On Volumes along Subvarieties of Line Bundles with Nonnegative Kodaira–Iitaka Dimension

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1. Introduction

We study the restricted volume along subvarieties of line bundles with nonnegative Kodaira–Iitaka dimension. Our main interest is to compare it with a similar notion defined in terms of the asymptotic multiplier ideal sheaf, with which it coincides in the big case. We shall prove that the former is nonzero if and only if the latter is. We then study inequalities between them and prove that if they coincide on every very general curve then the line bundle must have zero Kodaira–Iitaka dimension or be big.

Let X be a smooth projective variety and L a divisor or a line bundle on X with nonnegative Kodaira–Iitaka dimension: $\kappa(L) \geq 0$. Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \mathrm{SBs}(L)$, where $\mathrm{SBs}(L) := \bigcap_{m>0} \mathrm{Bs}|mL|$ is the stable base locus. We denote by $H^0(X|V, mL) = \mathrm{Image}[H^0(X, mL) \rightarrow H^0(V, mL)]$ the image of restriction maps. The *restricted volume* of L along V is defined to be

$$\mathrm{vol}_{X|V}(L) = \limsup_{m \to \infty} \frac{h^0(X|V, mL)}{m^d/d!}.$$

Similarly, we define the *reduced volume* of L along V as follows:

$$\mu(V, L) = \limsup_{m \to \infty} \frac{h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)}{m^d/d!}.$$

Here $\mathcal{J}(\|mL\|) = \mathcal{J}(X, \|mL\|)$ is the asymptotic multiplier ideal sheaf of mL for every positive integer m [L, 11.1.2]. When L is big, $\mu(V, L) = \mathrm{vol}_{X|V}(L) > 0$ for any $V \not\subset \mathrm{NAmp}(L)$ [ELMNP3, 2.13; T3, 3.1], where

$$\mathrm{NAmp}(L) := \bigcap_{m>0} \mathrm{SBs}(mL - A)$$

for any given ample divisor A on X and is called the nonample locus of L (in [L, 10.3.2], this is denoted by $\mathbf{B}_+(L)$ and called the augmented base locus). In the big case, the restricted volume has played an important role in the proof of the boundedness of pluricanonical maps (cf. [HMc; T3; Ts2]) and the topic has been systematically studied by Ein, Lazarsfeld, Mustaţă, Nakamaye, and Popa in [ELMNP1; ELMNP2; ELMNP3; L]. On the other hand, very little is known in

the general case $\kappa(L) \geq 0$, and the present paper is an attempt to make the first basic steps in this direction—and also with the hope that a better understanding of the restricted volume in the case $L = K_X$ could possibly lead to further progress in the study of pluricanonical maps for varieties with positive Kodaira dimension (for an attempt to adapt the arguments of [HMc; T3; Ts2] to the case $\kappa(X) \geq 0$, see [Pa]; for results when $\kappa(X) \leq 2$, obtained with different techniques, see [VZ] and [To]).

Our main concern is about the relationship between $\mathrm{vol}_{X|V}(L)$ and $\mu(V, L)$ and the geometric meaning of their discrepancy for a line bundle L with $\kappa(L) \geq 0$. The basic relation $\mathfrak{b}(|mL|) \subset \mathcal{J}(\|mL\|)$ [L, 11.1.8], where $\mathfrak{b}(|mL|)$ is the base ideal of the linear system, leads to $\mathrm{vol}_{X|V}(L) \leq \mu(V, L)$. By definition, $\mathrm{vol}_{X|V}(L) > 0$ implies $\kappa(L) \geq \dim V$. However it is not clear at all that $\mu(V, L) > 0$ for $\dim V > 0$ implies $\kappa(L) > 0$. We have a natural product map

$$H^0(X|V, kL) \times H^0(X|V, mL) \to H^0(X|V, (k+m)L),$$

and also

$$\mathfrak{b}(|kL|) \cdot \mathfrak{b}(|mL|) \subset \mathfrak{b}(|(k+m)L|).$$

However, we do not know whether there exists a natural product map for $H^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)$ except when $V = X$, because we only have $\mathcal{J}(\|(k+m)L\|) \subset \mathcal{J}(\|kL\|) \cdot \mathcal{J}(\|mL\|)$ [L, 11.2.4]. So we do not know about a natural ring structure on $\bigoplus_{m \geq 0} H^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)$. In spite of these difficulties, we think it is worth studying $H^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)$, as well as $H^0(X|V, mL)$, because $\mu(V, L)$ is a direct generalization of the usual intersection number. In fact, in the case when L is semi-ample, $\mu(V, L) = L^d \cdot V$ for any $V \subset X$ (see Proposition 3.3 for a generalization to the case of a nef and abundant line bundle) whereas $\mathrm{vol}_{X|V}(L) \neq L^d \cdot V$ in general (see [ELMNP3, 5.10]). We first describe their asymptotic behaviors.

Tʜᴇᴏʀᴇᴍ 1.1.   *Let X be a smooth projective variety, L a line bundle on X with $\kappa(L) \geq 0$, and $f : X \to Y$ the Iitaka fibration associated to L. Let $V \subset X$ be a subvariety such that $V \not\subset \mathrm{SBs}(L)$. Let $q = \dim f(V) \geq 0$.*

(1) *Assume that V contains a general point of X. Then*

$$0 < \limsup_{m \to \infty} \frac{h^0(X|V, mL)}{m^q} < +\infty.$$

(2) *Assume that V contains a very general point of X. Then*

$$0 < \limsup_{m \to \infty} \frac{h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)}{m^q} < +\infty.$$

The following is the main consequence in this paper.

Cᴏʀᴏʟʟᴀʀʏ 1.2.   *Let X, L, and $f : X \to Y$ be as before. Let $V \subset X$ be a subvariety that contains a very general point of X.*

(1) *The following three conditions are equivalent*:

(o) *the map $f|_V : V \dashrightarrow f(V)$ is generically finite;*
(i) $\mathrm{vol}_{X|V}(L) > 0$;
(ii) $\mu(V, L) > 0$.

(2) *The condition* $\limsup_{m\to\infty} h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)/m = 0$ *implies the boundedness of* $h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)$ *as* $m \to \infty$.

Thus, the positivity of $\mathrm{vol}_{X|V}(L)$ and $\mu(V, L)$ are equivalent to each other, and hence the weaker condition $\mu(V, L) > 0$ also implies $\kappa(L) \geq \dim V$. As for Corollary 1.2(2) (which looks rather technical), it is the type of estimate appearing in the work of Nakayama [Na, V.1.12], where it is used to prove the abundance conjecture in the case $\kappa = 0$.

We then try to describe their differences or the ratio $\mu(V, L)/\mathrm{vol}_{X|V}(L)$ more precisely. In two extreme cases, it is known that they coincide. Let us recall the following.

