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An Explicit $\bar\partial$-Integration Formula for Weighted Homogeneous Varieties II. Forms of Higher Degree

J. Ruppenthal & E. S. Zeron

1. Introduction

Let Σ be a weighted homogeneous (singular) subvariety of $\mathbb{C}^n$. The main objective of this paper is to present a class of explicit integral formulas for solving the $\bar\partial$-equation $\omega = \bar\partial\lambda$ on the regular part of Σ, where ω is a $\bar\partial$-closed $(0,q)$-form with compact support and degree $q \geq 1$. Particular cases of these formulas yield L^p-bounded solution operators for $1 \leq p \leq \infty$ if Σ is a homogeneous and pure dimensional subvariety of $\mathbb{C}^n$ with an arbitrary singular locus.

As is well known, solving the $\bar\partial$-equation forms one of the main pillars of complex analysis; however, it also has deep consequences for algebraic geometry, partial differential equations, and other areas. For example, the classical Dolbeault theorem implies that the $\bar\partial$-equation can be solved in all degrees on a Stein manifold, and it is known that an open subset of $\mathbb{C}^n$ is Stein if and only if the $\bar\partial$-equation can be solved in all degrees (on that set). Nevertheless, it is usually not easy to produce an explicit operator for solving the $\bar\partial$-equation on a given Stein manifold, even if we know that it can be solved. The construction of explicit operators depends strongly on the geometry of the manifold on which the equation is considered. There exists a vast literature about this problem on smooth manifolds, both in books and papers (see e.g. [10; 11]).

The respective Dolbeault theory on singular varieties has been developed only recently. Let Σ be a singular subvariety of the space $\mathbb{C}^n$ and ω a bounded $\bar\partial$-closed differential form on the regular part of Σ. Fornæss, Gavosto, and Ruppenthal produced a general technique for solving the $\bar\partial$-equation $\omega = \bar\partial\lambda$ on the regular part of Σ, and they have successfully applied this technique to varieties defined by the formula $z^m = \prod_k w_k^{b_k}$ in $\mathbb{C}^n$; see [6; 9; 16]. Acosta, Solís, and Zeron have developed an alternative technique for solving the $\bar\partial$-equation (if ω is bounded) on the regular part of any singular quotient variety embedded in $\mathbb{C}^n$ that is generated by a finite group of unitary matrices—such as, for instance, hypersurfaces in $\mathbb{C}^3$ with only a rational double point singularity; see [1; 2; 21].

Nevertheless, the research on calculating explicit operators for solving the $\bar\partial$-equation $\omega = \bar\partial\lambda$ on the regular part of singular subvarieties $\Sigma \subset \mathbb{C}^n$ is still at a

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very early state; the techniques mentioned in the previous paragraph do not produce useful explicit formulas. In [19], Ruppenthal and Zeron proposed explicit operators for calculating solutions λ if Σ is a weighted homogeneous variety and ω is a $\bar{\partial}$-closed $(0,1)$-differential form with compact support. The weighted homogeneous varieties are analyzed, for they are a main model for classifying the singular subvarieties of $\mathbb{C}^n$. A detailed analysis of the weighted homogeneous varieties may be found in Chapter 2 (Section 4) and Appendix B of [4]. The main objective of the present paper is to improve the explicit operators originally developed in [19] for calculating solutions λ to the $\bar{\partial}$-equation $\omega = \bar{\partial}\lambda$ on the regular part of any weighted homogeneous variety Σ if ω is a $\bar{\partial}$-closed $(0,q)$-differential form with compact support and degree $q \geq 1$. Furthermore, we produce $\bar{\partial}$-solution operators with L^p-estimates for $1 \leq p \leq \infty$ if Σ is homogeneous with an arbitrary singular locus.

DEFINITION 1. Let $\beta \in \mathbb{Z}^n$ be a fixed integer vector with strictly positive entries $\beta_k \geq 1$. A holomorphic polynomial $Q(z)$ on $\mathbb{C}^n$ is said to be *weighted homogeneous* of degree $d \geq 1$ with respect to β if the following equality holds for all $s \in \mathbb{C}$ and $z \in \mathbb{C}^n$:

$$Q(s^\beta * z) = s^d Q(z) \tag{1}$$

with the action

$$s^\beta * (z_1, z_2, \dots, z_n) := (s^{\beta_1} z_1, s^{\beta_2} z_2, \dots, s^{\beta_n} z_n). \tag{2}$$

An algebraic subvariety Σ in $\mathbb{C}^n$ is said to be weighted homogeneous with respect to β if Σ is the zero locus of a finite number of weighted homogeneous polynomials $Q_k(z)$ of (perhaps different) degrees $d_k \geq 1$, but all of them with respect to the same fixed vector β.

Let $\Sigma \subset \mathbb{C}^n$ be any subvariety. We use the following notation throughout. The regular part $\Sigma^* = \Sigma_{\text{reg}}$ is the complex manifold consisting of the regular points of Σ, and it is always endowed with the induced metric; hence Σ^* is a Hermitian submanifold in $\mathbb{C}^n$ with corresponding volume element dV_Σ and induced norm $|\cdot|_\Sigma$ on the Grassmannian $\Lambda T^* \Sigma^*$. Thus, any Borel-measurable $(0,q)$-form ω on Σ^* admits a representation $\omega = \sum_J f_J d\overline{z_J}$, where the coefficients f_J are Borel-measurable functions on Σ^* that satisfy the inequality $|f_J(z)| \leq |\omega(z)|_\Sigma$ for all points $z \in \Sigma^*$ and multi-indexes $|J| = q$. Notice that such a representation is by no means unique; see [16, Lemma 2.2.1] for a more detailed treatment of this point. For $1 \leq p < \infty$, we also introduce the L^p-norm of a measurable $(0,q)$-form ω on an open set $U \subset \Sigma^*$ via the formula

$$\|\omega\|_{L^p_{0,q}(U)} := \left(\int_U |\omega|^p_\Sigma \, dV_\Sigma \right)^{1/p}.$$

We can now present the main result of this paper. We assume that the $\bar{\partial}$-differentials are calculated in the sense of distributions, for we work with Borel-measurable functions.

