

Global Solutions of Homogeneous Linear Partial Differential Equations of the Second Order

PEI-CHU HU & CHUNG-CHUN YANG

1. Introduction and Main Results

In 1995, we proved in [9] that the meromorphic solutions of the system of partial differential equations

$$\frac{\partial u}{\partial z_j} = a_j(z) + b_j(z)u + c_j(z)u^2, \quad j = 1, 2, \dots, m$$

(where a_j, b_j, c_j are polynomials on \mathbb{C}^m) are of finite positive order and are pseudo-prime. Li and Saleeby [13] characterized entire solutions in \mathbb{C}^m of first-order partial differential equations of the form

$$\frac{\partial u}{\partial z_j} = f_j(u), \quad j = 1, 2, \dots, m,$$

where the f_j are meromorphic functions in \mathbb{C} . Berenstein and Li [2] studied entire solutions in \mathbb{C}^m for first-order partial differential equations of the form

$$\frac{\partial u}{\partial z_j} = p(z)f(u), \quad j = 1, 2, \dots, m,$$

where p and f are entire or meromorphic functions in \mathbb{C}^m and \mathbb{C} , respectively. Li [12] also gave a complete description of entire solutions of the Fermat type partial differential equation

$$\left(\frac{\partial u}{\partial z_1}\right)^m + \left(\frac{\partial u}{\partial z_2}\right)^n = 1.$$

In this paper, we study meromorphic solutions of homogeneous linear partial differential equations of the second order in two independent complex variables,

$$a_0 \frac{\partial^2 u}{\partial t^2} + 2a_1 \frac{\partial^2 u}{\partial t \partial z} + a_2 \frac{\partial^2 u}{\partial z^2} + a_3 \frac{\partial u}{\partial t} + a_4 \frac{\partial u}{\partial z} + a_6 u = 0; \quad (1)$$

here $a_k = a_k(t, z)$ are holomorphic functions for $(t, z) \in \Sigma$, where Σ is a region on \mathbb{C}^2 .

Received August 26, 2008. Revision received November 17, 2008.

The work of the first author was partially supported by the Natural Science Foundation of China. Work of the second author was partially supported by a UGC Grant of Hong Kong, Project no. 604106.

When t and z are real variables, Hilbert's 19th problem conjectures that if all $a_k = a_k(t, z)$ are analytic on t and z , then any solution $u = u(t, z)$ of an elliptic equation of the form (1) also is analytic on its existing region; this was confirmed by Bernšteĭn [3] provided one knows that $u \in C^3$. Lewy [11], using the solvability of the initial value problem for hyperbolic equations, gave a simple proof by extending t and z to a domain of \mathbb{C}^2 . Petrovskii [14] then extended this result to general nonlinear elliptic systems. It is known also that all regular solutions of linear elliptic equations of the second order have bounded derivatives up to order k , provided all coefficients have bounded derivatives up to order k .

In this paper, we follow Lewy's idea of studying equation (1) on a region $\Sigma \subseteq \mathbb{C}^2$. Let $\mathcal{S}_{(P)}(\Sigma)$ be solutions of (1) satisfying some property (P) on Σ . It is natural to seek proper properties (P) determining the cardinal number of $\mathcal{S}_{(P)}(\Sigma)$. For example, $\mathcal{S}_{\text{hol}}(\Sigma)$ denotes holomorphic solutions of equation (1) on Σ . Then $\mathcal{S}_{\text{hol}}(\Sigma)$ is a vector space. When $\Sigma = \mathbb{C}^2$, equation (1) usually has many entire solutions on \mathbb{C}^2 ; that is, $\dim \mathcal{S}_{\text{hol}}(\mathbb{C}^2) > 0$. To explain matters clearly, here we examine the following special differential equation:

$$t^2 \frac{\partial^2 u}{\partial t^2} - z^2 \frac{\partial^2 u}{\partial z^2} + t \frac{\partial u}{\partial t} - z \frac{\partial u}{\partial z} + t^2 u = 0. \tag{2}$$

THEOREM 1.1. *The differential equation (2) has an entire solution $f(t, z)$ on \mathbb{C}^2 if and only if f is an entire function expressed by the series*

$$f(t, z) = \sum_{n=0}^{\infty} n! c_n J_n(t) z^n \tag{3}$$

such that

$$\limsup_{n \rightarrow \infty} |c_n|^{1/n} = 0, \tag{4}$$

where $J_n(t)$ is the first kind of Bessel function of order n . Moreover, the order $\text{ord}(f)$ of the entire function f satisfies

$$\rho \leq \text{ord}(f) \leq \max\{1, \rho\},$$

where

$$\rho = \limsup_{n \rightarrow \infty} \frac{2 \log n}{\log(1/|c_n|^{1/n})}. \tag{5}$$

