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Extremal Rational Elliptic Threefolds

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An elliptic fibration is a proper morphism $f: X \to Y$ of normal projective varieties whose generic fibre E is a regular curve of genus 1. The Mordell–Weil rank of such a fibration is defined to be the rank of the abelian group $\mathrm{Pic}^0 E$ of degree-0 line bundles on E. In particular, f is called *extremal* if its Mordell–Weil rank is 0.

The simplest nontrivial elliptic fibration is a rational elliptic surface $f: X \to \mathbf{P}^1$. There is a complete classification of extremal rational elliptic surfaces due to Miranda and Persson in characteristic 0 [14] and to Lang in positive characteristic [12; 13]. (See also Cossec and Dolgachev [4, Sec. 5.6].) The purpose of the present paper is to produce a corresponding classification of a certain class of extremal rational elliptic threefolds. For reference, the results are shown in Table 1.

Let us say a bit more about exactly which objects we are classifying. It is a classical fact that any rational elliptic surface is the blowup of $\mathbf{P}^2$ at the base locus (a 0-dimensional subscheme of degree 9) of a pencil of cubic curves. This description allows one to compute the Mordell–Weil rank in terms of reducibility properties of curves in the pencil [17, Thm. 5.2]. In dimension 3, the analogous situation is to consider a net (2-dimensional linear system) of quadric surfaces in $\mathbf{P}^3$. The base locus of such a net is a 0-dimensional subscheme of degree 8. We will see in what follows that, under a certain nondegeneracy assumption on the net, blowing up at the base locus gives an elliptic fibration $f: X \to \mathbf{P}^2$, and then we can compute the Mordell–Weil rank of f in terms of reducibility properties of quadrics in the net. To exploit this, we will consider in this paper only elliptic threefolds obtained by blowing up the base locus of a net of quadrics in $\mathbf{P}^3$. Table 1 gives a list of all nets of quadrics (up to projective equivalence) that give rise to extremal elliptic threefolds in this way.

The classification may be of interest for several reasons. First, it is a natural counterpart of the results of Miranda–Persson and Lang on extremal rational elliptic surfaces. It is perhaps surprising to see that the situation for threefolds, in which the classification contains only a small finite number of cases, is simpler than that for surfaces. Second, the method of proof uses the theory of root systems in an essential way. This gives a further demonstration of the strong connection—elaborated in [6] and [4]—between root systems and configurations of points in projective space. Finally, the classification provides "test specimens" for the cone

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Table 1 List of Extremal Nets

Root lattice	$\mathrm{Pic}^0(E)$	Type of net	Standard form
E_7	0	$\{8\}_1$	$Q_1 = Z^2$ $Q_2 = X(Y+W) + YW$ $Q_3 = XZ + (Y+W)^2$
A_7	$\mathbf{Z}/2\mathbf{Z}$	$\{8\}_2$	$Q_1 = YZ + W^2$ $Q_2 = XZ + YW$ $Q_3 = XW - Y^2 + Z^2$
		$\{4,4\}_1$	$Q_1 = ZW$ $Q_2 = XZ + YW$ $Q_3 = XY + Z^2 + W^2$
$D_6 \oplus A_1$	$\mathbf{Z}/2\mathbf{Z}$	$\{6,2\}$	$Q_1 = YZ$ $Q_2 = XZ + W^2$ $Q_3 = XY + Z^2$
		$\{4,4\}_2$	$Q_1 = XY$ $Q_2 = Z^2$ $Q_3 = (X+Y)Z + W^2$
$A_5 \oplus A_2$	$\mathbf{Z}/3\mathbf{Z}$	$\{5,3\}$	$Q_1 = YZ$ $Q_2 = XW + Z^2$ $Q_3 = XY + W^2$
		$\{3,3,2\}_1$	$Q_1 = YZ$ $Q_2 = X(Z+W)$ $Q_3 = XY + W^2$
$D_4 \oplus 3A_1$	$(\mathbf{Z}/2\mathbf{Z})^2$	$\{4,2,2\}$	$Q_1 = X(Y+Z)$ $Q_2 = YZ$ $Q_3 = (X+Y)Z + W^2$
$2A_3 \oplus A_1$	$\mathbf{Z}/4\mathbf{Z}$	$\{4,4\}_3$	$Q_1 = XY$ $Q_2 = XZ + W^2$ $Q_3 = YW + Z^2$
		$\{3,3,2\}_2$	$Q_1 = XY$ $Q_2 = ZW$ $Q_3 = (X+Y)Z + W^2$
		$\{2,2,2,2\}$	$Q_1 = XY$ $Q_2 = ZW$ $Q_3 = (X+Y)(Z+W)$
$7A_1$	$(\mathbf{Z}/2\mathbf{Z})^3$	$\{1,1,1,1,1,1,1,1\}$ (char $k = 2$ only)	$Q_1 = (X+Y+Z)W$ $Q_2 = (X+Y+W)Z$ $Q_3 = (X+Z+W)Y$

Notes: The root lattices and Mordell–Weil groups are obtained in Section 3. The admissible types of nets are obtained in Section 4. Standard forms are obtained in Section 5.

conjecture in birational geometry [17, Conj. 8.1]. That conjecture predicts that the threefolds appearing in the classification should be particularly simple from the point of view of birational geometry. (More precisely, they have finitely generated Cox ring.) We will not explore this direction in the present work, but we plan to do so in a forthcoming paper.

The main results of this paper are as follows. Theorem 2.1 relates the Mordell–Weil rank of an elliptic fibration obtained from a net of quadrics to reducibility properties of quadrics in the net. Theorem 3.2 shows that, for an extremal fibration, the configuration of reducible quadrics in the net is constrained by a (fixed) finite root system. These two theorems combine to yield Theorem 4.1, which gives a list of the possible configurations of reducible quadrics for an extremal fibration. In Section 5 we use the combinatorial data produced by Theorem 4.1 to determine all extremal nets up to projective equivalence. Finally, in Section 6 we relate our extremal elliptic threefolds to extremal quartic plane curves via the discriminant.

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NOTATION, CONVENTIONS, DEFINITIONS. We work throughout over an algebraically closed field k. In general the characteristic of k is not specified, though in some contexts we will exclude characteristics 2 and 3.

The term *extremal fibration* will always refer to an extremal elliptic fibration $f: X \to \mathbf{P}^2$ obtained by blowing up the base locus (in the sense described below) of a net of quadrics in $\mathbf{P}^3$ that satisfies Assumption 1. A net of quadrics is called *extremal* if the corresponding morphism $X \to \mathbf{P}^2$ is an extremal fibration.

If Q_1, Q_2, Q_3 are quadrics in $\mathbf{P}^3$, we write $\langle Q_1, Q_2, Q_3\rangle$ to denote the net they span; that is, $\langle Q_1, Q_2, Q_3\rangle = \{\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 : \lambda_i \in k, \lambda_1, \lambda_2, \lambda_3 \text{ not all } 0\}$. Similarly, $\langle Q_1, Q_2\rangle$ denotes the pencil spanned by Q_1 and Q_2.

A *basepoint* of a net N of quadrics can refer either to a point $p \in \mathbf{P}^3$ in the set-theoretic intersection $\bigcap_{Q\in N} Q$ of all quadrics in the net or to a common tangent direction of the net (of any order). If we intend only a point $p \in \bigcap_{Q\in N} Q$ then we will use the term $\mathbf{P}^3$*-basepoint*. The multiplicity of a $\mathbf{P}^3$-basepoint p_i will be denoted by m_i. A net N is of *type* $\{m_1, \dots, m_n\}$ if it has $\mathbf{P}^3$-basepoints $p_1, \dots, p_n$ of multiplicities $m_1, \dots, m_n$.

We will use the notation $X_{m_1,\dots,m_n}$ to denote a threefold obtained from $\mathbf{P}^3$ by blowing up at the base locus of any extremal net of type $\{m_1, \dots, m_n\}$. Note that for a given type $\{m_1, \dots, m_n\}$ there may exist nonisomorphic spaces $X_{m_1,\dots,m_n}$.

We abuse terminology by using the term *rank-2 quadric* to refer to a quadric in $\mathbf{P}^3$ that is the union of two distinct planes, even in characteristic 2.

We denote by h the pullback to X of the hyperplane divisor class on $\mathbf{P}^3$ and by e_i the pullback to X of the exceptional divisor E_i of the blowup of the basepoint p_i $(i = 1, \dots, 8)$. For brevity, we will denote the class $h - e_i - e_j - e_k - e_l$ by h_{ijkl} and the class $e_i - e_j$ by e_{ij} or sometimes (for clarity) $e_{i,j}$. We denote by l the class in $N_1(X)$ represented by the pullback of a line in $\mathbf{P}^3$ and by l_i the class of the pullback of a line in the exceptional divisor e_i.

1. Preliminaries

In this section we explain how to obtain an elliptic fibration from a net of quadrics in $\mathbf{P}^3$ under a certain nondegeneracy assumption on the net. We then point out some simple consequences of this assumption that we will use later in the paper.

First let us consider what restriction is needed on a net of quadrics in $\mathbf{P}^3$ to ensure that it gives an elliptic fibration as defined previously. Given any net with a chosen set of generators, say $N = \langle Q_1, Q_2, Q_3 \rangle$, we get a rational map $\mathbf{P}^3 \dashrightarrow \mathbf{P}^2$: explicitly, the map is $p \mapsto [Q_1(p), Q_2(p), Q_3(p)]$. This map is defined outside the base locus of N, so we would like to "blow up at the base locus" (in some sense) to get a morphism $f\colon X \to \mathbf{P}^2$ from a smooth threefold to $\mathbf{P}^2$. Furthermore, since we are interested in elliptic fibrations, we want the generic fibre of f to be a smooth curve of genus 1. If the base locus of the net is reduced (i.e., if it consists of eight distinct points) then we can blow up these eight points in the usual way, and we do in fact get an elliptic fibration. But the condition of reduced base locus is too restrictive for our purposes—it is proved in [16] that there is only one such net that gives an extremal fibration—so we would like to relax it as much as possible.

Consider, however, the net spanned by the following three quadrics in $\mathbf{P}^3$ with homogeneous coordinates $[X, Y, Z, W]$:

$$Q_1 = X(X - W), \quad Q_2 = Y(Y - W), \quad Q_3 = ZW.$$

This net has four basepoints of multiplicity 1 at $[X, Y, Z, W] = [0, 0, 0, 1]$, $[1, 0, 0, 1]$, $[0, 1, 0, 1]$, $[1, 1, 0, 1]$ and one basepoint of multiplicity 4 at $p = [0, 0, 1, 0]$. Therefore we get a rational map $\mathbf{P}^3 \dashrightarrow \mathbf{P}^2$ defined outside these five points. We want to resolve the indeterminacy of this rational map to get a morphism $f\colon X \to \mathbf{P}^2$ that is an elliptic fibration. Suppose we are in the characteristic-0 case: then we can blow up along points and curves to get a morphism (though not uniquely). Bertini's theorem then tells us that the general fibre of f is smooth. On the other hand, the general fibre is birational to a quartic curve $C = Q \cap Q'$, the intersection of two quadrics in the net. One can check that any such C is singular at p and hence is rational. Therefore the general fibre of f is rational.

Since we are interested only in elliptic fibrations, we want to exclude troublesome examples like this one. What went wrong? The problem is that the differentials dQ_1 and dQ_2 are both zero at p, so no intersection $Q \cap Q'$ of two quadrics in the net can be smooth at p. Since the generic fibre of $f\colon X \to \mathbf{P}^2$ is birational to a singular quartic of the form $Q \cap Q'$ (a rational curve), we never get an elliptic fibration in this case. Therefore, in what follows we assume that all nets of quadrics in $\mathbf{P}^3$ satisfy the following assumption.

ASSUMPTION 1. There exist quadrics Q, Q' in the net such that the intersection $Q \cap Q'$ is smooth at the base locus of the net. Equivalently, for each $\mathbf{P}^3$-basepoint p of the net, there is at most one quadric in the net singular at p.

Under this assumption we obtain an elliptic fibration as follows. Choose a quartic curve of the form $C = Q \cap Q'$ that is smooth at the base locus and a quadric Q'', not in the pencil spanned by Q and Q', that also is smooth at the base locus. (This is possible since smoothness at a given point is an open condition on quadrics.) Since C is smooth, its higher tangent directions uniquely define the basepoints infinitely near to any multiple basepoint of the net. Blowing up repeatedly at these basepoints, we obtain a threefold X on which the proper transforms of C and Q'' are disjoint, and hence a morphism $f: X \to \mathbf{P}^2$.

For $f: X \to Y$ the blowup of a point in a smooth variety of dimension n, we have the formula $K_X = f^*(K_Y) + (n-1)E$, where E is the exceptional divisor of the blowup. Applying this in the case where X is obtained from $\mathbf{P}^3$ by blowing up eight points, we get $K_X = -4h + 2e_1 + \cdots + 2e_8$. So the class $-\frac{1}{2}K_X = 2h - e_1 - \cdots - e_8$ is represented by the proper transform on X of any quadric in the net smooth at the base locus. This means that the morphism $f: X \to \mathbf{P}^2$ from the previous paragraph is the same as the one given by the basepoint-free linear system $\left|-\frac{1}{2}K_X\right|$. The generic fibre E of f need not be smooth, but it is a regular scheme. Also, adjunction tells us the canonical bundle K_E is trivial, so E has arithmetic genus 1. Hence f is an elliptic fibration, as claimed.

Remark. It is customary to refer to a fibration as above whose generic fibre is regular but not smooth as a *quasi-elliptic fibration,* but since the arguments of this paper apply equally well in both the elliptic and quasi-elliptic cases, we abuse terminology and refer to both as elliptic fibrations. Many facts about quasi-elliptic fibrations are known: for instance, they exist only if the base field has characteristic 2 or 3; also, the geometric generic fibre $E(\overline{k(\mathbf{P}^2)})$ is always a cuspidal rational curve [4, Prop. 5.1.2]. Note that the final net in Table 1, which is extremal only in characteristic 2, gives a quasi-elliptic fibration.

Remark. It is a classical fact that the fibrations $f: X \to \mathbf{P}^2$ correspond to nets of cubic curves in the plane. In one direction, projecting from one basepoint of our net N of quadrics transforms the net of quartic curves in $\mathbf{P}^3$ dual to N to a net of cubic curves in $\mathbf{P}^2$ with seven basepoints; in the other, blowing up the seven basepoints of such a net and taking the universal family $\mathcal{X}$ of elliptic curves over the resulting surface, we get an elliptic fibration $\mathcal{X} \to \mathbf{P}^2$ birational to our original fibration $f: X \to \mathbf{P}^2$. For more details on this correspondence see [5, Sec. 6.3.3].

Here are some straightforward consequences of Assumption 1.

Lemma 1.1. *Given a net of quadrics satisfying Assumption 1, no three of the basepoints are collinear and no five are coplanar. More precisely, suppose X is the threefold obtained from such a net by blowing up its base locus as just described. Then no class $l - \sum_{k=1}^{3} l_{i_k}$ in $N_1(X)$ or $h - \sum_{k=1}^{5} e_{j_k}$ in $N^1(X)$ is represented by an effective cycle.*

Proof. For any choice of distinct indices we have $-K_X \circ \left(l - \sum_{k=1}^{3} l_{i_k}\right) = -1$ (where $\circ$ denotes intersection of cycles on X), but this is impossible for an effective cycle since $-K_X$ is basepoint-free.

For the second claim, suppose there were an effective cycle $h - \sum_{k=1}^{5} e_{j_k}$ in $N^1(X)$; its image in $\mathbf{P}^3$ would be a plane P. Choose any quartic curve $C = Q \cap Q'$, an intersection of two quadrics in the net, that is smooth at the base locus; such a curve exists by Assumption 1. Its proper transform $\tilde{C}$ on X has class $4l - \sum_{i=1}^{8} l_i$. Therefore $\left(h - \sum_{k=1}^{5} e_{j_k}\right) \circ \tilde{C} = -1$, implying that any such C is contained in P. But smoothness of C at a finite set of points is an open condition on Q and Q', so this is impossible. □

LEMMA 1.2. *Given a net of quadrics satisfying Assumption 1, we have the following facts.*

- *There is at most one double plane in the net.*
- *There are at most n irreducible cones with vertices at basepoints of the net, where n is the number of distinct $\mathbf{P}^3$-basepoints of the net.*
- *There are finitely many rank-2 quadrics in the net.*

Proof. Any double plane is singular at all $\mathbf{P}^3$-basepoints, so by Assumption 1 we get the first claim. For the second, Assumption 1 implies there is at most one cone with vertex at a given $\mathbf{P}^3$-basepoint p_i.

For the final claim, suppose there is a curve of rank-2 quadrics in the net. Then every pencil in the net contains a reducible quadric; hence the pencil's base locus is a reducible quartic in $\mathbf{P}^3$. But each fibre of $f\colon X \to \mathbf{P}^2$ is birational to the base locus of some pencil in the net and so must be reducible. This contradicts regularity of the generic fibre. □

2. Rank of the Elliptic Fibration

In this section, we derive a formula for the rank of an elliptic fibration $f\colon X \to \mathbf{P}^2$ obtained from a net of quadrics in $\mathbf{P}^3$ in terms of the number of distinct $\mathbf{P}^3$-basepoints of the net and the number of quadrics of rank 2 in the net. This generalizes [17, Thm. 7.2], which gives the formula for a net with eight distinct $\mathbf{P}^3$-basepoints.