THEOREM 2 (Main). *Let Σ be a weighted homogeneous subvariety of $\mathbb{C}^n$ with respect to a given vector $\beta \in \mathbb{Z}^n$, where $n \geq 2$ and all entries $\beta_k \geq 1$. Consider the class of all $(0, q)$-forms ω given by $\sum_J f_J d\overline{z_J}$, where $q \geq 1$, the coefficients f_J are all Borel-measurable functions in Σ, and $z_1, \ldots, z_n$ are the Cartesian coordinates of $\mathbb{C}^n$. Let $\sigma \geq -q$ be any fixed integer. Then the operator $\mathbf{S}_q^\sigma$ is well-defined on Σ for all forms ω that are essentially bounded and have compact support*:

$$\mathbf{S}_q^\sigma \omega(z) := \sum_{|J|=q} \frac{\aleph_J}{2\pi i} \int_{u \in \mathbb{C}} f_J(u^\beta * z) \frac{u^\sigma(\overline{u^{\beta_J}})\, d\bar{u} \wedge du}{\bar{u}(u-1)}, \tag{3}$$

where

$$\aleph_J = \sum_{j \in J,\, K = J \setminus \{j\}} \frac{\beta_j \overline{z_j} d\overline{z_K}}{\operatorname{sgn}(j, K)} \quad \textit{and} \quad \beta_J = \sum_{j \in J} \beta_j. \tag{4}$$

Observe that the multi-indexes J and K are both ordered in an ascending way and that $\operatorname{sgn}(j, K)$ is the sign of the permutation used for arranging the elements of the q-tuple (j, K) in ascending order. Finally, the form $\mathbf{S}_q^\sigma(\omega)$ is a solution of the $\bar{\partial}$-equation $\omega = \bar{\partial}\mathbf{S}_q^\sigma(\omega)$ on the regular part of $\Sigma \setminus \{0\}$ whenever ω is also $\bar{\partial}$-closed on the regular part of $\Sigma \setminus \{0\}$.

According to Definition 1, the origin of $\mathbb{C}^n$ is, in general, a singular point of Σ and so the regular parts of Σ and $\Sigma \setminus \{0\}$ coincide. We will prove Theorem 2 in Section 2 of this paper. Similar techniques and a slight modification of equations (3) and (4) can also be used to produce a $\bar{\partial}$-solution operator with L^p-estimates on homogeneous subvarieties with arbitrary singular locus.

THEOREM 3 (L^p-estimates). *Let Σ be a pure d-dimensional homogeneous (cone) subvariety of $\mathbb{C}^n$, where $n \geq 2$ and each entry $\beta_k = 1$ in Definition 1. Fix a real number $1 \leq p \leq \infty$ and an integer $1 \leq q \leq d$. Consider the class $L^p_{0,q}(\Sigma)$ of all $(0, q)$-forms ω given by $\sum_J f_J d\overline{z_J}$, where the coefficients f_J are all L^p-integrable functions in Σ and where $z_1, \ldots, z_n$ are the Cartesian coordinates of $\mathbb{C}^n$. Choose $\sigma \in \mathbb{Z}$ to be the smallest integer such that*

$$\sigma \geq \frac{2d-2}{p} + 1 - q. \tag{5}$$

Then the operator $\mathbf{S}_q^\sigma(\omega)$ is well-defined almost everywhere on Σ for all forms ω that lie in $L^p_{0,q}(\Sigma)$ and have compact support on Σ:

$$\mathbf{S}_q^\sigma \omega(z) := \sum_{|J|=q} \frac{\aleph_J}{2\pi i} \int_{u \in \mathbb{C}} f_J(uz) \frac{u^\sigma \overline{u^q}\, d\bar{u} \wedge du}{\bar{u}(u-1)}, \tag{6}$$

where

$$\aleph_J = \sum_{j \in J,\, K = J \setminus \{j\}} \frac{q \overline{z_j} d\overline{z_K}}{\operatorname{sgn}(j, K)}.$$

The form $\mathbf{S}_q^\sigma(\omega)$ is a solution of the $\bar{\partial}$-equation $\omega = \bar{\partial}\mathbf{S}_q^\sigma(\omega)$ on the regular part of $\Sigma \setminus \{0\}$ whenever ω is also $\bar{\partial}$-closed on the regular part of $\Sigma \setminus \{0\}$. Finally, if *we assume that the support of ω is contained in an open ball B_R of radius $R > 0$ and center at the origin, then there exists a strictly positive constant $C_\Sigma(R, \sigma)$ that does not depend on ω and such that*

$$\|\mathbf{S}_q^\sigma(\omega)\|_{L^p_{0,q-1}(\Sigma\cap B_R)} \leq C_\Sigma(R,\sigma)\cdot\|\omega\|_{L^p_{0,q}(\Sigma)}. \tag{7}$$

The case $p = \infty$ in Theorem 3 is a corollary of Theorem 2 because the formulas (6) and (3) coincide in the homogeneous case (where all coefficients $\beta_J = q$). We will give the full proof of Theorem 3 in Section 3.

