first $f \in \mathcal{N}_0^{\varphi}$ and assume that $z \in K$, where $K \subset \mathbb{C}$ is compact. Then, by (1.2),

$$\begin{aligned}(f \circ \phi_a)^{\#}(z) = \frac{f^{\#}(\phi_a(z))}{\varphi(|a|)} &= o(|\phi_a(z)|)\mathcal{R}_a(z), \quad |a| \to 1^-, \\ &= o(|\phi_a(z)|), \quad |a| \to 1^-,\end{aligned}$$

for all $z \in K$. Since (1.1) implies that $|\phi_a(z)|/|a| \to 1$ uniformly on K as $|a| \to 1^-$, it follows that $(f \circ \phi_a)^{\#}$ converges to zero uniformly on compact subsets of $\mathbb{C}$ as $|a| \to 1^-$.

Conversely, let $(f \circ \phi_a)^{\#}$ converge uniformly to zero on compact subsets of $\mathbb{C}$ as $|a| \to 1^-$, and assume to the contrary of the assertion that $f \notin \mathcal{N}_0^{\varphi}$. Then there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ such that $\lim_{n\to\infty}|z_n| = 1$ and $f^{\#}(z_n)/\varphi(|z_n|) \to c > 0$ as $n \to \infty$. However, by the assumption we have $f^{\#}(z_n)/\varphi(|z_n|) = (f \circ \phi_{z_n})^{\#}(0) \to 0$ as $n \to \infty$. This is clearly a contradiction, and so $f \in \mathcal{N}_0^{\varphi}$ as desired.

6. Proof of Theorem 4

Let first $f \in \mathcal{N}^{\varphi}$, and assume to the contrary of the assertion that there exist distinct points $Z, W \in f(\mathbb{D})$ such that (2.4) fails. Denote by $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ the infinite sequences of preimages of Z and W, respectively. Then, by passing to subsequences if necessary, we may assume that $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ satisfy

$$\rho(z_n, w_n)\varphi(|z_n|)(1 - |z_n|) \to 0, \quad n \to \infty. \tag{6.1}$$

Put $f_n := f \circ \phi_{z_n}$ and $u_n := (w_n - z_n)\varphi(|z_n|)$, so that $f_n(0) = Z$ and $f_n(u_n) = W$ for all $n \in \mathbb{N}$. Now (1.1) and (6.1) imply that $\rho(z_n, w_n) \to 0$ as $n \to \infty$, so every point on the arc joining z_n and w_n tends to the boundary as $n \to \infty$. Further, $\rho(z_n, w_n) \leq 1/2$ for all sufficiently large n. It follows that $|1 - \bar{z}_n w_n| \leq 4(1 - |z_n|)$ for any such n. Therefore,

$$|u_n| \leq 4\rho(z_n, w_n)\varphi(|z_n|)(1 - |z_n|) \to 0, \quad n \to \infty. \tag{6.2}$$

Then, by (1.2),

$$\begin{aligned}\chi(Z, W) = \chi(f_n(0), f_n(u_n)) &\leq |u_n| \int_0^1 f_n^{\#}(tu_n)\, dt \\ &= |u_n| \int_0^1 f^{\#}(\phi_{z_n}(tu_n))|\phi'_{z_n}(tu_n)|\, dt \\ &= |w_n - z_n| \int_0^1 f^{\#}(\phi_{z_n}(tu_n))\, dt \leq 2\|f\|_{\mathcal{N}^{\varphi}}|w_n - z_n|\varphi(|z_n|)\end{aligned}$$

for all sufficiently large n. Because $|w_n - z_n|\varphi(|z_n|) = |u_n| \to 0$ as $n \to \infty$, this together with (6.2) yields the contradiction $Z = W$.

A proof of the assertion for strongly φ-normal functions can be constructed by slightly modifying these arguments. We omit the details.

7. Proof of Theorem 7

Let first $f \in \mathcal{N}^{\varphi}$. Assume to the contrary of the assertion that there are sequences $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ tending to the boundary such that (2.6) is satisfied but $\alpha := \lim_{n\to\infty} f(z_n) \neq \lim_{n\to\infty} f(w_n) =: \beta$. By following the reasoning in the proof of Theorem 4 with appropriate modifications, we obtain $|u_n| \to 0$, $f_n(0) \to \alpha$, and $f_n(u_n) \to \beta$ as $n \to \infty$ and further that $\chi(\alpha,\beta) \leq \lim_{n\to\infty}|u_n| = 0$, which is the desired contradiction. Note that this part of the proof does not require the assumptions (1.1) and (1.2).

