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Maximal Operator for Pseudodifferential Operators with Homogeneous Symbols

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1. Introduction

The class S^0 is a basic class of pseudodifferential operators that has been investigated by many authors. For example, it is quite fundamental that the pseudodifferential operators with symbol S^0 are L^2-bounded (see [14]). However, given that $L^2 \simeq \dot{F}^0_{22}$ (where $\dot{F}^0_{22}$ is the homogeneous Triebel–Lizorkin space), it seems there is no need to assume $\sup_{x \in \mathbb{R}^n, |\xi| \le 1} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| < \infty$ for all multiindices α, β. Indeed, Grafakos and Torres established that it suffices to assume

$$c_{\alpha,\beta}(a) := \sup_{x \in \mathbb{R}^n, \xi \in \mathbb{R}^n} |\xi|^{|\alpha|-|\beta|} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| < \infty \tag{1}$$

for all multiindices α, β. Denote by $a(x,D)^\sharp$ the formal adjoint of $a(x,D)$. It is natural to assume that

$$a(x,D)^\sharp 1(x) = 0, \tag{2}$$

since one must postulate some moment condition on atoms for $\dot{F}^0_{22}$ when considering the atomic decomposition (see [2; 15]).

We shall assume that $a \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ is a function satisfying (1) and (2). In [5], Grafakos and Torres established that

$$f \in \mathcal{S}_0 \mapsto \int_{\mathbb{R}^n} a(x,\xi) \exp(2\pi i x \cdot \xi) \mathcal{F}^{-1} f(\xi)\, d\xi$$

extends to an L^2-bounded operator, where $\mathcal{S}_0$ denotes the closed subspace of $\mathcal{S}$ that consists of the functions with vanishing moment of any order.

We seek to obtain a maximal estimate related to this operator. To formulate our results, we need some notation. Given $a, \xi \in \mathbb{R}^n$ and $\lambda > 0$, define

$$\begin{aligned} T_a f(x) &:= f(x-a), \\ M_\xi f(x) &:= \exp(2\pi i \xi \cdot x) f(x), \\ D_\lambda f(x) &:= \lambda^{-n/2} f(\lambda^{-1} x). \end{aligned}$$

We use $A \lesssim_{X,Y,\dots} B$ to denote that there exists a constant $c > 0$, depending only on the parameters $X, Y, \dots$, such that $A \le cB$. If the constant c depends only on

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$c_{\alpha,\beta}(a)$ and the dimension n, we simply write $A \lesssim B$. If the two-sided estimate $A \lesssim_{X,Y,\dots} B \lesssim_{X,Y,\dots} A$ holds, we write $A \simeq_{X,Y,\dots} B$.

In this paper we establish the following result.

THEOREM 1.1. *Suppose that* $1 < p < \infty$. *Then*

$$\left\| \sup_{\xi \in \mathbb{R}^n} |M_{-\xi} a(x,D) M_\xi f| \right\|_p \lesssim_p \|f\|_p.$$

This theorem will be established via the following statement, which we shall struggle to prove.

THEOREM 1.2. *The following estimate holds*:

$$\left| \left\{ x \in \mathbb{R}^n : \sup_{\xi \in \mathbb{R}^n} |M_{-\xi} a(x,D) M_\xi f(x)| > \lambda \right\} \right| \lesssim \frac{1}{\lambda^2} \int_{\mathbb{R}^n} |f(x)|^2 \, dx.$$

Let $a(x,\xi) = m(\xi)$ with m homogeneous of degree 0. In this case, Theorem 1.2 was covered by Pramanik and Terwilleger [13] and Theorem 1.1 was covered by Grafakos, Tao, and Terwilleger [4].

We now give an example of the function a for which Theorem 1.2 is applicable. It is well known that pseudodifferential operators with symbol $S^0_{1,1}$ are not L^2-bounded. However, a slight transformation of $S^0_{1,1}$ comes about naturally in the following context. Let $\Delta_j : \mathcal{S}' \to \mathcal{S}'$ be the jth Littlewood–Paley operator; that is, $\Delta_j f = \mathcal{F}^{-1}[\Theta(2^{-j}\cdot) \cdot \mathcal{F}f]$ for some appropriate smooth function Θ. Set $S_j = \sum_{k<j-4} \Delta_k$.

If the aim is to show that $f \cdot g \in B^{s_0}_{p_0 q_0}$ if $f \in B^{s_1}_{p_1 q_1}$ and $g \in B^{s_2}_{p_2 q_2}$, then one is led to investigate the operator given by

$$P(f,g) = \sum_{j=-\infty}^{\infty} S_j(f) \Delta_j(g).$$

If Θ is appropriately localized (say, $\mathrm{supp}(\Theta) \subset \{1 \le |\xi| \le 4\}$) then we see that P_f satisfies the assumption of Theorem 1.2, where P_f is given by $P_f(g) = P(f,g)$. This is a traditional method proposed by Bony [1].

Before we consider another example for which Theorem 1.2 is applicable, recall how a counterexample showing that pseudodifferential operators with symbol in $S^0_{1,1}$ can be unbounded on L^2. To construct this example, we choose the same function Θ used in the previous paragraph. Then define

$$\tilde{a}(x,\xi) = \sum_{j=1}^{\infty} \exp(-2\pi i \cdot 4^j x) \Theta(4^{-j}\xi).$$

It is not difficult to see that $\tilde{a} \in S^0_{1,1} \cap \dot{S}^0_{1,1}$. By modifying Θ we can assume that $\Theta \equiv 1$ on $\frac{5}{3} \le |\xi| \le \frac{7}{3}$. Define

$$f_N(x) = \sum_{j=4}^{N} \frac{1}{j} \exp(2\pi i \cdot 2^j x_1)\mathcal{F}^{-1}\Theta\left(\frac{1}{4}x\right) \tag{3}$$

for $N \geq 4$. Since $\mathcal{F}f_N(x) = 4^n \sum_{j=4}^{N} \frac{1}{j}\Theta(4\xi - 2^j(1,0,0,\ldots,0))$, it follows that terms in the sum appearing in (3) are orthogonal to one another. Hence $\{f_N\}_{N\geq 4}$ forms a bounded subset in L^2. However, $\tilde{a}(x, D)$ completely undoes this orthogonality, since

$$\tilde{a}(x, D) f_N(x) = \sum_{j=4}^{N} \frac{1}{j}\mathcal{F}^{-1}\Theta\left(\frac{1}{4}x\right).$$

This example will lead us to the conclusion that pseudodifferential operators with symbol $S^0_{1,1}$ are not L^2-bounded.

However, a similar example will satisfy the assumption in Theorem 1.2. Indeed, if we define

$$a(x,\xi) = \sum_{j=1}^{\infty} \exp(-10\pi i \cdot 4^j x)\Theta(4^{-j}\xi),$$

then a simple calculation shows

$$\langle 1, a(x, D)\varphi\rangle = \sum_{j=1}^{\infty} \frac{1}{j}\langle \delta_{(5\pi 2^j,0,0,\ldots,0)}, \mathcal{F}\varphi\rangle = 0.$$

Moreover, we still have $a \in \dot{S}^0_{1,1}$. Therefore, a is an example to which we can apply Theorem 1.2.

In this paper we prove Theorems 1.1 and 1.2. In Section 3 we obtain a formula of the Fourier multiplier. The formula will be a simplification of results in [13] and enables us to extend those results. What is new about this formula is that there is no need to take the average over the time space, as discussed in Section 3. We investigate an estimate of Cotlar type in Section 4, and in In Section 5 we prove Theorem 1.2. Our proof parallels the one in [13], so we will invoke their notation and results. Finally, in Section 6 we consider an extension to L^p $(1 < p < \infty)$ of Theorem 1.2.

2. Preliminaries

The following notation will be used throughout.

2.1. Notation for Cubes

We begin with some notation for $\mathbb{R}^n$.

DEFINITION 2.1.

1. Denote $\{0, 1, 2, \ldots\}$ by $\mathbb{N}_0$.
2. Equip $\mathbb{R}^n$ with the lexicographic order $\ll$; namely, define

$$x = (x_1, x_2, \ldots, x_n) \ll y = (y_1, y_2, \ldots, y_n), \quad x \neq y$$

if and only if $x_1 = y_1, x_2 = y_2, \ldots, x_{j-1} = y_{j-1}$ and $x_j < y_j$ for some $j = 1, 2, \ldots, n$.

3. Let $\mathbf{1} := (1, 1, \ldots, 1)$.

The following notation will be used for dyadic cubes.

DEFINITION 2.2.

1. A *dyadic cube* is a cube of the form

$$Q_{\nu m} := \prod_{j=1}^{n} \left[\frac{m_j}{2^\nu}, \frac{m_j + 1}{2^\nu} \right)$$

for $m = (m_1, m_2, \ldots, m_n)$ and $\nu \in \mathbb{Z}$; its center and the side-length are defined by $c(Q_{\nu m}) := \left(\frac{2m_1+1}{2^{\nu+1}}, \ldots, \frac{2m_n+1}{2^{\nu+1}} \right)$ and $\ell(Q_{\nu m}) := 2^{-\nu}$, respectively.

2. A dyadic cube Q may be bisected into 2^n cubes of equal length, labeled $Q_{(1)}, Q_{(2)}, \ldots, Q_{(2^n)}$, such that

$$c(Q_{(1)}) \ll c(Q_{(2)}) \ll \cdots \ll c(Q_{(2^n)}).$$

Dyadic cubes are assumed to be open, but we assume that cubes (see next definition) are closed.