THEOREM 2.1. *Suppose $f\colon X \to \mathbf{P}^2$ is an elliptic fibration arising from a net of quadrics in $\mathbf{P}^3$. Then the rank ρ of the Mordell–Weil group of the generic fibre of f is given by*

$$\rho = n - d - 1,$$

where n is the number of distinct $\mathbf{P}^3$-basepoints of the net and d the number of quadrics of rank 2 in the net. In particular, f is extremal if and only if $d = n - 1$.

Proof. The rank of an elliptic threefold $f\colon X \to S$ is given by the Shioda–Tate–Wazir formula [9, Thm. 2.3]. Let us derive this formula in our case $S = \mathbf{P}^2$. To do this, we imitate the proof of [17, Thm. 7.2]. We have a surjective homomorphism $r\colon \operatorname{Pic} X \to \operatorname{Pic} E$ given by restriction of divisors, so $\operatorname{rank}\operatorname{Pic} E = \operatorname{rank}\operatorname{Pic} X - \operatorname{rank}\ker r$. Since we know that $\operatorname{Pic} E = \operatorname{Pic}^0 E \oplus \mathbf{Z}$, this gives $\operatorname{rank}\operatorname{Pic}^0 E = \operatorname{rank}\operatorname{Pic} X - \operatorname{rank}\ker r - 1 = 8 - \operatorname{rank}\ker r$. So we need to calculate the rank of the kernel of the restriction homomorphism.

The kernel of r is generated by the classes of all irreducible divisors in X that do not map onto $\mathbf{P}^2$ under f. If λ is the class of a line in $\mathbf{P}^2$, then $f^*(\lambda) = -\frac{1}{2}K_X$, so the pullback of any irreducible divisor in $\mathbf{P}^2$ is a multiple of $-\frac{1}{2}K_X$. Therefore the kernel of the restriction homomorphism is generated by $-\frac{1}{2}K_X$ together with r_F classes for every irreducible divisor F in $\mathbf{P}^2$ whose preimage in X consists of $r_F + 1$ irreducible components, say $\sum_{j=1}^{r_F+1} m_{F_j} D_{F_j}$. I claim that the divisors D_{F_j} for any F and $1 \leq j \leq r_F$ are linearly independent in $\operatorname{Pic} X \otimes \mathbf{Q}$. This follows from the corresponding fact about a morphism from a surface to a curve [2, Lemma II.8.2] by restricting to the inverse image of a general line in $\mathbf{P}^2$. So the Mordell–Weil group $\operatorname{Pic}^0 E$ has rank $8-1-\sum r_F$. We must show this can be written as $n - d - 1$, where n is the number of distinct $\mathbf{P}^3$-basepoints of the net and d the number of rank-2 quadrics in the net.

The map $f\colon X \to \mathbf{P}^2$ is given by resolving the indeterminacy of the rational map $\mathbf{P}^3 \dashrightarrow \mathbf{P}^2 : p \mapsto [Q_1(p), Q_2(p), Q_3(p)]$, where Q_i is any (fixed) basis for the net of quadrics. So a fibre of f is (at least away from the base locus of the net) the intersection $Q \cap Q'$ of two quadrics in the net and hence is a quartic curve. Let us refer to the corresponding quartic curve $Q \cap Q'$ in $\mathbf{P}^3$ as the *pseudofibre* of f over the given point.

If the intersection $Q \cap Q'$ is smooth at the base locus, then the pseudofibre $Q \cap Q'$ is isomorphic to the corresponding fibre of f. If such a fibre contains a line, then this must be the line through two of the basepoints p_i. So there are only finitely many fibres smooth at the base locus that contain a line. The only other possibility for a reducible pseudofibre smooth at the base locus is that it be the union $C_1 \cup C_2$ of two smooth conic curves in $\mathbf{P}^3$. But each curve C_i is contained in a plane P_i in $\mathbf{P}^3$; the union $P_1 \cup P_2$ is therefore a rank-2 quadric in the net that is smooth at the base locus.

Note this implies in particular that if a reducible divisor Δ in $\mathbf{P}^3$ contains a pseudofibre smooth at the base locus and maps to a curve in $\mathbf{P}^2$, then in fact it maps to a line in $\mathbf{P}^2$. To see this, assume without loss of generality that Δ is a union of pseudofibres. Every pseudofibre contained in Δ and smooth at the base locus is contained in some rank-2 quadric Q, whose image in $\mathbf{P}^2$ is a line, and Lemma 1.2 shows there are finitely many such Q. These pseudofibres are dense in Δ, so the image of Δ is contained in a finite union of lines in $\mathbf{P}^2$. If different pseudofibres were contained in different rank-2 quadrics, the image of Δ would be a union of distinct lines and hence reducible, but this contradicts our assumption. Therefore the image of Δ in $\mathbf{P}^2$ is a line, as required.

So the only possibilities for reducible pseudofibres that are smooth at the base locus are exactly those described in [17]. Let us therefore consider pseudofibres $Q \cap Q'$ that are not smooth at the base locus.

Suppose Q is a quadric in the net smooth at the base locus, and suppose a pseudofibre $Q \cap Q'$ is singular at a $\mathbf{P}^3$-basepoint p_i. This means that the differentials dQ and dQ' are linearly dependent at p_i, so (multiplying by a constant if necessary) $d(Q - Q') = 0$ at p_i. By Assumption 1, this implies that $Q - Q'$ is the unique quadric Q_i in the net singular at p_i or, put another way, that $Q' = \lambda Q + \mu Q_i$. So the pseudofibre $Q \cap Q'$ is singular at p_i if and only if Q' belongs to the pencil $\lambda Q + \mu Q_i$, implying that $Q \cap Q' = Q \cap Q_i$.

Now suppose $C \subset \mathbf{P}^2$ is a curve over which all pseudofibres of f are singular at a $\mathbf{P}^3$-basepoint p_i. Fix a quadric Q in the net that is smooth at the base locus. Over any point of $f(Q) \cap C$ the pseudofibre of f is singular at p_i. Over a point $q \in f(Q) \cap C$ the pseudofibre is an intersection $Q \cap Q'$, and by the previous paragraph we can take $Q' = Q_i$. Therefore $q = f(Q) \cap f(Q_i)$. This holds for all $q \in f(Q) \cap C$, so we have $f(Q) \cap C = f(Q) \cap f(Q_i)$. Since this is true for any quadric Q in the net smooth at p_i (which Q constitute a Zariski-open set in the net), we must have $C = f(Q_i)$. We conclude that the only subvarieties of $\mathbf{P}^2$ over which all the pseudofibres of f are singular at the base locus are the lines $f(Q_i)$, the images of the finitely many quadrics Q_i in the net singular at the base locus.

Suppose D is a reducible effective divisor in X whose image $f(D) \subset \mathbf{P}^2$ is an irreducible curve C; without loss of generality, we can assume $D = f^{-1}(f(D))$—that is, D is a union of fibres. Contracting the exceptional divisors E_i in X, the image of D is an effective divisor $\Delta \in \mathbf{P}^3$. If some pseudofibre contained in Δ is smooth at the base locus, then (as explained before) Δ must be a supported on a rank-2 quadric in the net. If the pseudofibre over every point of C is singular at the base locus then the previous paragraph implies that C must be one of the lines $f(Q_i)$ in $\mathbf{P}^2$, so Δ is supported on Q_i.

We therefore have three types of contribution to the rank of $\ker r$: first, the class $-\frac{1}{2}K_X$; second, reducible quadrics in the net smooth at the base locus, each of which adds 1 to the rank of the kernel; third, the quadrics Q_i singular at the base locus. Let us analyze the contribution of these Q_i to the rank of the kernel.

First, suppose Q_i is an irreducible reduced cone with vertex at p_i. The corresponding divisor $f^{-1}(f(Q_i))$ on X has m_i components in total—namely, the class of the proper transform of the cone together with $m_i - 1$ classes of the form $e_{j,j+1}$. The preimage of any line in $\mathbf{P}^2$ has class $-\frac{1}{2}K_X$ in $\operatorname{Pic} X$, so the classes of these m_i components sum to $-\frac{1}{2}K_X$. Therefore Q_i contributes $m_i - 1$ to the rank of $\ker r$.

Next suppose that Q_i is a rank-2 quadric in the net singular at the base locus. The singular locus of Q_i is a line in $\mathbf{P}^3$ and therefore contains at most two basepoints of the net by Lemma 1.1. The corresponding divisor $f^{-1}(f(Q_i))$ on X has $2+(m_i-1)$ components if Q_i is singular at one basepoint p_i and $2 + (m_i - 1) + (m_j - 1)$ components if Q_i is singular at two basepoints p_i and p_j. Again, in both cases the classes of these components sum to $-\frac{1}{2}K_X$. So in the first case we get a contribution of $1 + (m_i - 1)$ to the rank of $\ker r$ and in the second case a contribution of $1 + (m_i - 1) + (m_j - 1)$.

Finally, consider the case of a nonreduced quadric Q_i—that is, a double plane $2P$. In this case, all quadrics in the net except Q_i must be smooth at the base locus, by Assumption 1. The (reduced) plane P passes through some subset of the basepoints, including all of the $\mathbf{P}^3$-basepoints (which are therefore all multiple). The proper transform of P on X has class $h - e_{i_1} - \cdots - e_{i_j}$ in $\operatorname{Pic} X$ for some set of distinct indices. Therefore the proper transform of Q_i on X has class $2(h - e_{i_1} - \cdots - e_{i_j})$. On the other hand, this proper transform must be disjoint from some smooth fibre C that has class $4l - \sum_i l_i$. We conclude that j, the number of indices in the expression for the class of P, must be equal to 4. Again, the divisor $f^{-1}(f(Q_i))$ has class $-\frac{1}{2}K_X = 2h - \sum_i e_i$. We can rewrite this as a sum of effective classes as follows:

$$2h - \sum_i e_i = 2(h - e_{i_1} - \cdots - e_{i_j}) + \sum_{p_k} \sum_{p_l} e_{l,l+1} + R,$$

where the first sum is taken over the $\mathbf{P}^3$-basepoints p_k and the second over all basepoints p_l infinitely near to p_k, except the highest, and where R is a sum of terms of the form $e_{l,l+1}$ that have already appeared in sum. The number of distinct terms in this sum is $1 + \sum_{p_i}(m_i - 1)$, with the sum taken over all $\mathbf{P}^3$-basepoints p_i. Hence the contribution to the rank of $\ker r$ is $\sum_{\text{all } \mathbf{P}^3\text{-basepoints } p_i}(m_i - 1)$.

(It may help to think about the fibre of f over a general point of $f(Q_i)$; this is one of the degenerations of elliptic curves described by Kodaira in [11]. For instance, if our net has a single basepoint of multiplicity 8 and a double plane $Q_i = 2P$, then the fibre over the generic point of $f(Q_i)$ is a curve of type III* in Kodaira's notation.)

Let us now show that the preceding arguments together give the formula claimed. In the case of no double plane in the net, the total contribution to the rank of $\ker r$ from quadrics singular at the base locus is

$$\sum_{p_i}(m_i - 1) + \sum_{p_j} 1 + (m_j - 1) + \sum_{p_k, p_l} 1 + (m_n - 1) + (m_l - 1),$$

where the first sum is taken over multiple $\mathbf{P}^3$-basepoints at which the singular quadric is an irreducible cone, the second over multiple basepoints at which the singular quadric is rank-2 singular at one basepoint, and the third is taken over pairs of multiple $\mathbf{P}^3$-basepoints both lying on the singular locus of the same rank-2 quadric. Since every multiple $\mathbf{P}^3$-basepoint is of one of these three types, summing yields

$$d_{\text{sing}} + \sum_{\text{multiple } \mathbf{P}^3\text{-basepoints } p_i} (m_i - 1),$$

where d_{sing} is the number of rank-2 quadrics in the net singular at the base locus. Finally—including rank-2 quadrics smooth at the base locus, each of which contributes 1 to the rank, and the class $-\frac{1}{2}K_X$—we get

$$\operatorname{rank} \ker r = 1 + d + \sum_{\text{multiple } \mathbf{P}^3\text{-basepoints } p_i} (m_i - 1).$$

In the case of a double plane in the net, we know that all rank-2 quadrics in the net must be smooth at the base locus (hence each contributes 1 to the rank of $\ker r$) and also that there are no cones in the net with vertex at a basepoint. So using the formula from a few paragraphs back and including $-\frac{1}{2}K_X$ again, we get $\operatorname{rank} \ker r = 1 + d + \sum_{\text{multiple } \mathbf{P}^3\text{-basepoints } p_i}(m_i - 1)$. (Recall that in this case all $\mathbf{P}^3$-basepoints are multiple, so we are summing over the same set as before.)

Now computing the rank ρ of $\operatorname{Pic}^0 E$ as $\rho = 8 - \operatorname{rank}\ker r$, we get in both cases

$$\begin{aligned}\rho &= 8 - \Bigg(1 + d + \sum_{\text{multiple } \mathbf{P}^3\text{-basepoints } p_i} (m_i - 1)\Bigg)\\ &= 7 - d - \sum_{\text{all } \mathbf{P}^3\text{-basepoints } p_i} (m_i - 1)\\ &= 7 - d - 8 + n\\ &= n - d - 1\end{aligned}$$

as claimed. □

3. Extremal Fibrations and Root Systems

In this section, we will show that the possibilities for an extremal fibration are constrained by a certain root system. Together with the rank formula from Section 2, this will lead to a combinatorial classification of extremal fibrations in Section 4.

More precisely, suppose $f: X \to \mathbf{P}^2$ is an extremal fibration. Call an irreducible divisor in X *vertical* if it is mapped by f to a curve in $\mathbf{P}^2$; we saw in the previous section that the only vertical divisors are components of divisors $f^{-1}(L)$, where L is a line in $\mathbf{P}^2$. We will prove that the possible configurations of vertical divisors are constrained by maximal-rank subsystems of the root system E_7. Before explaining this, let us state the following lemma. A proof can be found for instance in [7, Thm. 6.1.2, Table 5].

LEMMA 3.1. *The only root subsystems of E_7 of finite index are the following*: (a) E_7, (b) A_7, (c) $D_6 \oplus A_1$, (d) $A_5 \oplus A_2$, (e) $D_4 \oplus 3A_1$, (f) $2A_3 \oplus A_1$, (g) $7A_1$.

We define a bilinear form denoted by $\cdot$ on $\operatorname{Pic} X$ as follows:

$$\begin{aligned}&\operatorname{Pic} X \otimes \operatorname{Pic} X \to \mathbf{Z}\\ &\qquad D_1 \otimes D_2 \mapsto D_1 \cdot D_2 := D_1 \circ D_2 \circ \left(-\frac{1}{2}K_X\right),\end{aligned}$$

where, as before, $\circ$ denotes intersection of algebraic cycles on X. For any $D \in \operatorname{Pic} X$, we have $D \cdot \left(-\frac{1}{2}K_X\right) = D \circ \left(4l - \sum_i l_i\right)$, so a divisor belongs to the corank-1 sublattice $K_X^{\perp}$ if and only if it has degree 0 on any fibre of f. That means the surjection $r: \operatorname{Pic} X \to \operatorname{Pic} E$ restricts to a surjection $r: K_X^{\perp} \to \operatorname{Pic}^0 E$. So the latter group is finite—that is, f is extremal—if and only if the kernel of r has finite index in $K_X^{\perp}$. But the kernel of r is generated by the classes of vertical divisors. So given an extremal fibration X, the lattice $\operatorname{Vert}(X) \subset \operatorname{Pic} X$ spanned by classes of vertical divisors must be a finite-index sublattice of $K_X^{\perp}$.

It is easy to check that the vectors $h_{1234}, e_{12}, e_{23}, \ldots, e_{78}$ form a system of simple roots of $K_X^{\perp}$ under the bilinear form defined previously and hence that $K_X^{\perp}$ is isomorphic to the affine root system $\tilde{E}_7$. At first sight, the appearance of root systems in this context may seem surprising, but there is an explanation. The preceding definition shows that $D_1 \cdot D_2$ actually computes the intersection number of the curves $D_1 \cap Q$ and $D_2 \cap Q$ inside Q, the proper transform of a general quadric in the net. Now $f|_Q\colon Q \to f(Q) \cong \mathbf{P}^1$ is a rational elliptic surface, so classical results [2, p. 201] on elliptic surfaces tell us that the intersection form on the classes of curves lying in fibres of $f|_Q$ defines the structure of a root system. Therefore the original form defined on Pic X also defines a root system. (For an extensive discussion of the connection between point sets in projective space and root systems, see [6, Chap. 5].)

Define the *radical* Rad Λ of a lattice Λ to be the subgroup of elements $\lambda \in \Lambda$ such that $\lambda \cdot x = 0$ for all $x \in \Lambda$. Then $\mathrm{Rad}(K_X^\perp)$ is spanned by the class $-\frac{1}{2}K_X$, and $K_X^\perp/\mathrm{Rad}(K_X^\perp) \cong \tilde{E}_7/\mathrm{Rad}(\tilde{E}_7) \cong E_7$. For any extremal fibration X, the sublattice $\mathrm{Vert}(X) \subset K_X^\perp$ spanned by classes of vertical divisors contains the class $-\frac{1}{2}K_X$, so $\mathrm{Vert}(X)/\bigl(-\frac{1}{2}K_X\bigr)$ injects into E_7 as a subsystem of finite index.