The obstructions to solving the $\bar\partial$-equation with L^p-estimates on subvarieties of $\mathbb{C}^n$ are not completely understood in general. An L^2-solution operator (for forms with noncompact support) is known only in the case where Σ is a complete intersection (more precisely, a Cohen–Macaulay space) of pure dimension ≥ 3 with only isolated singularities. This operator was constructed by Fornæss, Øvrelid, and Vassiliadou in [8] via an extension theorem for $\bar\partial$-cohomology groups originally presented by Scheja [20]. Usually, the L^p-results come with some obstructions to the solvability of the $\bar\partial$-equation. Different situations have been analyzed in the works of Diederich, Fornæss, Øvrelid, Ruppenthal, and Vassiliadou, where it is shown that the $\bar\partial$-equation is solvable with L^p-estimates for forms lying in a closed subspace of finite codimension of the vector space of all the $\bar\partial$-closed L^p-forms if the variety has only isolated singularities [3; 5; 8; 12; 17]. Moreover, in [7] the $\bar\partial$-equation is solved locally with some weighted L^2-estimates for forms that vanish to a sufficiently high order on the (arbitrary) singular locus of the given varieties.

There is a second line of research about the $\bar\partial$-operator on complex projective varieties (see [13; 14] for the state of the art and further references). Although that area clearly has much in common with the topic of $\bar\partial$-equations on analytic subvarieties of $\mathbb{C}^n$, it is a somewhat different theory because of the strong global tools that cannot be used in the (local) situation of Stein spaces (owing to the lack of compactness).

Because the estimates in Theorem 3 are given only for homogeneous varieties, in Section 4 we propose a useful technique for generalizing the estimates in Theorem 3 so as to consider weighted homogeneous subvarieties instead of homogeneous ones.

2. Proof of Main Theorem

The following result will be needed. We use $L^1_{p,q}$ to denote the class of all the (p,q)-forms with L^1-integrable coefficients, so that the differentials are calculated in the sense of distributions.

THEOREM 4. *Let $U \subset \mathbb{C}^m$ be open, $2 \leq q \leq m$, and let $\omega \in L^1_{0,q}(U)$ be a $\bar\partial$-closed form with compact support along the first coordinate z_1—that is, such that* $\operatorname{supp}(\omega) \cap F_y$ *is compact in $U \cap F_y$ for all fibres $F_y = \mathbb{C}\times\{y\}$ with $y \in \mathbb{C}^{m-1}$. Assume that ω is given by*

$$\omega = \sum_{|J|=q,\, 1\notin J} [a_J] d\overline{z_J} + \sum_{|K|=q-1,\, 1\notin K} [a_{1,K}] d\overline{z_1} \wedge d\overline{z_K},$$

where the multi-indexes J and K are both ordered in an ascending way. The operator

$$\mathbf{S}_q(\omega) := \sum_{|K|=q-1,\, 1\notin K} \mathbf{I}[a_{1,K}] d\overline{z_K}$$

with

$$\mathbf{I} f(z_1, \dots, z_m) := \frac{1}{2\pi i} \int_{t\in\mathbb{C}} f(t, z_2, \dots, z_m) \frac{d\bar t \wedge dt}{t - z_1}$$

is defined almost everywhere in U and satisfies $\omega = \bar\partial \mathbf{S}_q(\omega)$.

Notice that $\mathbf{S}_q(\omega)$ is well-defined in U if ω is essentially bounded and has compact support along the first coordinate z_1.

Proof of Theorem 4. It is clear that the restrictions $(a_{1,K})|_{F_y}$ are all L^1-integrable on the intersections $U \cap F_y$ for almost every fibre F_y, so that $\eta := \mathbf{S}_q(\omega)$ is defined almost everywhere in U; see [15, Apx. B] or [11; 16]. We need only show that $\bar\partial\eta = \omega$. The assumption $\bar\partial\omega = 0$ implies that the following equation holds for every multi-index $|J| = q$ with $1 \notin J$:

$$\frac{\partial [a_J]}{\partial \overline{z_1}} = \sum_{j\in J,\, K=J\setminus\{j\}} \operatorname{sgn}(j, K) \frac{\partial [a_{1,K}]}{\partial \overline{z_j}}. \tag{8}$$

The function $\operatorname{sgn}(j, K)$ is the sign of the permutation used for ordering the elements of the q-tuple (j, K) in an ascending way. A direct application of the inhomogeneous Cauchy integral formula in one complex variable, when combined with the fact that ω has compact support along the first coordinate, yields the following identity for every multi-index $|K| = q - 1$ with $1 \notin K$:

$$\bar\partial(\mathbf{I}[a_{1,K}]) = [a_{1,K}] d\overline{z_1} \wedge d\overline{z_K} + \sum_{j\notin K,\, j\neq 1} \mathbf{I}\left[\frac{\partial [a_{1,K}]}{\partial \overline{z_j}}\right] d\overline{z_j} \wedge d\overline{z_K}.$$

Therefore,

$$\bar\partial \mathbf{S}_q(w) = \sum_{|K|=q-1,\, 1\notin K} [a_{1,K}] d\overline{z_1} \wedge d\overline{z_K} + \sum_{|J|=q,\, 1\notin J} \mathbf{I}(b_J) d\overline{z_J},$$

where

$$b_J := \sum_{j\in J,\, K=J\setminus\{j\}} \operatorname{sgn}(j, J) \frac{\partial [a_{1,K}]}{\partial \overline{z_j}}.$$

Recall that the multi-indexes J and K are both ordered in an ascending way and that $\operatorname{sgn}(j, K)$ is the sign of the permutation used for arranging the elements of the q-tuple (j, K) in ascending order. Equation (8) implies that $\bar\partial \mathbf{S}_q(w)$ is equal to ω because a_K has compact support along the first coordinate, so

$$\mathbf{I}(b_J) = \mathbf{I}\left(\frac{\partial [a_J]}{\partial \overline{z_1}}\right) = a_J. \qquad \square$$

We may now proceed with the proof of the Main Theorem.