DEFINITION 2.3. A *cube* is a subset in $\mathbb{R}^n$ of the form

$$Q(x, r) := \left\{ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : \max_{i=1,2,\ldots,n} |x_i - y_i| \le r \right\}$$

for $x = (x_1, x_2, \ldots, x_n)$ and $r > 0$. The center and the side-length of $Q = Q(x, r)$ are given (respectively) by

$$c(Q) := x, \qquad \ell(Q) := 2r.$$

Given $\kappa > 0$ and a cube $Q = Q(x, r)$, we define $\kappa Q := Q(x, \kappa r)$.

2.2. Notation on Tiles and Trees

The key tool for our analysis is a decomposition technique using trees. The notion of trees can be traced back to the seminal papers [7; 8; 10; 11; 12].

DEFINITION 2.4 [7; 8; 10; 11; 12; 13].

1. A *tile* is a cross product of the form $s = Q_{\nu m} \times Q_{-\nu m'}$ with $\nu \in \mathbb{Z}$ and $m, m' \in \mathbb{Z}^n$. Given such a tile s, we define $I_s := Q_{\nu m}$ and $\omega_s := Q_{-\nu m'}$. The set of all tiles is denoted by $\mathbb{D}$.
2. Let $u, v \in \mathbb{D}$. Then $u \le v$ if and only if $I_u \subset I_v$ and $\omega_u \supset \omega_v$.
3. A *tree* is a pair $(\mathbb{T}, t)$ such that $\mathbb{T} \subset \mathbb{D}$ is a finite subset of $\mathbb{D}$ and $t \in \mathbb{D}$ is a tile with $t \ge s$ for all $s \in \mathbb{T}$. Define $\omega_{\mathbb{T}} := \omega_t$ and $I_{\mathbb{T}} := I_t$.
4. Let $1 \le i \le 2^n$. A tree $(\mathbb{T}, t)$ is called an *i-tree* if $\omega_{t(i)} \subset \omega_{s(i)}$ for all $s \in \mathbb{T}$.

Occasionally t is called a *top* of $\mathbb{T}$. Note that the top of $\mathbb{T}$ is not unique in general. In this paper, to avoid confusion, when we call a pair $(\mathbb{T}, t)$ a tree we are specifying the top.

2.3. *Notation for Auxiliary Functions*

We assume throughout that $\Phi \in \mathcal{S}$ is a function satisfying

$$\chi_{Q(9/100)} \le \Phi \le \chi_{Q(1/10)}.$$

DEFINITION 2.5.

1. $\varphi := \mathcal{F}^{-1}\Phi$.
2. $\Psi := \Phi - \Phi(2\cdot)$.
3. Given a cube Q, define $\Phi_Q(\xi) := \Phi\left(\frac{\xi - c(Q)}{\ell(Q)}\right)$.
4. Given a tile $s \in \mathbb{D}$, define $\varphi_s(x) := M_{c(\omega_{s(1)})} T_{c(I_s)} D_{\ell(I_s)} \varphi(x)$ (cf. [13]).

The following property is easily shown.

LEMMA 2.6.

1. *If Q is a cube, then*
$$\chi_{(27/25)Q} \le \Phi_{6Q} \le \chi_{(6/5)Q}. \tag{4}$$
2. *If s is a tile, then*
$$\mathcal{F}\varphi_s = T_{c(\omega_{s(1)})} M_{-c(I_s)} D_{\ell(\omega_s)} \Phi. \tag{5}$$
In particular, $\mathrm{supp}(\mathcal{F}\varphi_s) \subset \frac{1}{5}\omega_{s(1)}$.

According to (4), Φ_{6Q} is almost the same as χ_Q. Meanwhile, (5) implies that the frequency support of φ_s is concentrated near $c(\omega_{s(1)})$.

The following lemma is easy to show by using the Plancherel theorem.

LEMMA 2.7. *Let $\xi \in \mathbb{R}^n$. Then*

$$\left(\sum_{s \in \mathbb{D} : \omega_{s(2^n)} \ni \xi} |\langle f, \varphi_s \rangle_{L^2}|^2 \right)^{1/2} \lesssim \|f\|_2.$$

Next, we consider the model operator.

DEFINITION 2.8. The (model) dyadic operator is given by

$$A_{\xi,\mathbb{P}} f(x) := \sum_{s \in \mathbb{P} : \omega_{s(2^n)} \ni \xi} \langle f, \varphi_s \rangle_{L^2} \varphi_s, \quad \mathbb{P} \subset \mathbb{D}, \ \xi \in \mathbb{R}^n.$$

LEMMA 2.9 [13]. *$A_{\xi,\mathbb{P}}$ is L^2-bounded uniformly over $\mathbb{P} \subset \mathbb{D}$ and $\xi \in \mathbb{R}^n$:*

$$\|A_{\xi,\mathbb{P}} : B(L^2)\| \lesssim 1,$$

where $B(L^2)$ denotes the set of all bounded linear operators in L^2.

Proof. It is convenient to rely on the molecular decomposition described in [15]. When we consider that decomposition for $\dot{F}^0_{22} \simeq L^2$, we must consider the moment condition for molecules. However, this requirement is satisfied by virtue of φ_s having frequency support outside the origin. Meanwhile, the condition for the coefficients is satisfied because of Lemma 2.7. An alternative proof uses Lemma 2.7 and the almost-orthogonality. □

2.4. Integral Kernel of $a(x, D)$

We define

$$a_j(x,D)f(x) := \int_{\mathbb{R}^n\times\mathbb{R}^n} a(x,\xi)\Psi(2^{-j}\xi)\exp(2\pi i\xi\cdot(x-y))f(y)\,dy\,d\xi, \quad j\in\mathbb{Z}^n,$$

where $\Psi = \Phi - \Phi(2\cdot)$ (see Definition 2.5(1)). Then we have

$$a_j(x,D)f(x) = \int k_j(x,x-z)f(z)\,dz.$$

The integral kernel can be written as

$$k_j(x,z) := \int_{\mathbb{R}^n} a(x,\xi)\Psi(2^{-j}\xi)\exp(2\pi i\xi\cdot z)\,d\xi.$$

Using integration by parts yields the following estimate.

LEMMA 2.10. *Let $\alpha,\beta\in(\mathbb{N}_0)^n$ and $L\in\mathbb{N}_0$. Then*

$$|\partial_x^\alpha\partial_z^\beta k_j(x,z)| \lesssim_{\alpha,\beta,L} 2^{j(n+|\alpha|+|\beta|)-2jL}|z|^{-2L}.$$

A direct consequence of this lemma is that

$$\sum_{j=-\infty}^{\infty}|\partial_x^\alpha\partial_z^\beta k_j(x,z)| \lesssim_{\alpha,\beta} |z|^{-(n+|\alpha|+|\beta|)}.$$

Proof of Lemma 2.10. Assuming that the frequency support of Ψ is compact and does not contain 0, we have

$$\partial_x^\alpha\partial_z^\beta k_j(x,z) = \int_{\mathbb{R}^n}\partial_x^\alpha a(x,\xi)(2\pi i\xi)^\beta\Psi(2^{-j}\xi)\exp(2\pi i\xi\cdot z)\,d\xi.$$

From (1) we deduce that

$$|\Delta_\xi^L[\partial_x^\alpha a(x,\xi)(2\pi i\xi)^\beta\Psi(2^{-j}\xi)]| \lesssim_{\alpha,\beta,L} 2^{j(|\alpha|+|\beta|-2L)}. \tag{6}$$

An integration by parts then yields

$$\begin{aligned}\partial_x^\alpha\partial_z^\beta k_j(x,z) &= \frac{1}{(-4\pi^2|z|^2)^L}\int_{\mathbb{R}^n}\partial_x^\alpha a(x,\xi)(2\pi i\xi)^\beta\Psi(2^{-j}\xi)\Delta_\xi^L\exp(2\pi i\xi\cdot z)\,d\xi\\ &= \frac{1}{(-4\pi^2|z|^2)^L}\int_{\mathbb{R}^n}\Delta_\xi^L[\partial_x^\alpha a(x,\xi)(2\pi i\xi)^\beta\Psi(2^{-j}\xi)]\exp(2\pi i\xi\cdot z)\,d\xi.\end{aligned}$$

If we insert (6) and take into account the size of support, we obtain

$$|\partial_x^\alpha\partial_z^\beta k_j(x,z)| \lesssim |z|^{-2L}2^{j(|\alpha|+|\beta|-2L)}\int_{2^{-j}} d\xi \lesssim |z|^{-2L}2^{j(n+|\alpha|+|\beta|-2L)}.$$

This concludes the proof. □

Let us set $k(x,z) := \sum_{j=-\infty}^{\infty} k_j(x,z)$ and write $a(x,D)$ as

$$a(x,D)f(x) = \int k(x,x-z)f(z)\,dz, \quad x \notin \mathrm{supp}(f)$$

in terms of the integral kernel. Recall that $a(x,D)$ has been shown to be L^2-bounded (see [5]). As a consequence, we have

$$\int_{\mathbb{R}^n} \sup_{\varepsilon>0} |a(x,D)[\chi_{\mathbb{R}^n \setminus Q(x,\varepsilon)} f](x)|^2\,dx \lesssim \int_{\mathbb{R}^n} |f(x)|^2\,dx, \tag{7}$$

which is also known as the maximal estimate of the truncated singular integral operator (see [14]).

3. Simplified Phase Decomposition Formula and Some Reductions of Theorem 1.2

In this section we obtain a simplified phase decomposition formula. We follow the notation of Section 2.

