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Spectral Characteristics and Stable Ranks for the Sarason Algebra $H^\infty + C$

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0. Introduction

We prove a corona-type theorem with bounds for the Sarason algebra $H^\infty + C$ and determine its spectral characteristics, thus continuing a line of research initiated by N. Nikolski. We also determine the Bass, the dense, and the topological stable ranks of $H^\infty + C$.

To fix our setting, let A be a commutative unital Banach algebra with unit e and let $M(A)$ be its maximal ideal space. The following concept of spectral characteristics was introduced by Nikolski [15]. For $a \in A$, let $\hat{a}$ denote the Gelfand transform of a. We let

$$\delta(a) = \min_{t \in M(A)} |\hat{a}(t)|.$$

Note that $\delta(a) \le \|\hat{a}\|_\infty \le \|a\|_A$. When $a = (a_1, \dots, a_n) \in A^n$ we define

$$\delta_n(a) = \min_{t \in M(A)} |\hat{a}(t)|,$$

where $|\hat{a}(t)| = \sum_{j=1}^n |\hat{a}_j(t)|$ for $t \in M(A)$, and we let

$$\|a\|_{A^n} = \max\{\|a_1\|_A, \dots, \|a_n\|_A\}.$$

Typically, one defines $|\hat{a}(t)| = |\hat{a}(t)|_2 := \big(\sum_{j=1}^n |\hat{a}_j(t)|^2\big)^{1/2}$ and $\|a\|_{A^n} = \|a\|_2 := \big(\sum_{j=1}^n \|a_j\|_A^2\big)^{1/2}$. Our later calculations will be easier, though, with the present definition.

Let δ be a real number satisfying $0 < \delta \le 1$. We are interested in finding, or bounding, the functions

$$c_1(\delta, A) = \sup\{\|a^{-1}\|_A : \|a\|_A \le 1,\ \delta(a) \ge \delta\}$$

and

$$c_n(\delta, A) = \sup\left\{\inf\left\{\|b\|_{A^n} : \sum_{j=1}^n a_j b_j = e\right\},\ \|a\|_{A^n} \le 1,\ \delta_n(a) \ge \delta\right\} \tag{0.1}$$

when A is the Sarason algebra $H^\infty + C$. If a is not invertible, we define $\|a^{-1}\| = \infty$.

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It should be clear that $1 \leq c_n(\delta, A) \leq c_{n+1}(\delta, A)$ and, if $0 < \delta' \leq \delta \leq 1$, then $c_n(\delta, A) \leq c_n(\delta', A)$. This implies the existence of a *critical constant,* denoted here by $\delta_n(A)$, such that

$$c_n(\delta, A) = \infty \text{ for } 0 < \delta < \delta_n(A) \quad \text{and} \quad c_n(\delta, A) < \infty \text{ for } \delta_n(A) < \delta \leq 1.$$

It is clear that if A is a uniform algebra, then $\delta_1(A) = 0$ and $c_1(\delta, A) = 1/\delta$. It is not known for which uniform algebras $\delta_n(A) > 0$.

If $A = H^\infty$, the algebra of bounded holomorphic functions in the unit disk $\mathbb{D}$, then the famous Carleson corona theorem tells us that $\delta_n(H^\infty) = 0$ for each n. Estimates for $c_n(\delta, H^\infty)$ were given by (among others) Nikolski [14], Rosenblum [17], and Tolokonnikov [25]. The best-known estimate today seems to appear in [27] and [28]; see also [29]. Here, for the lower bound, δ is close to 0 and κ is a universal constant:

$$\kappa\delta^{-2} \log\log\left(\frac{1}{\delta}\right) \leq c_n^{(2)}(\delta, H^\infty) \leq \frac{1}{\delta} + \frac{17}{\delta^2}\log\frac{1}{\delta},$$

where $c_n^{(2)}(\delta, A)$ denotes the spectral characteristic $c_n(\delta, A)$ described previously whenever defined with the Euclidean norms $|\cdot|_2$ and $\|\cdot\|_2$.

The structure of the paper is as follows. In Section 1 we consider the problem of solving Bezout equations in the Sarason algebra $H^\infty + C$. Indeed, we prove that the corona theorem with bounds holds in $H^\infty + C$ (i.e., $\delta_n(H^\infty + C) = 0$ for each n). We also give explicit estimates of the associated spectral characteristics.

In Section 2 we present different notions of stable ranks that are relevant to the topic of this paper. We also show the relationships between these various notions of stable ranks.

In Section 3 we determine the Bass and topological stable ranks of $H^\infty + C$. These results can be considered as a generalization of the corona theorem for H^∞. In particular, we investigate whether any Bezout equation $af + bg = 1$ in $H^\infty + C$ admits a solution where a itself is invertible. We also show that, on $(H^\infty + C)$-convex sets in $M(H^\infty + C)$, zero-free functions admit $(H^\infty + C)$-invertible approximants. This will be used to determine the dense stable rank of $H^\infty + C$.

1. Spectral Characteristics of the Sarason Algebra $H^\infty + C$

Since we are dealing only with uniform algebras A, we shall identify the elements in A with their Gelfand transform $\hat{f}$.

Let $L^\infty(\mathbb{T})$ denote the algebra of essentially bounded, Lebesgue measurable functions on the unit circle $\mathbb{T}$. Then a *Douglas algebra* is a uniformly closed subalgebra of $L^\infty(\mathbb{T})$ that properly contains the algebra, H^∞, of (boundary values of) bounded analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We refer the reader to the book by Garnett [7] for items and results not explicitly defined here.

The simplest example of a Douglas algebra is the algebra $H^\infty + C$ of sums of (boundary values of) bounded analytic functions and complex-valued continuous functions on $\mathbb{T}$. This algebra is frequently called the Sarason algebra because it was first shown by Sarason that this space is a closed subalgebra of $L^\infty(\mathbb{T})$; see [22]. He showed that (on $\mathbb{T}$) $H^\infty + C$ is the uniform closure of the set of functions $\{f\bar{z}^n : f \in H^\infty, n \in \mathbb{N}\}$. Thus, $H^\infty + C = [H^\infty, \bar{z}]$, the closed algebra generated by H^∞ and the monomials $\bar{z}^n$.