Proof of Main Theorem. We follow the proof originally presented in [19], so here we point out only the main elements. Let $\{Q_k\}$ be the set of polynomials that define the algebraic variety Σ as its zero locus. The definition of weighted homogeneous varieties implies that the polynomials $Q_k(z)$ are all weighted homogeneous with respect to the same fixed vector β. Equation (1) automatically yields that every point $s^\beta * z$ lies in Σ for all $s \in \mathbb{C}$ and $z \in \Sigma$, so each coefficient $f_J(\cdot)$ in equation (3) is well-evaluated in Σ. Moreover, the coefficients $\beta_k \geq 1$ and $\beta_J \geq q$ for all indexes k and multi-indexes J of degree q. Fixing any point $z \in \Sigma$, the given hypotheses imply that the following Borel-measurable functions are all essentially bounded and have compact support in $\mathbb{C}$:

$$u \mapsto f_J(u^\beta * z).$$

Hence, the operator $\mathbf{S}_q^\sigma(\omega)$ in (3)–(4) is well defined on Σ for each fixed integer $\sigma \geq -q$ and for all forms ω that are essentially bounded and have compact support. We shall prove that $\mathbf{S}_q^\sigma(\omega)$ is also a solution of the equation $\omega = \bar{\partial}\mathbf{S}_q^\sigma(\omega)$ if the $(0,q)$-form ω is $\bar{\partial}$-closed. We may suppose, without loss of generality and in light of the given hypotheses, that the regular part of Σ does not contain the origin. Let $\xi \neq 0$ be any fixed point in the regular part of Σ. We shall suppose for simplicity that the first entry $\xi_1 \neq 0$, so we may define the following mapping and subvariety:

$$\begin{aligned} \eta(y) &:= (y_1/\xi_1)^\beta * (\xi_1, y_2, y_3, \ldots, y_n) \quad \text{for } \ y \in \mathbb{C}^n, \\ Y &:= \{\hat{y} \in \mathbb{C}^{n-1} : Q_k(\xi_1, \hat{y}) = 0 \ \forall k\}. \end{aligned} \tag{9}$$

The action $s^\beta * z$ was given in (2). We have that $\eta(\xi) = \xi$ and that the following identities hold for all $s \in \mathbb{C}$ and $\hat{y} \in \mathbb{C}^{n-1}$ (recall equation (1) and the fact that Σ is the zero locus of the polynomials $\{Q_k\}$):

$$\begin{aligned} Q_k(\eta(s,\hat{y})) &= (s/\xi_1)^{d_k} Q_k(\xi_1, \hat{y}), \\ \eta(\mathbb{C}^* \times Y) &= \{z \in \Sigma : z_1 \neq 0\}. \end{aligned} \tag{10}$$

The symbol $\mathbb{C}^*$ stands for $\mathbb{C} \setminus \{0\}$. The mapping $\eta(y)$ is locally a biholomorphism whenever the first entry $y_1 \neq 0$. Whence the point ξ lies in the regular part of the variety $\mathbb{C} \times Y$, because $\xi = \eta(\xi)$ also lies in the regular part of Σ and $\xi_1 \neq 0$. Thus, we can find a biholomorphism

$$\pi = (\pi_2, \ldots, \pi_n) \colon U \to Y \subset \mathbb{C}^{n-1}$$

defined from an open and bounded domain U in $\mathbb{C}^m$ onto an open set in the regular part of Y and such that $\pi(\zeta)$ is equal to $(\xi_2, \ldots, \xi_n)$ for some $\zeta \in U$. Consider the following holomorphic mapping defined for all points $s \in \mathbb{C}$ and $x \in U$:

$$\Pi(s,x) := s^\beta * (\xi_1, \pi(x)) = \eta(s\xi_1, \pi(x)) \in \Sigma. \tag{11}$$

The image $\Pi(\mathbb{C} \times U)$ will be known as a *generalized cone* from now on. Observe that $\Pi(\mathbb{C}^* \times U)$ lies in the regular part of Σ, since $\pi(U)$ is contained in the regular part of Y. The mapping $\Pi(s,x)$ is locally a biholomorphism whenever $s \neq 0$ because η is also a local biholomorphism for $y_1 \neq 0$, and the image $\Pi(1,\zeta)$ is equal to ξ. Hence, recalling the form ω and the operator $\mathbf{S}_q^\sigma(\omega)$ defined in (3)–(4), we need only prove that the pull-back $\Pi^*\omega$ is equal to the differential $\bar\partial\Pi^*\mathbf{S}_q^\sigma(\omega)$ inside $\mathbb{C}^* \times U$ in order to conclude that the $\bar\partial$-equation $\omega = \bar\partial\mathbf{S}_q^\sigma(\omega)$ holds in a neighborhood of ξ in Σ. We can use equations (2) and (11) to calculate the pull-back $\Pi^*\omega$ when ω is given by $\sum_J f_J d\overline{z_J}$. To simplify the notation, let $\pi_1(x) := \xi_1$ for all $x \in U$, so that $d\pi_1 = 0$. Then

$$
\begin{aligned}
[\Pi^*\omega](s,x) = & \sum_{|J|=q} f_J(\Pi(s,x))\overline{s^{\beta_J}} \bigwedge_{j\in J} d\overline{\pi_j(x)} \\
& + \sum_{|J|=q,\, j\in J} \frac{f_J(\Pi)\beta_j \overline{s^{\beta_J-1}\pi_j(x)}}{\operatorname{sgn}(j, J\setminus\{j\})} d\bar s \wedge \bigwedge_{k\in J\setminus\{j\}} d\overline{\pi_k(x)}.
\end{aligned} \tag{12}
$$