Therefore, given any extremal fibration X, the root system $\mathrm{Vert}(X)/\bigl(-\frac{1}{2}K_X\bigr)$ must be one of the seven listed in Lemma 3.1. What does this tell us about the possible configurations of vertical divisors? We have noted that a vertical divisor in X must map to a line in $\mathbf{P}^2$. Given any line $L \subset \mathbf{P}^2$, the divisor $f^*(L) \subset X$ has class $-\frac{1}{2}K_X$ in Pic X. Suppose that $f^*(L) = -\frac{1}{2}K_X = \sum_{i=1}^k m_i D_i$ with D_i (distinct) irreducible and effective divisors, m_i natural numbers, and $k > 1$. The classes D_i $(i = 1, \dots, k)$ are linearly independent in $\mathrm{Pic}\, X \otimes \mathbf{Q}$ and hence span a sublattice Pic X of rank k that is contained in $\mathrm{Vert}(X)$. Passing to the quotient $\mathrm{Vert}(X)/\bigl(-\frac{1}{2}K_X\bigr) \subset E_7$, the images of these classes span a sublattice $\Lambda(L)$ of rank $k-1$. Moreover, by restricting to the preimage of a general line in $\mathbf{P}^2$, one can check that each such class has $D_i^2 = -2$, so in fact their images span a subsystem.

By connectedness of the fibres of f, the Dynkin diagram of $\Lambda(L)$ is connected. Conversely, if D_1 and D_2 are components of $f^*(L_1)$ and $f^*(L_2)$ with the L_i distinct lines in $\mathbf{P}^2$, we have $D_1 \cdot D_2 = 0$ because the restrictions of the D_i to the preimage of a general line in $\mathbf{P}^2$ lie in different fibres and hence are disjoint. So the connected components Γ_i of the Dynkin diagram of $\mathrm{Vert}(X)/\bigl(-\frac{1}{2}K_X\bigr)$ correspond exactly to the subsystems spanned by classes of divisors lying over the finitely many lines L_i in $\mathbf{P}^2$ for which $f^*(L_i)$ is reducible. Note also that the number of nodes of Γ_i is 1 less than the number of components of $f^*(L_i)$, since the classes of those components sum to $-\frac{1}{2}K_X \equiv 0$ in $\mathrm{Vert}(X)/\bigl(-\frac{1}{2}K_X\bigr)$.

The upshot is that to determine the possible configurations of f-vertical divisors in X, we need to determine all graphs obtainable from the Dynkin diagrams of the subsystems in Lemma 3.1 by adding one node to each connected component. There is one extra condition: given a line $L \subset \mathbf{P}^2$ and the corresponding lattice $\Lambda(L) \subset \mathrm{Vert}(X)$ spanned by classes of irreducible components of $f^*(L)$, we know that $\Lambda(L)$ is negative semi-definite but not negative definite. (It contains $-\frac{1}{2}K_X$, which has square 0.) Consequently it is isomorphic to an affine root system of rank $k-1$. So, we must add our nodes in such a way that each component of the resulting graph is the Dynkin diagram of some affine root system. (See for instance [10] for a classification of these.) The result is the following.

1. E_7: Here we are adding just one node. The only possible outcome is $\tilde{E}_7$.
2. A_7: Adding one node, we can get either $\tilde{A}_7$ or $\tilde{E}_7$.

3. $A_5 \oplus A_2$: For $n \leq 6$, the only allowed way to add a node to A_n yields $\tilde{A}_n$. So in this case we get $\tilde{A}_5 \oplus \tilde{A}_2$. (Here the symbol $\oplus$ simply means the disjoint union of graphs.)
4. $2A_3 \oplus A_1$: As before, we get $2\tilde{A}_3 \oplus \tilde{A}_1$.
5. $D_6 \oplus A_1$: The only allowed way to add a node to D_n ($n \geq 4$) yields $\tilde{D}_n$. So here we get $\tilde{D}_6 \oplus \tilde{A}_1$.
6. $D_4 \oplus 3A_1$: As before, we get $\tilde{D}_4 \oplus 3\tilde{A}_1$.
7. $7A_1$: As before, we get $7\tilde{A}_1$.

We can summarize our results as follows.

THEOREM 3.2. *Suppose $f\colon X \to \mathbf{P}^2$ is an extremal fibration. Then the lattice* $\mathrm{Vert}(X)/\big(-\frac{1}{2}K_X\big)$ *is isomorphic to a finite-index subsystem of E_7. A choice of finite-index subsystem determines the configuration of f-vertical divisors on X, and all possibilities are realized.*

Proof. We have already proved the first claim. It remains to verify the second and third claims.

For the second claim, we must show that the finite-index subsystem $\mathrm{Vert}(X)/\big(-\frac{1}{2}K_X\big) \subset E_7$ determines the configuration of vertical divisors uniquely. In light of the preceding discussion, all we need show is that if $\mathrm{Vert}(X)/\big(-\frac{1}{2}K_X\big) \cong A_7$ then the configuration of vertical divisors is not $\tilde{E}_7$. If the configuration were $\tilde{E}_7$, we would have $\mathrm{Vert}(X)/\big(-\frac{1}{2}K_X\big) = \tilde{E}_7/\big(-\frac{1}{2}K_X\big) = E_7$, contrary to assumption. So the configuration of vertical divisors is uniquely determined by a choice of subsystem.

The last claim will be verified in Sections 4 and 5. In Section 4 we will determine the combinatorial possibilities for a net of quadrics whose associated configuration of f-vertical divisors is a given graph Γ on this list. Then in Section 5 we will exhibit standard forms for each permitted type of net, which shows in particular that they exist. □

COROLLARY 3.3. *Suppose that $f\colon X \to \mathbf{P}^2$ is an extremal fibration with generic fibre E. Then the Mordell–Weil group* $\mathrm{Pic}^0 E$ *is determined by the configuration of vertical divisors and is given by Table 2.* (*The types corresponding to a given configuration will be derived in Section 4.*)

Proof. We know from the earlier discussion that

$$\mathrm{Pic}^0 E \cong \tilde{E}_7/\mathrm{Vert}(X) \cong E_7/\big(\mathrm{Vert}(X)/\big(-\tfrac{1}{2}K_X\big)\big).$$

Theorem 3.2 shows that the sublattice $\mathrm{Vert}(X)/\big(-\frac{1}{2}K_X\big)$ is determined by the configuration of vertical divisors. Moreover, computing the quotients of E_7 by its seven finite-index sublattices is straightforward and gives the results shown. □

Combined, Theorem 3.2 and Corollary 3.3 are an analogue of [4, Thm. 5.6.2], which classifies the possible configurations of reducible fibres on an extremal rational elliptic surface. It is perhaps surprising—and certainly pleasant—that the result for threefolds is no more complicated than that for surfaces.

Table 2

Vertical divisors	$\mathrm{Pic}^0 E$	Types
$\tilde{E}_7$	0	$\{8\}_1$
$\tilde{A}_7$	$\mathbf{Z}/2\mathbf{Z}$	$\{8\}_2$, $\{4,4\}_1$
$\tilde{D}_6 \oplus \tilde{A}_1$	$\mathbf{Z}/2\mathbf{Z}$	$\{6,2\}$, $\{4,4\}_2$
$\tilde{A}_5 \oplus \tilde{A}_2$	$\mathbf{Z}/3\mathbf{Z}$	$\{5,3\}$, $\{3,3,2\}_1$
$2\tilde{A}_3 \oplus \tilde{A}_1$	$\mathbf{Z}/4\mathbf{Z}$	$\{4,4\}_3$, $\{3,3,2\}_2$, $\{2,2,2,2\}$
$\tilde{D}_4 \oplus 3\tilde{A}_1$	$(\mathbf{Z}/2\mathbf{Z})^2$	$\{4,2,2\}$
$7\tilde{A}_1$	$(\mathbf{Z}/2\mathbf{Z})^3$	$\{1,1,1,1,1,1,1,1\}$

4. Combinatorial Classification

In this section we use the list of possible configurations of vertical divisors from Section 3 together with the rank formula of Theorem 2.1 to determine the possible types of an extremal net. In fact, the list gives us more information: given an extremal net with its type and configuration of vertical divisors, we can say exactly which classes $D \in \mathrm{Pic}(X)$ are represented by vertical divisors.

THEOREM 4.1. *Suppose $f: X \to \mathbf{P}^2$ is an extremal fibration given by a net N of quadrics in $\mathbf{P}^3$. Then the type of N and the classes of irreducible vertical divisors in X are* (*up to permutation of indices*) *one of the cases shown in Figure 1.*

Note that for some types $\{m_1, \dots, m_n\}$ we get several possible configurations of reducible divisors: we use a subscript (as $\{m_1, \dots, m_n\}_i$) to distinguish between these.

Before proving the theorem, we need some facts about the structure of the diagram that describes the configuration of vertical divisors. For brevity, let us denote by Γ_X the diagram of irreducible vertical divisors on an extremal fibration X and by $h^0(\Gamma_X)$ the number of connected components of Γ_X.

LEMMA 4.2. *Suppose X is an extremal fibration. If Γ_X has a component γ of type $\tilde{A}_1$, then the nodes of γ are either* (a) *the class c_i of a cone with vertex p_i, a basepoint of the net with multiplicity 2, and the class $e_{i,i+1}$ or* (b) *the classes h_{abcd} and h_{ijkl} of two planes whose union is a rank-2 quadric in the net smooth at the base locus* (*so that $\{\{a,b,c,d\},\{i,j,k,l\}\}$ is a partition of $\{1,\dots,8\}$*).

In the first case we will say the $\tilde{A}_1$ component is conical; *in the second we will say it is* smooth.

Proof. Note that irreducible vertical divisors D_i and D_j are nodes of an $\tilde{A}_1$-component if and only if $D_i \cdot D_j = 2$. To prove the lemma, we simply need to consider the intersection numbers of different types of vertical divisors.

First consider the class of a double plane. In any net containing a double plane, all other quadrics are smooth at the base locus (by Assumption 1). So the only

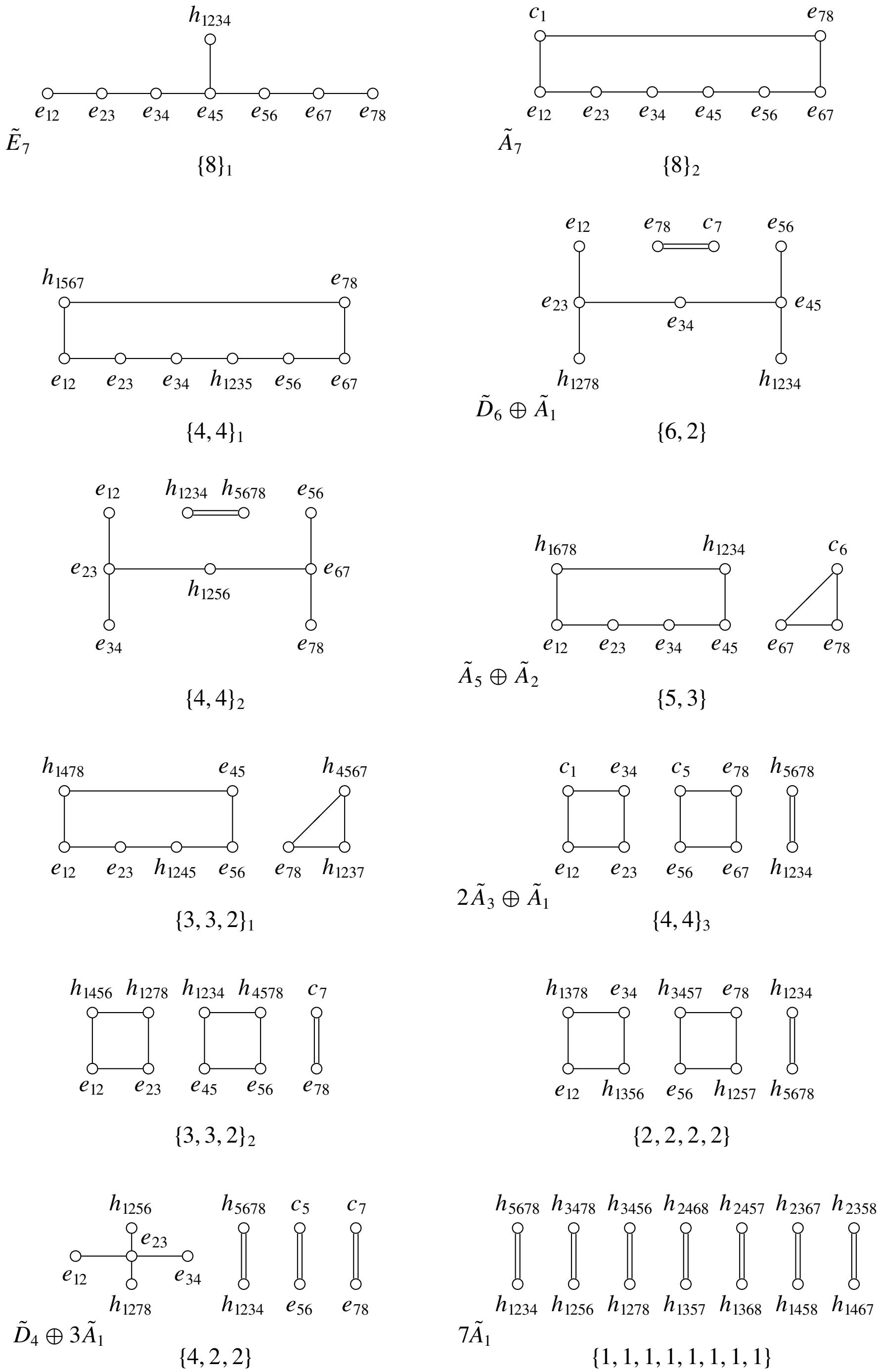

Figure 1 Configurations of vertical divisors on extremal fibrations

types of vertical divisors are classes of planes h_{abcd} and divisors $e_{i,i+1}$. But given any plane h_{abcd} that is a component of a quadric in the net, at least one of $\{a,b,c,d\}$ is the index of a $\mathbf{P}^3$-basepoint. If h_{ijkl} is the class of a double plane, then all indices of $\mathbf{P}^3$-basepoints are contained in $\{i,j,k,l\}$. So $\#(\{a,b,c,d\} \cap \{i,j,k,l\}) \geq 1$ and hence $h_{abcd} \cdot h_{ijkl} \leq 1$. Also, $h_{abcd} \cdot e_{i,i+1} = -1$, 0, or 1 for any i. So the class of a double plane cannot be a node of a component of type $\tilde{A}_1$.

Next consider h_{abcd}, the class of a component of a rank-2 quadric Q in the net that is singular at the base locus, say at p_a. Then the other component of Q has class h_{aijk} for some indices i,j,k, and we have $h_{abcd} \cdot h_{aijk} \leq 1$. Also, $h_{abcd} \cdot e_{i,i+1} = -1$, 0, or 1 for any i. Finally, the class of a singular cone with vertex at p_i is $2h - 2e_i - \sum_{k \neq i,j} e_k$ (where p_j is the highest-order basepoint infinitely near to p_i). Calculating then yields $h_{abcd} \cdot c_i \leq 1$. So h_{abcd} cannot be a node of component of type $\tilde{A}_1$.

Next consider the class c_i of a singular cone with vertex at p_i. The argument in the previous paragraph shows that $c_i \cdot h_{abcd} \leq 1$ for any class h_{abcd}. If c_ι is the class of a cone with vertex at another basepoint p_ι, then $c_i \cdot c_\iota = \left(2h - 2e_i - \sum_{k \neq i,j} e_k\right) \cdot \left(2h - 2e_\iota - \sum_{k \neq \iota,\lambda} e_k\right)$. But the first sum includes a term e_ι (since p_ι is not infinitely near to p_i) and the second includes e_i. So this is $8 - 2 - 2 - \#(\{1,\dots,8\} - \{i,j,\iota,\lambda\}) = 0$. Also, $c_i \cdot e_{j,j+1} = 2$ if and only if c_i contains a term $-2e_j$ but no term e_{j+1}—that is, if and only if $i = j$ and p_i is a basepoint of multiplicity 2. This gives the first case of the lemma.

Next consider the class h_{abcd} of a component of a quadric Q in the net smooth at the base locus. We have seen already that $h_{abcd} \cdot c_i < 2$ and $h_{abcd} \cdot e_{i,i+1} < 2$ for all i. Also, clearly $h_{abcd} \cdot h_{ijkl} = 2$ if and only if $\{a,b,c,d\} \cap \{i,j,k,l\} = \emptyset$—that is, if and only if h_{ijkl} is the class of the other component of Q. This gives the second case.

Finally, consider a class $e_{i,i+1}$. The only case not yet dealt with is $e_{i,i+1} \cdot e_{j,j+1}$. Again, one can check that the only possible values are -2, 0, and 1. □

The following lemma was already proved in the discussion preceding Theorem 3.2. We repeat it here to fix notation and to emphasise the role of Theorem 2.1 in the classification argument that follows.

LEMMA 4.3. *Let X be an extremal fibration. Then the number of components $h^0(\Gamma_X)$ is equal to $A + B + C + D$, where*

- A = *number of double planes in the net*;
- B = *number of rank-2 quadrics in the net singular at some* $\mathbf{P}^3$*-basepoint*;
- C = *number of rank-2 quadrics in the net smooth at the base locus*;
- D = *number of cones in the net with vertex at some* $\mathbf{P}^3$*-basepoint.*

In particular, since $B + C = d = n - 1$ in the notation of Theorem 2.1, we have $n \leq h^0(\Gamma_X) + 1$.