Let $M(H^\infty)$ denote the maximal ideal space of H^∞. It is well known that the spectrum of $M(H^\infty + C)$ can be identified with the corona of H^∞—namely, the set $M(H^\infty) \setminus \mathbb{D}$; see [7, p. 377]. We denote by

$$Z(f) = \{m \in M(H^\infty) : f(m) = 0\}$$

the zero set of a function in H^∞. We will also need the notion of pseudo-hyperbolic distance, $\rho(x,m)$, of two points $m, x \in M(H^\infty)$. Recall that

$$\rho(x,m) = \sup\{|f(m)| : \|f\|_\infty \leq 1,\ f(x) = 0\}$$

and that for $z, w \in \mathbb{D}$ we obtain $\rho(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|$, where we identify the point evaluation functional $f \mapsto f(z)$ with the point z itself.

Also, for a set E in $M(H^\infty)$, we let $\rho(E,x) = \inf\{\rho(e,x) : e \in E\}$ be the pseudo-hyperbolic distance of E to a point $x \in M(H^\infty)$. Similarly, $\rho(E,U) = \inf\{\rho(E,u) : u \in U\} = \inf\{\rho(e,u) : e \in E,\ u \in U\}$. It is well known that for closed sets E the distance functions $\rho(\cdot,\cdot)$ and $\rho(\cdot,E)$ are lower semicontinuous on $M(H^\infty)$; see [8] and [9]. In particular, if $\rho(E,x) > \eta > 0$ then there exists an open set U containing x such that $\rho(E,U) > \eta$.

Finally, for a point $m \in M(H^\infty)$, let $P(m) = \{x \in M(H^\infty) : \rho(x,m) < 1\}$ denote the Gleason part associated with m. For example, $\mathbb{D}$ itself is the Gleason part asociated with the origin. By Hoffman's theory, there exists a map L_m of $\mathbb{D}$ onto $P(m)$ such that $\hat{f} \circ L_m$ is analytic for all $f \in H^\infty$. If (a_β) is any net in $\mathbb{D}$ that converges to m, then L_m is given by $L_m(z) = \lim \frac{a_\beta + z}{1+\overline{a_\beta}z}$, where the limit is taken in the topological product space $M(H^\infty)^{\mathbb{D}}$.

1.1. The Corona Property for $H^\infty + C$

Our first theorem is based on Axler's result that any function $u \in L^\infty$ can be multiplied by a Blaschke product into $H^\infty + C$. See [1]. Using this result, it is possible to prove the following assertion.

THEOREM 1.1. *The corona theorem with bounds holds on $H^\infty + C$.*

Proof.

Step 1. We first consider H^∞-corona data on $M(H^\infty + C)$. The goal is to find $H^\infty + C$ solutions.

Let $f_1, \ldots, f_n \in H^\infty$ satisfy $\|f_j\|_\infty \leq 1$ and $|f_1| + \cdots + |f_n| \geq \delta > 0$ on $M(H^\infty + C)$. In particular, $|f_1| + \cdots + |f_n| \geq \delta > 0$ a.e. on $\mathbb{T}$. Hence

$$1 = \sum_{j=1}^{n} f_j \frac{\overline{f_j}}{\sum_{k=1}^{n} |f_k|^2} \quad \text{a.e. on } \mathbb{T}.$$

Note that the functions $\overline{f_j}/\sum_k |f_k|^2$ belong to $L^\infty(\mathbb{T})$. By the Axler multiplier theorem [1], there exists a Blaschke product B such that for all $j \in \{1, \dots, n\}$

$$\Phi_j := B\frac{\overline{f_j}}{\sum_k |f_k|^2} \in H^\infty + C.$$

So, a.e. on $\mathbb{T}$, we have $B = \sum_j \Phi_j f_j$. Moreover, $\|\Phi_j\|_\infty \le n/\delta^2$.

We know that, by continuity, there exists an annulus $A_r := \{r \le |z| < 1\}$ such that $|f_1| + \cdots + |f_n| \ge \delta/2$ on A_r. Choose a tail B_1 of B such that $|B_1| \ge \delta/2$ on $\{|z| \le r\}$. Let $b_1 = B/B_1$. Note that $\bar{b}_1$, viewed as a function on $\mathbb{T}$, belongs to $H^\infty + C$.

Now consider the ideal $I(f_1, \dots, f_n, B_1)$ in H^∞. We obviously have that $|f_1| + \cdots + |f_n| + |B_1| \ge \delta/2$ on $\mathbb{D}$. Hence, by the H^∞-corona theorem, there exists a constant depending on δ, $C(\delta)$, and functions $x_1, \dots, x_n, t \in H^\infty$ with

$$\|x_1\|_\infty + \cdots + \|x_n\|_\infty + \|t\|_\infty \le C(\delta)$$

such that $1 = \sum_{j=1}^n x_j f_j + tB_1$ on $\mathbb{D}$, and thus almost everywhere on $\mathbb{T}$ we have $1 = \sum_{j=1}^n x_j f_j + tB_1$. Switching again to $H^\infty + C$, we see that a.e. on $\mathbb{T}$

$$B_1 = \bar{b}_1 B = \sum_{j=1}^n (\bar{b}_1\Phi_j) f_j.$$

Hence

$$1 = \sum_{j=1}^n x_j f_j + t\left(\sum_{j=1}^n (\bar{b}_1\Phi_j) f_j\right) = \sum_{j=1}^n f_j(x_j + t\bar{b}_1\Phi_j),$$

where $\sum_{j=1}^n \|x_j + t\bar{b}_1\Phi_j\|_\infty \le C(\delta)(1 + n^2/\delta^2) =: \chi(\delta)$. Since $H^\infty + C$ is an algebra, the functions $\varphi_j := x_j + t\bar{b}_1\Phi_j$ belong to $H^\infty + C$. Thus, we have found a solution with bounds to the Bezout equation $\sum_{j=1}^n \varphi_j f_j = 1$ a.e. on $\mathbb{T}$ or, equivalently, on $M(H^\infty + C)$.

Step 2. We now look at general $H^\infty + C$ corona data.