Recall that $\beta_J = \sum_{j\in J}\beta_j \geq q \geq -\sigma$, the multi-index J is ordered in an ascending way, and $\operatorname{sgn}(\alpha_1,\dots,\alpha_q)$ is the sign of the permutation used for arranging the elements of the q-tuple $(\alpha_1,\dots,\alpha_q)$ in ascending order. The given hypotheses on ω yield that the pull-back $\Pi^*\omega$ is $\bar\partial$-closed and bounded in $\mathbb{C}^* \times U$; hence it is also $\bar\partial$-closed in $\mathbb{C} \times U$ (see [16, Lemma 4.3.2] or [21, Lemma (2.2)]). The same argument applies to the $\bar\partial$-closed and essentially bounded form

$$
s^{\sigma+1}[\Pi^*\omega](s,x) \in L^\infty_{0,q}(\mathbb{C} \times U). \tag{13}
$$

The open set U is bounded in $\mathbb{C}^m$. By the use of [16, Lemma 7.2.2, p. 186] or [18, Lemma 3.6] it then follows from (12) and (13) that $s^\sigma\Pi^*\omega$ is both $L^1_{0,q}(\mathbb{C}\times U)$ and $\bar\partial$-closed in $\mathbb{C} \times U$. It is easy to see that each coefficient $f_J(\Pi(s,x))$ has compact support with respect to the first coordinate s, so we can apply Theorem 4 to $t^\sigma[\Pi^*\omega](t,x)$ and calculate the form

$$
\mathbf{S}_q(t^\sigma\Pi^*\omega) = \sum_{|J|=q} \frac{\Theta_J}{2\pi i} \int_{t\in\mathbb{C}} f_J(\Pi(t,x)) \frac{t^\sigma(\overline{t^{\beta_J}})\, d\bar t \wedge dt}{\bar t(t-s)} \tag{14}
$$

with

$$
\Theta_J = \sum_{j\in J} \frac{\beta_j\overline{\pi_j(x)}}{\operatorname{sgn}(j, J\setminus\{j\})} \bigwedge_{k\in J\setminus\{j\}} d\overline{\pi_k(x)}.
$$

Theorem 4 now implies that

$$
s^\sigma[\Pi^*\omega](s,x) = \bar\partial\mathbf{S}_q(t^\sigma[\Pi^*\omega](t,x)).
$$

Hence, we need only verify that the form $\mathbf{S}_q(t^\sigma\Pi^*\omega)/s^\sigma$ is equal to the pull-back $\Pi^*\mathbf{S}_q^\sigma(\omega)$ of the form defined in (3) in order to conclude that $\omega = \bar\partial\mathbf{S}_q^\sigma(\omega)$, as desired. We begin by calculating the pull-back $\Pi^*\aleph$ of the differential form $\aleph_J$ given in (4). Notice that $\pi_1(x) \equiv \xi_1$ so that $d\pi_1 = 0$, and recall equations (2) and (11). We have

$$\Pi^*\aleph_J = \sum_{j\in J} \frac{\beta_j \overline{s^{\beta_J}\pi_j(x)}}{\operatorname{sgn}(j, J\setminus\{j\})} \bigwedge_{k\in J\setminus\{j\}} d\overline{\pi_k(x)}$$
$$+ \sum_{j,k\in J,\, j\neq k} \frac{\beta_j\beta_k \overline{s^{\beta_J-1}\pi_j(x)\pi_k(x)d\bar{s}}}{\operatorname{sgn}(j, J\setminus\{j\})\operatorname{sgn}(k, J\setminus\{j,k\})} \wedge \bigwedge_{i\in J\setminus\{j,k\}} d\overline{\pi_i(x)}. \quad (15)$$

Suppose that $J = (\alpha_1, \ldots, \alpha_a, j, \dot\alpha_1, \ldots, \dot\alpha_b, k, \ddot\alpha_1, \ldots, \ddot\alpha_c)$; then

$$\operatorname{sgn}(j, J\setminus\{j\})\operatorname{sgn}(k, J\setminus\{j,k\}) = (-1)^a(-1)^{a+b},$$
$$\operatorname{sgn}(k, J\setminus\{k\})\operatorname{sgn}(j, J\setminus\{j,k\}) = (-1)^{a+b+1}(-1)^a.$$

Hence the last sum in equation (15) vanishes and so the pull-back $\Pi^*\aleph_J$ is identically equal to $\overline{s^{\beta_J}}\Theta_J$, with Θ_J as defined for (14). Finally, we can calculate the pull-back of the form $\mathbf{S}_q^\sigma(\omega)$ given in (3), noticing that $\Pi(us, x)$ is equal to $u^\beta * \Pi(s, x)$:

$$\Pi^*\mathbf{S}_q^\sigma(w) = \sum_{|J|=q} \frac{\overline{s^{\beta_J}}\Theta_J}{2\pi i} \int_{u\in\mathbb{C}} f_J(\Pi(us, x)) \frac{u^\sigma(\overline{u^{\beta_J}})\, d\bar{u}\wedge du}{\bar{u}(u-1)}. \quad (16)$$

The change of variables $t = us$ yields that the form $\mathbf{S}_q(t^\sigma\Pi^*\omega)/s^\sigma$ in (14) is equal to the identity (16), so $\omega = \bar\partial\mathbf{S}_q^\sigma(\omega)$ as desired. □