Proof of Theorem 4.1. We saw in the previous section that the graph Γ_X of irreducible vertical divisors on an extremal fibration X must be one of the seven graphs in Figure 1. To prove the theorem, we will consider each of these graphs Γ in turn and determine for which types $\{m_1, \dots, m_n\}$ of nets there can exist an extremal net of that type with configuration of vertical divisors equal to Γ. This process rests on several earlier results. First, Theorem 2.1 tells us how many rank-2 quadrics an extremal net of a given type must contain. Next, Lemma 4.2 narrows down the possibilities for a component of type $\tilde{A}_1$ in any of the graphs. Finally, Lemma 4.3 allows us to ignore types $\{m_1, \dots, m_n\}$ with more than $h^0(\Gamma) + 1$ distinct $\mathbf{P}^3$-basepoints.

For the purposes of the proof, let us introduce some terminology. A *simple chain* is a connected graph consisting of nodes $n_1, \dots, n_k$, edges (of multiplicity 1) joining n_i to n_{i+1} for $i = 1, \dots, k-1$, and no other edges. A *simple k-chain* is a simple chain with k nodes.

For any net of quadrics in $\mathbf{P}^3$, we adopt the following convention in labeling its basepoints. First choose a $\mathbf{P}^3$-basepoint and call it p_1. If p_1 has multiplicity m_1, then we define p_2 to be the basepoint in the exceptional divisor E_1, p_3 to be the basepoint in the exceptional divisor E_2, and so on up to p_{m_1}. We then choose p_{m_1+1} to be another $\mathbf{P}^3$-basepoint and repeat the procedure until we have exhausted all basepoints. So, for instance, if we have a net of type $\{5, 2, 1\}$ then its $\mathbf{P}^3$-basepoints will be labeled p_1, p_6, and p_8.

Suppose $Q = P_1 \cup P_2$ is a rank-2 quadric in an extremal net with $\mathbf{P}^3$-basepoints $p_1, \dots, p_{i_k}$. We will use the (somewhat imprecise) notation $Q = 1^{m_1} 2^{m_2} \cdots k^{m_n} + 1^{\mu_1} 2^{\mu_2} \cdots k^{\mu_k}$ to indicate that the plane P_1 (resp. P_2) has intersection multiplicities with a smooth quartic $C = Q_1 \cap Q_2$ (Q_1, Q_2 quadrics that, together with Q, span the net) equal to $m_1, \dots, m_n$ (resp. $\mu_1, \dots, \mu_k$) at $p_1, \dots, p_{i_k}$. We refer to such an expression as the *multiplicity data* of Q. Note that there are various constraints on multiplicity data for rank-2 quadrics in the net. For one, the sums $\sum m_i$ and $\sum \mu_j$ of exponents appearing in each term must always be 4, since any plane in $\mathbf{P}^3$ intersects a quartic curve with multiplicity 4. Also, the "intersection" of the two terms must consist of at most two basepoints, since if two planes in $\mathbf{P}^3$ share three noncollinear points p_i then they are equal. So, for example, an expression of the form $Q = 1^2 2^2 + 1^2 2^1 3^1$ is not permitted.

Now let us consider each graph Γ from Figure 1 in turn.

1. First consider the case $\Gamma = 7\tilde{A}_1$. I claim that the only possible type in this case is $\{1, 1, 1, 1, 1, 1, 1, 1\}$. To see this, note that the base locus of any net contains at most four multiple basepoints. So at most four of the $\tilde{A}_1$-components of Γ are conical, hence at least three are smooth. So there are at least three rank-2 quadrics in the net smooth at the base locus. I claim that any set $\{Q_1, Q_2, Q_3\}$ of three such quadrics must span the net.

If not, the third quadric would belong to the pencil spanned by the other two; rescaling, we could write $Q_3 = Q_1 + Q_2$. By assumption, $Q_1 = L_1 \Lambda_1$ and $Q_2 = L_2 \Lambda_2$, which are products of linear forms. I claim that the set $\{L_1, \Lambda_1, L_2\}$ is linearly independent. If not, we could write $\alpha L_1 + \beta \Lambda_1 + \gamma L_2 = 0$. None of the coefficients in this relation can be zero, since by assumption the components of Q_1 and Q_2 are all distinct (they give distinct elements of $\mathrm{Pic}(X)$). So we see that $L_1 = L_2 = 0$ implies $\Lambda_1 = 0$, meaning that $Q_1 \cap Q_2$ contains a line $L_1 = \Lambda_1 = 0$

along which Q_1 is singular. Intersecting with any other Q' in the net but not in the pencil $\langle Q_1, Q_2\rangle$, we would get a point in the base locus at which Q_1 is singular, which contradicts the fact that Q_1 gives an $\tilde{A}_1$-component of smooth type. We conclude that L_1, Λ_1, L_2 are linearly independent. So changing coordinates, we can assume that $Q_1 = XY$ and $Q_2 = ZL$, where L is a nonzero linear form that is not a multiple of X, Y, or Z. If the coefficient of W in L is zero, then both Q_1 and Q_2 are singular at $[0,0,0,1]$, violating Assumption 1. So L must have nonzero coefficient of W; hence by changing coordinates $W \mapsto L$ we get $Q_3 = XY + ZW$, which is not reducible. This contradicts our assumption, and so we conclude that any such set $\{Q_1, Q_2, Q_3\}$ must span the net.

This means that, locally near each basepoint, the base locus $Q_1 \cap Q_2 \cap Q_3$ of the net is given by the intersection of three planes. If there were a multiple basepoint p_i, then the intersection of the three planes at p_i would not be transverse and hence would not be proper. So no multiple basepoint can exist, and the net must have type $\{1,1,1,1,1,1,1,1\}$.

Assume now we have an extremal net of type $\{1,1,1,1,1,1,1,1\}$. Since there are no multiple basepoints, the seven rank-2 quadrics in the net are smooth at the base locus. We must show that the classes of components of these quadrics are (up to permutation of indices) as shown in Figure 1.

To see this, note first that there are at most three classes of the form h_{12ij}. If there were four or more, we would have to choose at least eight indices from the set $\{3,4,5,6,7,8\}$. Hence at least one index would be repeated—say (by relabeling) the index 3. Then there would be two classes of the form h_{123j}, which is impossible. So there at most three classes of the form h_{12ij} and hence, by symmetry, at most three classes of the form h_{abij} for any pair $\{a,b\} \subset \{1,\dots,8\}$.

For each rank-2 quadric Q in the net, a given basepoint lies in exactly one component of Q; so, given an index $a \in \{1,\dots,8\}$, exactly 7 of the 14 classes h_{ijkl} in the graph have $a \in \{i,j,k,l\}$. Consider the 7 classes h_{aijk}: there are 21 indices to choose from $\{1,\dots,8\} - \{a\}$, with each index appearing at most three times (by the previous paragraph). The only possibility is that each index appears exactly three times.

Thus for any pair $\{a,b\} \in \{1,\dots,8\}$ there are exactly three nodes of the graph that have the form h_{abij}. Since no two classes h_{abij} can share three indices, each index in the set $\{1,\dots,8\} - \{a,b\}$ appears in exactly one of these classes. Geometrically this means that, given three basepoints p_a, p_b, p_c of the net, the plane spanned by these three points is a component of a rank-2 quadric in the net and contains a fourth basepoint p_d of the net.

We can relabel basepoints if necessary so that p_4 is the fourth basepoint on the plane spanned by $\{p_1, p_2, p_3\}$, p_6 is the fourth basepoint on the plane spanned by $\{p_1, p_2, p_5\}$, and p_7 is the fourth basepoint on the plane spanned by $\{p_1, p_3, p_5\}$. This gives the classes h_{1234}, h_{1256}, and h_{1357} (and, since every node in the graph determines the node to which it is connected, the three classes joined to these) appearing in the diagram.

To determine the remaining classes, consider the plane spanned by $\{p_1, p_2, p_7\}$. No two classes h_{ijkl} can share three indices, so this plane cannot contain p_3, p_4,

p_5, or p_6. Therefore its fourth basepoint must be p_8, so there is a node h_{1278}. Similar arguments show we must have nodes h_{1368}, h_{1458}, and h_{1467}. Since every node in the graph determines the node to which it is connected, this completes the proof that the nodes of the graph (possibly after permuting indices) must be the configuration in Figure 1 labeled $\{1,1,1,1,1,1,1,1\}$.

2. The next case is $\Gamma = \tilde{D}_4 \oplus 3\tilde{A}_1$. Here $h^0(\Gamma) = 4$, so we need only consider types with at most five basepoints. Also, note that if we had a basepoint p_1 of multiplicity 5 or more then we would have effective divisors $e_{12}, \ldots, e_{45}$. This would imply that there is a subgraph of Γ that is a simple 4-chain. But Γ has no such subgraph, so we need not consider types with basepoints of multiplicity 5 or more. The remaining types are $\{4,4\}$, $\{4,3,1\}$, $\{4,2,2\}$, $\{4,2,1,1\}$, $\{4,1,1,1,1\}$, $\{3,3,2\}$, $\{3,3,1,1\}$, $\{3,2,2,1\}$, $\{3,2,1,1,1\}$, $\{2,2,2,2\}$, and $\{2,2,2,1,1\}$.

(i) Type $\{4,4\}$: We can rule out this possibility as follows. We know $h^0(\Gamma) = A+B+C+D = A+D+n-1 = A+D+1$. But $A \leq 1$ and $D \leq 2$ by Lemma 1.2, and $A = 1$ implies $D = 0$ (since if there is a double plane in the net, all other quadrics must be smooth at the base locus). Therefore $h^0(\Gamma_X) \leq 3$ for this type of net, so it does not yield Γ.

(ii) Type $\{4,3,1\}$: Since this type has no basepoint of multiplicity 2, Lemma 4.2 says there is no conical $\tilde{A}_1$-component. So all three of the $\tilde{A}_1$-components are smooth, implying there are at least three rank-2 quadrics in the net. This is impossible by Theorem 2.1, so this type does not give Γ.

(iii) Type $\{4,2,2\}$: The nodes e_{12}, e_{23}, e_{34} form a simple 3-chain that must be contained in the $\tilde{D}_4$-component. The nodes e_{56} and e_{78} are disjoint from this chain, and from each other, so they must belong to two distinct $\tilde{A}_1$-components—which are therefore conical, with nodes e_{56}, c_5 and e_{78}, c_7. Since a conical $\tilde{A}_1$-component comes from a basepoint of multiplicity exactly 2, the third such component must be smooth. So there must be a rank-2 quadric in the net smooth at the base locus. Clearly, the only possibility for the multiplicity data is $Q = 1^4 + 2^2 3^2$. The corresponding nodes of the diagram are h_{1234} and h_{5678}. The other rank-2 quadric in the net must therefore be singular at the base locus, and its components must give the other two nodes in the $\tilde{D}_4$-component. Suppose a class h_{abcd} has $h_{abcd} \cdot e_{12} = 0$, $h_{abcd} \cdot e_{23} = 1$, and $h_{abcd} \cdot e_{34} = 0$. Then the set $\{a,b,c,d\}$ contains 1 and 2 but not 3 or 4. Also, we have $h_{abcd} \cdot e_{56} = h_{abcd} \cdot e_{78} = 0$, so $\{a,b,c,d\}$ must intersect both $\{5,6\}$ and $\{7,8\}$ in either zero or two elements. The only two possibilities are h_{1256} and h_{1278}. This gives the configuration in Figure 1 labeled $\{4,2,2\}$.

(iv) Type $\{4,2,1,1\}$: Here there is only one basepoint of multiplicity 2 and so at most one conical $\tilde{A}_1$-component. It is easy to see that the only possibility for a smooth $\tilde{A}_1$-component is $Q = 1^4 + 2^2 3^1 4^1$, so we cannot obtain the remaining two such components. Hence a net of this type cannot yield Γ.

(v) Type $\{4,1,1,1,1\}$: Here there are no basepoints of multiplicity 2, so all the $\tilde{A}_1$-components must be smooth. But again the only possibility is $Q = 1^4 + 2^1 3^1 4^1 5^1$, so we cannot get Γ from a net of this type.

(vi) Types $\{3,3,2\}$ and $\{3,3,1,1\}$: These types have two *disjoint* simple 2-chains with nodes e_{12}, e_{23} and e_{34}, e_{45}. But there is no way to embed two such chains disjointly in Γ, so these types cannot yield Γ.

(vii) Type $\{3,2,2,1\}$: In this case there are two basepoints of multiplicity 2, giving nodes e_{45}, e_{67}, which are disjoint from the 2-chain with nodes e_{12}, e_{23} and from each other. So these must give two distinct conical $\tilde{A}_1$-components. Also, there are three rank-2 quadrics in the net, giving six more nodes. Adding all these up gives twelve nodes in total, whereas Γ has only eleven nodes. So this type cannot yield Γ.

(viii) Type $\{3,2,1,1,1\}$: This is similar to the previous case. The basepoint of multiplicity 3 gives a simple 2-chain with nodes e_{12}, e_{23} that must be contained in the $\tilde{D}_4$-component. The basepoint of multiplicity 2 gives a node e_{45} disjoint from this, so it must be a node of a conical $\tilde{A}_1$-component. The net has four rank-2 quadrics, giving eight more nodes. Adding these up yields twelve nodes, so this type cannot give Γ.

(ix) Type $\{2,2,2,2\}$: We cannot have three smooth $\tilde{A}_1$-components, for the same reason as in the case $\Gamma = 7\tilde{A}_1$. So one of these components must be conical; without loss of generality, we have a cone c_1. Then all other quadrics in the net are smooth at p_1. In particular, the three rank-2 quadrics in the net are all smooth at p_1. But then exactly the same argument as in the case $\Gamma = 7\tilde{A}_1$ shows the intersection is not proper.

(x) Type $\{2,2,2,1,1\}$: Just as in the previous case we must have a conical $\tilde{A}_1$-component, so the four rank-2 quadrics in the net must all be smooth at p_1, say. Again this implies that the intersection is not proper.

3. The next graph to consider is $\Gamma = 2\tilde{A}_3 \oplus \tilde{A}_1$. It has $h^0(\Gamma) = 3$, so we need only consider nets with at most four basepoints. A basepoint of multiplicity at least 5 would give a simple 4-chain embedded in Γ, so we know that all basepoints have multiplicity at most 4. The remaining types are $\{4,4\}$, $\{4,3,1\}$, $\{4,2,2\}$, $\{4,2,1,1\}$, $\{3,3,2\}$, $\{3,3,1,1\}$, $\{3,2,2,1\}$, and $\{2,2,2,2\}$.

(i) Type $\{4,4\}$: There is no basepoint of multiplicity 2, so the $\tilde{A}_1$-component must be smooth. Its nodes are therefore h_{1234} and h_{5678}. Also there are two simple 3-chains with nodes e_{12}, e_{23}, e_{34} and e_{56}, e_{67}, e_{78}. If the net had a double plane, it would have class h_{1256}. This node would be joined to e_{23} and e_{67}, giving a component of Γ with at least seven nodes, so such a double plane cannot exist. Since the unique rank-2 quadric in the net is smooth at the base locus, we must have cones c_1 and c_5, and these give all nodes of Γ. The resulting diagram is shown in Figure 1 and labeled $\{4,4\}_3$.

(ii) Type $\{4,3,1\}$: Again the $\tilde{A}_1$-component must be smooth. Since there is only one such component, the other reducible quadric in the net is singular at some basepoint. If it were smooth at p_1, then its multiplicity data would be $Q = 1^4 + 2^2 3^1$ and hence would be smooth at the base locus. This is impossible, so Q must be singular at p_1. If it were smooth at p_5, then its multiplicity data would be $Q = 1^3 3^1 + 1^1 2^3$ and so the corresponding nodes would be h_{1238} and h_{1567}. The first node would be joined to e_{34} and the second to e_{12}. This would give a component of Γ with at least five nodes, which does not exist. Finally, if Q were singular at both p_1 and p_5 then it would look like $Q = 1^3 2^1 + 1^1 2^2 3^1$, giving nodes h_{1235} and h_{1568}. But the first node would be joined to both e_{34} and e_{56}, giving a component with at least six nodes, which again is impossible. So this type does not yield Γ.

(iii) Type $\{4, 2, 2\}$: We have a simple 3-chain with nodes e_{12}, e_{23}, e_{34}—which must be contained in one of the $\tilde{A}_3$-components—and two other nodes e_{56}, e_{78} that are disjoint from this 3-chain and from each other. If we assume first that the $\tilde{A}_1$-component is conical, then the remaining four nodes are the components of the two rank-2 quadrics in the net. So there must be a class h_{abcd} having intersection 1 with both e_{12} and e_{34} and intersection 0 with e_{23}, which is impossible. So we can assume the $\tilde{A}_1$-component is smooth; hence its nodes are h_{1234} and h_{5678}. There are three more nodes; two are components of the other rank-2 quadric in the net. If the third were the class of a double plane, it would be h_{1257}; this would have intersection 1 with e_{23}, which would therefore have degree 3. Since the simple 3-chain is contained in an $\tilde{A}_3$-component and since all nodes of that component have degree 2, that is impossible. So the last node must be the class of a cone, hence c_5 or c_7. But either choice would yield another double edge of the graph, which is impossible. So this type cannot yield Γ.