Let $F_1, \dots, F_n \in H^\infty + C$ satisfy $\|F_j\|_\infty \le 1$ and $\sum_{j=1}^n |F_j| \ge \delta > 0$ on $M(H^\infty + C)$. Let $\chi(\delta)$ be the function described previously. We uniformly approximate F_j by functions of the form $\bar{z}^{N_j} f_j$; say

$$\sum_{j=1}^n \|F_j - \bar{z}^{N_j} f_j\|_\infty \le \varepsilon = \varepsilon(\delta) := \min\{\delta/4, [4\chi(\delta/2)]^{-1}\},$$

where $f_j \in H^\infty$ and $\|f_j\|_\infty \le 1$.

Then $\sum_{j=1}^n |f_j| \ge \delta/2$ on $M(H^\infty + C)$. We apply Step 1 to the functions f_j and find $H^\infty + C$ functions φ_j and a constant $\chi(\delta/2)$ that bounds the sum of the norms of these functions and such that $1 = \sum_{j=1}^n \varphi_j f_j$. Now

$$u := \sum_{j=1}^n z^{N_j}\varphi_j F_j = \sum_{j=1}^n z^{N_j}\varphi_j(F_j - \bar{z}^{N_j} f_j) + \sum_{j=1}^n \varphi_j f_j$$
$$= \gamma + 1,$$

where $\gamma := \sum_{j=1}^n z^{N_j}\varphi_j(F_j - \bar{z}^{N_j} f_j) \in H^\infty + C$ is majorized by $\varepsilon \sum_{j=1}^n \|\varphi_j\|_\infty \le \varepsilon\chi(\delta/2) \le 1/2$. Thus $|u| \ge 1/2$ on $M(H^\infty + C)$; hence u is invertible in $H^\infty + C$ with $\|u^{-1}\|_\infty \le 2$. We conclude that

$$1 = \sum_{j=1}^n (u^{-1} z^{N_j}\varphi_j) F_j,$$

where the coefficients are bounded by $2\chi(\delta/2)$. □

An immediate corollary (at least for the upper bound) is the following.

COROLLARY 1.2. *For every integer n we have* $\delta_n(H^\infty + C) = 0$ *and, for* δ *close to* 0,

$$\kappa\delta^{-2}\log\left(\log\left(\frac{1}{\delta}\right)\right) \le c_n(\delta, H^\infty + C) \le 2\chi\left(\frac{\delta}{2}\right),$$

where $\chi(\delta) = (1 + n^2/\delta^2)C(\delta)$ *with* $C(\delta)$ *the best constant in the* H^∞*-corona problem and where* κ *is an absolute constant.*

Proof. It remains to verify the lower estimate. Here we use a result of Treil [27, p. 484] that tells us that there exist two finite Blaschke products B_1 and B_2 satisfying $|B_1| + |B_2| \ge \delta > 0$ on $\mathbb{D}$ such that for any solution $(g_1, g_2) \in (H^\infty)^2$ of the Bezout equation $g_1 B_1 + g_2 B_2 = 1$ we have $\|g_1\|_\infty \ge \kappa\delta^{-2}\log(\log(1/\delta))$ whenever $\delta > 0$ is close to 0. Now, of course, this does not give us an example in $H^\infty + C$, since $1 = \bar{B}_1 B_1 + 0B_2$ is a solution with coefficients bounded by 1. We proceed to the following modification.

Let m be a thin point in $M(H^\infty + C)$—that is, a point lying in the closure of a thin interpolating sequence, say $(z_n) = (1 - 1/n!)$. Since the associated Blaschke product b satisfies $(1 - |z_n|^2)|b'(z_n)| \to 1$, Schwarz's lemma implies that $(b \circ L_m)(z) = e^{i\theta} z$ for every $z \in \mathbb{D}$. Now consider the functions

$$f_1 = B_1 \circ (e^{-i\theta} b) \quad \text{and} \quad f_2 = B_2 \circ (e^{-i\theta} b).$$

Clearly $|f_1| + |f_2| \ge \delta$ on $\mathbb{D}$ and hence, viewed as functions in $H^\infty + C$, we have $|f_1| + |f_2| \ge \delta$ on $M(H^\infty + C)$. Let $(h_1, h_2) \in (H^\infty + C)^2$ be a solution of $h_1 f_1 + h_2 f_2 = 1$ in $H^\infty + C$. Since $f_j \circ L_m = B_j$, we get that $1 = (h_1 \circ L_m)B_1 + (h_2 \circ L_m)B_2$ in $\mathbb{D}$. Thus, by Treil's result mentioned before,

$$\|h_1\|_\infty \ge \|h_1 \circ L_m\|_\infty \ge \kappa\delta^{-2}\log(\log(1/\delta)).$$

Thus,

$$c_n(\delta, H^\infty + C) \ge c_2(\delta, H^\infty + C) \ge \kappa\delta^{-2}\log(\log(1/\delta)).$$ □

2. Several Notions of Stable Ranks

Let A be a commutative unital ring. An n-tuple $(a_1, \dots, a_n) \in A^n$ is said to be *invertible* (or unimodular) if there is a solution $(x_1, \dots, x_n) \in A^n$ of the Bezout equation $\sum_{j=1}^n a_j x_j = 1$. Of course, this is equivalent to saying that the ideal generated by the a_j is the whole ring. The set of all invertible n-tuples in A is denoted by $U_n(A)$.

An $(n+1)$-tuple $(a_1, \dots, a_n, a_{n+1}) \in U_{n+1}(A)$ is said to be *n-reducible* (or simply reducible) in A if there exists $(x_1, \dots, x_n) \in A^n$ such that $(a_1 + x_1 a_{n+1}, \dots, a_n + x_n a_{n+1})$ is an invertible n-tuple in A^n. It can be shown that if every invertible n-tuple in A is reducible, then every invertible $(n+1)$-tuple is reducible (see e.g. [30]). The smallest integer $n \in \{1, 2, \dots\}$ for which every invertible $(n+1)$-tuple is reducible is called the *Bass stable rank* of A and is denoted by $\operatorname{bsr}(A)$. This notion was introduced in K-theory.

For algebras of continuous or analytic functions, this has been studied for example by Corach and Larotonda [3], Rupp [18; 19; 20; 21], Suárez [23; 24], and Vasershtein [30]. It was shown by Jones, Marshall, and Wolff [10] that the Bass stable rank of the disk algebra $A(\mathbb{D})$ is 1. Later, simpler proofs were given by Corach, Suárez [4], and Rupp [18; 19]. Treil [26] later showed that the Bass stable rank of H^∞ is 1.