3. L^p-Estimates

We prove Theorem 3 in this section. Recall that Σ is a pure d-dimensional homogeneous (cone) subvariety of $\mathbb{C}^n$ with arbitrary singular locus, so that $n \geq 2$ and each entry $\beta_k = 1$ in Definition 1. Moreover, given a fixed real number $1 \leq p \leq \infty$ and an integer $1 \leq q \leq d$, we consider the class $L^p_{0,q}$ of all $(0, q)$-forms ω expressed as $\sum_J f_J d\overline{z_J}$, where the coefficients f_J are all L^p-integrable functions in Σ and where $z_1, \ldots, z_n$ are the Cartesian coordinates of $\mathbb{C}^n$. Assume that the support of each form $\omega \in L^p_{0,q}$ is contained in the open ball B_R of radius $R > 0$ and center at the origin. Fix $\sigma \in \mathbb{Z}$ to be the smallest integer such that

$$\sigma \geq \frac{2d-2}{p} + 1 - q. \quad (17)$$

We begin by showing that $\mathbf{S}_q^\sigma$ in (6) defines a bounded operator

$$\mathbf{S}_q^\sigma : L^p_{0,q}(\Sigma\cap B_R) \to L^p_{0,q-1}(\Sigma\cap B_R), \quad (18)$$

where

$$\mathbf{S}_q^\sigma\omega(z) = \sum_{|J|=q} \frac{\aleph_J}{2\pi i} \int_{u\in\mathbb{C}} f_J(uz) \frac{u^\sigma\overline{u^q}\, d\bar{u}\wedge du}{\bar{u}(u-1)} \quad (19)$$

with

$$\aleph_J = \sum_{j\in J,\, K=J\setminus\{j\}} \frac{q\overline{z_j}d\overline{z_K}}{\operatorname{sgn}(j, K)}.$$

Recall that the multi-indexes J and K are both ordered in an ascending way and that $\operatorname{sgn}(j, K)$ is the sign of the permutation used for arranging the elements of the q-tuple (j, K) in ascending order. Observe that the case $p = \infty$ in (18) is a corollary of Theorem 2 because the formulas (3) and (19) coincide when the variety Σ is homogeneous (so that all $\beta_J = q$). Hence, we can suppose from now on that $p < \infty$, and we only need to prove that the following inequality holds for every multi-index $|J| = q$ and $j \in J$ in order to conclude that (18) and (7) hold:

$$\int_{z\in\Sigma\cap B_R}\left|\overline{z_j}\int_{|u|<R/\|z\|}\frac{f_J(uz)u^{\sigma+q}}{(u-1)u}\,dV_{\mathbb{C}}\right|^p dV_\Sigma \lesssim \|\omega\|^p_{L^p_{0,q}(\Sigma)}. \tag{20}$$

Notice that the support of $f_J(z)$ is contained in the open ball B_R of radius R and that $dV_{\mathbb{C}}$ and dV_Σ are the respective volume forms on $\mathbb{C}$ and Σ. Furthermore, we may use the variable u instead of its complex conjugate $\bar{u}$ because we work under an absolute-value sign. Let $\delta < 1$ be any fixed real number. It is easy to deduce the existence of a finite real constant M_1 such that the following inequalities hold for all complex numbers $\hat{t}$ and $\hat{w}$:

$$\int_{|w|<R}\frac{dV_{\mathbb{C}}(w)}{|\hat{t}|^\delta|w-\hat{t}|}\leq M_1 \quad\text{and}\quad \int_{|t|<R}\frac{dV_{\mathbb{C}}(t)}{|t|^\delta|\hat{w}-t|}\leq M_1.$$

Hence, the generalized Young inequality for convolution integrals yields that the modified Cauchy–Pompeiu formula defines an L^p-bounded operator (see e.g. [15, Apx. B] with $s = 1$ and $\delta < 1$):

$$\int_{|t|<R}\left|\int_{|w|<R}\frac{h(w)\,dw\wedge d\bar{w}}{(w-t)|t|^\delta}\right|^p dV_{\mathbb{C}}(t)\lesssim\int_{|t|<R}|h(t)|^p\,dV_{\mathbb{C}}. \tag{21}$$

Moreover, let $\tilde{\Sigma}$ be the projective variety associated to Σ in the space $\mathbb{CP}^{n-1}$, for Σ is a pure d-dimensional homogeneous subvariety of $\mathbb{C}^n$. We also use the fact that any integral on Σ can be decomposed as a pair of nested integrals on $\mathbb{C}$ and $\tilde{\Sigma}$; that is:

$$\int_{z\in\Sigma}\Phi(z)\,dV_\Sigma(z)=\int_{[z]\in\tilde{\Sigma}}\int_{t\in\mathbb{C}}\Phi(\dot{z}t)|t|^{2d-2}\,dV_{\mathbb{C}}(t)\,dV_{\tilde{\Sigma}}([z]),$$

where $\dot{z}\in\Sigma$ is any representative of $[z]\in\tilde{\Sigma}$ with $\|\dot{z}\|=1$. Finally, since $\sigma\in\mathbb{Z}$ is the smallest integer that satisfies (17), we have that the constant

$$\delta := \sigma+q-1+\frac{2-2d}{p}\quad\text{satisfies }\ 0\leq\delta<1. \tag{22}$$