To prove the converse, assume to the contrary that $f \notin \mathcal{N}^{\varphi}$. Then, by Theorem 6 with $\beta = 0$, there exist a sequence $\{a_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ tending to the boundary and a sequence $\{\rho_n\}_{n=1}^{\infty}$ of positive real numbers such that $\varphi(|a_n|)\rho_n \to 0$ and the sequence $\{f_n(\xi)\}_{n=1}^{\infty} := \{f(a_n + \rho_n\xi)\}_{n=1}^{\infty}$ of functions converges spherically uniformly on each compact subset of $\mathbb{C}$ to a nonconstant meromorphic function g as $n \to \infty$. Then, in particular, $f_n(0) \to g(0) =: Z$ as $n \to \infty$. Take $w \in D(0,r)$ such that $g(w) = W \neq Z$. From Hurwitz's theorem it follows that, for any given $r \in (0,1)$, all but a finite number of the functions f_n assume the value W in $D(0,r)$. Hence there exists a sequence $\{w_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ such that $w_n \to w$ as $n \to \infty$ and $f_n(w_n) = W$ for all sufficiently large n. Putting $b_n := a_n + \rho_n w_n$, we obtain

$$\begin{aligned}\rho(a_n,b_n)\varphi(|b_n|)(1-|b_n|) &\leq \frac{\rho_n|w_n|}{1-|b_n|}\varphi(|b_n|)(1-|b_n|)\\ &= \varphi(|a_n|)\rho_n|w_n|\frac{\varphi(|a_n+\rho_n w_n|)}{\varphi(|a_n|)}\end{aligned}$$

for all sufficiently large n. Since $\varphi(|a_n|)\rho_n \to 0$, $|w_n| \to |w| \in (0,1)$, and $\varphi(|a_n+\rho_n w_n|)/\varphi(|a_n|) \to 1$ as $n \to \infty$, it follows that

$$\rho(a_n,b_n)\varphi(|b_n|)(1-|b_n|) \to 0, \quad n \to \infty,$$

but $Z = \lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n) = W$. This proves the "if" part of the assertion.

That $f \in \mathcal{N}_0^{\varphi}$ implies (2.7) can be proved in the same way that we proved the second assertion of Theorem 4. Once again we omit the details.

To prove the converse, assume to the contrary that $f \notin \mathcal{N}_0^{\varphi}$. Then by Theorem 6 there exist $R > 0$, a sequence $\{a_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ tending to the boundary, and a sequence $\{\rho_n\}_{n=1}^{\infty}$ of positive real numbers such that $\varphi(|a_n|)\rho_n < 1/R$ for all n and the sequence $\{f_n(\xi)\} := \{f(a_n + \rho_n\xi)\}$ of functions converges on each compact subset of $D(0,R)$ to a nonconstant meromorphic function g. Then, in particular, $f_n(0) \to g(0) := Z$ as $n \to \infty$. Take $w \in D(0,r)$ such that $g(w) = W \neq Z$. As in the proof of the first assertion for $\mathcal{N}^{\varphi}$, we obtain a sequence $\{w_n\}$ of points in $\mathbb{D}$ such that $w_n \to w$ as $n \to \infty$ and $f_n(w_n) = W$ for all sufficiently large n. Putting $b_n := a_n + \rho_n w_n$, we have

$$\rho(a_n,b_n)\varphi(|b_n|)(1-|b_n|) \leq \frac{|w_n|}{R}\frac{\varphi(|a_n+\rho_n w_n|)}{\varphi(|a_n|)}$$

for all sufficiently large n. Since

$$|w_n| \to |w| \in (0,1) \quad \text{and} \quad \varphi(|a_n + \rho_n w_n|)/\varphi(|a_n|) \to 1 \quad \text{as } n \to \infty,$$

it follows that (2.7) is satisfied but $Z = \lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n) = W$. This proves the "if" part of the second assertion.

8. Proof of Theorem 8

Let $f \in \mathcal{N}^{\varphi}$, and suppose to the contrary that there exist sequences $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} f(z_n) \in \mathbb{C}$ and $\lim_{n\to\infty} f(w_n) = \infty$ but