A notion related to the Bass stable rank is that of the topological stable rank. Let A be a commutative unital Banach algebra. The smallest integer n for which the set $U_n(A)$ of invertible n-tuples is dense in A^n is called the *topological stable rank* of A, denoted by $\operatorname{tsr}(A)$. This notion was introduced by Rieffel [16] in the study of C^*-algebras.

Finally, we recall two additional notions of stable ranks. Let $\mathscr{B}$ be the class of all commutative unital Banach algebras over a field $\mathbb{K}$. We will always assume that algebra homomorphisms f between members of $\mathscr{B}$ are continuous and satisfy $f(1_A) = 1_B$. Also, if $f\colon A \to B$ is an algebra homomorphism, then $\underline{f}$ will denote the associated map given by $\underline{f}\colon (a_1, \dots, a_n) \mapsto (f(a_1), \dots, f(a_n))$ from A^n to B^n.

By [6, p. 542], the *dense stable rank* $\operatorname{dsr}(A)$ of $A \in \mathscr{B}$ is the smallest integer n such that for every $B \in \mathscr{B}$ and every algebra homomorphism $f\colon A \to B$ with dense image the induced map $U_n(A) \to U_n(B)$ has dense image. If there is no such n, we write $\operatorname{dsr}(A) = \infty$. We note that if for some $n \in \mathbb{N}$, all $B \in \mathscr{B}$ and all homomorphisms $f\colon A \to B$ with dense image the set $\underline{f}(U_n(A))$ is dense in $U_n(B)$, then $\underline{f}(U_{n+1}(A))$ is dense in $U_{n+1}(B)$ (see [6, p. 543]).

The *surjective stable rank* $\operatorname{ssr}(A)$ of $A \in \mathscr{B}$ is the smallest integer n such that for every $B \in \mathscr{B}$ and every surjective algebra homomorphism $f\colon A \to B$ the induced map of $U_n(A) \to U_n(B)$ is surjective, too. Again, if there is no such n, then we write $\operatorname{ssr}(A) = \infty$. Let us point out that the assumption $\underline{f}(U_n(A)) = U_n(B)$ for some n, all $B \in \mathscr{B}$, and all surjective homomorphisms $f\colon A \to B$ implies that $\underline{f}(U_{n+1}(A)) = U_{n+1}(B)$. This works similarly to the proof for the corresponding statement for denseness in [6, p. 543].

In fact, let $b := (b_1, \dots, b_{n+1}) \in U_{n+1}(B)$. Consider for $I = \overline{(b_{n+1})}$ (the closure of the principal ideal generated by b_{n+1}) the quotient algebra $\tilde{B} := B/I$ and the quotient mapping $\pi\colon B \to \tilde{B}$. Then

$$(b_1 + I, \dots, b_n + I) \in U_n(\tilde{B}).$$

By our hypothesis, since πf is surjective, there exists $a := (a_1, \dots, a_n) \in U_n(A)$ such that $\pi f(a_j) = \pi b_j$ for $1 \le j \le n$. Choose $\varepsilon > 0$ so that every perturbation $(a_1 - r_1, \dots, a_n - r_n)$ of a with $\|r_j\|_A < \varepsilon$ is in $U_n(A)$ again. Using the open mapping theorem, let $\eta = \eta(\varepsilon)$ be such that

$$\{y \in B : \|y\|_B < \eta\} \subseteq f(\{x \in A : \|x\|_A < \varepsilon\}).$$

Since $f(a_j) - b_j \in I$, there exist $k_j \in B$ such that $\|f(a_j) - b_j - k_j b_{n+1}\|_B < \eta$ $(j = 1, \dots, n)$. Since f is surjective, we may choose $r_j, x_j \in A$, $\|r_j\|_A < \varepsilon$, and $a_{n+1} \in A$ such that $f(r_j) = y_j := f(a_j) - b_j - k_j b_{n+1}$, $f(x_j) = k_j$ $(j = 1, \dots, n)$, and $f(a_{n+1}) = b_{n+1}$. Then

$$(a_1 - r_1, \dots, a_n - r_n) \in U_n(A)$$

and so

$$(a'_1, \dots, a'_{n+1}) := (a_1 - r_1 - x_1 a_{n+1}, \dots, a_n - r_n - x_n a_{n+1}, a_{n+1}) \in U_{n+1}(A).$$

Moreover, $f(a'_j) = f(a_j) - f(x_j) f(a_{n+1}) - f(r_j) = f(a_j) - k_j b_{n+1} - y_j = b_j$ for $1 \le j \le n$ and $f(a'_{n+1}) = b_{n+1}$. Hence $\underline{f}(U_{n+1}(A)) = U_{n+1}(B)$.

Next, we present relations between these notions of stable rank. Most of these relations are known and can be found in the papers of Corach and Larotonda [3] and Suárez [23; 24]. Many of the proofs in these papers are based on far-reaching concepts and techniques from algebraic geometry, such as Serre fibrations and homotopy classes. For the reader's convenience we present short direct proofs of some of these facts.

Proposition 2.1 [3, p. 293]. *Suppose that $U_n(A)$ is dense in A^n. Then the stable rank of A is less than n. Namely,* $\mathrm{bsr}(A) \le \mathrm{tsr}(A)$.

Proof. Let $(f_1, \dots, f_n, h) \in U_{n+1}(A)$. Then there exist $x_j \in A$ and $x \in A$ such that $1 = \sum_{j=1}^n x_j f_j + xh$. Since $U_n(A)$ is dense in A^n, for every $\varepsilon > 0$ there exists $(u_1, \dots, u_n) \in U_n(A)$ such that $\|u_j - x_j\|_A < \varepsilon$. Also, $x = \sum_{j=1}^n h_j u_j$ for some $h_j \in A$ because $(u_1, \dots, u_n)$ is invertible. Hence

$$\begin{aligned}\sum_{j=1}^n u_j(f_j + h_j h) &= \sum_{j=1}^n u_j f_j + xh \\ &= \left(\sum_{j=1}^n x_j f_j + xh\right) + \sum_{j=1}^n (u_j - x_j) f_j = 1 + u,\end{aligned}$$

where we have defined $u := \sum_{j=1}^n (u_j - x_j) f_j$. Moreover, we have $\|u\|_A \le \varepsilon \sum_{j=1}^n \|f_j\|_A$. Hence, for $\varepsilon > 0$ small enough, $1 + u$ is invertible in A and so $(f_1 + h_1 h, \dots, f_n + h_n h) \in U_n(A)$. □

The following lemma is due to Corach and Suárez [4; 5].