PROPOSITION 1.3. (1) [ELMNP3, 2.13; T3, 3.1]. *Assume* $\kappa(L) = \dim X$. *Then* $\mu(V, L) = \mathrm{vol}_{X|V}(L) > 0$ *for any* $V \not\subset \mathrm{NAmp}(L)$.
(2) [T1, 1.2]. $\kappa(L) = 0$ *if and only if* $\mu(C, L) = 0$ (*then, in particular,* $\mathrm{vol}_{X|C}(L) = 0$) *for any curve* $C \not\subset \mathrm{SBs}(L)$.

We can read [T1, 1.2] as Proposition 1.3(2) because $\mu(C, L) = \|L; C\|$, where $\|L; C\|$ is the "intersection number" in [T1, 2.7] (see Proposition 2.5 in this paper). Moreover, by using the arguments in [T1, 3.1], we can show that $\kappa(L) = 0$ if and only if $\mu(V, L) = \mathrm{vol}_{X|V}(L) = 0$ for any subvarieties $V \not\subset \mathrm{SBs}(L)$. We can show that these are the only cases when the two invariants are equal as follows.

THEOREM 1.4. *Let X be a smooth projective variety and L a line bundle on X with $\kappa(L) \geq 0$. Let $x \in X$ be a very general point. Assume $\mu(C, L) = \mathrm{vol}_{X|C}(L)$ for any curve C passing through x. Then either $\kappa(L) = 0$ or $\kappa(L) = \dim X$.*

In case L is semi-ample, this is quite easy. Our proof consists, in fact, of trying to generalize the argument in this case. We show that an inequality $\mu(C, L) \geq \delta\, \mathrm{vol}_{X|C}(L)$ holds for every curve $C \not\subset \mathrm{SBs}(L)$ when the map $f|_C: C \dashrightarrow f(C)$ is finite of degree δ.

Our methods in this paper depend on a careful study of various multiplier ideal sheaves and dimension counting arguments. As mentioned in [L, 11.1.10], we do not know whether the definition of the asymptotic multiplier ideal $\mathcal{J}(\|L\|)$ is in the final form or not. This paper does not give a definitive answer on this. However, we hope some results in this paper will help us to understand it.

NOTATION AND CONVENTIONS. Throughout this paper, we let X be a smooth projective variety, L a divisor or a line bundle on X with $\kappa(L) \geq 0$, and $f: X \to Y$ the Iitaka fibration associated to L ([I] or [L, 2.1.33]). The Iitaka fibration is defined only up to birational equivalence. If a subvariety $V \subset X$ with $V \not\subset \mathrm{SBs}(L)$ is given then we take a birational morphism $\pi: X' \to X$ from a smooth projective variety X' with a projective morphism $f': X' \to Y'$ to a smooth projective variety Y', so that π is isomorphic over the generic point of V and f' is birational to the Iitaka fibration $f: X \to Y$ associated to L. Then we understand $\dim f(V)$ to be $\dim f'(V')$ and also $f|_V: V \dashrightarrow f(V)$ to be $f'|_{V'}: V' \to f'(V')$. A curve in a general fiber of $f|_V: V \dashrightarrow f(V)$ will be a curve C whose strict transform $C' \subset V'$ is contained in a general fiber of $f'|_{V'}$. By a *general* (resp. *very general*)

point on X, we mean a point that belongs to the complement of a (resp. a countable union of) proper Zariski closed subset(s), which is determined by the divisor L.

2. Volumes along Subvarieties

We shall study volumes along subvarieties and prove Theorem 1.1.

2.1. Intermediate Restricted Volumes

We shall prove Theorem 1.1(1). Let $\dim f(V) = q$. We note that the space $H^0(X|V, mL)$ is unchanged under a birational morphism $\pi\colon X' \to X$ from a smooth projective variety X', which is isomorphic over the generic point of V (cf. the proof of [ELMNP3, 2.4]). We may assume, by taking an embedded resolution of V, that V is smooth and that there exists a projective morphism $f\colon X \to Y$ to a smooth projective variety Y, so that f is the Iitaka fibration associated to L.

Positivity: $\limsup h^0(X|V, mL)/m^q > 0$. Let A_Y be a very ample divisor on Y. We see that $0 < \limsup_\ell h^0(f^*A_Y, \ell L)/\ell^{\kappa(L)-1} < +\infty$. By the same argument as in Kodaira's lemma, we have $H^0(X, \ell L - f^*A_Y) \neq 0$ for some large ℓ. We take one such ℓ. Then $\ell L = f^*A_Y + E$ for some effective divisor E on X and $H^0(Y, mA_Y) \cong H^0(X, mf^*A_Y) \subset H^0(X, m\ell L)$ for any $m > 0$. If V contains a general point then we can assume $V \not\subset E$. (If L is big, this is equivalent to saying that $V \not\subset \mathrm{NAmp}(L)$.) Since A_Y is ample, the restriction map $H^0(Y, mA_Y) \to H^0(f(V), mA_Y)$ is surjective for every large m. Then we have an inclusion $H^0(X|V, m\ell L) \supset (f|_V)^* H^0(f(V), mA_Y)$ for every large m. Hence there exists a constant $c > 0$ such that $h^0(X|V, m\ell L) \geq cm^q$ for every large m.

Finiteness: $\limsup h^0(X|V, mL)/m^q < +\infty$. In case $q = d = \dim V$, this is well known. We may assume $q < d$. Suppose to the contrary that

$$\limsup h^0(X|V, mL)/m^q = +\infty.$$

We take a sufficiently general complete intersection $W \subset V$ of $\dim W = q$ and $f(W) = f(V)$. By the same argument as in Kodaira's lemma, the restriction map $H^0(X|V, mL)(\subset H^0(V, mL)) \to H^0(W, mL)$ has a nontrivial kernel for large m. This means that there exists a nonzero $s \in H^0(X, mL)$ such that $s|_V$ is not zero and vanishes along W. We may take m so large that the map $\Phi_{|mL|}\colon X \dashrightarrow \Phi_{|mL|}(X)$ is birational to the Iitaka fibration $f\colon X \to Y$. Since $f|_V$ is not generically finite, it follows that $(\mathrm{div}\, s)|_V$, where $s \in H^0(X, mL)$ must be in the direction of the ruling $f|_V\colon V \to f(V)$ plus some another fixed divisor $F_m|_V$ independent of $s \in H^0(X, mL)$. On the other hand, $W \subset V$ can be in arbitrary direction and $f(W) = f(V)$. The vanishing of $s|_V$ along W imposes the vanishing of $s|_V$ on V. This is a contradiction, which proves Theorem 1.1(1).

REMARK 2.1. Theorem 1.1(1) can be read as follows. Let X, L, $f : X \to Y$, and $V \subset X$ be as in Theorem 1.1(1). Let p be an integer with $0 \leq p \leq d$. Then the following two conditions are equivalent:

(0) $\dim f(V) = p$;
(1) $0 < \lim \sup_m h^0(X|V, mL)/m^p < +\infty$.