(iv) Type $\{4, 2, 1, 1\}$: First assume the $\tilde{A}_1$-component is conical. Then there are no rank-2 quadrics in the net smooth at the base locus, so all three rank-2 quadrics in the net are singular at some basepoint. No quadric in the net is singular at a basepoint of multiplicity 1, so each of the three rank-2 quadrics is singular at one of the two multiple basepoints. But then two quadrics must be singular at the same basepoint, which contravenes Assumption 1. So this type cannot yield Γ.

(v) Type $\{3, 3, 2\}$: Here we have two disjoint simple 2-chains, with nodes e_{12}, e_{23} and e_{45}, e_{56}, and a node e_{78} not joined to either. It follows that e_{78} must be a node of the $\tilde{A}_1$-component, which is therefore conical with second node c_7. The four remaining nodes are the components of the two rank-2 quadrics in the net, so each is of type h_{abcd}. Consider the node h_{abcd} of this type joined to e_{12}: it is not joined to e_{23}, so the set $\{a, b, c, d\}$ contains 1 but not 2 or 3. It is also disjoint from e_{45} and e_{56}, so $\{a, b, c, d\}$ either contains or is disjoint from $\{4, 5, 6\}$. But it cannot be disjoint from a set of five elements, so we must have $\{a, b, c, d\} = \{1, 4, 5, 6\}$. Similar arguments show that the remaining node in this component must be h_{1278} and that the two missing nodes in the other component are h_{1234} and h_{4578}. This gives the configuration shown in Figure 1 and labeled $\{3, 3, 2\}_2$.

(vi) Type $\{3, 3, 1, 1\}$: There is no basepoint of multiplicity 2, so the $\tilde{A}_1$-component must be smooth. Hence its nodes (possibly after swapping p_7 and p_8) are h_{1237} and h_{4568}. We have two simple 2-chains with nodes e_{12}, e_{23} and e_{45}, e_{56}; the four remaining nodes must be the components of the two remaining rank-2 quadrics in the net. If a class h_{abcd} is joined to e_{12} but not e_{23}, then 1 belongs to $\{a,b,c,d\}$, but 2 and 3 do not. Also, h_{abcd} is not connected to e_{45} or e_{56}, so either $\{4,5,6\} \subset \{a,b,c,d\}$ or the two sets are disjoint. They cannot be disjoint because 2 and 3 are not in $\{a,b,c,d\}$, either; therefore, the node connected to e_{12} is h_{1456}. But then $h_{1456} \cdot h_{1237} = 1$, which gives an illegal edge of the graph. So this type does not yield Γ.

(vii) Type $\{3,2,2,1\}$: Again we have a simple 2-chain with nodes e_{12}, e_{23} and two nodes e_{45}, e_{67} not connected to that chain or each other. If the $\tilde{A}_1$-component were conical, we would have five nodes. The components of the three rank-2 quadrics in the net would give another six, making eleven altogether, which is a contradiction. So the $\tilde{A}_1$-component must be smooth. The only possibility for the multiplicity data of the corresponding rank-2 quadric is $Q = 1^3 4^1 + 2^2 3^2$, so the nodes of this $\tilde{A}_1$-component must be h_{1238} and h_{4567}.

The 2-chain must be contained in an $\tilde{A}_3$-component, so there must be a node joined to e_{12} but not e_{23}. Since p_1 has multiplicity 3, the class of a cone in the net with vertex at p_1 would be $c_1 = 2h - 2e_1 - e_2 - e_4 - \cdots - e_8$, which would give $c_1 \cdot e_{23} = 1$. So the node in question must be the class of a plane h_{abcd}. By the same logic as before, the set $\{a,b,c,d\}$ contains 1 but not 2 or 3, and since $h_{abcd} \cdot e_{45} = h_{abcd} \cdot e_{67} = 0$, it must contain or be disjoint from the sets $\{4,5\}$ and $\{6,7\}$. So after relabeling basepoints if necessary, it is h_{1458}. One can check that the final node in that component of Γ must be h_{1267}.

There are two remaining nodes with classes h_{abcd} and h_{ijkl}, which must both be joined to e_{45} and e_{67} but to no other nodes. So $\{a,b,c,d\}$ and $\{i,j,k,l\}$ both contain 4 and 6 but not 5 or 7. Since neither node is joined to e_{12} or e_{23}, the two sets must also contain or be disjoint from $\{1,2,3\}$. But neither is possible, so this type does not yield Γ.

(viii) Type $\{2,2,2,2\}$: As before, if we have a conical $\tilde{A}_1$-component, say with node c_1, then the three rank-2 quadrics in the net are smooth at c_1 and so the intersection is not proper. Hence the $\tilde{A}_1$-component must be smooth. Possibly relabeling basepoints, its nodes are h_{1234} and h_{5678}. The four components of the remaining rank-2 quadrics in the net (which must be singular at the base locus) give the four remaining nodes. Suppose e_{12} and e_{56} belonged to the same $\tilde{A}_3$-component. Then there would be a node h_{abcd} joined to e_{12} and e_{56} and to no other nodes. So the set $\{a,b,c,d\}$ must contain 1 and 5 but not 2 or 6; also, it must either contain or be disjoint from $\{3,4\}$ and $\{7,8\}$. If it contained $\{3,4\}$, then the intersection $h_{abcd} \cdot h_{1234}$ would be -1; if it were disjoint from $\{3,4\}$, the intersection would be 1. Since h_{1234} belongs to the $\tilde{A}_1$-component, neither is possible. The same argument shows that e_{12} and e_{78} cannot belong to the same component. Therefore, one $\tilde{A}_3$-component contains e_{12} and e_{34}, and the other contains e_{56} and e_{78}.

So there are two nodes h_{abcd} joined to e_{12} and e_{34} and no other nodes: it is not hard to see they must be h_{1356} and h_{1378}. Similarly, there are two nodes joined to e_{56} and e_{78} and no other nodes: they must be h_{1257} and h_{3457}. This gives the configuration shown in Figure 1 and labeled $\{2,2,2,2\}$.

4. The next graph to consider is $\Gamma = \tilde{A}_5 \oplus \tilde{A}_2$. This has $h^0(\Gamma) = 2$, so we need only consider types with at most three basepoints. Also, the maximum length of a simple chain embedded in this graph is 5, so there can be no basepoint of multiplicity more than 6. Also note that if a net has a basepoint p_i of multiplicity at least 5, then all rank-2 quadrics in the net must be singular at that basepoint (otherwise we would have a smooth quadric intersecting a plane with multiplicity at least 5 at p_i). If the net has three basepoints then it has two rank-2 quadrics, which therefore must both be singular at p_i. But this contradicts Assumption 1. So we can ignore the types satisfying these two conditions—namely, $\{6,1,1\}$ and $\{5,2,1\}$. This leaves the following types to be considered: $\{6,2\}$, $\{6,1,1\}$, $\{5,3\}$, $\{5,2,1\}$, $\{4,4\}$, $\{4,3,1\}$, $\{4,2,2\}$, and $\{3,3,2\}$.

(i) Type $\{6,2\}$: The unique rank-2 quadric in the net must have multiplicity data $Q = 1^4 + 1^2 2^2$, so it is singular at p_1 and smooth at p_7. Because it is singular at p_1, there is no double plane in this net; because it is smooth at p_7, there must be a cone in the net with vertex at p_7. But this would give a node joined to e_{78} by a double edge, which Γ does not possess. So this type does not yield Γ.

(ii) Type $\{5,3\}$: We have a simple 4-chain with nodes $e_{12}, \dots, e_{45}$ and a simple 2-chain with nodes e_{67}, e_{78} that is disjoint from it. The longer chain must be contained in the $\tilde{A}_5$-component and the shorter one in the $\tilde{A}_2$-component. The third node of the $\tilde{A}_2$-component cannot be a class h_{abcd}, since we cannot have $h_{abcd} \cdot e_{67} = h_{abcd} \cdot e_{78} = 1$. (If we did, we would have $h_{abcd} \cdot (e_6 - e_8) = 2$, which is impossible.) So it must be the class of a cone and hence c_6. The two remaining nodes of the $\tilde{A}_5$-component must be the components of the unique rank-2 quadric Q in the net. The multiplicity data must be $Q = 1^4 + 1^1 3^3$, so these nodes are h_{1234} and h_{1678}. This gives the configuration shown in Figure 1 and labeled $\{5,3\}$.

(iii) Type $\{4,4\}$: We know that $h^0(\Gamma_X) = A + B + C + D = A + D + (n-1) = A + D + 1$, so to get $h^0 = 2$ we need $A + D = 1$. First suppose $A = 0$. There is a unique rank-2 quadric Q in the net; the only possibilities for the multiplicity data are $Q = 1^4 + 2^4$ and $Q = 1^3 2^1 + 1^1 2^3$. So Q is singular at neither or both of the basepoints. If neither, then there must be cones in the net with vertices at both basepoints and hence $D = 2$; if both, then there are no cones singular at the base locus and hence $D = 0$. Neither case gives $h^0 = 2$. On the other hand, if $A = 1$ then the unique reducible reduced quadric in the net must be smooth at the base locus and so Γ_X must have an $\tilde{A}_1$-component, which Γ does not possess. Hence this type does not yield Γ.

(iv) Type $\{4,3,1\}$: We have a simple 3-chain with nodes e_{12}, e_{23}, e_{34} that must be contained in the $\tilde{A}_5$-component. So the simple 2-chain with nodes e_{56}, e_{67} must be contained in the $\tilde{A}_2$-component. There are four more nodes, which are therefore the components h_{abcd} of the two rank-2 quadrics in the net. One of these must be the last node of the $\tilde{A}_2$-component, so must have $h_{abcd} \cdot e_{56} = h_{abcd} \cdot e_{67} = 1$. As before this is impossible, so this type does not yield Γ.

(v) Type $\{4,2,2\}$: We have a simple 3-chain with nodes e_{12}, e_{23}, e_{34} and two nodes e_{56}, e_{78} not joined to this chain or to each other. Possibly after relabeling, e_{78} is a node of the $\tilde{A}_2$-component. Again counting nodes, the remaining two nodes of this component must be the classes h_{abcd} and h_{ijkl} of components of rank-2 quadrics in the net. As before, $\{a,b,c,d\}$ and $\{i,j,k,l\}$ must both contain or be disjoint from $\{1,2,3,4\}$. So these index sets must be $\{1,2,3,4\}$ and $\{5,6,7,8\}$ and therefore $h_{abcd} \cdot h_{ijkl} = 2$, which is impossible. So this type does not yield Γ.

(vi) Type $\{3,3,2\}$: Here we have two simple 2-chains, with nodes e_{12}, e_{23} and e_{45}, e_{56}, and a node e_{78} disjoint from these chains. The four remaining nodes must be classes h_{abcd} of components of rank-2 quadrics in the net.

Suppose first that e_{78} is a node of the $\tilde{A}_2$-component. The other two nodes of that component must be classes h_{abcd} and h_{ijkl}, where $\{a,b,c,d\}$ and $\{i,j,k,l\}$ must both contain or be disjoint from $\{1,2,3\}$ and $\{4,5,6\}$ and must both contain 7 but not 8, so they are h_{1237} and h_{4567}. What of the other two nodes? One is connected to e_{12} but not e_{23}: its class is h_{abcd}, where the index set contains 1 but not 2 or 3. If this node is joined to e_{45} then the index set contains 4 but not 5 or 6, so the class must be h_{1478}. The other node is then h_{1245}. This gives the configuration shown in Figure 1 and labeled $\{3,3,2\}_1$.

If the node connected to e_{12} is also connected to e_{56}, then the index set contains 4 and 5 but not 6 as well as neither or both of 7 and 8. But this is impossible, since we know it contains 1 but not 2 or 3.

Next suppose that e_{78} belongs to the $\tilde{A}_5$-component. There is no way to embed two simple 2-chains and one other node disjointly in this component, so in this case one of the 2-chains must belong to the $\tilde{A}_2$-component. Say it is the chain with nodes e_{45}, e_{56}: then there is a class h_{abcd} with $h_{abcd} \cdot e_{45} = h_{abcd} \cdot e_{56} = 1$, which again is impossible, and similarly for the other 2-chain.

5. The next graph to consider is $\tilde{D}_6 \oplus \tilde{A}_1$. Again we need only consider types with no more than three basepoints and multiplicities no greater than 6. Also, as before we can ignore types $\{6,1,1\}$ and $\{5,2,1\}$. The remaining types are $\{6,2\}$, $\{5,3\}$, $\{4,4\}$, $\{4,3,1\}$, $\{4,2,2\}$, and $\{3,3,2\}$.

(i) Type $\{6,2\}$: We have a simple 5-chain with nodes $e_{12}, \dots, e_{56}$ and a node e_{78} disjoint from it. A simple 5-chain in $\tilde{D}_6$ is joined to all nodes of that component, so e_{78} must be a node of the $\tilde{A}_1$-component, which is therefore conical, with the other node equal to c_7. The multiplicity data of the unique rank-2 quadric in the net must be $Q = 1^4 + 1^2 2^2$, so the corresponding nodes are h_{1234} and h_{1278}. This gives the configuration shown in Figure 1 and labeled $\{6,2\}$.

(ii) Type $\{5,3\}$: Here there is a simple 4-chain with nodes $e_{12}, \dots, e_{45}$ and a disjoint simple 2-chain with nodes e_{67}, e_{78}. But it is impossible to embed these two chains disjointly in Γ. So this type does not yield Γ.

(iii) Type $\{4,4\}$: We saw before that this type has $h^0(\Gamma) = 2$ only if there is a double plane in the net. Such a plane has class h_{1256}. We have nodes e_{12}, e_{23},

e_{34}, e_{56}, e_{67}, and e_{78}; together with h_{1256}, these form the $\tilde{D}_6$-component. The unique rank-2 quadric in the net is smooth; hence it gives the $\tilde{A}_1$-component with nodes h_{1234} and h_{5678}. So this type yields the diagram shown in Figure 1 and labeled $\{4,4\}_2$.

(iv) Type $\{4,3,1\}$: This type has no basepoint of multiplicity 2, so the $\tilde{A}_1$-component must be smooth, with nodes h_{1234} and h_{5678}. We have a simple 3-chain with nodes e_{12}, e_{23}, e_{34} and a disjoint simple 2-chain with nodes e_{56}, e_{67}. The two remaining nodes must be the classes h_{abcd} of the components of the second rank-2 quadric in the net. There is a unique way (up to graph isomorphism) to embed a simple 3-chain and a simple 2-chain disjointly in $\tilde{D}_6$; hence, for $\{i,j\} = \{5,6\}$ or $\{7,8\}$, one of the classes h_{abcd} must satisfy $h_{abcd} \cdot e_{ij} = 1$ and $h_{abcd} \cdot D = 0$ for all other nodes D of the graph. In either case $\{a,b,c,d\}$ contains exactly two of $\{5,6,7\}$. Also, it cannot contain $\{1,2,3,4\}$ and so must be disjoint from it. But then $\{a,b,c,d\}$ contains at most three elements—a contradiction. So this type cannot yield Γ.

(v) Type $\{4,2,2\}$: This graph has a single $\tilde{A}_1$-component, so there is some rank-2 quadric in the net singular at the base locus (and therefore, by Assumption 1, no double plane). The multiplicity data of such a rank-2 quadric has the form $Q = 1^i 2^j 3^k + 1^{4-i} 2^{2-j} 3^{2-k}$. From this we see that Q cannot be singular at both basepoints of multiplicity 2, for if it were then we would have $Q = 1^2 2^1 3^1 + 1^2 2^1 3^1$, which cannot occur. So there must be cones in the net with vertices at p_5 and p_7; hence there must be nodes c_5 and c_7 in Γ. These nodes are joined to e_{56} and e_{78} (respectively) by double edges; since there is only 1 double edge in Γ, we get a contradiction. So this type cannot yield Γ.

(vi) Type $\{3,3,2\}$: We have two disjoint simple 2-chains with nodes e_{12}, e_{23} and e_{45}, e_{56} and a node e_{78} disjoint from both chains. Any union of two disjoint simple 2-chains in $\tilde{D}_6$ is joined to every node, so we conclude that the node e_{78} must belong to the $\tilde{A}_1$-component. This component is then conical with node c_7; then, together with the components of the two rank-2 quadrics in the net, we get ten nodes rather than nine. So this type does not yield Γ.

6. The next graph to consider is $\tilde{A}_7$. Here we need only consider types with at most two basepoints, so the possible types are $\{8\}$, $\{7,1\}$, $\{6,2\}$, $\{5,3\}$, and $\{4,4\}$.

(i) Type $\{8\}$: We have seven nodes $e_{12}, \ldots, e_{78}$. There are no rank-2 quadrics in the net, so the only issue is whether the quadric singular at the basepoint is a double plane or a cone. If it were a double plane then it would have class h_{1234}, meaning that the node e_{45} in the graph would have degree 3. The graph $\tilde{A}_7$ has no such node, so the final node must be a cone c_1. Hence the only possibility is the configuration shown in Figure 1 and labeled $\{8\}_2$.

(ii) Type $\{7,1\}$: The unique rank-2 quadric must have multiplicity data $Q = 1^4 + 1^3 2^1$, so the corresponding nodes must be h_{1234} and h_{1238}. These have intersection -1, which is impossible. So this type cannot yield Γ—indeed, it cannot occur at all.