3.1. Simplified Phase Decomposition Formula

DEFINITION 3.1. The model operator $A_{\eta,l}$ of the lth generation is defined by

$$A_{\eta,l}f(x) := \sum_{s\in\mathbb{D}:\,\omega_{s(2^n)}\ni\eta,\,|I_s|=2^{ln}} \langle f,\varphi_s\rangle_{L^2}\varphi_s.$$

LEMMA 3.2. *There exists a function $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ such that*

$$\lim_{N\to\infty} \int_{Q_N} M_{-\eta}A_{\eta,l}M_\eta f \,\frac{d\eta}{|Q_N|} = \mathcal{F}^{-1}[m(2^l\cdot)\cdot\mathcal{F}f]$$

for any sequence of cubes $\{Q_N\}_{N\in\mathbb{N}}$ such that $2Q_N \subset Q_{N+1}$ for all $N \in \mathbb{N}$, where the convergence takes place in the strong topology of L^2.

Proof. Because the family of operators

$$\left\{\int_{Q_N} M_{-\eta}A_{\eta,l}M_\eta \,\frac{d\eta}{|Q_N|}\right\}_{N\in\mathbb{N}}$$

is uniformly bounded in $B(L^2)$, we can assume that $f \in \mathcal{S}_0$ in order to investigate the limit as $N \to \infty$. Consider

$$\mathcal{F}\left(\int_{Q_N} M_{-\eta}A_{\eta,l}M_\eta \,\frac{d\eta}{|Q_N|}\right)\mathcal{F}^{-1}f = \int_{Q_N} \mathcal{F}M_{-\eta}A_{\eta,l}M_\eta\mathcal{F}^{-1}f\,\frac{d\eta}{|Q_N|}.$$

We denote by $Q_l = Q_l(\eta)$ the unique dyadic cube with $\ell(Q_l) = 2^{-l}$ such that $\eta \in Q_{l(2^n)}$, if such a cube exists. Assuming the existence of such a Q_l, we can use the Fourier expansion and (5) to obtain

$$
\begin{aligned}
&\mathcal{F}M_{-\eta}A_{\eta,l}M_\eta\mathcal{F}^{-1}f\\
&\quad=\sum_{s\in\mathbb{D}:\,\omega_{s(2^n)}\ni\eta,\,|I_s|=2^{ln}}\langle M_\eta\mathcal{F}^{-1}f,\varphi_s\rangle_{L^2}\mathcal{F}M_{-\eta}\varphi_s\\
&\quad=\sum_{s\in\mathbb{D}:\,\omega_{s(2^n)}\ni\eta,\,|I_s|=2^{ln}}\langle f,\mathcal{F}M_{-\eta}\varphi_s\rangle_{L^2}\mathcal{F}M_{-\eta}\varphi_s\\
&\quad=\sum_{s\in\mathbb{D}:\,\omega_{s(2^n)}\ni\eta,\,|I_s|=2^{ln}}\langle f,T_{c(\omega_{s(1)})-\eta}M_{c(I_s)}D_{\ell(\omega_s)}\Phi\rangle_{L^2}T_{c(\omega_{s(1)})-\eta}M_{c(I_s)}D_{\ell(\omega_s)}\Phi\\
&\quad=f\cdot\left|\Phi\left(\frac{\cdot+\eta-c(Q_{l(1)})}{\ell(Q_l)}\right)\right|^2.
\end{aligned}
$$

Inserting this equality, we obtain

$$
\mathcal{F}\left(\int_{Q_N}M_{-\eta}A_{\eta,l}M_\eta\,\frac{d\eta}{|Q_N|}\right)\mathcal{F}^{-1}f=m_l\cdot f,
$$

where

$$
m_l:=2^{ln}\int_{Q_{l+1,\mathbf{1}}}|\Phi(2^l(\cdot+\eta-2^{-2}\mathbf{1}))|^2\,d\eta=\int_{(\mathbf{1}/2)+Q(1/4)}|\Phi(2^l\cdot+\zeta)|^2\,d\zeta.
$$

Hence, we have the desired result with $m:=\int_{(\mathbf{1}/2)+Q(1/4)}|\Phi(\cdot+\zeta)|^2\,d\zeta$. □

COROLLARY 3.3. *With the same notation as in Lemma 3.2, define*

$$
M(\xi):=\sum_{l=-\infty}^{\infty}m(2^l\xi). \tag{8}
$$

Then

$$
\lim_{L\to\infty}\sum_{l=-L}^{L}\left(\lim_{N\to\infty}\int_{Q_N}M_{-\eta}A_{\eta,l}M_\eta f\,\frac{d\eta}{|Q_N|}\right)=\mathcal{F}^{-1}(M\cdot\mathcal{F}f),
$$

where the convergence takes place in the strong topology of L^2.

With this result, we can obtain a (simpler) decomposition of the phase space. Recall that $\mathrm{SO}(n)$ denotes the set of all orthogonal matrices with determinant 1. Since $\mathrm{SO}(n)$ is compact, it carries the normalized Haar measure μ. We define $\rho\colon \mathrm{SO}(n)\to U(L^2)$ as the unitary representation of $\mathrm{SO}(n)$; namely, we define

$$
\rho(A)f:=f(A^{-1}\cdot),\quad f\in L^2.
$$

COROLLARY 3.4. *With the same notation as in Lemma 3.2, let $\alpha>0$ be a constant given by*

$$
\alpha:=\int_{\mathrm{SO}(n)}\int_0^1 M(2^\kappa A\xi)\,d\kappa\,d\mu
$$

for $\xi\in\mathbb{R}^n\setminus\{0\}$. Then

$$
\begin{aligned}
&\alpha\,\mathrm{id}_{L^2}\\
&\quad=\int_{\mathrm{SO}(n)}\int_0^1\left(\sum_{l=-\infty}^{\infty}\lim_{N\to\infty}\int_{Q_N}\rho(A^{-1})D_{2^{-\kappa}}M_{-\eta}A_{\eta,l}M_\eta D_{2^\kappa}\rho(A)\,\frac{d\eta}{|Q_N|}\right)d\kappa\,d\mu,
\end{aligned}
$$

where all the convergences take place in the strong topology of L^2.

REMARK 3.5.

1. In view of (8), α does not depend on the ξ that appears in the formula defining α.
2. In [13], Pramanik and Terwilleger considered the average of

$$\rho(A^{-1})D_{2^{-\kappa}}T_{-y}M_{-\eta}A_{\eta,l}M_{\eta}T_yD_{2^{\kappa}}\rho(A).$$

However, as Corollary 3.4 here shows, there is no need to take the average over the time space $\mathbb{R}^n_y$. We shall take full advantage of this fact in the course of proving Theorem 1.2.

3.2. Some Reductions of Theorem 1.2

Corollary 3.4 is the simplified phase decomposition formula, which is beautiful in its own right. However, in this paper we discretize the formula. More precisely, we proceed as follows.

PROPOSITION 3.6. *Let* $\{A_n\}_{n\in\mathbb{N}}$ *and* $\{\kappa(n)\}_{n\in\mathbb{N}}$ *be dense subsets of* $\mathrm{SO}(n)$ *and* $[0,1]$, *respectively, such that* $A_1=\mathrm{id}_{\mathbb{R}^n}$ *and* $\kappa(1)=0$. *Then*

$$m_K(\xi) := \sum_{k_1,k_2=1}^{K} M(2^{\kappa(k_1)}A_{k_2}^{-1}\xi) \tag{9}$$

satisfies the following conditions, provided K is sufficiently large:

1. $c_{\alpha,\beta}(m_K)<\infty$ *for all* $\alpha,\beta\in(\mathbb{N}_0)^n$;
2. $\inf_{\xi\in\mathbb{R}^n\setminus\{0\}} m_K(\xi)>0$.

Proof. If $\alpha\neq 0$, then $c_{\alpha,\beta}(m_K)=0$. Therefore, to establish part 1, we may assume that $\alpha=0$. Note that $\partial^\beta M(\xi)=\sum_{l=-\infty}^{\infty}2^{l|\beta|}\partial^\beta m(2^l\xi)$. Consequently,

$$|\partial^\beta M(\xi)|\le\sum_{l=-\infty}^{\infty}2^{l|\beta|}|\partial^\beta m(2^l\xi)|\le|\xi|^{-\beta}\sum_{l=-\infty}^{\infty}|2^l\xi|^{|\beta|}|\partial^\beta m(2^l\xi)|.$$

Now that the function $\tilde M_\beta(\xi)=\sum_{l=-\infty}^{\infty}|2^l\xi|^{|\beta|}|\partial^\beta m(2^l\xi)|$ satisfies $\tilde M_\beta(2\xi)=\tilde M_\beta(\xi)$, we have

$$\sup_{\xi\in\mathbb{R}^n}|\tilde M_\beta(\xi)|=\sup_{1\le|\xi|\le 2}|\tilde M_\beta(\xi)|\lesssim_\beta 1.$$

Hence it follows that $|\partial^\beta M(\xi)|\lesssim_\beta|\xi|^{-|\beta|}$. This establishes part 1 of the proposition, since we are considering a finite sum with respect to k_1 and k_2.