2.2. Reduced Volumes

Here we collect basic properties of reduced volumes.

LEMMA 2.2 (Homogeneity) [ELMNP3, 3.3]. *Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \mathrm{SBs}(L)$. Then $\mu(V, pL) = p^d \mu(V, L)$ for every positive integer p.*

Lemma 2.2 allows to define the quantity $\mu(V, L)$ for $\mathbb{Q}$-divisors.

LEMMA 2.3 (Projection formula). *Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \mathrm{SBs}(L)$. Let $\pi : X' \to X$ be a birational morphism from a smooth projective variety X'. Let $V' \subset X'$ be a subvariety of $\dim V' = d$ with $f(V') = V$ and with $V' \not\subset \mathrm{Exc}(\pi)$ the exceptional locus of π. Then $\mu(V, L) = \mu(V', \pi^*L)$.*

Proof. Let $e = e(L) \geq 1$ be the exponent of L, which is the smallest positive integer such that $h^0(X, meL) \neq 0$ for all integer $m > 0$ [L, 2.1.1]. We denote by $L' = \pi^*L$. We see that $e(L) = e(L')$. We take a sufficiently large p such that $\mathcal{J}(\|mL\|) = \mathcal{J}\big(\frac{m}{ep} \cdot |epL|\big)$ and $\mathcal{J}(\|mL'\|) = \mathcal{J}\big(\frac{m}{ep} \cdot |epL'|\big)$ [L, 11.1.5]. We note the basic relations

$$\mathcal{J}\Big(\frac{m}{ep} \cdot |epL'|\Big) \subset \pi^{-1}\mathcal{J}\Big(\frac{m}{ep} \cdot |epL|\Big) \cdot \mathcal{O}_{X'} \quad \text{and}$$

$$\mathcal{J}\Big(\frac{m}{ep} \cdot |epL|\Big) = \pi_*\Big(\mathcal{J}\Big(\frac{m}{ep} \cdot |epL'|\Big) \otimes K_{X'/X}\Big)$$

(cf. [L, 9.5.8] and [L, 9.2.33], resp.). Here $K_{X'/X}$ is the relative canonical bundle of π. Since $\mathcal{J}\big(\frac{m}{ep} \cdot |epL'|\big) \otimes K_{X'/X}$ is torsion free, the natural homomorphism

$$\pi^*\Big(\pi_*\Big(\mathcal{J}\Big(\frac{m}{ep} \cdot |epL'|\Big) \otimes K_{X'/X}\Big)\Big) \to \mathcal{J}\Big(\frac{m}{ep} \cdot |epL'|\Big) \otimes K_{X'/X}$$

induces a homomorphism

$$\pi^{-1}\Big(\pi_*\Big(\mathcal{J}\Big(\frac{m}{ep} \cdot |epL'|\Big) \otimes K_{X'/X}\Big)\Big) \cdot \mathcal{O}_{X'} \to \mathcal{J}\Big(\frac{m}{ep} \cdot |epL'|\Big) \otimes K_{X'/X},$$

which is generically an isomorphism because π is birational. Moreover, since $\pi^{-1}\big(\pi_*(\mathcal{J}\big(\frac{m}{ep} \cdot |epL'|\big) \otimes K_{X'/X})\big) \cdot \mathcal{O}_{X'}$ is torsion free, the last homomorphism is injective. Putting everything together, we have

$$\mathcal{J}(\|mL'\|) \subset \pi^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{O}_{X'} \to \mathcal{J}(\|mL'\|) \otimes K_{X'/X},$$

where the last homomorphism is injective. Since $K_{X'/X}$ is independent of m, it is not difficult to see that

$$\mu(V', L') = \limsup \frac{h^0(V', \mathcal{O}_{V'}(mL') \otimes \mathcal{J}(\|mL'\|)|_{V'} \otimes K_{X'/X}|_{V'})}{m^d/d!}.$$

Hence we obtain

$$\mu(V', L') = \limsup \frac{h^0(V', \mathcal{O}_{V'}(mL') \otimes \pi^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{O}_{V'})}{m^d/d!}.$$

The right-hand side is, in fact, $\mu(V, L)$ by Lemma 2.4, so we are done. □

LEMMA 2.4. *Let* $V \subset X$ *be a subvariety of* $\dim V = d > 0$ *such that* $V \not\subset$ SBs(L), *and let* $v: V' \to V$ $(\subset X)$ *be a birational morphism from a proper variety* V'. *Then, as* $m \to \infty$, *one has*

(1) $\overline{\lim} \dfrac{h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)}{m^d/d!} = \overline{\lim} \dfrac{h^0(V', \mathcal{O}_{V'}(mv^*L) \otimes v^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{O}_{V'})}{m^d/d!},$

(2) $\underline{\lim} \dfrac{h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)}{m^d/d!} = \underline{\lim} \dfrac{h^0(V', \mathcal{O}_{V'}(mv^*L) \otimes v^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{O}_{V'})}{m^d/d!}.$

Proof. (1) We denote by $L' = v^*L$. Let $\mathcal{I} \subset \mathcal{O}_V$ be the annihilator of $v_*\mathcal{O}_{V'}/\mathcal{O}_V$, and let $\mathcal{I}' = v^{-1}\mathcal{I} \cdot \mathcal{O}_{V'} \subset \mathcal{O}_{V'}$. Then we have

$$H^0(V', \mathcal{O}_{V'}(mL') \otimes v^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{I}') \subset v^*H^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)$$

as subspaces of $H^0(V', \mathcal{O}_{V'}(mL'))$; in particular,

$$h^0(V', \mathcal{O}_{V'}(mL') \otimes v^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{I}') \le h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V).$$

Since $\dim(\operatorname{Supp} \mathcal{O}_{V'}/\mathcal{I}') < d$, by an exact sequence argument we have

$$\overline{\lim} \frac{h^0(V', \mathcal{O}_{V'}(mL') \otimes v^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{O}_{V'})}{m^d}$$
$$= \overline{\lim} \frac{h^0(V', \mathcal{O}_{V'}(mL') \otimes v^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{I}')}{m^d}.$$

Hence we obtain

$$\overline{\lim} \frac{h^0(V', \mathcal{O}_{V'}(mL') \otimes v^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{O}_{V'})}{m^d}$$
$$\le \overline{\lim} \frac{h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)}{m^d}.$$

The converse of this inequality follows from this elementary fact:

$$h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V) \le h^0(V', \mathcal{O}_{V'}(mL') \otimes v^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{O}_{V'}).$$

Thus we obtain our equality.