(iii) Type $\{6,2\}$: Here the unique rank-2 quadric has multiplicity data $Q = 1^4 + 1^2 2^2$, so the corresponding nodes are h_{1234} and h_{1278}. But $h_{1278} \cdot e_{23} = 1$ and so e_{23} has degree 3, which again is impossible for this graph. So this type does not yield Γ.

(iv) Type $\{5,3\}$: This type has $h^0(\Gamma_X) = A+B+C+D = A+D+(n-1) = A+D+1$. In this case $h^0(\tilde{A}_7) = 1$, so we must have $A = D = 0$. Therefore, the unique rank-2 quadric in the net must be singular at both basepoints. But the only possible multiplicity data is $Q = 1^4 + 1^1 2^3$, so Q is smooth at one basepoint. Hence this type does not yield Γ, or indeed any graph with $h^0 = 1$.

(v) Type $\{4,4\}$: This type has $h^0(\Gamma_X) = A+B+C+D = A+D+(n-1) = A+D+1$. In this case $h^0(\tilde{A}_7) = 1$, so we must have $A = D = 0$. Therefore, the unique rank-2 quadric in the net must be singular at both basepoints. The multiplicity data of this quadric must be $Q = 1^3 2^1 + 1^1 2^3$, so the corresponding nodes are h_{1235} and h_{1567}. Together with the 3-chains e_{12}, e_{23}, e_{34} and e_{56}, e_{67}, e_{78}, these give the configuration shown in Figure 1 and labeled $\{4,4\}_1$. Note that this argument shows in fact that any net of type $\{4,4\}$ with $h^0(\Gamma_X) = 1$ must have $\Gamma_X = \tilde{A}_7$.

7. The final graph to consider is $\tilde{E}_7$. Here we need only consider types with at most two basepoints. We have already shown that the type $\{7,1\}$ cannot occur, that the type $\{5,3\}$ cannot give $h^0(\Gamma_X) = 1$, and that a net of type $\{4,4\}$ with $h^0(\Gamma_X) = 1$ must have $\Gamma_X = \tilde{A}_7$. So the only types we need to consider are $\{8\}$ and $\{6,2\}$.

(i) Type $\{8\}$: As for the case $\Gamma = \tilde{A}_7$, the only issue is whether the quadric singular at the basepoint is a cone or a double plane. We saw that a cone gives $\Gamma_X = \tilde{A}_7$, so it must be a double plane with class h_{1234}. Hence the configuration is as shown in Figure 1 and labeled $\{8\}_1$.

(ii) Type $\{6,2\}$: We have a simple 5-chain with nodes $e_{12}, \dots, e_{56}$. The unique rank-2 quadric in the net must have multiplicity data $Q = 1^4 + 1^2 2^2$, so the corresponding nodes are h_{1234} and h_{1278}. But then the nodes e_{23} and e_{45} both have degree 3, which is impossible in $\tilde{E}_7$. So this type does not yield Γ. $\square$

5. Standard Forms for Extremal Nets

The aim of this section is to find standard forms for extremal nets of the possible types $\{m_1, \dots, m_n\}$ determined in Theorem 4.1. More precisely, for each possible configuration $\{m_1, \dots, m_n\}_i$ of irreducible vertical divisors shown in Figure 1, we give a unique standard form for extremal nets whose associated configuration is $\{m_1, \dots, m_n\}_i$.

Note on Characteristic. We must note at this point that some of the arguments used to obtain the standard forms listed next are not valid in characteristics 2 and 3. Therefore, *we claim only that these standard forms exist for nets in* $\mathbf{P}^3_k$ *where* $\operatorname{char} k = 0$ *or* $p \geq 5$. On the other hand, it is straightforward to check that in each case (except the last) the given net has the configuration of vertical divisors claimed, and that the net satisfies Assumption 1, for all characteristics. So our standard forms prove the existence of extremal nets with each possible configuration, except $\{1,1,1,1,1,1,1,1\}$, in all characteristics.

1. $\{8\}_1$: The standard form is $Q_1 = Z^2$, $Q_2 = X(Y+W) + YW$, $Q_3 = XZ + (Y+W)^2$.
2. $\{8\}_2$: The standard form is $Q_1 = YZ + W^2$, $Q_2 = XZ + YW$, $Q_3 = XW - Y^2 + Z^2$.
3. $\{6,2\}$: The standard form is $Q_1 = YZ$, $Q_2 = XZ + W^2$, $Q_3 = XY + Z^2$.
4. $\{5,3\}$: The standard form is $Q_1 = YZ$, $Q_2 = XW + Z^2$, $Q_3 = XY + W^2$.
5. $\{4,4\}_1$: The standard form is $Q_1 = ZW$, $Q_2 = XZ + YW$, $Q_3 = XY + Z^2 + W^2$.
6. $\{4,4\}_2$: The standard form is $Q_1 = XY$, $Q_2 = Z^2$, $Q_3 = (X+Y)Z + W^2$.
7. $\{4,4\}_3$: The standard form is $Q_1 = XY$, $Q_2 = XZ + W^2$, $Q_3 = YW + Z^2$.
8. $\{4,2,2\}$: The standard form is $Q_1 = X(Y+Z)$, $Q_2 = YZ$, $Q_3 = XZ + W^2$.
9. $\{3,3,2\}_1$: The standard form is $Q_1 = XY$, $Q_2 = ZW$, $Q_3 = (X+Y)Z + W^2$.
10. $\{3,3,2\}_2$: The standard form is $Q_1 = YZ$, $Q_2 = X(Z+W)$, $Q_3 = XY + W^2$.
11. $\{2,2,2,2\}$: The standard form is $Q_1 = XY$, $Q_2 = ZW$, $Q_3 = (X+Y)(Z+W)$.
12. $\{1,1,1,1,1,1,1,1\}$: Extremal nets of this type exist only in characteristic 2 and have standard form $Q_1 = (X+Y+Z)W$, $Q_2 = (X+Y+W)Z$, $Q_3 = (X+Z+W)Y$.

The remainder of this section gives a detailed derivation of the standard forms listed above.

1. $\{8\}_1$: First suppose we have a net of type $\{8\}$ that contains a double plane. I claim we can put it in standard form $Q_1 = Z^2$, $Q_2 = XY + XW + YW$, $Q_3 = XZ + (Y+W)^2$. To see this, first apply a projective transformation moving the unique basepoint to $[X,Y,Z,W] = [1,0,0,0]$. Next, applying an element of $\mathrm{PGL}(3) \subset \mathrm{PGL}(4)$ fixing p_1, we can move the double plane so that (set-theoretically) it becomes the plane $\{Z = 0\}$. This gives Q_1 the form we claimed.

Next consider Q_2. I claim that we can choose Q_2 to be an irreducible reduced cone with vertex not lying on Q_1. To see this, consider the subset $S \subset \mathbf{P}^3$ consisting of all singular points of all quadrics in the net. I claim S is not contained in $\{Z = 0\}$.

Suppose it were, and assume first that the set of singular quadrics spans the net. Choose two singular quadrics Q, Q' that, together with Q_1, span the net. By assumption, Q and Q' are singular at some point of $\{Z = 0\}$. The intersection $Q_1 \cap Q \cap Q'$ is a single eightfold point p_1, which means that both $Q_1 \cap Q$ and $Q_1 \cap Q'$ must be quadruple lines meeting at p_1. It is then not difficult to see that we can find a quadric in the pencil spanned by Q and Q' that is singular at p_1. But this violates Assumption 1.

On the other hand, suppose that the set of singular quadrics is contained in a pencil. This pencil is spanned by Q_1 and any other singular quadric Q_2, which by assumption is a cone with vertex lying in the plane $\{Z = 0\}$. We can move the vertex to $p_2 = [0, 1, 0, 0]$ without changing Q_1 or p_1. Adding a multiple of Q_1 to Q_2 does not change the differential at a point of $\{Z = 0\}$, so every quadric in the pencil (hence every singular quadric in the net) is singular at p_2 (and nowhere else, unless it is $Q = Q_1$). Now choose any smooth quadric Q_3 in the net, and consider the intersection $Q_1 \cap Q_2 \cap Q_3$. (Note that Q_3 is not contained in the pencil $\langle Q_1, Q_2\rangle$, so this intersection is the set of $\mathbf{P}^3$-basepoints of the net.) If $Q_1 \cap Q_2$ consisted (as a set) of two distinct lines $L_1 \cup L_2$ in $\{Z = 0\}$, with $L_1 = \{Z = W = 0\}$ the line through p_1 and p_2, then $L_2 \cap Q_3$ would give another basepoint of the net, contradicting our assumption. So $Q_1 \cap Q_2$ must be a double line $\{Z = W = 0\}$. Therefore the form defining Q_2 looks like $Q_2 = \alpha XZ + \beta Z^2 + \gamma ZW + \delta W^2$. Subtracting a multiple of Q_1, we can assume $\beta = 0$; since Q_2 is an irreducible cone, neither α nor δ is zero. We can therefore scale X and W to obtain $\alpha = \delta = 1$ without changing Q_1, p_1, or p_2. Now the restriction of the form Q_3 to the line $\{Z = W = 0\}$ must have a double root at p_1, so the coefficient of XY in Q_3 must be zero. The coefficient of X^2 in Q_3 is also zero, since Q_3 passes through p_1. Finally, we can subtract multiples of Q_1 and Q_2 from Q_3 to make the coefficients of XZ and Z^2 zero without changing anything else. Note also that since p_2 is not a basepoint of the net, the coefficient ε of Y^2 in Q_3 is nonzero. But now computing the determinant of a general member of the net $\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3$, we see that the discriminant locus is defined by a degree-4 polynomial in the λ_i that is different from λ_3^4—specifically, the coefficient of $\lambda_2^3\lambda_3$ equals $-\varepsilon$, which is nonzero. In other words, the set of singular quadrics in the net is not contained in the pencil $\{\lambda_3 = 0\} = \langle Q_1, Q_2\rangle$, which contradicts our assumption.

So without loss of generality, we can choose Q_2 to be an irreducible reduced cone with vertex not lying on Q_1. Applying a projective transformation fixing p_1 and Q_1, we can bring this vertex to the point $p_2 = [0, 0, 1, 0]$. This implies that, in the equation defining Q_2, each monomial containing Z has coefficient zero. Since Q_2 passes through p_1, the coefficient of X^2 is zero also; hence $Q_2 = b_2XY + c_2XW + d_2Y^2 + e_2YW + f_2W^2$ for some coefficients $b_2, \dots, f_2$.

Next we can change coordinates in the plane $\{Z = 0\}$ without affecting p_1, p_2, or Q_1. So, choose any two points in $Q_1 \cap Q_2$ that do not span a line through p_1: we can move these to $[0, 1, 0, 0]$ and $[0, 0, 0, 1]$. In these coordinates $d_2 = f_2 = 0$, so we have $Q_2 = b_2XY + c_2XW + e_2YW$. Now Q_2 is an irreducible reduced cone with vertex not lying on $\{Z = 0\}$, so its intersection with this plane must be a smooth conic. So none of b_2, c_2, e_2 can be zero. Dividing by b_2, we can write $Q_2 = XY + c_2XW + e_2YW$. Changing coordinates $W \mapsto c_2W$, we get $Q_2 = XY + XW + e_2YW$. Finally, changing coordinates $Y \mapsto e_2^{-1}Y$ and $W \mapsto e_2^{-1}W$, we get $Q_2 = e_2^{-1}(XY + XW + YW)$. (None of these coordinate changes affect p_1, p_2, Q_1, or the two points fixed previously.) So we have $Q_2 = XY + XW + YW$, as claimed.

Finally we must deal with Q_3. First suppose it is a general quadric in the net: it has the form $Q_3 = b_3XY + c_3XZ + d_3XW + e_3Y^2 + f_3YZ + g_3YW + h_3Z^2 + i_3ZW + j_3W^2$. We know that the plane curves $Q_2 \cap \{Z = 0\}$ and $Q_3 \cap \{Z = 0\}$ must have an intersection point of multiplicity 4 at p_1. This means the following.

Suppose we restrict to the affine chart $\{X = 1\}$ inside $\{Z = 0\}$. Then, on Q_2, we can express Y (say) as a power series in W: $Y = p(W)$. Now substituting $Y = p(W)$ into the equation for Q_3, we get a power series $q_3(W)$, and the condition that p_1 has multiplicity 4 means that q_3 vanishes to order 4 at $W = 0$. Since we already know that Q_3 vanishes at p_1, this gives three additional equations in the coefficients of Q_3—namely, $d_3 = b_3$, $g_3 = b_3 + 2e_3$, and $j_3 = e_3$. Applying these conditions and replacing Q_3 by $Q_3 - b_3Q_2 - h_3Q_1$, we can assume $Q_3 = e_3W^2 + 2e_3WY + e_3Y^2 + i_3WZ + c_3XZ + f_3YZ$, which simplifies to $Q_3 = e_3(Y + W)^2 + Z(c_3X + f_3Y + i_3W)$. From this we see that e_3 cannot be zero, for then Q_3 would be reducible. So dividing across, we can assume $e_3 = 1$. Moreover, we see that the differential dQ_3 at p_1 is just $c\,dZ$, so by Assumption 1 we have $c \neq 0$. Hence, changing coordinates $Z \mapsto cZ$, we can assume $c = 1$. So we get $Q_3 = (Y + W)^2 + Z(X + f_3Y + i_3W)$. Finally, for a given choice of coefficients f_3, i_3, it is straightforward to find a projective transformation that takes the net spanned by Q_1, Q_2, and this Q_3 to the net spanned by the standard quadrics described previously.

2. $\{8\}_2$: The next case is a net of type $\{8\}$ that does not contain a double plane. The unique quadric Q_1 in the net that is singular at p_1 is then a reduced irreducible cone. Again putting $p_1 = [1, 0, 0, 0]$, we see that Q_1 is the cone over a smooth conic in $\{X = 0\} \cong \mathbf{P}^2$. Standard arguments about smooth quadrics show that we can change coordinates so that $Q_1 = YZ + W^2$.

Since p_1 is a multiple basepoint, there must be some tangent line $L \subset T_{p_1}\mathbf{P}^3$ shared by all quadrics in the net. Let p_2 be the unique point of $Q_1 \cap \{X = 0\}$ such that $\overline{p_1p_2}$ has tangent direction L at p_1. We can apply a projective transformation in $\mathrm{PGL}(3) \subset \mathrm{PGL}(4)$ to bring p_2 to the point $[0, 1, 0, 0]$.

Now, by Assumption 1, for any choice of generators Q_2, Q_3 of the net, the differentials dQ_2 and dQ_3 are linearly independent at p_1. So given a plane $P \subset T_{p_1}\mathbf{P}^3$ containing the line L, we can find a quadric Q in the net with P as its tangent plane at p_1. In particular we can choose Q so that its embedded tangent plane at p_1 intersects Q_1 in a double line. Moreover, this property is unchanged if we replace Q by $Q + \lambda Q_1$ (for any $\lambda \in k$). So without loss of generality, we can assume that Q_2 has embedded tangent plane intersecting Q_1 in a double line $2L$ and that the coefficient of W^2 in Q_2 is zero. This means that Q_2 is given by a form $Q_2 = XZ + b_2Y^2 + c_2YZ + d_2YW + e_2Z^2 + f_2ZW$. (We know the coefficient of XZ is nonzero, since Q_2 is smooth at p_1, so we can divide across by that coefficient.)

Now consider Q_3. We know that it passes through p_1 and that its tangent space at p_1 contains the line L. This implies that the coefficients of the monomials X^2 and XY in Q_3 vanish. Also, subtracting appropriate multiples of Q_1 and Q_2, we can assume that the coefficients of W^2 and XZ in Q_3 also vanish. Finally, since Q_3 is smooth at p_1, the coefficient of XW must be nonzero, so we can assume it is 1. Putting these facts together, we obtain $Q_3 = XW + e_3Y^2 + f_3YZ + g_3YW + h_3Z^2 + i_3ZW + j_3W^2$.

We can now use the power-series method explained in the previous case to obtain equations in the coefficients of Q_2 and Q_3. These are as follows: $b_2 = 0, b_3 = 0$, $e_3 = -d_2, g_3 = c_2 + c_3d_2 - g_2, i_3 = e_2 + c_3f_2$, and $j_3 = f_2 + f_3 + c_3(-c_2 + g_2)$.

Things seem pretty bleak, but actually our standard form is close at hand. Let us return to $Q_2 = XZ + b_2Y^2 + c_2YZ + d_2YW + e_2Z^2 + f_2ZW$. We have $b_2 = 0$ (since p_2 lies on q_2) and so—applying projective transformations that fix p_1, p_2, and Q_1—we can put Q_2 in the form $Q_2 = XZ + YW$. We can then substitute $d_2 = 1$ and $c_2 = e_2 = f_2 = 0$ into the previous equations; the result is that we solve for e_3, f_3, g_3, i_3, and j_3 in terms of c_3. Explicitly, we get $Q_3 = WX - Y^2 + c_3(WY + XZ) + h_3Z^2$. Finally, applying projective transformations that fix p_1, p_2, and Q_1 and that map Q_2 to some quadric in the pencil $\langle Q_1, Q_2\rangle$, we can put Q_3 in the form $Q_3 = XW - Y^2$ or $Q_3 = XW - Y^2 + Z^2$, according as $h_3 = 0$ or not. One can compute that if $Q_3 = XW - Y^2$ then the base locus of the net is not 0-dimensional, so the standard form is as claimed.