We now turn to the proof of part 2. Since $\{A_n\}_{n\in\mathbb{N}}$ and $\{\kappa(n)\}_{n\in\mathbb{N}}$ are the respective dense subsets of $\mathrm{SO}(n)$ and $[0,1]$, it follows that

$$\{1\le|\xi|\le 2\}\subset\bigcup_{k_1=1}^{\infty}\bigcup_{k_2=1}^{\infty}\bigcup_{l=-\infty}^{\infty}\mathrm{Int}\{\mathrm{supp}[m(2^{l+\kappa(k_j)}A_{k_2}^{-1}\cdot)]\}.$$

Hence, we can find a large K such that

$$\{1\le|\xi|\le 2\}\subset\bigcup_{k_1=1}^{K}\bigcup_{k_2=1}^{K}\bigcup_{l=-\infty}^{\infty}\mathrm{Int}\{\mathrm{supp}[m(2^{l+\kappa(k_j)}A_{k_2}^{-1}\cdot)]\}$$

by virtue of the compactness of $\{1 \le |\xi| \le 2\}$. With this choice of K, we have $m_K(\xi) > 0$ for $1 \le |\xi| \le 2$. Since the support of each summand in the sum defining m_K is disjoint, it follows that m_K is continuous. Hence $m_K(\xi) > \delta$, $1 \le |\xi| \le 2$, for some $\delta > 0$. In view of the periodicity $m_K(2\xi) = m_K(\xi)$, we have $\inf_{\xi \in \mathbb{R}^n \setminus \{0\}} m_K(\xi) = \inf_{1 \le |\xi| \le 2} m_K(\xi) > \delta$. $\square$

Given Proposition 3.6, we set $b(x,\xi) := a(x,\xi)/m_K(\xi)$. Then

$$a(x,D) = \sum_{k_1,k_2=1}^{K} \sum_{l=-\infty}^{\infty} \lim_{N\to\infty} \int_{K_N} b(x,D)\rho(A_{k_1}^{-1}) D_{2^{-\kappa(k_2)}} M_{-\eta} A_{\eta,l} M_\eta D_{2^{\kappa(k_2)}} \rho(A_{k_1}) \frac{d\eta}{|K_N|}.$$

Let us consider the summand for $k_1 = k_2 = 1$ (the other summand can be dealt with similarly). We shall deal with

$$\sum_{l=-\infty}^{\infty} \lim_{N\to\infty} \int_{K_N} b(x,D) M_{-\eta} A_{\eta,l} M_\eta \frac{d\eta}{|K_N|},$$

which is equal to

$$\sum_{l=-\infty}^{\infty} \lim_{N\to\infty} \int_{K_N} M_{-\eta} b(x,D-\eta) A_{\eta,l} M_\eta \frac{d\eta}{|K_N|}.$$

Recall that the main theorem concerns the conjugated modulation. So, we are led to consider

$$\sum_{l=-\infty}^{\infty} \lim_{N\to\infty} \int_{K_N} M_{-\xi-\eta} b(x,D-\eta) A_{\eta,l} M_{\eta+\xi} \frac{d\eta}{|K_N|} = \sum_{l=-\infty}^{\infty} \lim_{N\to\infty} \int_{K_N} M_{-\eta} b(x,D-\eta+\xi) A_{\eta-\xi,l} M_\eta \frac{d\eta}{|K_N|}.$$

Here the equality holds by virtue of Lemma 3.2.

Define a norm by

$$\|f : L^{2,\infty}\|^* := \sup_E |E|^{-1/2} \int_E |f|.$$

Here E in sup runs over all the nonempty bounded measurable sets. Then, the weak-L^2 quasi-norm is equivalent to this norm (see [3]). Furthermore, if f is locally square integrable, then

$$\|f : L^{2,\infty}\|^* \simeq \sup_E |E|^{-1/2} \left| \int_E f \right|. \tag{10}$$

In view of Proposition 3.6, the functions a and b enjoy the same property:

$$c_{\alpha,\beta}(a) \simeq_{\alpha,\beta} c_{\alpha,\beta}(b)$$

for all $\alpha, \beta \in (\mathbb{N}_0)^n$. Hence, it is sufficient to show that

$$\sup_E |E|^{-1/2} \int_E \sup_{\xi \in \mathbb{R}^n} \left| \sum_{l=-L}^{L} a(x,D-\xi) A_{\xi,l} f(x) \right| dx \lesssim \|f\|_2.$$

Since there exists a measurable mapping $N\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\sup_{\xi\in\mathbb{R}^n}\left|\sum_{l=-L}^{L} a(x,D-\xi)A_{\xi,l}f(x)\right| \le 2\left|\sum_{l=-L}^{L} a(x,D-\xi)A_{\xi,l}f(x)|_{\xi=N(x)}\right|,$$

we have only to show that

$$\sup_E |E|^{-1/2}\left|\int_E \sum_{l=-L}^{L} a(x,D-\xi)A_{\xi,l}f(x)|_{\xi=N(x)}\,dx\right| \lesssim \|f\|_2.$$

Taking into account Lemma 3.8 (to follow), we conclude that

$$a(x,D-\xi)\sum_{l=-L}^{L} A_{\xi,l}f(x) = \lim_{M\to\infty} \sum_{\substack{s\in\mathbb{D}:\omega_{s(2^n)}\ni\xi \\ 2^{-L}\le\ell(I_s)\le 2^L,\,|c(I_s)|\le M}} \langle f,\varphi_s\rangle_{L^2}\, a(x,D-\xi)\varphi_s$$

converges pointwise. Hence, we need only establish that

$$\sup_{\mathbb{P}\subset\mathbb{D}}\left|\sum_{s\in\mathbb{P}}\langle f,\varphi_s\rangle_{L^2}\int_{\mathbb{R}^n}\chi_{N^{-1}[\omega_{s(2^n)}]\cap E}(x)a(x,D-\xi)\varphi_s(x)|_{\xi=N(x)}\,dx\right| \lesssim |E|^{1/2}\|f\|_2,$$

where $\mathbb{P}\subset\mathbb{D}$ runs over any finite set. Finally, by scaling we can assume that $|E|\le 1$. We refer to [13, p. 780] for more details of this dilation technique.

With this in mind, we shall prove the following in Section 5.

THEOREM 3.7 (Basic estimate). *Let $N\colon \mathbb{R}^n \to \mathbb{R}^n$ be a measurable mapping and E a bounded measurable subset whose volume is less than 1. Then*

$$\sum_{s\in\mathbb{D}}\left|\langle f,\varphi_s\rangle_{L^2}\int_{\mathbb{R}^n}\chi_{N^{-1}[\omega_{s(2^n)}]\cap E}(x)a(x,D-\xi)\varphi_s(x)|_{\xi=N(x)}\,dx\right| \lesssim \|f\|_2.$$

In this paper we fix a measurable mapping $N\colon \mathbb{R}^n \to \mathbb{R}^n$ and a bounded measurable set E with volume less than 1. To simplify the notation, we define $E_{s(2^n)} := N^{-1}[\omega_{s(2^n)}]\cap E$ and

$$\psi_s^{\xi}(x) := a(x,D-\xi)\varphi_s(x), \qquad \psi_s^{N(\cdot)}(x) := \psi_s^{\xi}(x)|_{\xi=N(x)}.$$

As for ψ_s^{ξ}, we have the following pointwise estimate.

LEMMA 3.8. *Let $\xi\in\omega_{s(2^n)}$ and $\alpha\in(\mathbb{N}_0)^n$. Then we have*

$$|\partial^\alpha\psi_s^{\xi}(x)| \lesssim_L |I_s|^{-1/2}\left(1+\frac{|x-c(I_s)|}{\ell(I_s)}\right)^{-L}$$

for all $L\in\mathbb{N}$.

Proof. We have only to deal with the case $\alpha = 0$, because the passage to the higher derivatives follows immediately from the calculation for $\alpha = 0$. With ξ fixed, we can integrate by parts. More precisely, we proceed as follows. We need to show that

$$|\psi_s^\xi(x)| \lesssim_L |I_s|^{-1/2} \frac{\ell(I_s)^{2L}}{|x-c(I_s)|^{2L}} \tag{11}$$

for all $L \in \mathbb{N}_0$.

First, we write out $\psi_s^\xi(x)$ in full:

$$\psi_s^\xi(x) = |I_s|^{1/2} \int_{\mathbb{R}^n} \exp(2\pi i(x-c(I_s))\cdot\eta) a(x,\eta-\xi)\,\Phi(\ell(I_s)(\eta-c(\omega_{s(1)})))\,d\eta.$$

Now we use

$$\Delta_\eta^L \exp(2\pi i(x-c(I_s))\cdot\eta) = (-4\pi^2|x-c(I_s)|^2)^L \exp(2\pi i(x-c(I_s))\cdot\eta)$$

to integrate by parts and use the triangle inequality for integrals. The result is

$$|\psi_s^\xi(x)| \le \frac{|I_s|^{1/2}}{(-4\pi^2|x-c(I_s)|^2)^L} \int_{\mathbb{R}^n} |\Delta^L[a(x,\eta-\xi)\,\Phi(\ell(I_s)(\eta-c(\omega_{s(1)})))]|\,d\eta. \tag{12}$$

Now we invoke the assumption (1) and use that $\eta \in \omega_{s(2)}$ and $\Psi(\ell(I_s)(\cdot - c(\omega_{s(1)})))$ is supported on $\frac{1}{5}\omega_{s(1)}$ to conclude that

$$|\Delta^L[a(x,\eta-\xi)\,\Phi(\ell(I_s)(\eta-c(\omega_{s(1)})))]| \lesssim \ell(I_s)^{-2L}\chi_{\omega_s}(\eta).$$

If we insert this pointwise inequality to (12), we obtain (11). □

An immediate corollary of this estimate is the following.

COROLLARY 3.9. *Let $\mathbb{P}$ be a finite subset of $\mathbb{D}$. Then*

$$\left\| \sum_{s\in\mathbb{D}} \langle f,\varphi_s\rangle \psi_s^\xi \right\|_2 \lesssim \|f\|_2,$$

where the implicit constant does not depend on $\xi \in \mathbb{R}^n$ or $s \in \mathbb{D}$.

Proof. This is another application of molecular decomposition. Here we need the assumption (2) for the molecules to satisfy the moment condition. □

Following the notation in [9], we define

$$\mathrm{Sum}(\mathbb{P}) := \sum_{s\in\mathbb{P}} |\langle f,\varphi_s\rangle_{L^2}| \cdot |\langle \psi_s^{N(\cdot)}, \chi_{E_{s(2^n)}}\rangle_{L^2}|$$

for $\mathbb{P} \subset \mathbb{D}$. We shall establish

$$\mathrm{Sum}(\mathbb{P}) \lesssim \|f\|_2 \tag{13}$$

for any finite subset $\mathbb{P}$ instead of proving Theorem 3.7 directly.