(2) is obtained by substituting "$\underline{\lim}$" for "$\overline{\lim}$" in the proof of (1). □

When the subvariety V is a curve we have a more explicit description of the reduced volume, which will be used in the proof of Theorem 1.1(2). Let us first recall the definition of $\|L; C\|$ [T1, 2.7]. Let $\mathcal{J} \subset \mathcal{O}_C$ be an ideal sheaf. For the normalization $v: C' \to C$, we define $\deg_C \mathcal{J}$ as the degree of the invertible sheaf $v^{-1}\mathcal{J} \cdot \mathcal{O}_{C'}$. Then $mL \cdot C + \deg_C \mathcal{J}(\|mL\|)|_C \ge 0$ for any $m > 0$ [T1, 2.6(1)], and we can define

$$\|L; C\| = L \cdot C + \limsup_{m \to \infty} \frac{1}{m} \deg_C \mathcal{J}(\|mL\|)|_C.$$

PROPOSITION 2.5. *Let $C \subset X$ be a curve with $C \not\subset \mathrm{SBs}(L)$. Then $\mu(C, L) = \|L; C\|$ holds.*

Proof. The proof will proceed in the same way as in [T2, 3.1]. Let $\nu: C' \to C \subset X$ be the normalization. We consider a family of invertible sheaves $\{\mathcal{G}_m\}_{m \in \mathbb{N}}$ on C', where $\mathcal{G}_m = \mathcal{O}_{C'}(m\nu^*L) \otimes \nu^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{O}_{C'}$ with degree $d_m := \deg \mathcal{G}_m \geq 0$ [T1, 2.6(1)]. By the subadditivity $\mathcal{J}(\|(\ell + m)L\|) \subset \mathcal{J}(\|\ell L\|) \cdot \mathcal{J}(\|mL\|)$ [DEL; L, 11.2.4], it follows that $d_{\ell+m} \leq d_\ell + d_m$. Then, by [T2, 3.4], their limits $\lim_{m \to \infty} h^0(C', \mathcal{G}_m)/m$ and $\lim_{m \to \infty} d_m/m$ exist and coincide: $\lim h^0(C', \mathcal{G}_m)/m = \lim d_m/m$. By definition, $\deg_C(\mathcal{O}_C(mL) \otimes \mathcal{J}(\|mL\|)|_C) = \deg_{C'}(\mathcal{O}_{C'}(m\nu^*L) \otimes \nu^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{O}_{C'})$. Hence

$$\lim_{m \to \infty} \frac{h^0(C', \mathcal{O}_{C'}(m\nu^*L) \otimes \nu^{-1}\mathcal{J}(\|mL\|) \cdot \mathcal{O}_{C'})}{m}$$
$$= \lim_{m \to \infty} \frac{\deg_C(\mathcal{O}_C(mL) \otimes \mathcal{J}(\|mL\|)|_C)}{m}.$$

Then, by Lemma 2.4, we obtain our assertion. $\qquad\square$

2.3. Intermediate Reduced Volumes

We shall prove Theorem 1.1(2). We need a refinement of [T1, 3.1].

LEMMA 2.6. *Assume $\kappa(L) = 0$. Let $C \subset X$ be a curve with $C \not\subset \mathrm{SBs}(L)$. Then $h^0(C, \mathcal{O}_C(mL) \otimes \mathcal{J}(\|mL\|)|_C)$ and $\deg_C(\mathcal{O}_C(mL) \otimes \mathcal{J}(\|mL\|)|_C)$ are bounded as $m \to \infty$.*

Proof. Let $e = e(L) \geq 1$ be the exponent of L. Since $\kappa(L) = 0$, there exists a nonzero effective divisor $D \in |eL|$ such that $|meL|$ is generated by mD for any m. In general we have $\mathcal{J}(\|mL\|) = \mathcal{J}\left(\frac{1}{pe}|pemL|\right)$ for sufficiently large p [L, 11.1.5] and $\mathcal{J}\left(\frac{1}{pe}|pemL|\right) = \mathcal{J}\left(\frac{1}{pe}pmD\right) = \mathcal{J}\left(\frac{m}{e}D\right)$ [L, 9.2.26].

We have at least $\mathcal{J}(\|mL\|) = \mathcal{J}\left(\frac{m}{e}D\right) \subset \mathcal{J}(\lfloor m/e \rfloor D) = \mathcal{O}_X(-\lfloor m/e \rfloor D)$. Here $\lfloor a \rfloor$ denotes the integral part of a nonnegative number a. Then it is enough to bound $h^0(C, \mathcal{O}_C(mL - \lfloor m/e \rfloor D))$ and $\deg_C \mathcal{O}_C(mL - \lfloor m/e \rfloor D)$. Let $\nu: C' \to C$ be the normalization. We have $h^0(C, \mathcal{O}_C(mL - \lfloor m/e \rfloor D)) \leq h^0(C', \mathcal{O}_{C'}(m\nu^*L - \lfloor m/e \rfloor \nu^*D))$ and $\deg_{C'} \mathcal{O}_{C'}(m\nu^*L - \lfloor m/e \rfloor \nu^*D) = (m - e\lfloor m/e \rfloor)L \cdot C$. Since $0 \leq m - e\lfloor m/e \rfloor < e$, the invertible sheaves $\mathcal{O}_{C'}(m\nu^*L - \lfloor m/e \rfloor \nu^*D)$ have nonnegative bounded degrees as $m \to \infty$. The following sublemma implies our assertion. $\quad\square$

SUBLEMMA 2.7. *Let C be a smooth projective curve, and let $\{\mathcal{G}_m\}_m$ be a family of invertible sheaves on C with $\deg \mathcal{G}_m \geq 0$. Then, as $m \to \infty$, $\deg \mathcal{G}_m$ is bounded if and only if $h^0(C, \mathcal{G}_m)$ is bounded.*

Proof. We denote by g the genus of C, and we put $\chi(\mathcal{O}_C) = 1 - g$ and $d_m = \deg \mathcal{G}_m \geq 0$.

If d_m is unbounded, then for every $k > 0$ we have m_k such that $d_{m_k} > 2g - 2 + k$. Then, by the Riemann–Roch theorem and vanishing, $h^0(C, \mathcal{G}_m) = \deg \mathcal{G}_m + \chi(\mathcal{O}_C)$, which is unbounded.

Assume $d_m < b$ for any m. We claim that $h^0(C, \mathcal{G}_m) \le \max\{b, 2g - 1\} + \chi(\mathcal{O}_C)$. Take m. If $d_m > 2g - 2$, then Riemann–Roch and vanishing imply $h^0(C, \mathcal{G}_m) = \deg \mathcal{G}_m + \chi(\mathcal{O}_C) \le b + \chi(\mathcal{O}_C)$. If $d_m \le 2g - 2$, we take an effective divisor D_m on C with $\deg D_m + d_m = 2g - 1$. Then $h^0(C, \mathcal{G}_m) \le h^0(C, \mathcal{G}_m \otimes \mathcal{O}(D_m)) = 2g - 1 + \chi(\mathcal{O}_C)$. $\qquad\square$

For the finiteness we will need the following.