Lemma 2.2 [4, p. 636; 5, p. 608]. *Let A be a commutative unital Banach algebra. Then, for $g \in A$, the set*

$$R_n(g) = \{(f_1, \dots, f_n) \in A^n : (f_1, \dots, f_n, g) \text{ is reducible}\}$$

is open-closed inside

$$I_n(g) = \{(f_1, \dots, f_n) \in A^n : (f_1, \dots, f_n, g) \in U_{n+1}(A)\}.$$

In particular, for $n = 1$, if $\varphi: [0,1] \to I_1(g)$ is a continuous curve and $(\varphi(0), g)$ is reducible, then $(\varphi(1), g)$ is reducible.

A very useful characterization of the Bass stable rank is the following. Here the equivalence of items (2) and (3) was known (see [3]).

THEOREM 2.3. *Let A be a commutative unital Banach algebra. The following assertions are equivalent*:

(1) $\underline{\pi}(U_n(A))$ *is dense in* $U_n(A/I)$ *for every closed ideal* I *in* A;
(2) $\mathrm{bsr}(A) \le n$;
(3) $\underline{\pi}(U_n(A)) = U_n(A/I)$ *for every closed ideal* I *in* A.

Here $\pi : A \to A/I$ *is the canonical quotient mapping and* $\underline{\pi}$ *the associated map on* A^n.

Proof. (1) ⇒ (2): Let $(a_1, \dots, a_n, a_{n+1}) \in U_{n+1}(A)$. Consider the closure I of the ideal generated by a_{n+1}. Then A/I is a Banach algebra under the quotient norm and $(a_1 + I, \dots, a_n + I) \in U_n(A/I)$. By (1), there exists a sequence $(b_1^{(k)}, \dots, b_n^{(k)}) \in U_n(A)$ such that $\|\pi(b_j^{(k)}) - \pi(a_j)\|_{A/I} \to 0$ as $k \to \infty$ for $j = 1, \dots, n$. Hence there are $x_j^{(k)} \in A$ such that

$$\|a_j - b_j^{(k)} + x_j^{(k)} a_{n+1}\|_A \to 0.$$

Now for every k we have that the $(n+1)$-tuples

$$(b_1^{(k)} - x_1^{(k)} a_{n+1}, \dots, b_n^{(k)} - x_n^{(k)} a_{n+1}, a_{n+1})$$

are invertible and reducible, since

$$(b_j^{(k)} - x_j^{(k)} a_{n+1}) + x_j^{(k)} a_{n+1} = b_j^{(k)} \quad \text{and} \quad (b_1^{(k)}, \dots, b_n^{(k)}) \in U_n(A).$$

Using Lemma 2.2, which tells us that $R_n(a_{n+1})$ is closed inside $I_n(a_{n+1})$, and noticing that $b_j^{(k)} - x_j^{(k)} a_{n+1} \to a_j$ for $j = 1, \dots, n$, we see that $(a_1, \dots, a_n, a_{n+1})$ is reducible and so $\mathrm{bsr}(A) \le n$.

(2) ⇒ (3): This appears in [3, p. 296]. For the reader's convenience we present the argument. Let $(a_1 + I, \dots, a_n + I) \in U_n(A/I)$. Then there exist $y_1, \dots, y_n \in A$ and $b \in I$ such that $\sum_{j=1}^n y_j a_j = 1 + b$. Hence $(a_1, \dots, a_n, b) \in U_{n+1}(A)$ and so, by (2), there exist $x_1, \dots, x_n \in A$ such that

$$(a_1 + x_1 b, \dots, a_n + x_n b) \in U_n(A).$$

It is clear that $\pi(a_j + x_j b) = a_j + I$. Hence $\underline{\pi}(U_n(A)) = U_n(A/I)$.

(3) ⇒ (1): This is immediate. □

Parts of the following result appear without proof in [6, p. 542].

THEOREM 2.4. *If A is a commutative unital Banach algebra, then*

$$\mathrm{bsr}(A) = \mathrm{ssr}(A) \le \mathrm{dsr}(A) \le \mathrm{tsr}(A).$$

Proof. The assertion that $\mathrm{bsr}(A) = \mathrm{ssr}(A)$ follows from Theorem 2.3. Indeed, let $n = \mathrm{ssr}(A) < \infty$. Since for any closed ideal I the canonical map $\pi : A \to A/I$ is surjective, $\mathrm{ssr}(A) = n$ implies that $\underline{\pi}(U_n(A)) = U_n(A/I)$. Hence, by Theorem 2.3, $m := \mathrm{bsr}(A) \le n$. To show that $n \le m$, let $f : A \to B$ be a surjective homomorphism. Then the canonical injection $\check{f}\colon \tilde{A} = A/\mathrm{Ker}\, f \mapsto B$ is an algebra isomorphism and so $U_m(\tilde{A})$ is mapped onto $U_m(B)$ by $\check{f}$. Since $m = \mathrm{bsr}(A)$, by Theorem 2.3, $\underline{\pi}(U_m(A)) = U_m(A/\mathrm{Ker}\, f)$. Thus $\underline{f}(U_m(A)) = U_m(B)$. This means that $\mathrm{ssr}(A) \le m$. Altogether we have shown that $\mathrm{bsr}(A) = \mathrm{ssr}(A)$.

Now suppose that $\mathrm{dsr}(A) = n$. Consider the algebra $B := A/I$, where I is any closed ideal in A. Then the assertion that $\mathrm{bsr}(A) \le \mathrm{dsr}(A)$ follows from Theorem 2.3 when applied to the epimorphism $f = \pi$.