We can now use the results presented in the preceding paragraphs to calculate (20) and (7) with the change of variables $w = ut$:

$$
\begin{aligned}
&\int_{z\in\Sigma\cap B_R}\left|\overline{z_j}\int_{|u|<R/\|z\|}\frac{f_J(uz)u^{\sigma+q}}{(u-1)u}\,dV_{\mathbb{C}}\right|^p dV_\Sigma\\
&\quad\le\int_{[z]\in\tilde{\Sigma}}\int_{|t|<R}|t|^p\left|\int_{|u|<R/|t|}\frac{f_J(u\dot{z}t)u^{\sigma+q}}{(u-1)u}\,dV_{\mathbb{C}}\right|^p|t|^{2d-2}\,dV_{\mathbb{C}}\,dV_{\tilde{\Sigma}}\\
&\quad=\int_{[z]\in\tilde{\Sigma}}\int_{|t|<R}\left|\int_{|w|<R}\frac{f_J(w\dot{z})w^{\sigma+q-1}}{(w-t)t^{\sigma+q-2}}\cdot\frac{dV_{\mathbb{C}}}{|t|^2}\right|^p|t|^{2d-2+p}\,dV_{\mathbb{C}}\,dV_{\tilde{\Sigma}}\\
&\quad=\int_{[z]\in\tilde{\Sigma}}\int_{|t|<R}\left|\int_{|w|<R}\frac{f_J(w\dot{z})w^{\sigma+q-1}}{(w-t)|t|^{\delta}}\,dV_{\mathbb{C}}\right|^p dV_{\mathbb{C}}\,dV_{\tilde{\Sigma}}\\
&\quad\lesssim\int_{[z]\in\tilde{\Sigma}}\int_{|t|<R}|f_J(t\dot{z})t^{\sigma+q-1}|^p\,dV_{\mathbb{C}}\,dV_{\tilde{\Sigma}}\\
&\quad=\int_{[z]\in\tilde{\Sigma}}\int_{|t|<R}|f_J(t\dot{z})|^p|t|^{p\delta+2d-2}\,dV_{\mathbb{C}}\,dV_{\tilde{\Sigma}}\\
&\quad\le\int_{z\in\Sigma\cap B_R}|f_J(z)|^pR^{p\delta}\,dV_\Sigma\lesssim\|f_J\|^p_{L^p(\Sigma)}\le\|\omega\|^p_{L^p_{0,q}(\Sigma)}<\infty.
\end{aligned}
$$

Here we have used (22) and (21) with $h(w)=f_J(w\dot{z})w^{\sigma+q-1}$. This completes the proof of equations (20) and (7).

Finally, notice that the operators $\mathbf{S}^\sigma_q(\omega)$ given in (3), (6), and (19) are all the same because the coefficients $\beta_J=q$ for every multi-index $|J|=q$. Therefore, we can show that the operator $\mathbf{S}^\sigma_q(\omega)$ satisfies the differential equation $\omega=\bar\partial\mathbf{S}^\sigma_q(\omega)$ by following step by step the proof presented in Section 2. We must first rewrite the pull-back given in (12), which is $\bar\partial$-closed in the product $\mathbb{C}^*\times U$:

$$
\begin{aligned}
[\Pi^*\omega](u,x)&=\sum_{|J|=q}f_J(\Pi(u,x))\overline{u^q}\bigwedge_{j\in J}d\overline{\pi_j(x)}\\
&\qquad+\sum_{|J|=q,\,j\in J}\frac{f_J(\Pi)q\overline{u^{q-1}\pi_j(x)}}{\operatorname{sgn}(j,J\setminus\{j\})}d\bar{u}\wedge\bigwedge_{k\in J\setminus\{j\}}d\overline{\pi_k(x)}. \qquad (23)
\end{aligned}
$$

We must also show that $u^\sigma[\Pi^*\omega](u,x)$ lies in $L^1_{0,q}(\mathbb{C}\times U)$, where U is a bounded domain in $\mathbb{C}^m$. Thus, we have that $u^\sigma\Pi^*\omega$ is also $\bar\partial$-closed in $\mathbb{C}\times U$ by [16, Lemma 7.2.2, p. 186] or [18, Lemma 3.6]. We can then apply Theorem 4 and follow step by step the proof presented in Section 2 from equation (14) to the end of that section.

Recall that the integer $\sigma\ge\frac{2d-2}{p}+1-q$. We begin by showing that the form $u^\sigma\Pi^*\omega$ lies in $L^p_{0,q}(\mathbb{C}\times U)$. Notice that $\Pi(u,x)$ is equal to $u(\xi_1,\pi(x))$ because each entry $\beta_k=1$ in (2) and (11). It is easy to calculate the pull-back of the volume form dV_Σ:

$$
\begin{aligned}
\Pi^*dV_\Sigma&=\sum_{|I|=|J|=d}\beta_{I,J}(z)dz_I\wedge d\overline{z_J}\Big|_{z=u(\xi_1,\pi(x))}\\
&=\Theta(x)|u|^{2d-2}[du\wedge d\bar{u}]\wedge\bigwedge_{k=1}^{d-1}[dx_k\wedge d\overline{x_k}]. \qquad (24)
\end{aligned}
$$

Recall that x lies in the bounded open set $U \subset \mathbb{C}^{d-1}$. Since Σ is a pure d-dimensional homogeneous (cone) subvariety of $\mathbb{C}^n$, it follows that the coefficients $\beta_{I,J}(z)$ are all invariant under the transformations $z \mapsto uz$ and so $\Theta(x)$ depends only on the values of $\pi(x)$ and all its partial derivatives (Θ is constant with respect to u). That Π is a biholomorphism from $\mathbb{C}^* \times U$ onto its image also implies that Θ cannot vanish. Hence, choosing a smaller set U if necessary, we can suppose that $|\Theta|$ is bounded from below by a constant $M_2 > 0$.