PROPOSITION 2.8. *Let $x \in X$ be a very general point, and let $C \subset X$ be a curve passing through x. If $\mu(C, L) = 0$, then $h^0(C, \mathcal{O}_C(mL) \otimes \mathcal{J}(\|mL\|)|_C)$ and $\deg_C(\mathcal{O}_C(mL) \otimes \mathcal{J}(\|mL\|)|_C)$ are bounded as $m \to \infty$.*

Proof. By virtue of Lemmas 2.3 and 2.4, possibly after taking a modification of X, we may assume that C is smooth. We may also assume that there exists a projective morphism $f : X \to Y$, to a smooth projective variety Y, that is birational to the Iitaka fibration associated to L. Let Y_0 be a countable union of subvarieties of Y such that X_y is smooth and $\kappa(X_y, L_y) = 0$ for any $y \in Y \setminus Y_0$, where $X_y = f^{-1}(y)$ and $L_y = L|_{X_y}$. Let $e_y = e(L_y)$ be the exponent of L_y for $y \in Y \setminus Y_0$. As in the proof of Lemma 2.6, for every $y \in Y \setminus Y_0$ we have $B_y \in |e_y L_y|$ such that $\mathcal{J}(X_y, \|mL_y\|) = \mathcal{J}\left(X_y, \frac{m}{e_y} B_y\right)$ for any m.

We fix m. Let $e = e(L)$ be the exponent of L. We take a sufficiently large integer $p = p_m$ such that $\mathcal{J}(X, \|mL\|) = \mathcal{J}\left(X, \frac{1}{pe}|pemL|\right)$ and then take a general member $D = D_m \in |pemL|$ such that $\mathcal{J}\left(X, \frac{1}{pe}|pemL|\right) = \mathcal{J}\left(X, \frac{1}{pe}D\right)$. By the generic restriction theorem [L, 9.5.35], there exists a subvariety $Y_m \subset Y$ such that $\mathcal{J}(X, \|mL\|)|_{X_y} = \mathcal{J}\left(X_y, \frac{1}{pe}D|_{X_y}\right)$ for any $y \in Y \setminus Y_m$.

We then take a very general point $y \in Y \setminus \bigcup_{m \ge 0} Y_m$. We can write $e = e_y q_y$ for a positive integer q_y. For every m, $|p_m em L_y| = |p_m m q_y e_y L_y| = \langle p_m m q_y B_y \rangle$. Hence $D_m|_{X_y} = p_m m q_y B_y$. Thus $\mathcal{J}\left(X_y, \frac{1}{p_m e} D_m|_{X_y}\right) = \mathcal{J}\left(X_y, \frac{m}{e_y} B_y\right)$. Finally, we have $\mathcal{J}(X, \|mL\|)|_{X_y} = \mathcal{J}(X_y, \|mL_y\|)$ for every $m > 0$.

By [T1, 1.3], $\mu(C, L) = 0$ entails that C is contained in a fiber $X_y = f^{-1}(y)$, where $y \in Y$ is also very general. In particular, $\mathcal{O}_C(mL) \otimes \mathcal{J}(X, \|mL\|)|_C = \mathcal{O}_C(mL_y) \otimes \mathcal{J}(X_y, \|mL_y\|)|_C$. Since we know the boundedness properties for L_y by Lemma 2.6, we have our assertion. $\qquad\square$

Proof of Theorem 1.1(2). The positivity,

$$\limsup_m \frac{h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)}{m^q} > 0,$$

follows from $h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V) \ge h^0(X|V, mL)$ and Theorem 1.1(1). We shall prove the finiteness, $\limsup_m h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)/m^q < +\infty$. In the case $q = d$ this is well known (by [I] or [L, 2.1.38]); hence we assume $q < d$. The proof proceeds by induction on $d - q \ge 0$. The first step, $d - q = 0$, is already completed.

We assume Theorem 1.1(2) to be true for any subvariety $W \subset X$ containing a very general point of X with $\dim W \leq d - 1$ and $\dim f(W) = q \ (\leq d - 1)$. Let $V \subset X$ be a subvariety containing a very general point of X with $\dim V = d$ and $\dim f(V) = q$. Let A be a very ample Cartier divisor on V. Let k be a positive integer. We take a general member $W_k \in |kA|$ such that $f(W_k) = f(V)$, W_k is smooth where V is, and $\dim \mathrm{Sing}\, W_k = \dim \mathrm{Sing}\, V - 1 \leq d - 2$. For every $m > 0$, we consider a restriction map $r_m \colon H^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V) \to H^0(W_k, \mathcal{O}_{W_k}(mL) \otimes \mathcal{J}(\|mL\|)|_{W_k})$. By the induction hypothesis, we know that

$$0 < \limsup_m \frac{h^0(W_k, \mathcal{O}_{W_k}(mL) \otimes \mathcal{J}(\|mL\|)|_{W_k})}{m^q} < +\infty.$$

If $\limsup_m h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)/m^q = +\infty$, then there exists a positive integer m_k such that the map r_{m_k} has a nontrivial kernel. We take such m_k and a nonzero $s_k \in \ker r_{m_k}$. We follow the same process for every k.

We can find a curve C in a very general fiber of $f|_V \colon V \dashrightarrow f(V)$ such that $C \not\subset W_k$, C intersects W_k where V (and hence W_k) is smooth, and $s_k|_C \not\equiv 0$ for all $k > 0$. Since $\dim f(C) = 0$, we have $\|C; L\| = 0$ by [T1, 1.3]. From Proposition 2.5 we deduce that $\mu(C, L) = 0$. Let $\nu \colon C' \to C \ (\subset V)$ be the normalization. Then $\nu^*(s_k|_C)$ defines a nonzero element of

$$H^0(C', \mathcal{O}_{C'}(m_k \nu^* L - k \nu^* A) \otimes \nu^{-1} \mathcal{J}(\|m_k L\|) \cdot \mathcal{O}_{C'}).$$

In particular,

$$\deg_C(\mathcal{O}_C(m_k L) \otimes \mathcal{J}(\|m_k L\|)|_C) \equiv \deg_{C'}(\mathcal{O}_{C'}(m_k \nu^* L) \otimes \nu^{-1} \mathcal{J}(\|m_k L\|) \cdot \mathcal{O}_{C'})$$
$$\geq kA \cdot C$$

and hence is unbounded. It may be that $\limsup_k m_k/k = 0$, but this contradicts Proposition 2.8. Thus $\limsup_m h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)/m^q < +\infty$. $\quad\square$

Proof of Corollary 1.2. (1) Assume (o) (resp. (i), resp. (ii)). Then $q = \dim f(V)$ in Theorem 1.1 must be $q = d = \dim V$. Then by Theorem 1.1, we have (i) and (ii) (resp. (o) and (ii), resp. (o) and (i)).

(2) Assume $\limsup_m h^0(V, mL \otimes \mathcal{J}(\|mL\|))/m = 0$. By Theorem 1.1(2) we have $q = \dim f(V) = 0$, and then Theorem 1.1(2) with $q = 0$ implies the boundedness of $h^0(V, mL \otimes \mathcal{J}(\|mL\|))$. $\quad\square$

3. Relations among Various Volumes along Subvarieties

3.1. Proof of Theorem 1.4

To prove Theorem 1.4, we introduce another, more geometric notion.