Next we suppose that $\mathrm{tsr}(A) = n$. To show that $\mathrm{dsr}(A) \le \mathrm{tsr}(A)$, we note that if $f\colon A \to B$ has dense image then $\underline{f}\colon A^n \to B^n$ has dense image as well. Now, the continuity of f and the density of $U_n(A)$ in A^n imply that

$$\overline{\underline{f}(U_n(A))} \supseteq \underline{f}(\overline{U_n(A)}) = \underline{f}(A^n).$$

Therefore,

$$U_n(B) \subseteq B^n = \overline{\underline{f}(A^n)} \subseteq \overline{\underline{f}(U_n(A))}.$$

Since $\underline{f}(U_n(A)) \subseteq U_n(B)$, we finally obtain that $\underline{f}(U_n(A))$ is dense in $U_n(B)$. Therefore $\mathrm{dsr}(A) \le n = \mathrm{tsr}(A)$. □

It is not known whether always $\mathrm{bsr}(A) = \mathrm{dsr}(A)$. For $A = H^\infty$, for instance, we have $\mathrm{bsr}(H^\infty) = \mathrm{dsr}(H^\infty) = 1$ (see [13; 23; 26]) and $\mathrm{tsr}(H^\infty) = 2$ (see [24]); for $A = H^\infty_{\mathbb{R}} = \{f \in H^\infty : \overline{f(\bar{z})} = f(z)\}$ we have $\mathrm{bsr}(H^\infty_{\mathbb{R}}) = \mathrm{dsr}(H^\infty_{\mathbb{R}}) = \mathrm{tsr}(H^\infty_{\mathbb{R}}) = 2$ (see [12]).

Our next result, which seems to be new, gives a version of Lemma 2.2 with bounds.

Proposition 2.5. *Let (f, g) be an invertible pair in the commutative unital Banach algebra A. Suppose that f_n converges to f and that there exist a constant $K \ge 1$ and $u_n \in A$ such that $f_n + u_n g$ is invertible with*

$$\|u_n\|_A + \|f_n + u_n g\|_A + \|(f_n + u_n g)^{-1}\|_A \le K.$$

Then there is an $h \in A$ such that $f + hg$ is invertible and

$$\|h\|_A + \|f + hg\|_A + \|(f + hg)^{-1}\|_A \le 8K.$$

Proof. Let $(x, y) \in A^2$ be such that $1 = xf + yg$. Then

$$f_n + u_n g = f(xf_n + yg) + (y(f_n - f) + u_n)g.$$

Since $\|f_n - f\|_A \to 0$ we may choose n_0 so big that for all $n \ge n_0$ the elements $xf_n + yg$ are invertible in A, such that

$$\|xf_n + yg\|_A \le 2 \quad \text{and} \quad \|(xf_n + yg)^{-1}\|_A \le 2,$$

and such that

$$\|f - f_n\|_A \le \min\{1, \|y\|_A^{-1}\}.$$

If we let

$$h = (y(f_n - f) + u_n)(xf_n + yg)^{-1},$$

then $f + hg$ is invertible. Noticing that by hypothesis $\|u_n\|_A \le K$, we get

$$\|h\|_A \le 2(1+K),$$

$$\|(f_n + u_n g)(xf_n + yg)^{-1}\|_A \le 2K,$$

and

$$\|(xf_n + yg)(f_n + u_n g)^{-1}\|_A \le 2K.$$

Since $K \ge 1$,

$$\|h\|_A + \|f + hg\|_A + \|(f + hg)^{-1}\|_A \le 8K. \qquad \square$$

3. The Stable Ranks of $H^\infty + C$

It is the aim of this section to determine the stable ranks defined previously for the Sarason algebra $H^\infty + C$. Toward this end let us recall the following theorem of Treil [26], which tells us in particular that $\mathrm{bsr}(H^\infty) = 1$.

THEOREM 3.1. *There exists a constant $C(\delta)$ depending only on $\delta \in]0,1[$ such that for every pair (f,g) of elements in the unit ball of H^∞ satisfying $|f| + |g| \ge \delta > 0$ in $\mathbb{D}$ there are functions $u, h \in H^\infty$ with u invertible in H^∞ satisfying $1 = uf + hg$ and $\|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \le C(\delta)$.*

The following well-known result can easily be deduced from Treil's theorem.

PROPOSITION 3.2. *There exists a constant $C(\delta)$ depending only on δ such that, for every $f_1, \dots, f_n$ in H^∞ satisfying $1 \ge \sum_{j=1}^n |f_j| \ge \delta > 0$ in $\mathbb{D}$, there exist $a_j, t_j \in H^\infty$ bounded by $C(\delta)$ such that*

$$1 = \sum_{j=1}^{n-1} a_j (f_j + t_j f_n).$$

Proof. By the H^∞-corona theorem [2], there is a constant $C_1(\delta)$ such that the Bezout equation $\sum_{j=1}^n x_j f_j = 1$ admits a solution $(x_1, \dots, x_n) \in (H^\infty)^n$ with

$$\sum_{j=1}^n \|x_j\|_\infty \le C_1(\delta).$$

We may assume that $C_1(\delta) \ge 1$. Now

$$1 \le \|x_1\|_\infty |f_1| + \left| \sum_{j=2}^n x_j f_j \right| \le C_1(\delta) \left(|f_1| + \left| \sum_{j=2}^n x_j f_j \right| \right);$$

hence

$$2 + C_1(\delta) \ge |f_1| + \left| \sum_{j=2}^n x_j f_j \right| \ge \frac{1}{C_1(\delta)} := \varepsilon > 0.$$

By Treil's theorem there is a constant $C_2(\varepsilon)$ and $u, v \in H^\infty$ such that u is invertible in H^∞ and $\|u\|_\infty + \|u^{-1}\|_\infty + \|v\|_\infty \le C_2(\varepsilon)$ with

$$1 = uf_1 + v \left(\sum_{j=2}^n x_j f_j \right).$$

The latter equation can be rewritten as

$$1 = u(f_1 + u^{-1}vx_n f_n) + \sum_{j=2}^{n-1} vx_j(f_j + 0 \cdot f_n).$$

It is clear that the functions $a_1 := u$, $t_1 := u^{-1}vx_n$, $a_j := vx_j$, and $t_j := 0$ for $j = 2, \dots, n-1$ are bounded by a constant $C(\delta)$ depending only on δ. □

The following theorem, given by Laroco [11, p. 819], will be essential for our determination of the stable rank of $H^\infty + C$.

THEOREM 3.3. *Let $f \in H^\infty$. Then, for every $\varepsilon > 0$, there exist a Blaschke product B and an outer function v, invertible in H^∞, such that*

$$\|f - Bv\|_\infty < \varepsilon \quad \textit{and} \quad |v| \ge \varepsilon/4 \ \textit{ on } \partial\mathbb{D}$$

as well as $\|v\|_\infty \le 1 + \|f\|_\infty$.