4. Cotlar-type Estimate

In this section we obtain a Cotlar-type estimate. We let

$$a_{\eta,\tau,\ell}(x,\xi) := a(x,\xi-\eta)\,\Phi\left(\frac{\xi-\tau}{6\ell}\right), \qquad a_{s,\eta}(x,\xi) := a_{\eta,c(\omega_s),\ell(\omega_s)}(x,\xi)$$

for $\ell > 0$, $\eta, \tau \in \mathbb{R}^n$, and $s \in \mathbb{D}$. To formulate our result, we use the maximal operator $M_{\geq b}$ given by

$$M_{\geq b} f(x) := \sup_{r \geq b} \frac{1}{r^n} \int_{Q(x,r)} |f(y)|\, dy = \sup_{r \geq b} \frac{1}{r^n} \int_{Q(r)} |f(x+y)|\, dy$$

for $b > 0$. We prove the following estimate.

PROPOSITION 4.1. *Let $u, v \in \mathbb{D}$ with $u \leq v$. Suppose that $y \in \mathbb{R}^n$ and $\eta_0, \eta_1 \in \omega_v$. Then*

$$\begin{aligned} &|a_{v,\eta_0}(x,D) f(y) - a_{u,\eta_0}(x,D) f(y)| \\ &\qquad \lesssim \inf_{z \in Q(y,\ell(I_u))} \Big(M_{\geq \ell(I_u)} f(z) + \sup_{\varepsilon > 0} |a(x, D-\eta_1)[\chi_{\mathbb{R}^n \setminus Q(z,\varepsilon)} f](z)| \Big). \end{aligned}$$

4.1. Maximal Operator $M_{\geq b}$

In this section we frequently use the following estimates.

LEMMA 4.2.

1. *Let $a > 0$ and $L > n$. Then*

$$\int_{\mathbb{R}^n \setminus Q(x,a)} \frac{a^{L-n} |f(y)|}{|x-y|^L}\, dy \lesssim_L M_{\geq a} f(x). \tag{14}$$

2. *Let $b > a > 0$. Then*

$$\int_{Q(x,b) \setminus Q(x,a)} \frac{|f(y)|}{b|x-y|^{n-1}}\, dy \lesssim M_{\geq a} f(x). \tag{15}$$

Proof. For the proof of (15), we may assume that $a = 2^{-l} b$ for some $l \in \mathbb{N}$ by replacing a with a number slightly less than a. Both cases can be proved easily by decomposing

$$\int_{\mathbb{R}^n \setminus Q(x,a)} = \sum_{j=1}^{\infty} \int_{Q(x,2^j a) \setminus Q(x,2^{j-1} a)},$$

$$\int_{Q(x,b) \setminus Q(x,a)} = \sum_{j=1}^{l} \int_{Q(x,2^{1-j} b) \setminus Q(x,2^{-j} b)}.$$

Using this decomposition, we can prove (14) and (15) easily. We omit the details. $\square$

LEMMA 4.3. *Let $a, b > 0$, $s \in \mathbb{D}$, and $y, y^*, \eta, \tau \in \mathbb{R}^n$. Then*

$$|a_{\eta,\tau,a\ell(\omega_s)}(x,D)[\chi_{Q(y,\ell(I_s))} f](y^*)| \lesssim_{a,b} M_{\geq \ell(I_s)} f(y) \tag{16}$$

and

$$|(a(x,D-\eta) - a_{\eta,\tau,\ell(\omega_s)}(x,D))[\chi_{\mathbb{R}^n \setminus Q(y,\ell(I_s))} f](y)| \lesssim M_{\geq \ell(I_s)} f(y) \tag{17}$$

whenever $|y - y^*| \lesssim_b \ell(I_s)$.

Proof. By the triangle inequality we have

$$\text{LHS of (16)} \le \int_{Q(y,\ell(I_s))} \left(\int_{\mathbb{R}^n} \left| \Phi\left(\frac{\xi - c(\omega_s)}{6a\ell(\omega_s)} \right) \right| d\xi \right) |f(z)|\, dz,$$

from which we easily obtain (16).

As for (17), we decompose

$$a(x, D-\eta) - a_{\eta,\tau,\ell(\omega_s)}(x,D) = \sum_{j=1}^{\infty} a_{\eta,\tau,2^j\ell(\omega_s)}(x,D) - a_{\eta,\tau,2^{j-1}\ell(\omega_s)}(x,D).$$

Observe that the integral kernel $k_j(x,z)$ of $a_{\eta,\tau,2^j\ell(\omega_s)}(x,D) - a_{\eta,\tau,2^{j-1}\ell(\omega_s)}(x,D)$ has the naive bound

$$|k_j(x,z)| \lesssim_L (2^j\ell(\omega_s))^{n-2L}|x-z|^{-2L} \tag{18}$$

for each $L \in \mathbb{N}$. This inequality is summable if $L \ge n$, and we obtain

$$\text{LHS of (17)} \lesssim \int_{\mathbb{R}^n \setminus Q(y,\ell(I_s))} \frac{|f(z)|\, dz}{\ell(I_s)^{n-2L}|z-y|^{2L}} \lesssim M_{\ge \ell(I_s)} f(y).$$

This shows (17), completing the proof of the lemma. □

The following estimate can be obtained by using the same ideas.

LEMMA 4.4. *Suppose that $u \le v$ and $\eta \in \omega_v$. Then*

$$|(a_{\eta,\eta,\ell(\omega_u)}(x,D) - a_{\eta,\eta,\ell(\omega_v)}(x,D))[\chi_{\mathbb{R}^n \setminus Q(y,\ell(I_v))} f](y)| \lesssim M_{\ge \ell(I_u)} f(y).$$

Proof. Let $N = \log_2 \ell(\omega_u) - \log_2 \ell(\omega_v)$. Then

$$a_{\eta,\tau,\ell(\omega_u)}(x,D) - a_{\eta,\tau,\ell(\omega_v)}(x,D) = \sum_{j=1}^{N} a_{\eta,\tau,2^j\ell(\omega_v)}(x,D) - a_{\eta,\tau,2^{j-1}\ell(\omega_v)}(x,D),$$

which yields

$$\begin{aligned} &|a_{\eta,\tau,\ell(\omega_u)}(x,D) f(y) - a_{\eta,\tau,\ell(\omega_v)}(x,D) f(y)| \\ &\qquad \le \sum_{j=1}^{\infty} |a_{\eta,\tau,2^j\ell(\omega_v)}(x,D) f(y) - a_{\eta,\tau,2^{j-1}\ell(\omega_v)}(x,D) f(y)|. \end{aligned}$$

Now we have only to appeal to (18) and the same argument as before works here. □

LEMMA 4.5. *Let $s \in \mathbb{D}$ and $\eta \in \omega_s$. Then*

$$|(a_{s,\eta}(x,D) - a_{\eta,\eta,\ell(\omega_s)}(x,D)) f(y)| \lesssim M_{\ge \ell(I_s)} f(y)$$

for all $y \in \mathbb{R}^n$.

Proof. First, we write the left-hand side out in full:

$$\begin{aligned} &(a_{s,\eta}(x,D) - a_{\eta,\eta,\ell(\omega_s)}(x,D)) f(y) \\ &\qquad = \int_{\mathbb{R}^n} \exp(2\pi i (y-z)\cdot\xi) A(x,\xi,\eta;s) f(z)\, dz\, d\eta, \end{aligned}$$

where

$$A(x,\xi,\eta;s) = a(x,\xi-\eta)\left\{\Phi\left(\frac{\xi-c(\omega_s)}{6\ell(\omega_s)}\right) - \Phi\left(\frac{\xi-\eta}{6\ell(\omega_s)}\right)\right\}.$$

Note that if $|\xi-\eta| \le \frac{1}{100}\ell(\omega_s)$ then $\xi \in \frac{27}{25}\omega_s$, since $\eta \in \omega_s$. Hence it follows that

$$\Phi\left(\frac{\xi-c(\omega_s)}{6\ell(\omega_s)}\right) = \Phi\left(\frac{\xi-\eta}{6\ell(\omega_s)}\right) = 1.$$

With this in mind, we invoke again the assumption (1) and conclude that

$$|\Delta^L A(x,\xi,\eta;s)| \lesssim_L \ell(\omega_s)^{-2L}\chi_{10\omega_s}(x).$$

Inserting this formula and then integrating by parts, we obtain

$$|(a_{s,\eta}(x,D) - a_{\eta,\eta,\ell(\omega_s)}(x,D))f(y)| \lesssim \frac{1}{|I_s|}\int_{\mathbb{R}^n}\left(1+\frac{|y-z|^{2L}}{\ell(I_s)^{2L}}\right)|f(z)|\,dz.$$

Hence, a dyadic partition of this integral yields the desired result. □

4.2. *Proof of Proposition 4.1*

Fix a point $z \in Q(y,\ell(I_u))$. In view of Lemmas 4.3, 4.4, and 4.5, it is sufficient to prove Proposition 4.1 assuming that f is supported outside $Q(z,2\ell(I_u))$. Note that

$$M_{\ge\ell(I_u)}f(y) \simeq_\kappa M_{\ge\ell(I_u)}f(y^*)$$

whenever $|y-y^*| \le \kappa b$. We seek to establish that

$$|a_{\eta_0,\eta_0,\ell(I_v)}(x,D)f(y) - a_{\eta_0,\eta_0,\ell(I_u)}(x,D)f(y)| \\ \lesssim M_{\ge\ell(I_u)}f(y) + |a(x,D-\eta_1)f(z)|,$$

which immediately yields Proposition 4.1. For the time being, we concentrate on reducing the matter to the case when $\eta_0 = \eta_1$.