NOTATION 3.1. Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \mathrm{SBs}(L)$. Let m be a sufficiently large integer such that $\mathrm{Bs}|mL| = \mathrm{SBs}(L)$ and such that the rational map $\Phi_{|mL|} \colon X \dashrightarrow \mathbb{P} = \mathbb{P}^{N_m}$ with $N_m = \dim|mL|$ is birational to the Iitaka fibration $f \colon X \to Y$ associated to L. Let $\pi_m \colon X_m \to X$ be

a birational morphism from a smooth projective variety X_m such that $\pi_m^*|mL| = |M_m| + F_m$, where $|M_m|$ is base point free (the moving part) and F_m is the fixed part. Denote by $\psi_m = \Phi_{|M_m|} : X_m \to \mathbb{P}$ the induced morphism and by $\mathcal{O}(1)$ the hyperplane bundle on $\mathbb{P}$ such that $\mathcal{O}_{X_m}(M_m) = \psi_m^*\mathcal{O}(1)$. We can take π_m such that it is an isomorphism over the generic point of V. We then denote by $V_m \subset X_m$ the strict transform of V.

Definition 3.2 [ELMNP3, 2.6, 2.7].   Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \mathrm{SBs}(L)$. We define

$$\|L^d \cdot V\| := \limsup_{m \to \infty} \frac{M_m^d \cdot V_m}{m^d} = \limsup_{m \to \infty} \frac{\#(V \cap D_{m,1} \cap \cdots \cap D_{m,d} \setminus \mathrm{SBs}(L))}{m^d}.$$

Here $D_{m,1}, \ldots, D_{m,d} \in |mL|$ are general members. This number $\|L^d \cdot V\|$ is called the *asymptotic intersection number* of L and V [ELMNP3, 2.6] or the *asymptotic moving intersection number* for the right-hand side [ELMNP3, 2.7].

This $\|L \cdot C\|$ for curves is different from $\|L; C\|$ in [T1, 2.7] in general. In case L is big, $\|L^d \cdot V\| = \mu(V, L) = \mathrm{vol}_{X|V}(L)$ holds for any subvariety $V \subset X$ of $\dim V = d > 0$ with $V \not\subset \mathrm{NAmp}(L)$ [ELMNP3, 2.13; T3, 3.1]. In another ideal case, these quantities relate to each other as follows. (See [L, 2.3.17] for nef and abundant divisors.)

Proposition 3.3.   *Assume L is nef and abundant. Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that V contains a general point of X (in particular, $V \not\subset \mathrm{SBs}(L)$) and such that the map $f|_V : V \dashrightarrow f(V)$ is generically finite of degree δ. Then $\mu(V, L) = \delta\, \mathrm{vol}_{X|V}(L) = \|L^d \cdot V\|$.*

Proof. By Kawamata [K, 2.1], there exists a birational morphism $\pi : X' \to X$ from a smooth projective variety X', and a surjective morphism $f : X' \to Y$ with connected fibers to a smooth projective variety Y with a nef and big divisor L_Y on Y, such that $\pi^*L \sim_{\mathbb{Q}} f^*L_Y$, where $\sim_{\mathbb{Q}}$ denotes $\mathbb{Q}$-linear equivalence. Because V contains a general point, we may assume that π is an isomorphism over the generic point of V. Hence, using this fact and the homogeneity of $\mu(V, L)$, we may assume from the beginning that $X' = X$ and also $L = f^*L_Y$ for a nef and big divisor L_Y.

(1) We claim that $\mathrm{vol}_{X|V}(L) = \mathrm{vol}_{Y|f(V)}(L_Y)$. This follows because there are natural isomorphisms $H^0(Y, mL_Y) \cong H^0(X, mL)$ by pull-back for all m and hence $H^0(Y|f(V), mL_Y) \cong H^0(X|V, mL)$ for all m.

(2) We claim that $\|L^d \cdot V\| = \delta\|L_Y^d \cdot f(V)\|$. This follows from the alternative definition (Definition 3.2) of $\|L^d \cdot V\|$. By taking general members $D_{m,1}, \ldots, D_{m,d} \in |mL_Y|$ ($\cong |mL|$) for every large m, we have

$$\delta\|L_Y^d \cdot f(V)\| = \delta \limsup_m \frac{\#(f(V) \cap D_{m,1} \cap \cdots \cap D_{m,d} \setminus \mathrm{SBs}(L_Y))}{m^d}$$

$$= \limsup_m \frac{\#(V \cap f^*D_{m,1} \cap \cdots \cap f^*D_{m,d} \setminus \mathrm{SBs}(L))}{m^d} = \|L^d \cdot V\|.$$

(3) We claim that $\mu(V, L) = L^d \cdot V$. By Wilson [W, 2.2], since L_Y is nef and big, there exists an effective divisor D on Y such that the linear system $|mL_Y - D|$ is base point free for all sufficiently large m; hence so is $|mL - f^*D|$ for all sufficiently large m. Then it is not difficult to see that $\mathcal{J}(X, \|mL\|) = \mathcal{O}_X$ for all m. Then $\mu(V, L) = \limsup_m h^0(V, \mathcal{O}_V(mL))/(m^d/d!) = L^d \cdot V = \delta L_Y^d \cdot f(V)$. We can find this type of argument in [MoR, Sec. 2].

(4) For the nef and big L_Y, we know that $L_Y^d \cdot f(V) = \mathrm{vol}_{Y|f(V)}(L_Y) = \|L_Y^d \cdot f(V)\|$ (cf. [ELMNP3, 2.13]). Then $\delta L_Y^d \cdot f(V) = \delta \mathrm{vol}_{Y|f(V)}(L_Y) = \delta\|L_Y^d \cdot f(V)\|$. By (1) and (2), we have $L^d \cdot V = \delta \mathrm{vol}_{X|V}(L) = \|L^d \cdot V\|$. Then we have our assertion by (3). $\qquad\square$

REMARK 3.4. In a similar situation, if $L = f^*L_Y$ for a big (but not necessarily nef) divisor L_Y on Y, we have $\mu(V, L) \geq \delta \mathrm{vol}_{X|V}(L) = \|L^d \cdot V\|$. In fact, for L_Y big it is known that $\mathrm{vol}_{Y|f(V)}(L_Y) = \|L_Y^d \cdot f(V)\|$ [ELMNP3, 2.13]. The claims (1) and (2) in the preceding proof still hold because we do not require that L be nef. Then we have $\delta \mathrm{vol}_{X|V}(L) = \|L^d \cdot V\|$. Since $\mu(V, L) \geq \|L^d \cdot V\|$ in general (by Lemma 3.5 to follow), we have our assertion.