3. $\{6, 2\}$: In this case there are two basepoints of the net, so without loss of generality we can assume these are $p_1 = [1, 0, 0, 0]$ and $p_7 = [0, 1, 0, 0]$. There is a unique rank-2 quadric in the net that we know is singular at p_1 and smooth at p_7. Hence its equation is $Q_1 = (a_1Y + b_1Z + c_1W)(d_1Z + e_1W)$, where the linear forms $a_1Y + b_1Z + c_1W$ and $d_1Z + e_1W$ are linearly independent. We can apply projective transformations fixing p_1 and p_7 to make this $Q_1 = YZ$.

Now p_7 is a multiple basepoint of the net, so there must be some quadric in the net that is singular there. It cannot be a double plane, since this would also be singular at p_1 and, by Assumption 1, only one quadric in our net may be singular at a given basepoint. So we can take Q_2 to be an irreducible reduced cone with vertex p_7. Such a cone is given by a form with no monomials involving Y: we can write it as $Q_2 = a_2XZ + b_2XW + e_2Z^2 + f_2ZW + g_2W^2$. Note that a_2 and b_2 cannot both be zero, since Q_2 cannot be singular at p_1.

Next, applying projective transformations fixing p_1, p_7, and Q_1, we can put Q_2 in the form $Q_{2a} = XW + Z^2$ or $Q_{2b} = XZ + W^2$ according to whether $b_2 \neq 0$ or $b_2 = 0$. (Note that no projective transformation fixing p_1, p_7, and Q_1 takes the pencil $\langle Q_1, Q_{2a}\rangle$ to the pencil $\langle Q_1, Q_{2b}\rangle$; any such transformation would have to take Q_{2a} to Q_{2b}, since they are the only quadrics in each pencil that are singular at p_7, but it is easy to show that no such transformation fixing p_1, p_7, and Q_1 exists.)

What of Q_3? We know it is a quadric containing p_1 and p_7, so the coefficients of X^2 and Y^2 in Q_3 must be zero. Moreover, we can subtract a multiple of Q_1 to make the coefficient of YZ equal to zero, too. If $Q_2 = Q_{2a}$, we can subtract a multiple of Q_2 to make the coefficient of Z^2 equal zero; if $Q_2 = Q_{2b}$, we can arrange that the coefficient of W^2 be zero. So we get two possibilities: $Q_{3a} = b_3XY + c_3XZ + d_3XW + g_3YW + i_3ZW + j_3W^2$ or $Q_{3b} = b_3XY + c_3XZ + d_3XW + g_3YW + h_3Z^2 + i_3ZW$.

Our combinatorial classification showed that the curves $Q_2 \cap \{Y = 0\}$ and $Q_3 \cap \{Y = 0\}$ must have an intersection point of order 4 at p_1. As before, we can translate this condition into constraints on the coefficients of Q_3. In both cases, we get the conditions $c_3 = d_3 = i_3 = 0$.

If $Q_2 = Q_{2a}$ and $Q_3 = Q_{3a}$, we have $Q_{3a} = b_3XY + g_3YW + j_3W^2$. We can then apply projective transformations fixing p_1, p_7, Q_1, and Q_2 to put it in the form $Q_{3a} = XY + W^2$. But now we note the following: the intersection $Q_{2a} \cap \{Z = 0\}$

is a reducible conic XW, and $Q_{3a} \cap \{Z = 0\}$ is a smooth conic whose tangent line at p_7 is $\{X = 0\}$. So these two curves have intersection multiplicity 3 at p_7, meaning that this net is actually of type $\{5, 3\}$.

It remains to consider the case where $Q_2 = Q_{2b}$ and $Q_3 = Q_{3b}$. In this case we get $Q_{3b} = b_3XY + g_3YW + h_3Z^2$, and admissible projective transformations put this in one of two forms: $Q_{3b} = XY + Z^2$ (if $g_3 = 0$) or $Q'_{3b} = XY + YW + Z^2$ (if $g_3 \neq 0$). But in fact the resulting nets are projectively equivalent: the projective transformation $\phi \in \mathrm{PGL}(4, k)$ with matrix

$$\phi = \begin{pmatrix} 1 & 0 & -\frac{1}{4} & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

fixes p_1, p_7, Q_1, and Q_2, and one can write $Q'_{3b} = \frac{1}{4}\phi(Q_1) + \phi(Q_{3b})$. So ϕ maps the net $\langle Q_1, Q_2, Q_{3b}\rangle$ to the net $\langle Q_1, Q_2, Q'_{3b}\rangle$. Hence all extremal nets of type $\{6, 2\}$ have the standard form claimed.

4. $\{5, 3\}$: The argument in this case goes through exactly as in the previous one. We have two basepoints, which we can choose to be $p_1 = [1, 0, 0, 0]$ and $p_6 = [0, 1, 0, 0]$; there is a unique rank-2 quadric Q_1 in the net, which we can transform to $Q_1 = YZ$; and there is a unique quadric in the net Q_2 that is singular at p_6. Exactly the same argument as before shows that we can put this in the form $Q_2 = XW + Z^2$ and then put Q_3 in the form $XY + W^2$. So this type has the standard form we claimed.

5. $\{4, 4\}_1$: In this case the net has a single rank-2 quadric Q_1 with multiplicity data $1^3 2^1 + 1^1 2^3$. We can apply projective transformations to put the basepoints at $p_1 = [1, 0, 0, 0]$ and $p_5 = [0, 1, 0, 0]$ and to put Q_1 in the form $Q_1 = ZW$. Moreover, without loss of generality, the plane $\{Z = 0\}$ has the correct tangent direction at p_1 and $\{W = 0\}$ has the correct tangent direction at p_5.

Now take two other quadrics Q_2 and Q_3 that, together with Q_1, span the net. We can write down quadratic forms defining these quadrics:

$$Q_2 = a_2XY + b_2XZ + c_2XW + d_2YZ + e_2YW + f_2Z^2 + g_2ZW + h_2W^2,$$

$$Q_3 = a_3XY + b_3XZ + c_3XW + d_3YZ + e_3YW + f_3Z^2 + g_3ZW + h_3W^2.$$

Since Q_1 is singular at both basepoints, the differentials dQ_2 and dQ_3 must be linearly independent at the basepoints by Assumption 1. In affine coordinates $\{X = 1\}$ near p_1, their tangent spaces are $T_{p_1}Q_2 = \{a_2Y + b_2Z + c_2W = 0\}$ and $T_{p_1}Q_3 = \{a_3Y + b_3Z + c_3W = 0\}$. When restricted to the plane $\{Z = 0\}$ these tangent spaces must coincide, which means that $a_2Y + c_2W$ and $a_3Y + c_3W$ are linearly dependent. On the other hand, when restricted to the plane $W = 0$ the tangent spaces are transverse, implying that $a_2Y + b_2Z$ and $a_3Y + b_3Z$ are linearly independent. The analogous argument near p_5 states that $a_2X + d_2Z$ and $a_3X + d_3Z$ are linearly dependent whereas $a_2X + e_2W$ and $a_3X + e_3W$ are linearly independent. In particular we see that neither a_2 nor a_3 can be zero, so without loss of generality we can divide through the two forms Q_2 and Q_3 to get $a_2 = a_3 = 1$. Then linear dependence implies $c_2 = c_3$ and $d_2 = d_3$. Now scaling Z and W (which does not affect Q_1), we can assume that $c_2 = c_3 = 1$ and $d_2 = d_3 = 1$.

Next consider the intersection $Q_2 \cap Q_3 \cap \{Z = 0\}$; this should consist of p_1 with multiplicity 3 and p_5 with multiplicity 1. Setting $Z = 0$ and $X = 1$ in the forms defining Q_2 and Q_3 and then setting $a_2 = a_3 = c_2 = c_3 = d_2 = d_3 = 1$ as explained before, we get the forms

$$q_2 = Y + W + e_2 YW + h_2 W^2,$$

$$q_3 = Y + W + e_3 YW + h_3 W^2.$$

Setting $q_3 = 0$, again we can solve for W as a power series in Y. Up to terms of order 4, this is $W = Y(-1 + (e_3 - h_3)Y)$. Substituting this into q_2 yields $q_2(Y) = Y^2(-e_2 + e_3 + h_2 - h_3)$, and this vanishes to order 3 at $Y = 0$ if and only if $e_2 - e_3 = h_2 - h_3$.

Consider similarly the intersection $Q_2 \cap Q_3 \cap \{W = 0\}$; this should consist of a simple point at p_1 and a triple point at p_5. Setting $W = 0$ and $Y = 1$ in the forms defining Q_2 and Q_3, we get

$$q_2 = X + Z + b_2 XZ + f_2 Z^2,$$

$$q_3 = X + Z + b_3 XZ + f_3 Z^2.$$

By exactly the same reasoning as before, we get the equation $b_2 - b_3 = f_2 - f_3$. So putting all these facts together and then subtracting multiples of Q_1 from Q_2 and Q_3 to eliminate the monomials ZW in each, we can write our quadrics as

$$\begin{aligned} Q_2 &= XY + b_2 XZ + XW + YZ + e_2 YW + f_2 Z^2 + h_2 W^2, \\ Q_3 &= XY + b_3 XZ + XW + YZ + e_3 YW + (f_2 - b_2 + b_3) Z^2 \\ &\quad + (h_2 - e_2 + e_3) W^2. \end{aligned}$$

But now

$$\begin{aligned} Q := Q_2 - Q_3 &= (b_2 - b_3) XZ + (e_2 - e_3) YW + (b_2 - b_3) Z^2 + (e_2 - e_3) W^2 \\ &= (b_2 - b_3) Z(X + Z) + (e_2 - e_3) W(Y + W). \end{aligned}$$

Neither of the coefficients can be zero, since Q is not reducible. Scaling X and Z together and Y and W together, we can put Q in the form $Q = Z(X + Z) + W(Y + W)$; after changing coordinates $X \mapsto X + Z$ and $Y \mapsto Y + W$, this becomes $Q = XZ + YW$. (Note that none of these changes affect p_1, p_5, or Q_1.) We will take Q to be the second generator of our net, so we rename it Q_2; that is, we define $Q_2 := XZ + YW$. Observe that the intersection $Q_1 \cap Q_2$ is a union of lines with total degree 4: the line $\{Y = Z = 0\}$, the line $\{X = W = 0\}$, and the line $\{Z = W = 0\}$ with multiplicity 2. Any other quadric Q that, together with Q_1 and Q_2, spans the net must pass through p_1 and p_5 and hence must intersect $\{Z = W = 0\}$ transversely at those two points. So the correct tangent direction at p_1 is the line $\{Y = Z = 0\}$ and that at p_5 is the line $\{X = W = 0\}$.

Now suppose Q_3 is any other quadric that, together with Q_1 and Q_2, spans the net. Since it passes through p_1 and p_5, it has the form

$$Q_3 = a_3 XY + b_3 XZ + c_3 XW + d_3 YZ + e_3 YW + f_3 Z^2 + g_3 ZW + h_3 W^2.$$

We can subtract arbitrary multiples of Q_1 and Q_2 without affecting anything, so we may assume that the coefficient b_3 of XZ and the coefficient g_3 of ZW both vanish. We know that if we restrict to the plane $Z = 0$, then the tangent line to Q_3 at p_1 should be the line $\{Y = 0\}$. So $a_3 \neq 0$ and $c_3 = 0$. Similarly restricting to $\{W = 0\}$, the tangent line should be $\{X = 0\}$ and so $d_3 = 0$. Thus (dividing across by a_3) we get $Q_3 = XY + e_3YW + f_3Z^2 + h_3W^2$. Neither of the coefficients f_3 and h_3 can be zero: if $f_3 = 0$, then Q_1, Q_2, Q_3 all contain the line $\{X = W = 0\}$; if $h_3 = 0$, they all contain $\{Y = Z = 0\}$. By assumption our net has base locus of dimension 0, so this is forbidden. With this restriction it is not difficult to see that, for any values of the coefficients e_3, f_3, h_3, the net $\langle Q_1, Q_2, Q_3 = XY + e_3YW + f_3Z^2 + h_3W^2\rangle$ is projectively equivalent to the net $\langle Q_1, Q_2, Q_3 = XY + Z^2 + W^2\rangle$, and this gives the standard form claimed.

6. $\{4,4\}_2$: In this case the unique rank-2 quadric is smooth at both basepoints, so we can move the basepoints to $p_1 = [1,0,0,0]$ and $p_5 = [0,1,0,0]$ and then transform the rank-2 quadric to $Q_1 = XY$. In this case the net contains a double plane $Q_2 = L^2$ (where L is a homogeneous linear form). It passes through both $[1,0,0,0]$ and $[0,1,0,0]$, so the coefficients of both X and Y in L must vanish; hence, by changing coordinates $Z \mapsto L(Z,W)$ (and $W \mapsto Z$ if $L = W$), we can assume that $Q_2 = Z^2$.

Now consider the third generator of the net, which by Assumption 1 must be a quadric smooth at both basepoints. We can write it as $Q_3 = b_3XY + c_3XZ + d_3XW + f_3YZ + g_3YW + h_3Z^2 + i_3ZW + j_3W^2$ (the coefficients of X^2 and Y^2 are zero because Q_3 passes through p_1 and p_5). Moreover, $Q_1 \cap Q_2 \cap Q_3$ should have two points of multiplicity 4 at p_1 and p_5: this implies that the double lines $Q_2 \cap \{Y = 0\}$ and $Q_2 \cap \{X = 0\}$ should be tangent to the curves $Q_3 \cap \{Y = 0\}$ and $Q_3 \cap \{X = 0\}$ at p_1 and p_5, respectively. In suitable affine coordinates near these points, the tangent lines to Q_3 inside these planes are defined by $c_3z + d_3w$ and $f_3z + g_3w$, respectively, so we get $d_3 = g_3 = 0$ (and c_3, f_3 nonzero). Finally, upon replacing Q_3 by $Q_3 - b_3Q_1 - h_3Q_2$, we get $Q_3 = c_3XZ + f_3YZ + i_3ZW + j_3W^2$.

Note that of the four coefficients of Q_3, only i_3 can be zero: if c_3 were, Q_3 would be a cone with vertex p_1; if f_3 were, it would be a cone with vertex p_5; if j_3 were, Q_3 would be divisible by Z, which would make it a second rank-2 quadric in the net. If $i_3 = 0$, we get $Q_3 = c_3XZ + f_3YZ + j_3W^2$; changing variables $X \mapsto c_3X$, $Y \mapsto f_3Y$, and $W \mapsto \sqrt{j_3}W$, we get $Q_3 = XZ + YZ + W^2$ (without changing p_1, p_5, Q_1, or Q_2). If i_3 is nonzero, then we can do a similar rescaling of variables to make $c_3 = f_3 = i_3 = j_3 = 1$ and $Q_3 = XZ + YZ + ZW + W^2$. But then replacing Q_3 by $Q_3 + Q_2/4$ yields $Q_3 = XZ + YZ + W^2 + WZ + (Z/2)^2$, and finally changing variables $W \mapsto W + Z/2$ we get $Q_3 = XZ + YZ + W^2$ again. So a net of this type containing a double plane can always be put in this standard form, as claimed.

7. $\{4,4\}_3$: Again the unique rank-2 quadric in the net is smooth at both basepoints. We can move the basepoints by projective transformations to $p_1 = [1,0,0,0]$ and $p_5 = [0,1,0,0]$ and then transform the rank-2 quadric to $Q_1 = XY$.

The net contains no double plane. Therefore the unique quadrics in the net singular at the basepoints p_1 and p_5 must be irreducible reduced cones with vertices at p_1 and p_5. Call these Q_2 and Q_3, respectively; then $C_2 := Q_2 \cap \{Y = 0\}$ must be a reducible conic curve (i.e., the union of two lines in the plane $\{Y = 0\}$, which may be equal), and similarly for $C_3 := Q_3 \cap \{X = 0\}$. On the other hand, $\Gamma_3 := Q_3 \cap \{X = 0\}$ and $\Gamma_2 := Q_2 \cap \{Y = 0\}$ are smooth conic curves in those planes, each meeting the reducible conic in the same plane in a single point of multiplicity 4. It follows that C_2 (resp. C_3) is a double line that is tangent at p_1 (resp. p_5) to the smooth conic Γ_3 (resp. Γ_2).

Let us write $Q_2 = a_2YZ + b_2YW + c_2Z^2 + d_2ZW + e_2W^2$ and $Q_3 = a_3XZ + b_3XW + c_3Z^2 + d_3ZW + e_3W^2$. The restriction of Q_2 (resp. Q_3) to $\{Y = 0\}$ (resp. $\{X = 0\}$) is a double line, so we get $d_2 = \pm 2\sqrt{c_2e_2}$ and $d_3 = \pm 2\sqrt{c_3e_3}$. Rewriting, we have $Q_2 = Y(a_2Z + b_2W) + (\gamma_2Z + \varepsilon_2W)^2$ and $Q_3 = X(a_3Z + b_3W) + (\gamma_3Z + \varepsilon_3W)^2$ for some choice of square roots γ_i, ε_i of c_i, e_i $(i = 1, 2)$. If the forms $\gamma_2Z + \varepsilon_2W$ and $\gamma_3Z + \varepsilon_3W$ were linearly dependent, then Q_2 and Q_3 would have an intersection point on the line $\{X = Y = 0\} \subset Q_1$, which is impossible since the net has only two basepoints. Therefore they must be linearly independent, so we can change variables in Z and W to make $Q_2 = Y(a_2Z + b_2W) + Z^2$ and $Q_3 = X(a_3Z + b_3W) + W^2$. Now $Q_2 \cap \{X = 0\}$ should be tangent to the double line $Q_3 \cap \{X = 0\} = W^2$, so we get $a_2 = 0$; an identical argument gives $b_3 = 0$. Rescaling via $Y \mapsto b_2Y$ and $X \mapsto a_3X$, we get $Q_2 = YW + Z^2$ and $Q_3 = XZ + W^2$. Finally, we can swap Q_2 and Q_3, and our net has the standard form we claimed.