LEMMA 4.6. *Let $u \le v \in \mathbb{D}$ and $\eta_0,\eta_1 \in \omega_v$. Set*

$$A_{\eta_0,\eta_1,u,v}(x,D) := \sum_{k,l=0,1}(-1)^{k+l}a_{\eta_k,\eta_l,\ell(I_u)}(x,D).$$

Then we have

$$|A_{\eta_0,\eta_1,u,v}(x,D)[\chi_{Q(y,\ell(I_v))}f](y)| \lesssim M_{\ge\ell(I_u)}f(y).$$

Proof. Note that $A_{\eta_0,\eta_1,u,v}(x,D)$ can be written as

$$A_{\eta_0,\eta_1,u,v}(x,D)[\chi_{Q(y,\ell(I_v))}f](y) \\ = \sum_{j=1}^{\log_2(\ell(I_v)/\ell(I_u))}\int_{Q(y,\ell(I_v))\setminus Q(y,\ell(I_u))}\left(\int_{\mathbb{R}^n}\alpha_j(y,y^*,\xi;\eta_0,\eta_1)\,d\xi\right)f(y^*)\,dy^*,$$

where

$$\alpha_j(y,y^*,\xi;\eta_0,\eta_1) \\ := -a(y,\xi)\Psi\left(\frac{\xi}{3\cdot 2^{j+1}\ell(\omega_u)}\right) \\ \times\{\exp(2\pi i(\xi+\eta_0)\cdot(y-y^*)) - \exp(2\pi i(\xi+\eta_1)\cdot(y-y^*))\}.$$

An integration by parts yields

$$|\alpha_j(y, y^*, \xi; \eta_0, \eta_1)| \lesssim_L |y - y^*|^{1-2L} \ell(\omega_v)(2^j \ell(\omega_u))^{n-2L}$$

for all $L \in \mathbb{N}$. If $L > n/2$, then this inequality is summable over $j \in \mathbb{N}$ and we have

$$\sum_{j=1}^{\infty} |\alpha_j(y, y^*, \xi; \eta_0, \eta_1)| \lesssim_L \ell(I_u)^{n+1} |y - y^*|^{-2n-1}.$$

Inserting this estimate and invoking (14), we obtain

$$|A_{u,v,\eta_0,\eta_1}(x, D) f(y)| \lesssim \int_{\mathbb{R}^n \setminus Q(y,\ell(I_u))} \frac{\ell(I_u)|f(y^*)|}{|y - y^*|^{n+1}}\, dy^* \lesssim M_{\geq \ell(I_u)} f(y).$$

This completes the proof. □

COROLLARY 4.7. *Suppose that $u \leq v$ and $\eta_0, \eta_1 \in \omega_v$. Then*

$$|(a_{u,\eta_0}(x, D) - a_{v,\eta_0}(x, D) - a_{u,\eta_1}(x, D) + a_{v,\eta_1}(x, D)) f(y)| \lesssim M_{\geq \ell(I_u)} f(y).$$

Proof. Combine Lemmas 4.3, 4.4, and 4.6. □

In view of Corollary 4.7, we can and do assume that $\eta_0 = \eta_1 = \eta \in \omega_v$ for the proof of Proposition 4.1.

LEMMA 4.8. *Let $s \in \mathbb{D}$. Then*

$$|a_{s,\eta}(x, D) f(y)| \lesssim M_{\geq \ell(I_s)} f(y) + |a(x, D - \eta) f(y)|$$

for all $y \in \mathbb{R}^n$.

Proof. This is an immediate consequence of (16) and (17). □

Proposition 4.1 will be proved completely once we establish the following.

LEMMA 4.9. *Let $s \in \mathbb{D}$. Then*

$$|a(x, D - \eta) f(y)| \lesssim M_{\geq \ell(I_s)} f(y) + |a(x, D - \eta) f(z)|$$

for all $z \in Q(y, \ell(I_s))$.

Proof. We shall control

$$|a_{\eta,\eta,\ell(\omega_s)}(x, D) f(y) - a_{\eta,\eta,\ell(\omega_s)}(x, D) f(z)|,$$

which is sufficient by virtue of (17). Note that

$$a_{\eta,\eta,\ell(\omega_s)}(x, D) f(y) - a_{\eta,\eta,\ell(\omega_s)}(x, D) f(z) = \int k(z^*) f(z^*)\, dz^*,$$

where

$$k(z^*) := \int_{\mathbb{R}^n} a(y, \xi - \eta) \Phi\left(\frac{\xi - \eta}{6\ell(\omega_s)}\right) \exp(2\pi i (y - z^*) \cdot \xi)\, d\xi$$
$$- \int_{\mathbb{R}^n} a(z, \xi - \eta) \Phi\left(\frac{\xi - \eta}{6\ell(\omega_s)}\right) \exp(2\pi i (z - z^*) \cdot \xi)\, d\xi.$$

Let us define

$$k_j(z^*) := -\int_{\mathbb{R}^n} a(y,\xi-\eta)\Psi\left(\frac{\xi-\eta}{3\cdot 2^{3-j}\ell(\omega_s)}\right)\exp(2\pi i(y-z^*)\cdot\xi)\,d\xi$$
$$+\int_{\mathbb{R}^n} a(z,\xi-\eta)\Psi\left(\frac{\xi-\eta}{3\cdot 2^{3-j}\ell(\omega_s)}\right)\exp(2\pi i(z-z^*)\cdot\xi)\,d\xi.$$

It then follows that $k=\sum_{j=1}^{\infty} k_j$.

A simple calculation now yields

$$|k_j(z^*)| \lesssim_L \ell(I_s)(2^{-j}\ell(\omega_s))^{n+1-2L}|y-z^*|^{-2L}$$

for all $L\in\mathbb{N}$. Interpolating this inequality with $L=0$ and $n+1$, we obtain

$$|k_j(z^*)| \lesssim_\theta \ell(I_s)(2^{-j}\ell(\omega_s))^{1-\theta}|y-z^*|^{-n-\theta}$$

for $0<\theta<1$ and hence

$$\sum_{j=1}^{\infty}|k_j(z^*)| \lesssim_\theta \ell(\omega_s)^{-\theta}|y-z^*|^{-n-\theta}.$$

As a result, we obtain

$$|a_{\eta,\eta,\ell(\omega_s)}(x,D)f(y)-a_{\eta,\eta,\ell(\omega_s)}(x,D)f(z)| \lesssim M_{\geq\ell(I_s)}f(y).$$

This is the desired result. □

5. Proofs of Theorems 1.2 and 3.7

In this section we shall prove Theorem 1.2 and Theorem 3.7, which are reduced to establishing (13).

5.1. *Review of* Size *and* Count

Definition 5.1 [8; 12; 13].

1. The density of a tile $s\in\mathbb{D}$ is defined by

$$\mathrm{dense}(s) := \int_{E\cap N^{-1}[\omega_s]}\left(1+\frac{|x-c(I_s)|}{\ell(I_s)}\right)^{-20n}\frac{dx}{|I_s|} \leq \left(\frac{2}{19}\right)^n.$$

2. Define

$$\mathrm{size}(\mathbb{T}_0) := \left(\sum_{s\in\mathbb{T}_0}\frac{|\langle f,\varphi_s\rangle_{L^2}|^2}{|I_t|}\right)^{1/2}$$

for an i-tree $(\mathbb{T}_0,t)$ with $2\leq i\leq 2^n$.

Definition 5.2 [8; 12; 13]. Let $\mathbb{P}$ be a subset of $\mathbb{D}$. Then define

$$\mathrm{Dense}(\mathbb{P}) := \sup_{s\in\mathbb{P}}\mathrm{dense}(s);$$

$$\mathrm{Size}(\mathbb{P}) := \sup\{\mathrm{size}(\mathbb{T}_0) : \mathbb{T}_0\subset\mathbb{P} \text{ and } (\mathbb{T}_0,t) \text{ is an } i\text{-tree with } 2\leq i\leq 2^n\};$$

$$\mathrm{Count}(\mathbb{P}) := \inf\left\{\sum_{j=1}^{J_0}|I_{t_j}| : \text{each } (\mathbb{T}_j,t_j) \text{ is a tree and } \mathbb{P}=\bigcup_{j=1}^{J_0}\mathbb{T}_j \text{ as a set}\right\}.$$

We now invoke the following crucial lemmas.

LEMMA 5.3 [13, Density lemma, Lemma 1]. *There exists a constant α with the following property: Any finite subset $\mathbb{T}$ admits a partition such that*

$$\mathbb{T} = \mathbb{T}_{\text{light}} \coprod \mathbb{T}_{\text{heavy}}, \qquad \text{Dense}(\mathbb{T}_{\text{light}}) \le \frac{1}{4}\text{Dense}(\mathbb{T}),$$
$$\text{Count}(\mathbb{T}_{\text{heavy}}) \le \frac{\alpha}{\text{Dense}(\mathbb{T})}.$$

LEMMA 5.4 [13, Size lemma, Lemma 2]. *There exists a constant β with the following property: Any finite subset $\mathbb{T}$ admits a partition such that*

$$\mathbb{T} = \mathbb{T}_{\text{small}} \coprod \mathbb{T}_{\text{large}}, \quad \text{Size}(\mathbb{T}_{\text{small}}) \le \frac{1}{2}\text{Size}(\mathbb{T}), \quad \text{Count}(\mathbb{T}_{\text{large}}) \le \frac{(\beta\|f\|_2)^2}{\text{Size}(\mathbb{T})^2}.$$

If we combine the density lemma and the size lemma, we obtain the following.