We shall study relationships among three notions of volumes along subvarieties in case the divisor is neither big nor nef-abundant (i.e., in very bad situations) and prove Theorem 1.4 as a consequence.

LEMMA 3.5. *Let $V \subset X$ be a subvariety of* $\dim V = d > 0$ *such that* $V \not\subset$ $\mathrm{SBs}(L)$*. Then $\mu(V, L) \geq \|L^d \cdot V\|$.*

Proof. We use Notation 3.1. We take a sufficiently large m. We denote by ν_m: $V_m \to V \subset X$ the induced morphism and put $L' = \pi_m^*L$. Let k be a positive integer. We have $\mathcal{O}_{X_m}(-kF_m) = \mathfrak{b}(|mL'|)^k \subset \mathfrak{b}(|kmL'|)$, where $\mathfrak{b}(|\cdot|)$ stands for the base ideal of a linear system, $\mathfrak{b}(|kmL'|) \subset \mathcal{J}(\|kmL'\|)$ [L, 11.1.8(iv)], and $\mathcal{J}(\|kmL'\|) \subset \pi_m^{-1}\mathcal{J}(\|kmL\|) \cdot \mathcal{O}_{X_m}$ [L, 9.5.8]. Therefore $\mathcal{O}_{X_m}(kM_m) = \mathcal{O}_{X_m}(kmL' - kF_m) \subset \mathcal{O}_{X_m}(kmL') \otimes \pi_m^{-1}\mathcal{J}(\|kmL\|) \cdot \mathcal{O}_{X_m}$, and then $\mathcal{O}_{V_m}(kM_m) \subset \mathcal{O}_{V_m}(kmL') \otimes (\pi_m^{-1}\mathcal{J}(\|kmL\|) \cdot \mathcal{O}_{X_m})|_{V_m} = \mathcal{O}_{V_m}(kmL') \otimes \nu_m^{-1}\mathcal{J}(\|kmL\|) \cdot \mathcal{O}_{V_m}$.

Now we have

$$
\begin{aligned}
M_m^d \cdot V_m &= \limsup_{k\to\infty} \frac{h^0(V_m, \mathcal{O}_{V_m}(kM_m))}{k^d/d!} \\
&\leq \limsup_{k\to\infty} \frac{h^0(V_m, \mathcal{O}_{V_m}(kmL') \otimes \nu_m^{-1}\mathcal{J}(\|kmL\|) \cdot \mathcal{O}_{V_m})}{k^d/d!}.
\end{aligned}
$$

By Lemma 2.4, we know that the last term is $\mu(V, mL)$. Thus we have $M_m^d \cdot V_m/m^d \leq \mu(V, L)$, and letting $m \to \infty$ we have $\|L^d \cdot V\| \leq \mu(V, L)$. $\qquad\square$

LEMMA 3.6. *Let $V \subset X$ be a subvariety of* $\dim V = d > 0$ *such that* $V \not\subset$ $\mathrm{SBs}(L)$ *and the map $f|_V: V \dashrightarrow f(V)$ is generically finite of degree δ. Then $\delta \mathrm{vol}_{X|V}(L) \geq \|L^d \cdot V\|$.*

Proof. We use Notation 3.1. We may assume, by taking m to be large enough, that $\psi_m|_{V_m}: V_m \to \psi_m(V_m) \subset \mathbb{P}$ is generically finite of degree δ. We take one such m. We see that

$$\mathrm{vol}_{X|V}(L) = \mathrm{vol}_{X_m|V_m}(m\pi_m^* L)/m^d \geq \mathrm{vol}_{X_m|V_m}(M_m)/m^d.$$

We know by [ELMNP3, 2.5] that

$$\mathrm{vol}(V_m, M_m|_{V_m}) = \delta \, \mathrm{vol}(\psi_m(V_m), \mathcal{O}(1)|_{V_m}).$$

Since $\mathcal{O}(1)$ is ample, for every sufficiently large k we have $H^0(\psi_m(X_m), \mathcal{O}(k)) = H^0(\mathbb{P}|\psi_m(X_m), \mathcal{O}(k))$ and $H^0(\psi_m(V_m), \mathcal{O}(k)) = H^0(\mathbb{P}|\psi_m(V_m), \mathcal{O}(k))$; hence

$$H^0(\psi_m(V_m), \mathcal{O}(k)) = H^0(\psi_m(X_m)|\psi_m(V_m), \mathcal{O}(k)).$$

By pull-back, $H^0(\psi_m(X_m), \mathcal{O}(k)) \subset H^0(X_m, \mathcal{O}(kM_m))$. Thus

$$h^0(\psi_m(V_m), \mathcal{O}(k)) \leq h^0(X_m|V_m, \mathcal{O}(kM_m)).$$

Then

$$\begin{aligned}
\delta \, \mathrm{vol}_{X_m|V_m}(M_m) &= \delta \limsup_k h^0(X_m|V_m, \mathcal{O}(kM_m))/(k^d/d!) \\
&\geq \delta \limsup_k h^0(\psi_m(V_m), \mathcal{O}(k)|_{\psi_m(V_m)})/(k^d/d!) \\
&= \delta \, \mathrm{vol}(\psi_m(V_m), \mathcal{O}(1)|_{V_m}) \\
&= \mathrm{vol}(V_m, M_m|_{V_m}) = M_m^d \cdot V_m.
\end{aligned}$$

Hence $\delta \, \mathrm{vol}_{X|V}(L) \geq M_m^d \cdot V_m/m^d.$ $\square$

REMARK 3.7.   We have an additional remark in the preceding proof. Denote by $\psi_m \colon X_m \to Y_m := \psi_m(X_m) \subset \mathbb{P}$. Let $\nu_m \colon Y_m' \to Y_m$ be the normalization and $\psi_m' \colon X_m \to Y_m'$ the induced morphism (with connected fibers!). Then

$$\begin{aligned}
H^0(X_m, \mathcal{O}(kM_m)) &= H^0(X_m, \psi_m'^* \nu_m^* \mathcal{O}(k)) \\
&= H^0(Y_m', \nu_m^* \mathcal{O}(k)) \supset H^0(Y_m, \mathcal{O}(k)).
\end{aligned}$$

We see that $\mathrm{vol}(Y_m', \nu_m^* \mathcal{O}(1)) = \mathrm{vol}(Y_m, \mathcal{O}(1))$, since $\mathcal{O}(1)$ is ample. Hence

$$\limsup_k \frac{h^0(X_m, \mathcal{O}(kM_m))}{k^d/d!} = \mathrm{vol}(Y_m, \mathcal{O}(1))$$

with $d = \kappa(L)$.

For curves, we can show the converse of Lemma 3.6.