8. $\{4, 2, 2\}$: In this case we have three distinct basepoints. By Lemma 1.1 these do not lie on a line, so we can move them to $p_1 = [1, 0, 0, 0]$, $p_5 = [0, 1, 0, 0]$, and $p_7 = [0, 0, 1, 0]$. The combinatorial classification shows that Q_1 can be taken to be a rank-2 quadric $P_1 \cup P_2$, where P_1 is a plane passing through p_1 (but not through p_5 or p_7) and P_2 is a plane passing through p_5 and p_7 but not p_1. So we can write these as $P_1 = b_1Y + c_1Z + d_1W$ and $P_2 = a_2X + d_2W$ with $b_1, c_1, a_2 \neq 0$. After changing coordinates $X \mapsto a_2X + d_2W$, $Y \mapsto b_1Y + d_1W$, and $Z \mapsto c_1Z$ (which does not affect p_1, p_5, or p_7), we obtain $Q_1 = X(Y + Z)$.

Now for Q_2. It is a rank-2 quadric that consists of a plane Π_1 passing through p_1 and p_5 and a plane Π_2 passing through p_1 and p_7. So we have $\Pi_1 = c_1Z + d_1W$ and $\Pi_2 = b_2Y + d_2W$ with c_1 and b_2 nonzero; dividing out, we can assume these coefficients both equal 1. Each of these two planes should contain the tangent line at p_1 that is the first basepoint infinitely near to p_1; hence, in terms of embedded tangent spaces, that tangent line is the intersection $\Pi_1 \cap \Pi_2$. Moreover, we know that the plane P_1 defined previously must also contain that tangent line. This means that the lines $P_1 \cap \Pi_1 = \{-Y + d_1W = 0\}$ and $P_1 \cap \Pi_2 = \{Y + d_2W = 0\}$ are equal; hence $d_2 = -d_1$. Now applying the transformations $Y \mapsto Y - d_1W$ and $Z \mapsto Z + d_1W$, we get $Q_2 = YZ$ with p_1, p_5, p_7, and Q_1 unchanged.

Finally we must deal with Q_3. We know it passes through p_1, p_5, and p_7, so the coefficients of X^2, Y^2, and Z^2 must be zero. So write $Q_3 = a_3XY + b_3XZ + c_3XW + d_3YZ + e_3YW + f_3ZW + g_3W^2$. Moreover, we know the tangent direction that Q_3 must have at the three basepoints. At p_1, the correct tangent line is that shared by Π_1 and Π_2 from the preceding paragraph—namely, $\{Y = Z = 0\}$. Setting $X = 1$ in the equation of Q_3, we get $a_3Y + b_3Z + c_3W +$ (quadratic terms). So we get the condition $c_3 = 0$. Now consider p_5: the correct tangent direction there is that shared by the planes P_1 and Π_1, and that is $\{X = Z = 0\}$. Setting $Y = 1$ in the equation of Q_3, we get $a_3X + d_3Z + e_3W +$ (quadratic terms), so the condition we get is $e_3 = 0$. Finally looking at p_7, the correct tangent direction is that shared by P_1 and Π_2, and the same argument gives the condition $f_3 = 0$. So these three conditions give us $Q_3 = a_3XY + b_3XZ + d_3YZ + g_3W^2$. But now by replacing Q_3 by $Q_3 - d_3Q_2 - a_3Q_1$ we can eliminate the monomials YZ and XY, giving $Q_3 = b_3XZ + g_3W^2$. Neither coefficient can be zero: if b_3 were zero, then Q_3 would be a double plane and hence singular at p_1, but this would violate Assumption 1 because Q_1 is singular there; if g_3 were zero, then Q_3 would be a third rank-2 quadric in the net. So both are nonzero; dividing across by b_3 and scaling W (which does not affect the basepoints or Q_1, Q_2) we get $Q_3 = XZ + W^2$, as claimed.

9. $\{3,3,2\}_1$: Again we can put the three basepoints at $p_1 = [1,0,0,0]$, $p_4 = [0,1,0,0]$, and $p_7 = [0,0,1,0]$. In this case, the rank-2 quadrics in the net have multiplicity data $Q_1 = 1^3 3^1 + 2^3 3^1$ and $Q_2 = 1^2 2^2 + 1^1 2^1 3^2$. So they have equations $Q_1 = (b_1Y + d_1W)(a_2X + d_2W)$ and $Q_2 = (\gamma_1 Z + \delta_1 W)W$. None of the coefficients b_1, a_2, γ_1 can be zero, for otherwise the corresponding planes would pass through more basepoints than specified by the combinatorial classification. So by changing coordinates ($X \mapsto a_2X + d_2W$, $Y \mapsto b_1Y + d_1W$, and $Z' = \gamma_1 Z + \delta_1 W$) we obtain $Q_1 = XY$ and $Q_2 = ZW$.

Now consider Q_3, any quadric in the net that forms a basis together with Q_1 and Q_2. Such a Q_3 must pass through p_1, p_4, and p_7. Moreover, Q_1 is singular at one $\mathbf{P}^3$-basepoint and Q_2 is singular at the other two, so Q_3 is smooth at the base locus and has the correct tangent direction at each. But Q_1 and Q_2 define the correct tangent direction at p_1 and p_4. Applying these conditions to the quadratic form defining Q_3, we see that the coefficients of the monomials X^2, Y^2, Z^2, XW, and YW must all be zero. So we can write $Q_3 = a_3XY + b_3XZ + c_3YZ + d_3ZW + e_3W^2$.

These facts concerning the smoothness of Q_3 at the base locus (and its tangent directions there) hold for any quadric in the net outside the pencil spanned by Q_1 and Q_2. In particular they remain true if we replace Q_3 by $Q_3 - a_3Q_1 - d_3Q_2$. So without loss of generality we obtain $Q_3 = b_3XZ + c_3YZ + e_3W^2$. Now we see that e_3 must be nonzero, for otherwise Q_3 would be reducible. Also, in affine coordinates near p_1 and p_4, the tangent spaces to Q_3 are given by $b_3z = 0$ and $c_3z = 0$, respectively. Smoothness at these points tells us that b_3 and c_3 are nonzero. So all three coefficients are nonzero; scaling the coordinates gives $Q_3 = XZ + YZ + W^2$, as claimed.

10. $\{3,3,2\}_2$: The combinatorial classification tells us in this case that one of the rank-2 quadrics in the net (let us call it Q_1) is the union of a plane P_1 passing through p_1 and p_4 and a plane P_2 passing through p_1 and p_7. These are given by forms $P_1 = c_1Z + d_1W$ and $P_2 = b_2Y + d_2W$; exactly as in the previous case, we can transform these to $P_1 = Z$ and $P_2 = Y$. So $Q_1 = YZ$. Similarly. the other rank-2 quadric in the net (call it Q_2) is the union of a plane Π_1 through p_1 and p_4 and a plane Π_2 through p_4 and p_7; by exactly the same argument, we can put this in the form $Q_2 = X(Z + W)$.

What of Q_3? As in the previous case, we know that the coefficients of the monomials X^2, Y^2, and Z^2 in Q_3 must be zero. Also, just as before, we can compute the shared tangent directions of components of Q_1 and Q_2 at the basepoints: this tells us that the coefficients of YW and ZW in Q_3 are zero and that those of XZ and XW must be equal. So we get $Q_3 = a_3XY + b_3(XZ + XW) + c_3YZ + d_3W^2$. But now replacing Q_3 by $Q_3 - b_3Q_2 - c_3Q_1$, we get $Q_3 = a_3XY + d_3W^2$. Just as in the previous case, neither coefficient can be zero, so we can rescale via $X \mapsto a_3X$ and $(W, Z) \mapsto \sqrt{d_3}(W, Z)$ (without moving the basepoints or Q_1, Q_2) to get $Q_3 = XY + W^2$, as claimed.

11. $\{2, 2, 2, 2\}$: First note that the four $\mathbf{P}^3$-basepoints of the net cannot be coplanar. The proper transform of such a plane would have class h_{1357}, but the combinatorial classification shows there is an effective class h_{1257}; the corresponding planes in $\mathbf{P}^3$ must then be equal, which means that in fact the class $h-e_1-e_2-e_3-e_5-e_7$ would be effective, which is impossible.

We know also that no three of the $\mathbf{P}^3$-basepoints are collinear. So we can move them to the coordinate points of $\mathbf{P}^3$: $p_1 = [1, 0, 0, 0]$, $p_3 = [0, 1, 0, 0]$, $p_5 = [0, 0, 1, 0]$, $p_7 = [0, 0, 0, 1]$. The combinatorial classification shows that the multiplicity data of the rank-2 quadrics in the net are as follows: $Q_1 = 1^1 2^1 3^2 + 1^1 2^1 4^2$, $Q_2 = 1^2 3^1 4^1 + 2^2 3^1 4^1$, $Q_3 = 1^2 2^2 + 3^2 4^2$. But then the components of Q_1 and Q_2 are determined: we have $Q_1 = ZW$ and $Q_2 = XY$. We also get $Q_3 = (aX + bY)(cZ + dW)$ with a, b, c, d all nonzero. But then we can scale the coordinates (without changing the p_i or Q_1, Q_2) to get $Q_3 = (X + Y)(Z + W)$, as claimed.

12. $\{1, 1, 1, 1, 1, 1, 1, 1\}$: The combinatorial classification from Section 4 showed that the four points $\{p_1, p_2, p_3, p_5\}$ do not lie in a plane in $\mathbf{P}^3$, so we can move them to the coordinate points: $p_1 = [1, 0, 0, 0]$, $p_2 = [0, 1, 0, 0]$, $p_3 = [0, 0, 1, 0]$, $p_5 = [0, 0, 0, 1]$. We know that p_4 (resp. p_6, p_7) lies in the plane spanned by $\{p_1, p_2, p_3\}$ (resp. $\{p_1, p_2, p_5\}$, $\{p_1, p_3, p_5\}$); thus we have that $p_4 = [x_4, y_4, z_4, 0]$, $p_6 = [x_6, y_6, 0, w_6]$, and $p_7 = [x_7, 0, z_7, w_7]$ and that the coordinates x_i, y_j, z_k, w_l are all nonzero (since otherwise we would have three collinear basepoints, which is forbidden). Normalizing, we can write $p_4 = [1, y_4, z_4, 0]$, $p_6 = [1, y_6, 0, w_6]$, and $p_7 = [1, 0, z_7, w_7]$.

What of p_8? We know it does not belong to any of the planes $\{Y = 0\}$, $\{Z = 0\}$, or $\{W = 0\}$, since each of these already contains four basepoints. So it has coordinates $p_4 = [x_8, y_8, z_8, w_8]$ with $y_8z_8w_8 \neq 0$. On the other hand, we know that p_8 lies in the plane spanned by $\{p_2, p_3, p_5\}$, so it must have $x_8 = 0$. Applying the projective transformation $[X, Y, Z, W] \mapsto [X, Y/y_8, Z/z_8, W/w_8]$ to $\mathbf{P}^3$, we bring p_8 to $[0, 1, 1, 1]$ without moving p_1, p_2, p_3, p_5 or changing the form of p_4, p_6, p_7.

We know from the combinatorial classification that the points $\{p_1, p_4, p_5, p_8\}$ are coplanar. This is equivalent to the determinant of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & y_4 & z_4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

(whose rows are the homogeneous coordinates of the four points) *vanishing,* which occurs if and only if $y_4 = z_4$. Similar arguments show we must have $y_6 = w_6$ and $z_7 = w_7$.

Next, we use the fact that the points $\{p_1, p_4, p_6, p_7\}$ are coplanar. That means the determinant of the corresponding matrix must vanish: this determinant is $-2y_4y_6z_7$, and we know y_4, y_6, z_7 are all nonzero. This shows that an extremal net of this type can exist only if the characteristic of the base field is 2.

To find the standard form in the case of characteristic 2, we now use that the points $\{p_5, p_6, p_7, p_8\}$ are coplanar. Again we use vanishing of the determinant of the corresponding matrix: this determinant is $y_6 + z_7$, so we get $y_6 = z_7$. A similar argument shows that $y_4 = y_6$. So our points have coordinates $p_4 = [1, \xi, \xi, 0]$, $p_6 = [1, \xi, 0, \xi]$, and $p_7 = [1, 0, \xi, \xi]$ for some nonzero $\xi \in k$. Applying the projective transformation $[X, Y, Z, W] \mapsto [X, Y/\xi, Z/\xi, W/\xi]$, the points p_4, p_6, p_7 are transformed to $p_4 = [1,1,1,0]$, $p_6 = [1,1,0,1]$, $p_7 = [1,0,1,1]$ while the other five points are left fixed.

Finally, consider the equations of the planes containing four of the basepoints. The plane containing $\{p_1, p_2, p_3, p_4\}$ has equation $W = 0$ and the plane containing $\{p_5, p_6, p_7, p_8\}$ has equation $X+Y+Z = 0$. This gives a rank-2 quadric $Q_1 = (X+Y+Z)W = 0$ in the net. The plane containing $\{p_1, p_2, p_5, p_6\}$ has equation $Z = 0$ and the plane containing $\{p_3, p_4, p_7, p_8\}$ has equation $X + Y + W = 0$, giving a rank-2 quadric $Q_2 = (X + Y + W)Z = 0$ in the net. The plane containing $\{p_1, p_3, p_5, p_7\}$ has equation $Y = 0$ and the plane containing $\{p_2, p_4, p_6, p_8\}$ has equation $X + Z + W = 0$, giving a rank-2 quadric $Q_3 = (X + Z + W)Y = 0$ in the net. This gives the standard form claimed.

6. Extremal Fibrations and Extremal Quartics

In this section we assume that the characteristic of the ground field k is not 2. (In particular, our remarks do not apply to the extremal net of type $\{1,1,1,1,1,1,1,1\}$.) Suppose we are given a net N of quadrics in $\mathbf{P}^3$ with some fixed basis, say $N = \langle \lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 \rangle$. The *discriminant form* $\Delta_N = \det(\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3)$ defines a quartic curve in the plane $N \cong \mathbf{P}^2$. It seems reasonable to expect that extremality of the net N in the sense used heretofore should correspond to some extremality property of the quartic N.

To explain the correspondence, we first note that there is a natural connection between plane quartic curves and the root system E_7. To an isolated hypersurface singularity one can associate in a natural way a root system (see [1, Chap. 4] for details). For plane quartics, the ranks of the root systems associated to its various singular points sum to at most 7, and in this case the direct sum of the root systems is a rank-7 root subsystem of E_7. So one can hope that, for an extremal net N, the quartic Δ_N is extremal in the sense that the associated root system has rank 7. Indeed, it seems natural to expect in this case that the root system associated to N in Table 1 and that associated to Δ_N should in fact be the same. This is what we verify next.

Table 3 lists, for each type of extremal net N, a defining equation for its discriminant quartic Δ_N and the root system associated to the singularities of Δ_N. (See e.g. [3] for details on how to identify root systems of singularities from equations.) In the table, $\lambda_1, \lambda_2, \lambda_3$ are homogeneous coordinates on the net $N \cong \mathbf{P}^2$.

Table 3

Type	Δ_N	Singularities of Δ_N
$\{8\}_1$	$\lambda_2(4\lambda_1\lambda_2^2+\lambda_2\lambda_3^2+4\lambda_3^3)$	E_7
$\{8\}_2$	$\lambda_2^4+2\lambda_1\lambda_2^2\lambda_3+\lambda_1^2\lambda_3^2+4\lambda_3^4$	A_7
$\{4,4\}_1$	$(\lambda_2^2-\lambda_1\lambda_3+2\lambda_3^2)(\lambda_2^2-\lambda_1\lambda_3-2\lambda_3^2)$	A_7
$\{6,2\}$	$\lambda_2\lambda_3(\lambda_1\lambda_2-\lambda_3^2)$	D_6+A_1
$\{4,4\}_2$	$\lambda_1\lambda_3(\lambda_1\lambda_2-\lambda_3^2)$	D_6+A_1
$\{5,3\}$	$\lambda_2(\lambda_1^2\lambda_2-4\lambda_3^3)$	A_5+A_2
$\{3,3,2\}_1$	$\lambda_1(\lambda_1\lambda_2^2-4\lambda_3^3)$	A_5+A_2
$\{4,2,2\}$	$\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2)$	D_4+3A_1
$\{4,4\}_3$	$\lambda_2\lambda_3(\lambda_1^2-\lambda_2\lambda_3)$	$2A_3+A_1$
$\{3,3,2\}_2$	$\lambda_1\lambda_2(\lambda_1\lambda_2+4\lambda_3^2)$	$2A_3+A_1$
$\{2,2,2,2\}$	$\lambda_1\lambda_2(\lambda_1\lambda_2-4\lambda_3^2)$	$2A_3+A_1$

We observe that in each case the root system associated to Δ_N is the same as that associated to N in Table 1. It would be interesting to find an explanation for this correspondence.

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