COROLLARY 5.5 [3; 13]. *Any finite subset $\mathbb{P} \subset \mathbb{D}$ admits the following decomposition:*

1. $\mathbb{P} = \coprod_{j=-\infty}^{\infty} \mathbb{P}_j$.
2. *Set* $\mathbb{U}_j := \mathbb{P} \setminus \coprod_{k=j}^{\infty} \mathbb{P}_k$. *Then*

$$\text{Dense}(\mathbb{U}_j) \le 4^j; \tag{19}$$
$$\text{Size}(\mathbb{U}_j) \le 2^j\|f\|_2. \tag{20}$$

3. $\text{Count}(\mathbb{P}_j) \le (\alpha+\beta)4^{-j}$.

Here the constants α and β are from Lemmas 5.3 and 5.4, respectively.

Although the proof is essentially contained in [3; 13], we outline it here for the convenience of readers.

Proof of Corollary 5.5. Assume j_0 is large enough that

$$\text{Dense}(\mathbb{P}) \le 4^{j_0}, \qquad \text{Size}(\mathbb{P}) \le 2^{j_0}\|f\|_2.$$

We define $\mathbb{P}_j := \emptyset$ for $j \ge j_0$. Assume that $\mathbb{P}_k$, $k \ge j$, is defined such that

$$\text{Dense}(\mathbb{U}_j) \le 4^j, \qquad \text{Size}(\mathbb{U}_j) \le 2^j\|f\|_2.$$

We use Lemmas 5.3 and 5.4 to define $\mathbb{P}_{j-1}$ as follows:

$$\mathbb{P}_{j-1} := \begin{cases} \emptyset & \text{if } \text{Dense}(\mathbb{U}_j) \le 4^{j-1} \text{ and } \text{Size}(\mathbb{U}_j) \le 2^{j-1}\|f\|_2, \\ (\mathbb{U}_j)_{\text{large}} & \text{if } \text{Dense}(\mathbb{U}_j) \le 4^{j-1} \text{ and } \text{Size}(\mathbb{U}_j) > 2^{j-1}\|f\|_2, \\ (\mathbb{U}_j)_{\text{heavy}} & \text{if } \text{Dense}(\mathbb{U}_j) > 4^{j-1} \text{ and } \text{Size}(\mathbb{U}_j) \le 2^{j-1}\|f\|_2, \\ (\mathbb{U}_j)_{\text{heavy}} \cup (\mathbb{U}_j)_{\text{large}} & \text{if } \text{Dense}(\mathbb{U}_j) > 4^{j-1} \text{ and } \text{Size}(\mathbb{U}_j) > 2^{j-1}\|f\|_2. \end{cases}$$

By Lemmas 5.3 and 5.4, we see that $\text{Count}(\mathbb{P}_{j-1}) \le 4(\alpha+\beta)\cdot 4^{-j}$ in any case. □

Next we review the results that will be used in the proof of Theorem 1.2 and Theorem 3.7.

To prove (13), it suffices to establish the following statement.

THEOREM 5.6. *There exists a γ with the property that, for $(\mathbb{T}, t)$ a tree,*

$$\mathrm{Sum}(\mathbb{T}) \le \gamma \,\mathrm{Dense}(\mathbb{T})\,\mathrm{Size}(\mathbb{T})|I_t|. \tag{21}$$

In particular,

$$\mathrm{Sum}(\mathbb{P}) \le \gamma \,\mathrm{Dense}(\mathbb{P})\,\mathrm{Size}(\mathbb{P})\,\mathrm{Count}(\mathbb{P}). \tag{22}$$

We remark that (22) is an immediate consequence of (21). Indeed, to obtain (22) we need only decompose $\mathbb{P}$ into a sequence of trees and then add (21) over those trees. Furthermore, once we obtain (22), we have

$$\mathrm{Sum}(\mathbb{P}_j) \le 4\gamma(\alpha+\beta)\|f\|_2 \min(2^{-j}, 2^j)$$

in the notation in Corollary 5.5. This inequality is summable over $j \in \mathbb{Z}$ to yield Theorem 3.7 and hence Theorem 1.2.

By linearization, (21) amounts to establishing

$$\left|\sum_{s\in\mathbb{T}} \alpha_s \langle f, \varphi_s\rangle_{L^2} \cdot \langle \psi_s^{N(\cdot)}, \chi_{E_{s(2^n)}}\rangle_{L^2}\right| \lesssim \mathrm{Dense}(\mathbb{T})\,\mathrm{Size}(\mathbb{T})|I_t| \tag{23}$$

for all sequences $\{\alpha_s\}_{s\in\mathbb{T}} \subset \Delta(1) := \{z \in \mathbb{C} : |z| < 1\}$.

5.2. *Partition $\mathcal{J}(\mathbb{T})$ of $\mathbb{R}^n$ and Further Reduction*

To proceed, we consider a partition of $\mathbb{R}^n$ associated with a tree $\mathbb{T}$.

LEMMA 5.7 [8; 10; 13]. *Suppose that $\mathbb{T}$ is a tree. Define*

$$\mathcal{J}_0(\mathbb{T}) := \{Q \in \mathcal{D} : I_s \text{ is not contained in } 3Q \text{ for all } s \in \mathbb{T}\},$$

and define $\mathcal{J}(\mathbb{T})$ as the subfamily consisting of all cubes that are maximal with respect to inclusion. Then $\mathcal{J}(\mathbb{T})$ is a partition of $\mathbb{R}^n$.

It is not difficult to prove Lemma 5.7 by using the maximality of $\mathcal{J}(\mathbb{T})$. Along with this partition, (23) can be decomposed into

$$\sum_{J\in\mathcal{J}(\mathbb{T})}\left|\sum_{s\in\mathbb{T},\,|I_s|\le 2^n|J|} \alpha_s \langle f, \varphi_s\rangle_{L^2} \cdot \int_{J\cap E_{s(2^n)}} \psi_s^{N(\cdot)}(x)\,dx\right| \lesssim \mathrm{Dense}(\mathbb{T})\,\mathrm{Size}(\mathbb{T})|I_t|, \tag{24}$$

$$\sum_{J\in\mathcal{J}(\mathbb{T})}\left|\sum_{s\in\mathbb{T},\,|I_s|> 2^n|J|} \alpha_s \langle f, \varphi_s\rangle_{L^2} \cdot \int_{J\cap E_{s(2^n)}} \psi_s^{N(\cdot)}(x)\,dx\right| \lesssim \mathrm{Dense}(\mathbb{T})\,\mathrm{Size}(\mathbb{T})|I_t|. \tag{25}$$

Keeping Lemma 3.8 in mind, we can prove (24) completely analogously to the corresponding part in [13] (we omit the details). For the proof of (25) we use our simplified phase decomposition formula. Now we invoke the following result from [13].

LEMMA 5.8 [13, p. 795].

$$\left|J \cap \bigcup_{s\in\mathbb{T},\,|I_s|>2^n|J|} E_{s(2^n)}\right| \lesssim \mathrm{Dense}(\mathbb{T})|J|.$$

5.3. Conclusion of the Proof of Theorem 3.7

To establish (25), we obtain a pointwise estimate of

$$\sum_{s\in\mathbb{T},\,|I_s|>2^n|J|} \chi_{J\cap E_{s(2^n)}}(x)\alpha_s\langle f,\varphi_s\rangle_{L^2}\psi_s^{N(\cdot)}(x).$$

Toward that end, we set

$$F_1(x) := \sum_{s\in\mathbb{T}} \alpha_s\langle f,\varphi_s\rangle_{L^2}\varphi_s(x),$$

$$F_{2,J}(x) := \sum_{s\in\mathbb{T},\,|I_s|>2^n|J|} \chi_{J\cap E_{s(2^n)}}(x)\alpha_s\langle f,\varphi_s\rangle_{L^2}\psi_s^{N(\cdot)}(x).$$

The following lemma is easy to show with the help of Lemma 2.9.

LEMMA 5.9 [13]. $\int_{\mathbb{R}^n}|F_1(x)|^2\,dx \lesssim |I_t|\,\mathrm{Size}(\mathbb{T})^2$.

To obtain the pointwise estimate, we fix a point $x\in J$ such that $|I_s|>2^n|J|$ and $x\in E_{s(2^n)}$ for some $s\in\mathbb{T}$.

We define

$$\omega_+ = \omega_+(x;J) := \bigcup\{\omega_s : s\in\mathbb{T},\ x\in E_{s(2^n)},\ |I_s|>2^n|J|\},$$

$$\omega_- = \omega_-(x;J) := \bigcap\{\omega_{s(2^n)} : s\in\mathbb{T},\ x\in E_{s(2^n)},\ |I_s|>2^n|J|\}.$$

A geometric observation shows the following.

LEMMA 5.10 [13]. *Let $s\in\mathbb{T}$.*

1. *If ω_+ is a proper subset of ω_s, then $\frac{6}{5}\omega_+\cap\frac{1}{5}\omega_s=\emptyset$.*
2. *$\omega_-\subsetneq\omega_s\subset\omega_+$ if and only if $|I_s|>2^n|J|$. If this is the case, then $\frac{1}{5}\omega_s\subset\frac{27}{25}\omega_+\setminus\frac{6}{5}\omega_-$.*
3. *If ω_- contains ω_s, then $\frac{1}{5}\omega_s\subset\frac{6}{5}\omega_-$.*

In light of this observation, it follows that

$$F_{2,J}(x) = (a_{\xi,c(\omega_+),\ell(\omega_+)}(x,D) - a_{\xi,c(\omega_-),\ell(\omega_-)}(x,D))F_1(x)|_{\xi=N(x)}.$$

Let $\omega_+=\omega_u$ and $\omega_-=\omega_{v(2^n)}$ with $u,v\in\mathbb{T}$. We apply Proposition 4.1 with $\eta_0=N(x)$ and $\eta_1=c(\omega_{\mathbb{T}})$ to obtain

$$|F_{2,J}(\bar{x})| \lesssim M_{\geq\ell(J)}F_1(\bar{x}) + \inf_{z\in Q(\bar{x},\ell(J))}\sup_{\varepsilon>0}|a(x,D-c(\omega_{\mathbb{T}}))[\chi_{\mathbb{R}^n\setminus Q(z,\varepsilon)}F_1](z)|.$$

Let us set

$$F_3(\bar{x}) := MF_1(\bar{x}) + \sup_{\varepsilon>0}|a(x,D-c(\omega_{\mathbb{T}}))[\chi_{\mathbb{R}^n\setminus Q(\bar{x},\varepsilon)}F_1](\bar{x})|$$

for $\bar{x}\in\mathbb{R}^n$. Here M denotes the usual Hardy–Littlewood maximal operator.