LEMMA 3.8.   *Let $C \subset X$ be a curve such that $C \not\subset \mathrm{SBs}(L)$ and such that the map $f|_C \colon C \dashrightarrow f(C)$ is finite of degree δ. Then $\delta \, \mathrm{vol}_{X|C}(L) \leq \|L \cdot C\|$.*

Proof. We may assume C is smooth by taking an embedded resolution of C and using Lemma 2.3. Since $\dim f(C) > 0$, we know that $\mathrm{vol}_{X|C}(L) > 0$; in particular, we have $\limsup_m h^0(X|C, mL) = +\infty$. We use Notation 3.1 with $V = C$. We denote by $\nu_m \colon C_m' \to \psi_m(C_m) = \Phi_{|mL|}(C)$ ($\subset \mathbb{P}$) the normalization and by $\alpha_m \colon C_m \to C_m'$ the induced morphism. We note that $C_m \cong C$. We may assume, by taking m to be large enough, that the map α_m has degree δ. We have $M_m|_{C_m} = \alpha_m^* \nu_m^* \mathcal{O}(1)$ and $\deg M_m|_{C_m} = \delta \deg \nu_m^* \mathcal{O}(1)$. Then

$$H^0(X|C, mL) \cong H^0(X_m|C_m, M_m) \subset \alpha_m^* H^0(C'_m, \nu_m^* \mathcal{O}(1)).$$

(We note that there is an isomorphism $H^0(X_m|C_m, M_m) \xrightarrow{\sim} H^0(X|C, mL)$.)

We note that $\limsup_m h^0(X|C, mL) = +\infty$ implies $\limsup_m (\deg M_m|_{C_m}) = +\infty$. In fact, since $h^1(C_m, M_m|_{C_m}) = h^0(C_m, K_{C_m} - M_m|_{C_m}) \leq h^0(C, K_C)$ is bounded, it follows that if we had $\deg M_m|_{C_m} < d_0$ for all m then the Riemann–Roch theorem would imply that $h^0(C_m, M_m|_{C_m}) \geq h^0(X|C, mL)$ is also bounded. Thus we can take m so large that $\deg M_m|_{C_m} > \delta \deg K_C$. In particular, $\deg \nu_m^* \mathcal{O}(1) > \deg K_{C'_m}$. Then, by Riemann–Roch and vanishing, we have

$$h^0(C_m, M_m|_{C_m}) = \deg M_m|_{C_m} + \chi(\mathcal{O}_{C_m})$$

and

$$h^0(C'_m, \nu_m^* \mathcal{O}(1)) = \deg \nu_m^* \mathcal{O}(1) + \chi(\mathcal{O}_{C'_m}).$$

Therefore,

$$\delta h^0(C'_m, \nu_m^* \mathcal{O}(1)) = \delta(\deg \nu_m^* \mathcal{O}(1) + \chi(\mathcal{O}_{C'_m})) = \deg M_m|_{C_m} + \delta \chi(\mathcal{O}_{C'_m}).$$

Hence $\delta \frac{1}{m} h^0(X|C, mL) \leq \frac{1}{m} \deg M_m|_{C_m} + \frac{\delta}{m} \chi(\mathcal{O}_{C'_m}) \leq \frac{1}{m} \deg M_m|_{C_m} + \delta/m$. Since this holds for infinitely many m, by letting $m \to \infty$ we have $\delta \operatorname{vol}_{X|C}(L) \leq \|L \cdot C\|$. $\qquad \square$

The previous three lemmas immediately imply the following.

COROLLARY 3.9. *Let $C \subset X$ be a curve such that $C \not\subset \mathrm{SBs}(L)$ and the map $f|_C : C \dashrightarrow f(C)$ is finite of degree δ. Then $\mu(C, L) \geq \delta \operatorname{vol}_{X|C}(L) = \|L \cdot C\|$ holds.*

COROLLARY 3.10 (= Theorem 1.4). *Let $x \in X$ be a very general point. Assume $\mu(C, L) = \operatorname{vol}_{X|C}(L)$ for any curve C passing through x. Then either $\kappa(L) = 0$ or $\kappa(L) = \dim X$.*

Proof. Assume $0 < \kappa(L) < \dim X$. Then we can find a curve $C \subset X$—for example, as a general complete intersection over a general curve C' in Y, with $\deg(f|_C : C \dashrightarrow C') > 1$. By Corollary 3.9, we have $\mu(C, L) > \operatorname{vol}_{X|C}(L)$ and get a contradiction. $\qquad \square$

3.2. Concluding Remarks

Here are some remarks relevant to pursuing the previous arguments.

REMARK 3.11. There is a missing piece for better understanding the asymptotic property of linear series $\{|mL|\}_{m>0}$ for general L with $\kappa(L) \geq 0$. In case L is big, there exists an effective (very ample) divisor G such that $\mathfrak{b}_m(-G) \subset \mathcal{J}(\|mL\|)(-G) \subset \mathfrak{b}_m$ for all $m > 0$ [ELMNP3, 3.1; L, 11.2.21], where $\mathfrak{b}_m = \mathfrak{b}(|mL|)$ is the base ideal. This "uniformity" was crucial in the asymptotic study of big divisors. We would like to see whether or not this uniformity still holds when L is not big. A counterexample would also be interesting.

In any case, let us point out that one can argue as in the proof of [ELMNP3, 2.13] to obtain the following.

Pʀᴏᴘᴏꜱɪᴛɪᴏɴ 3.12. *Suppose there exists an effective divisor G on X such that $\mathfrak{b}_m(-G) \subset \mathcal{J}(\|mL\|)(-G) \subset \mathfrak{b}_m$ for all $m > 0$. Then $\|L^d \cdot V\| \geq \mu(V, L)$ (which implies $\|L^d \cdot V\| = \mu(V, L)$) for any subvariety V of $\dim V = d > 0$ such that $V \not\subset (\mathrm{SBs}(L) \cup \mathrm{Supp}\, G)$.*

It would also be interesting to understand whether the converse of Lemma 3.6 holds in general, as follows.

Qᴜᴇꜱᴛɪᴏɴ 3.13. Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \mathrm{SBs}(L)$ and the map $f|_V : V \dashrightarrow f(V)$ is generically finite of degree δ. In this case, does the inequality $\delta \operatorname{vol}_{X|V}(L) \leq \|L^d \cdot V\|$ hold?

Rᴇᴍᴀʀᴋ 3.14. Notice that if Question 3.13 is answered in the affirmative then we would have $\mu(V, L) \geq \delta \operatorname{vol}_{X|V}(L) = \|L^d \cdot V\|$ for V as in the question (cf. Corollary 3.9) and thus obtain a natural generalization of Theorem 1.4. Precisely the same arguments given in the proof of Corollary 3.10 would yield the following. Let $x \in X$ be a very general point, and assume $\mu(V, L) = \operatorname{vol}_{X|V}(L)$ for any d-dimensional subvariety V passing through x. Then either $\kappa(L) < d$ or $\kappa(L) = \dim X$.

Rᴇᴍᴀʀᴋ 3.15. A parallel analytic approach, in the spirit of [B], to the questions studied in this paper would be possible.

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