On the other hand, since $\Pi(u,x) = u(\xi_1, \pi)$ and since the support of each $f_J(z)$ is contained in a ball of radius $R > 0$ and center at the origin, we have that every $f_J(\Pi(u,x))$ vanishes if $|u\xi_1| > R$. Thus, equation (23) and the analysis performed in the preceding paragraphs imply that the form $u^\sigma \Pi^* \omega$ lies in $L^p_{0,q}(\mathbb{C} \times U)$, because the following inequalities hold for every multi-index J and exponent $b = 0, 1$:

$$\begin{aligned}\int_{\mathbb{C}\times U} |u^{\sigma+q-b} f_J(\Pi)|^p \, dV_{\mathbb{C}\times U} &\lesssim \int_{\mathbb{C}\times U} |f_J(\Pi)|^p \Theta(x) |u|^{2d-2} \, dV_{\mathbb{C}\times U} \\ &= \int_{\Pi(\mathbb{C}\times U)} |f_J|^p \, dV_\Sigma \le \|\lambda\|^p_{L^2_{0,1}(\Sigma)} < \infty.\end{aligned}$$

Recall that $p(\sigma + q - b) \ge 2d - 2$ because of the hypothesis in (5)–(17). Finally, the support of $\Pi^*\omega$ is bounded in $\mathbb{C} \times U$ because U is bounded and each $f_J(\Pi(u,x))$ vanishes if $|u\xi_1| > R$. Thus, we have that the form $u^\sigma \Pi^* \omega$ is $L^1_{0,q}$ and $\bar{\partial}$-closed in $\mathbb{C} \times U$ (see e.g. [16, Lemma 7.2.2, p. 186] or [18, Lemma 3.6]). We can then apply Theorem 4 and follow step by step the proof presented in Section 2 from equation (14) to the end of that section in order to conclude that the operator $\mathbf{S}^\sigma_q(\omega)$ satisfies the differential equation $\omega = \bar{\partial}\mathbf{S}^\sigma_q(\omega)$, as desired.

4. Weighted Homogeneous Estimates

We wish to close this paper by presenting a useful technique for generalizing the estimates given in Theorem 3, so as to consider weighted homogeneous subvarieties instead of cones. Let $\Sigma \subset \mathbb{C}^n$ be a weighted homogeneous subvariety defined as the zero locus of a finite set of polynomials $\{Q_k\}$. Thus, the polynomials $Q_k(z)$ are all weighted homogeneous with respect to the same vector $\beta \in \mathbb{Z}^n$, and each entry $\beta_k \ge 1$. Define the following holomorphic mapping:

$$\Phi\colon \mathbb{C}^n \to \mathbb{C}^n \quad \text{with} \quad \Phi(x) = (x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_n^{\beta_n}). \tag{25}$$

It is easy to see that each polynomial $Q_k(\Phi)$ is homogeneous, and so the subvariety $X \subset \mathbb{C}^n$ defined as the zero locus of $\{Q_k(\Phi)\}$ is a cone.

Consider a $(0,q)$-form ω given by the sum $\sum_J f_J d\overline{z_J}$, where the coefficients f_J are all Borel-measurable functions with compact support in Σ and where $z_1, \ldots, z_n$ are the Cartesian coordinates of $\mathbb{C}^n$. We may follow two different paths in order to solve the equation $\bar{\partial}\lambda = \omega$. First, we may calculate the pull-back,

$$\Phi^*\omega = \sum_{|J|=q} f_J(\Phi(x)) \Bigg[\prod_{j\in J} \beta_j \overline{x_j^{\beta_j - 1}} \Bigg] d\overline{x_J},$$

and then apply Theorems 2 and 3 on the cone X to obtain the following operators:

$$\mathbf{S}_q^\sigma(\Phi^*\omega) := \sum_{|J|=q} \frac{\widehat{\aleph_J}}{2\pi i} \int_{u\in\mathbb{C}} f_J(\Phi(ux)) \frac{u^\sigma(\overline{u^{\beta_J}})\, d\bar{u}\wedge du}{\bar{u}(u-1)} \tag{26}$$

with

$$\widehat{\aleph_J} = \sum_{j\in J,\, K=J\setminus\{j\}} \frac{\beta_j \overline{x_j^{\beta_j}}}{\operatorname{sgn}(j,K)} \Big[\prod_{k\in K} \beta_k \overline{x_k^{\beta_k - 1}}\Big] d\overline{x_K}.$$

Alternatively, we may use Theorem 2 on the weighted homogeneous variety Σ so as to get

$$\mathbf{S}_q^\sigma(\omega) := \sum_{|J|=q} \frac{\aleph_J}{2\pi i} \int_{u\in\mathbb{C}} f_J(u^\beta * z) \frac{u^\sigma(\overline{u^{\beta_J}})\, d\bar{u}\wedge du}{\bar{u}(u-1)} \tag{27}$$

with

$$\aleph_J = \sum_{j\in J,\, K=J\setminus\{j\}} \frac{\beta_j \overline{z_j} d\overline{z_K}}{\operatorname{sgn}(j,K)} \quad \text{and} \quad \beta_J = \sum_{j\in J} \beta_j.$$

Here we can easily verify that $\Phi^*\mathbf{S}_q^\sigma(\omega)$ is equal to $\mathbf{S}_q^\sigma(\Phi^*\omega)$. The main problem is that $\Phi^*\omega$ may not necessarily lie in $L^p_{0,q}(X)$ for $p < \infty$.

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J. Ruppenthal
Fachbereich C
Bergische Universität Wuppertal
Gaußstraße 20
D-42119 Wuppertal
Germany

ruppenthal@uni-wuppertal.de

E. S. Zeron
Departamento Matemáticas
CINVESTAV
Apartado Postal 14-740
México D.F., 07360
Mexico

eszeron@math.cinvestav.mx