We note that, because v is invertible, we actually have $|v| \ge \varepsilon/4$ on $\mathbb{D}$. We additionally need the following lemma.

LEMMA 3.4. *Let B be a Blaschke product and $g \in H^\infty$ with $\|g\|_\infty \le 1$. Suppose that $|B| + |g| \ge \delta > 0$ on $M(H^\infty + C)$. Then there exists a constant $C(\delta)$, depending only on δ, and functions h and u in $H^\infty + C$ with u invertible in $H^\infty + C$ such that $\|u\|_\infty$, $\|u^{-1}\|_\infty$, and $\|h\|_\infty$ are bounded by $C(\delta)$ and such that*

$$1 = uB + hg.$$

Proof. By continuity, there exists an $r > 0$ such that on $\{r \le |z| < 1\}$

$$|B| + |g| \ge \delta/2.$$

Let B^* be a tail of B such that $|B^*| \ge \delta/2$ on $\{|z| \le r\}$. Hence

$$|B^*| + |g| \ge \delta/2 \ \text{ on } M(H^\infty).$$

Let $b = B/B^*$. Note that b is a finite Blaschke product and hence $\bar{b} \in H^\infty + C$. By Treil's result [26] (here Theorem 3.1), there is a constant $C(\delta)$ and two functions $R, h \in H^\infty$, R invertible in H^∞, such that

$$1 = RB^* + hg$$

and

$$\|R\|_\infty + \|R^{-1}\|_\infty + \|h\|_\infty \le C(\delta/2).$$

Therefore, as $(H^\infty + C)$-functions,

$$1 = (R\bar{b})B + hg.$$ □

THEOREM 3.5. *The Bass stable rank of $H^\infty + C$ equals 1.*

Proof. Let (φ, ψ) be an invertible pair in $H^\infty + C$. We may assume that $\|\varphi\|_\infty \le 1$ and $\|\psi\|_\infty \le 1$. Since $H^\infty + C$ is the uniform closure of the set of functions $\{\bar{z}^n f \,:\, f \in H^\infty, n \in \mathbb{N}\}$ (see [7]), there exist $n \in \mathbb{N}$ and $f \in H^\infty$ such that

$\|\varphi - f\bar{z}^n\|_\infty < \varepsilon$. By Theorem 3.3, there is a Blaschke product B and a function v invertible in H^∞ such that $\|f - vB\|_\infty < \varepsilon$. Hence the set of functions

$$\{\bar{z}^n vB : n \in \mathbb{N},\ B \text{ Blaschke},\ v \text{ invertible in } H^\infty\}$$

is dense in $H^\infty + C$. By Lemma 2.2 and the fact that the factors $\bar{z}^n v$ are invertible in $H^\infty + C$, it suffices to show the reducibility of the pairs (B, ψ), where B is any Blaschke product such that $|B| + |\psi| \geq \delta > 0$ on $M(H^\infty + C)$.

To do this, we shall use Lemma 3.4. Choose $n \in \mathbb{N}$ and $g \in H^\infty$, $\|g\|_\infty \leq 1$, such that

$$\|\bar{z}^n g - \psi\|_\infty < \min\left\{\frac{\delta}{2}, \frac{1}{2C(\delta/2)}\right\},$$

where $C(\delta)$ is the constant from Lemma 3.4. Now consider the pair (B, g). We obviously have (on $M(H^\infty + C)$)

$$|B| + |g| = |B| + |\bar{z}^n g| \geq |B| + |\psi| - |\psi - \bar{z}^n g| \geq \delta/2.$$

By Lemma 3.4, there exists $u \in H^\infty + C$, u invertible, and $h \in H^\infty + C$ with $\|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \leq C(\delta/2)$ such that

$$1 = uB + hg.$$

Hence

$$\begin{aligned}|uB + (hz^n)\psi| &= |uB + hg + h(z^n\psi - g)| \\ &\geq 1 - \|h\|_\infty \|\psi - \bar{z}^n g\|_\infty \geq 1 - \|h\|_\infty \frac{1}{2C(\delta/2)} \geq \frac{1}{2}.\end{aligned}$$

Thus $uB + (hz^n)\psi$ is invertible in $H^\infty + C$. Hence $1 = xB + y\psi$, where $x \in H^\infty + C$ is invertible and

$$\max\{\|x\|_\infty, \|y\|_\infty\} \leq 2C(\delta/2) \quad \text{and} \quad \|x^{-1}\|_\infty \leq 2C^2(\delta/2).$$

This shows that the pair (B, ψ) is reducible in $H^\infty + C$. □

Combining Proposition 2.5 with the proof just given, we get the following extension of Theorem 3.5.

THEOREM 3.6. *There exists a constant $C(\delta)$ depending only on $\delta \in]0,1[$ such that for every pair (φ, ψ) of functions in the unit ball of $H^\infty + C$ satisfying $|\varphi| + |\psi| \geq \delta$ on $M(H^\infty + C)$ there is a solution $(u, v) \in (H^\infty + C)^2$ of the Bezout equation $u\varphi + v\psi = 1$, where u is invertible in $H^\infty + C$ and such that $\|u\|_\infty + \|u^{-1}\|_\infty + \|v\|_\infty \leq C(\delta)$.*

Proof. According to Theorem 1.1, let (x, y) be a solution in $H^\infty + C$ of the Bezout equation $x\varphi + y\psi = 1$ with $\|x\|_\infty + \|y\|_\infty \leq \tilde{\chi}(\delta)$. Using Theorem 3.3, we may choose a Blaschke product B and a function h invertible in H^∞ such that

$$\|\varphi - \bar{z}^\nu Bh\|_\infty < \sigma(\delta) := \min\{\delta/2, (2\tilde{\chi}(\delta))^{-1}\}$$

and $2 \geq |h| > \sigma(\delta)/4$. Since $|B| + |\psi| \geq \delta/4$ on $M(H^\infty + C)$, there exists by the proof of Theorem 3.5 a constant $C_1(\delta)$ and a function $q \in H^\infty + C$ such that $F := B + q\psi$ is invertible in $H^\infty + C$ and such that

$$\|q\|_\infty + \|F\|_\infty + \|F^{-1}\|_\infty \le C_1(\delta).$$

Let $f^* := \bar{z}^\nu hB$ and $u^* := \bar{z}^\nu hq$. Then we have that $v^* := f^* + u^*\psi$ is invertible in $H^\infty + C$ and

$$\|u^*\|_\infty + \|v^*\|_\infty + \|(v^*)^{-1}\|_\infty \le C_2(\delta) := 6C_1(\delta) + \frac{4}{\sigma(\delta)}C_1(\delta).$$