$$\rho(z_n, w_n)\varphi(|z_n|)(1 - |z_n|) \le C < \infty$$

for all $n \in \mathbb{N}$. Define $u_n := (w_n - z_n)\varphi(|z_n|)$, and assume for a moment that $C \le 1/4$. Then $\lim_{n\to\infty}|u_n| \le 2C \le 1/2$. By Theorem 3, the family $\{f_n(z) := f \circ \phi_{z_n} : n \in \mathbb{N}\}$ is normal in $\mathbb{D}$. By passing to a subsequence if necessary, we may assume that f_n converges uniformly on compact subsets of $\mathbb{D}$ either to an analytic function in $\mathbb{D}$ or to the constant ∞. However, the latter case can be excluded because, by assumption, $f(z_n) \to c \in \mathbb{C}$ as $n \to \infty$. Therefore, also $g(0) = c$ for the limit function g by the uniform convergence. It follows that, for a given $r \in (1/2, 1)$, there exist $C_r \in (0,\infty)$ and $N_r \in \mathbb{N}$ such that $|f_n(z)| \le C_r$ for all $z \in D(0,r)$ and $n \ge N_r$. Since $f(w_n) = f_n(u_n) \to \infty$ as $n \to \infty$ and since $|u_n| \le r$ for all sufficiently large n, we obtain a contradiction.

If $C > 1/4$ then $\lim_{n\to\infty}|u_n|/(4C) \le 1/2$. By (1.1) we can find an $N_C \in \mathbb{N}$ such that $z_n + 4Cz/\varphi(|z_n|) \in \mathbb{D}$ for all $n \ge N_C$. Since clearly $\mathcal{N}^{\varphi} = \mathcal{N}^{\varphi/(4C)}$, the family $\{f(z_n + 4Cz/\varphi(|z_n|)) : n \ge N_C\}$ is normal in $\mathbb{D}$. Proceeding as before and using the sequence $\{u_n/(4C)\}_{n=N_C}^{\infty}$ instead of $\{u_n\}_{n=1}^{\infty}$, we again obtain a contradiction.

9. Proof of Theorem 11

If $B_1/B_2 \in \mathcal{N}^{\varphi}$, then there exists an $C > 0$ such that

$$\left(\frac{B_1}{B_2}\right)^{\#}(z) = \frac{|B_1'(z)B_2(z) - B_1(z)B_2'(z)|}{|B_1(z)|^2 + |B_2(z)|^2} \le C\varphi(|z|) \tag{9.1}$$

for all $z \in \mathbb{D}$. Choose $z = z_n$ to obtain $|B_1'(z_n)| \le C|B_2(z_n)|\varphi(|z_n|)$. Since

$$\begin{aligned} B_1'(z) &= \sum_{j=1}^{\infty} \frac{|z_j|}{z_j}\left(\frac{|z_j|^2 - 1}{(1-\bar{z}_j z)^2} \prod_{k\neq j} \frac{|z_k|}{z_k}\frac{z_k - z}{1 - \bar{z}_k z}\right), \\ |B_1'(z_n)| &= \frac{1}{1-|z_n|^2}\prod_{k\neq n}\rho(z_k, z_n), \end{aligned} \tag{9.2}$$

and $\{z_n\}_{n=1}^{\infty}$ is uniformly separated, there exists a $\delta > 0$ such that

$$\prod_{k=1}^{\infty}\rho(w_k,z_n)\prod_{k\neq n}\rho(z_k,z_n)=|B_2(z_n)|\prod_{k\neq n}\rho(z_k,z_n)$$
$$\geq\frac{\left(\prod_{k\neq n}\rho(z_k,z_n)\right)^2}{C\varphi(|z_n|)(1-|z_n|^2)}\geq\frac{\delta^2/C}{\varphi(|z_n|)(1-|z_n|^2)}$$

for all $n\in\mathbb{N}$. Because of symmetric reasons this ensures that $\{z_n\}_{n=1}^{\infty}\cup\{w_n\}_{n=1}^{\infty}$ is uniformly φ-separated and hence φ-separated. Thus (1) implies (3) and (3) implies (2).

To prove that (2) implies (1), assume to the contrary of the assertion that $B_1/B_2\notin\mathcal{N}^{\varphi}$. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of points in $\mathbb{D}$ for which (1.3) fails. The equality in (9.1) and the Schwarz–Pick lemma give

$$\left(\frac{B_1}{B_2}\right)^{\#}(z)\leq\frac{1}{|B_1(z)|(1-|z|^2)}+\frac{1}{|B_2(z)|(1-|z|^2)}$$

for all $z\in\mathbb{D}$. Therefore, by passing to a subsequence if necessary, we have

$$|B_i(a_k)|\varphi(|a_k|)(1-|a_k|^2)\to 0,\quad k\to\infty, \tag{9.3}$$

for either $i=1$ or $i=2$. In fact, (9.3) holds for both indexes. Namely, if it is satisfied for $i=1$ and if the limit inferior equals $\gamma>0$ for $i=2$, then $|B_1(a_k)|/|B_2(a_k)|=o(1)$ as $k\to\infty$. It follows that

$$\left(\frac{B_1}{B_2}\right)^{\#}(a_k)\leq\frac{1}{|B_2(a_k)|(1-|a_k|^2)}\left(1+\left|\frac{B_1(a_k)}{B_2(a_k)}\right|\right)$$
$$=\frac{\varphi(|a_k|)}{\gamma}(1+o(1)),\quad k\to\infty.$$