In view of this result and the fact that $4\ell(J)\leq\ell(I_s)$, we obtain

$$\int|F_{2,J}(y)|\,dy \lesssim \left|J\cap\bigcup_{s\in\mathbb{T},\,|I_s|>2^n|J|}E_{s(2^n)}\right|\cdot\frac{1}{|J|}\int_J F_3(y)\,dy.$$

Hence it follows from the Hölder inequality that

$$\sum_J \int |F_{2,J}(y)|\,dy \lesssim \mathrm{Dense}(\mathbb{T}) \sum_{J\in\mathcal{J},\,|I_t|>2^n|J|} \sqrt{|J|\int_J F_3(y)^2\,dy}$$

$$\lesssim \mathrm{Dense}(\mathbb{T})\sqrt{\sum_{J\in\mathcal{J},\,|I_t|>2^n|J|} |J| \int_{\mathbb{R}^n} F_3(y)^2\,dy}.$$

Since M and $a(x, D)$ are both L^2-bounded, we obtain

$$\int_{\mathbb{R}^n} F_3(y)^2\,dy \lesssim \int_{\mathbb{R}^n} |F_1(y)|^2\,dy \lesssim |I_t|\,\mathrm{Size}(\mathbb{T})^2$$

from Lemma 5.9. Combining our observations yields

$$\sum_J \int |F_{2,J}(y)|\,dy \lesssim \mathrm{Dense}(\mathbb{T})\,\mathrm{Size}(\mathbb{T})|I_t|,$$

which gives the desired result.

6. Self-extension

With Theorem 1.2 established, we consider a self-extension of this result using the result in [4].

In this section we consider key estimates needed for the proof of Theorem 1.1.

THEOREM 6.1. *Suppose we are given two measurable sets E, F of finite measure such that $1 \le |E| \le 2^n$. Let us define*

$$\Omega := \{M\chi_F > 100^n|F|\}, \qquad E' := E \setminus \Omega.$$

1. *Assume that $\mathbb{P} \subset \mathbb{D}$ is a finite subset such that $I_s \subset \Omega$ for all $s \in \mathbb{P}$. Then*

$$\left|\sum_{s\in\mathbb{P}} \langle \chi_F, \varphi_s\rangle \cdot \langle \chi_{E'\cap N^{-1}[\omega_{s(2^n)}]}, \psi_s^{N(\cdot)}\rangle\right| \lesssim_{n,p} \min(1, |F|). \tag{26}$$

2. *Assume that $\mathbb{P} \subset \mathbb{D}$ is a finite subset such that $I_s \cap \Omega^c \neq \emptyset$ for all $s \in \mathbb{P}$. Then*

$$\mathrm{Size}(\chi_F; \mathbb{P}) := \sum_{s\in\mathbb{P}} \left|\langle \chi_F, \varphi_s\rangle \cdot \langle \chi_{E'\cap N^{-1}[\omega_{s(2^n)}]}, \psi_s^{N(\cdot)}\rangle\right|$$

$$\lesssim_{n,p} |F| \log\left(1 + \frac{1}{|F|}\right). \tag{27}$$

Once (26) and (27) are proved, we will have shown that

$$\|\mathcal{D}f\|_{p,\infty} \lesssim \|f\|_p, \quad 1 < p < \infty. \tag{28}$$

We can then interpolate (28) to obtain the desired L^p estimate:

$$\|\mathcal{D}f\|_p \lesssim \|f\|_p, \quad 1 < p < \infty.$$

If we use our new phase decomposition formula (see Corollary 3.4), then we finally obtain Theorem 1.1.

It is fairly easy to establish (26), which we do in Section 6.1; (27) is taken up in Section 6.2. The following result is the crux of the proof of Theorem 6.1.

THEOREM 6.2 [4]. *Let F be a measurable function, and assume that $\mathbb{P}$ is a finite set of tiles such that I_s intersects Ω^c for all $s \in \mathbb{P}$. Then*

$$\mathrm{Size}(\chi_F; \mathbb{P}) \lesssim \min(|F|, 1). \tag{29}$$

6.1. Proof of (26)

For the proof of (26), we can assume that $|F| \leq 1$. Otherwise Ω is empty and there is nothing to prove.

We define

$$\mathcal{F}_k := \{Q \in \mathcal{D} : k \text{ is the largest integer such that } 2^k Q \subset \Omega\},$$

where $\mathcal{D}$ denotes the set of all dyadic cubes in $\mathbb{R}^n$. Also, we decompose

$$\left|\sum_{s\in\mathbb{P}} \langle \chi_F, \varphi_s\rangle \cdot \langle \chi_{E'\cap N^{-1}[\omega_{s(2^n)}]}, \psi_s^{N(\cdot)}\rangle\right| \leq \sum_{k=0}^{\infty} \sum_{Q\in\mathcal{F}_k} S(Q),$$

where

$$S(Q) = \left|\sum_{s\in\mathbb{P},\, I_s=Q} \langle \chi_F, \varphi_s\rangle \cdot \langle \chi_{E'\cap N^{-1}[\omega_{s(2^n)}]}, \psi_s^{N(\cdot)}\rangle\right|.$$

Now observe that, if $I_s = Q$, then

$$|\langle \chi_F, \varphi_s\rangle| \leq \langle \chi_F, \rho_Q\rangle \lesssim 2^{k\gamma_0} \inf_{2^{k+1}Q} M\chi_F \lesssim 2^{k\gamma_0} \min(1, |F|)$$

for some $\gamma_0 > 0$. Furthermore, by Lemma 3.8 we have

$$|\langle \chi_{E'\cap N^{-1}[\omega_{s(2^n)}]}, \psi_s^{N(\cdot)}\rangle| \lesssim |Q|^{1/2} 2^{-\gamma k}.$$

Finally, for each $x \in \mathbb{R}^n$ and $Q \in \mathcal{D}$, we can find at most one $s \in \mathbb{D}$ such that $\omega_s \ni N(x)$ and $I_s = Q$. As a result, if we choose $\gamma > \gamma_0$ then we obtain the desired result.

6.2. Proof of (27)

Here we shall assume that $1 \leq |E| \leq 2^n$ by scaling.

LEMMA 6.3. *Let $A > 0$. Then*

$$\sum_{j=-\infty}^{\infty} 2^{-2j} \min(A, 2^j) \min(1, 2^{2j}) \lesssim 1 + A \min(1, -\log A).$$

Proof. We write $f(A) = \sum_{j=-\infty}^{\infty} 2^{-2j} \min(A, 2^j) \min(1, 2^{2j})$. It is trivial that

$$f(A) \leq \sum_{j=-\infty}^{\infty} 2^{-j} \min(1, 2^{2j}) \leq 3,$$

so let $A \leq 1$. Then

$$f(A) = \sum_{j<\log_2 A} 2^{jn} + \sum_{\log_2 A \le j \le 0} A + \sum_{j=1}^{\infty} A2^{-2jn} \le A(3 - \log_2 A),$$

which is the desired result. □

Corollary 5.5 allows us to partition $\mathbb{P}$ into a disjoint union $\{\mathbb{P}_j\}$, where $\mathbb{P}_j = \bigcup_{j\in G_j} \mathbb{T}_{jk}$ is a set of trees contained in $\mathbb{P}$ that satisfies (19), (20), and

$$\sum_{k\in G_j} |I_{\mathbb{T}_{jk}}| \le 2\,\mathrm{Count}(\mathbb{P}_j) \le 2(\alpha+\beta)4^{-j}.$$

Given Theorem 5.6, it is easy to establish (27).

Indeed, let $\mathbb{P} \subset \mathbb{D}$ be any set such that $I_s \cap \Omega^c \neq \emptyset$ for all $s \in \mathbb{P}$. Then we have

$$\mathrm{Sum}(\mathbb{P}) = \sum_{j,k} \mathrm{Sum}(\mathbb{T}_{jk}) \lesssim \sum_{j=-\infty}^{\infty} \sum_{k\in G_j} |I_{\mathbb{T}_{jk}}|\, \mathrm{Dense}(\mathbb{T}_{jk})\, \mathrm{Size}(\mathbb{T}_{jk})$$

by virtue of (21). If we use (19), (20), the inequality $\mathrm{Dense}(\mathbb{T}_{jk}) \le \frac{2^n}{19^n} \le 1$, and Theorem 6.1, then we obtain

$$\begin{aligned}\mathrm{Sum}(\mathbb{P}) &\lesssim \sum_{j=-\infty}^{\infty} \min(1, 2^{2j}) \min(1, |F|, 2^j) \sum_{k\in G_j} |I_{\mathbb{T}_{jk}}| \\ &\lesssim \sum_{j=-\infty}^{\infty} 2^{-2j} \min(1, 2^{2j}) \min(1, |F|, 2^j).\end{aligned}$$

Using Lemma 6.3 now yields

$$\mathrm{Sum}(\mathbb{P}) \lesssim |F| \log\left(1 + \frac{1}{|F|}\right).$$

This proves (27). Now that both (26) and (27) have been established, it follows that Theorem 1.1 is completely proved.

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