Now, as in the proof of Proposition 2.5, we see from

$$f^* + u^*\psi = \varphi(xf^* + y\psi) + (y(f^* - \varphi) + u^*)\psi$$

that $1 = u\varphi + v\psi$ with

$$\|u\|_\infty + \|u^{-1}\|_\infty + \|v\|_\infty \le C_3(\delta). \qquad \square$$

In [24], Suárez showed that $\mathrm{tsr}(H^\infty) = 2$. Using this result, we deduce the topological stable rank for $H^\infty + C$.

THEOREM 3.7. *The topological stable rank of $H^\infty + C$ is 2.*

Proof. First we show that the topological stable rank of $H^\infty + C$ is at most 2. Let $(\varphi_1, \varphi_2) \in (H^\infty + C)^2$. Approximate φ_j by functions of the form $\overline{z^{n_j}} f_j$, where $f_j \in H^\infty$, say $\|\overline{z^{n_j}} f_j - \varphi_j\|_\infty < \varepsilon$, $j = 1, 2$.

Since the topological stable rank of H^∞ is 2, there exist $g_j \in H^\infty$ such that $\|g_j - f_j\|_\infty \le \varepsilon$ and $(g_1, g_2) \in U_2(H^\infty)$. Obviously, $(\overline{z^{n_1}} g_1, \overline{z^{n_2}} g_2) \in U_2(H^\infty + C)$ and $\|\overline{z^{n_j}} g_j - \varphi_j\|_\infty < 2\varepsilon$. Thus $\mathrm{tsr}(H^\infty + C) \le 2$.

Let b be an (infinite) interpolating Blaschke product. Note that b is not invertible in $H^\infty + C$. Let $m \in M(H^\infty + C)$ be a zero of b and let L_m be the associated Hoffman map. Since $b \circ L_m$ is analytic but not identically zero, we see that b cannot be uniformly approximated on $M(H^\infty + C)$ by invertibles in $H^\infty + C$, say u_n, since otherwise $\|b \circ L_m - u_n \circ L_m\|_\infty \to 0$—a contradiction to Rouché's theorem. Thus $\mathrm{tsr}(H^\infty + C) \ge 2$. $\square$

Next we deal with the dense stable rank of $H^\infty + C$. Recall that if A is a commutative unital Banach algebra, then E is an A-convex subset of $M(A)$ if

$$\forall x \notin E \quad \exists f \in A : |f(x)| > \sup_E |f|.$$

We let $\hat{E}$ denote the A-convex hull of a closed set $E \subseteq M(A)$. This is given by

$$\hat{E} = \{m \in M(A) : |f(m)| \le \sup_E |f| \ \forall f \in A\}.$$

Note that E is A-convex if and only if $E = \hat{E}$. We say that E is a proper A-convex set if $E = \hat{E}$ and $\hat{E} \ne M(A)$.

THEOREM 3.8. *The dense stable rank of $H^\infty + C$ is 1.*

Proof. Let E be an $(H^\infty + C)$-convex subset of $M(H^\infty + C)$. By [13] it is sufficient to prove that any function $\varphi \in H^\infty + C$ that does not vanish on E can be uniformly approximated on E by invertible functions in $H^\infty + C$. Toward this end, we first approximate φ on $\mathbb{T}$ (hence on $M(H^\infty + C)$) by a function of the form $\bar{z}^n f$, where $n \in \mathbb{N}$ and $f \in H^\infty$, say $\|\varphi - \bar{z}^n f\|_\infty < \varepsilon/2$. By choosing ε sufficiently small, we see that f does not vanish on E, too. Let

$$\check{E} = \{m \in M(H^\infty) : |h(m)| \leq \sup_E |h| \ \forall h \in H^\infty\}$$

be the H^∞-convex hull of E. We show that $\check{E} \cap M(H^\infty + C) = E$. Toward this end, let $m_0 \in M(H^\infty + C) \setminus E$. Since E is $(H^\infty + C)$-convex, there exists a function $\psi \in H^\infty + C$ such that $|\psi(m_0)| > \sup_E |\psi|$. Uniformly approximating ψ by a function of the form $\bar{z}^n g$ shows that $|g(m_0)| > \sup_E |g|$ for some $g \in H^\infty$. Thus $m_0 \notin \check{E}$, the H^∞-convex hull of E. This shows that $\check{E} \cap M(H^\infty + C) = E$.

Now $\check{E}$ can be written as $\check{E} = E \cup S$, where $S = \check{E} \cap \mathbb{D}$. We probably have $S = \emptyset$; however, this isn't necessary for the rest of the proof. Note that $\check{E}$ is closed and $\bar{S} \setminus \mathbb{D} \subseteq E$. Recall that our function f described previously does not vanish on E; hence f can have only finitely many zeros in S. Write $f = qF$, where q is the finite Blaschke product formed with the zeros of f in S. Thus F does not vanish on $\check{E}$. By [13], for every $\varepsilon > 0$ there is an invertible function $h \in H^\infty$ such that $\sup_{\check{E}} |F - h| < \varepsilon/2$. Hence, by noticing that $|q| = 1$ on $M(H^\infty + C)$, we have $|qh - f| = |qh - qF| < \varepsilon/2$ on E. Therefore, on E,

$$\|\bar{z}^n qh - \varphi\|_\infty \leq \|\bar{z}^n (qh - f)\|_\infty + \|\varphi - \bar{z}^n f\|_\infty < \varepsilon.$$

Since $\bar{z}^n qh$ does not vanish on $M(H^\infty + C)$, that function is invertible in $H^\infty + C$. □

REMARK. Using Theorem 2.4 and Theorem 3.8 we get a second proof of the fact that $\mathrm{bsr}(H^\infty + C) = 1$ (see Theorem 3.5).

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