This clearly contradicts the original assumption on $\{a_k\}_{k=1}^{\infty}$, so (9.3) must be satisfied for both indexes. Passing to subsequences if necessary, we may assume that both $\rho(z_n,a_n)$ and $\rho(w_n,a_n)$ tend to zero as $n\to\infty$. This follows by (9.3) and [12, Thm. 1]. Since $\{z_n\}_{n=1}^{\infty}$ is uniformly separated, the triangular inequality and the Schwarz–Pick lemma yield

$$\delta\leq\prod_{k\neq n}\rho(z_k,z_n)=\rho\left(\prod_{k\neq n}\frac{z_k-z_n}{1-\bar{z}_kz_n},0\right)$$
$$\leq\rho\left(\prod_{k\neq n}\frac{z_k-z_n}{1-\bar{z}_kz_n},\prod_{k\neq n}\frac{z_k-z}{1-\bar{z}_kz}\right)+\prod_{k\neq n}\rho(z_k,z)\leq\rho(z_n,z)+\prod_{k\neq n}\rho(z_k,z)$$

for all $z\in\mathbb{D}$; in particular,

$$|B_1(a_n)|\geq\rho(z_n,a_n)(\delta-\rho(z_n,a_n)).$$

Hence $|B_1(a_n)|\geq\rho(z_n,a_n)\delta/2$ for all n large enough. This combined with (9.3) gives

$$\rho(z_n,a_n)\varphi(|a_n|)(1-|a_n|^2)\to 0,\quad n\to\infty. \tag{9.4}$$

Analogous reasoning for B_2 shows that (9.4) with z_n replaced by w_n holds. The triangular inequality then yields

$$\rho(z_n, w_n)\varphi(|a_n|)(1 - |a_n|^2) \to 0, \quad n \to \infty. \tag{9.5}$$

Since $\rho(a_n, z_n) \to 0$ as $n \to \infty$, we have

$$\frac{1 - |a_n|^2}{1 - |z_n|^2} \to 1, \quad n \to \infty. \tag{9.6}$$

If $|z_n| \leq |a_n|$, then $\varphi(|a_n|) \geq \varphi(|z_n|)$ by the monotonicity. If $|z_n| > |a_n|$, then (9.4) implies $|z_n - a_n| \leq 1/\varphi(|a_n|)$ for all sufficiently large n, and hence (1.2) yields

$$\varphi(|a_n|)/\varphi(|z_n|) \to 1 \tag{9.7}$$

as $n \to \infty$. Combining (9.5)–(9.7), it follows that

$$\rho(z_n, w_n)\varphi(|z_n|)(1 - |z_n|^2) \to 0, \quad n \to \infty,$$

and hence $\{z_n\}_{n=1}^{\infty} \cup \{w_n\}_{n=1}^{\infty}$ is not φ-separated. Thus (2) implies (1).

If $B_1/B_2 \in \mathcal{N}_0^{\varphi}$, then the foregoing arguments with C replaced by $o(1)$, as $n \to \infty$, show that $\{z_n\}_{n=1}^{\infty} \cup \{w_n\}_{n=1}^{\infty}$ is strongly uniformly φ-separated and hence strongly φ-separated.

To prove that (2′) implies (1′), assume to the contrary of the assertion that $B_1/B_2 \notin \mathcal{N}_0^{\varphi}$. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of points in $\mathbb{D}$ for which (1.4) fails. Following our previous reasoning, we deduce that

$$\lim_{k\to\infty} |B_i(a_k)|\varphi(|a_k|)(1 - |a_k|^2) \in (0, \infty), \quad i = 1, 2. \tag{9.8}$$

Further, passing to subsequences if necessary, we obtain

$$\lim_{n\to\infty} \rho(z_n, w_n)\varphi(|z_n|)(1 - |z_n|^2) < \infty;$$

hence $\{z_n\}_{n=1}^{\infty} \cup \{w_n\}_{n=1}^{\infty}$ is not strongly φ-separated. Thus (2′) implies (1′).

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R. Aulaskari
University of Eastern Finland
Department of Physics
and Mathematics
P.O. Box 111
80101 Joensuu
Finland

rauno.aulaskari@uef.fi

J. Rättyä
University of Eastern Finland
Department of Physics
and Mathematics
P.O. Box 111
80101 Joensuu
Finland

jouni.rattya@uef.fi