By definition, the order of f is defined by

$$\text{ord}(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ M(r, f)}{\log r},$$

where

$$\log^+ x = \begin{cases} \log x & \text{if } x \geq 1, \\ 0 & \text{if } x < 1, \end{cases}$$

and

$$M(r, f) = \max_{|t| \leq r, |z| \leq r} |f(t, z)|.$$

Valiron [17] showed that each transcendental entire solution of a homogeneous linear ordinary differential equation with polynomial coefficients is of finite positive

order. However, Theorem 1.1 shows that Valiron’s theorem is not true for general partial differential equations. For example, the equation

$$t^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial z^2} + t \frac{\partial u}{\partial t} = 0$$

has an entire solution $\exp(te^z)$ of infinite order.

If $0 < \lambda = \text{ord}(f) < \infty$, we define the type of f by

$$\text{typ}(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ M(r, f)}{r^\lambda}.$$

For the type of entire solutions of equation (2) we have an analogue of the Lindelöf–Pringsheim theorem, whose proof is essentially the same as that of determining the type for Taylor series of entire functions of one complex variable.

THEOREM 1.2. *If $f(t, z)$ is an entire solution of (2) defined by (3) and (4) such that $1 < \lambda = \text{ord}(f) < \infty$, then the type $\sigma = \text{typ}(f)$ satisfies*

$$e\lambda\sigma = 2^{-\lambda/2} \limsup_{n \rightarrow \infty} 2n |c_n|^{\lambda/(2n)}.$$

Next we introduce some properties (P) on \mathbb{C}^2 such that the cardinal number of $S_{(P)}(\mathbb{C}^2)$ is finite. In other words, we give some conditions that determine uniquely meromorphic solutions of (1) on \mathbb{C}^2 . Nevanlinna’s four-value theorem states that if two nonconstant meromorphic functions f and g on \mathbb{C} share four distinct values counting multiplicity, then g must be a fractional linear transformation of f , which is also called a Möbius transformation of f . Brosch [4] proved that if two nonconstant meromorphic functions f and g on \mathbb{C} share three distinct values c_1, c_2, c_3 counting multiplicity, if f is a solution of the differential equation

$$\left(\frac{dw}{dz}\right)^n = \sum_{j=0}^{2n} b_j(z)w^j := P(z, w)$$

such that b_0, b_1, \dots, b_{2n} ($b_{2n} \neq 0$) are small functions of f (grow more slowly than f), and if $P(z, c_i) \neq 0$ for $i = 1, 2, 3$, then $f = g$.

To state a generalization of Brosch’s result to partial differential equations, we abbreviate

$$u_t = \frac{\partial u}{\partial t}, \quad u_{tz} = \frac{\partial^2 u}{\partial t \partial z}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2},$$

and so on, and we set

$$\begin{aligned} Du &= a_0 u_t^2 + 2a_1 u_t u_z + a_2 u_z^2 \quad \text{and} \\ Lu &= a_0 u_{tt} + 2a_1 u_{tz} + a_2 u_{zz} + a_3 u_t + a_4 u_z. \end{aligned}$$

We make the following assumption.

(A) All coefficients a_i in (1) are polynomials, and when $a_6 = 0$ there are no nonconstant polynomials u satisfying the system

$$\begin{cases} Du = 0, \\ Lu = 0. \end{cases}$$

For technical reasons, here we study only meromorphic functions of finite orders. The order of a meromorphic function of several variables may be defined by using its Nevanlinna characteristic function (see [8; 16]).

THEOREM 1.3. *Suppose assumption (A) holds. Let $f(t, z)$ be a nonconstant meromorphic solution of (1) such that $\text{ord}(f) < \infty$, and let g be a nonconstant meromorphic function of finite order on \mathbb{C}^2 . If f and g share $0, 1, \infty$ counting multiplicity, then one of the following five statements holds:*

- (a) $g = f$;
- (b) $gf = 1$;
- (c) $a_6 = 0, gf = f + g$;
- (d) $a_6 = 0$, and there exist a constant $b \notin \{0, 1\}$ and a polynomial β such that

$$f = \frac{1}{b-1}(e^\beta - 1), \quad g = \frac{b}{b-1}(1 - e^{-\beta});$$

- (e) $a_6 \neq 0, f^2g^2 = 3fg - f - g$.

When $a_6 \neq 0$, case (b) may occur. For example, consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial u}{\partial t} - u = 0, \quad (6)$$

which has an entire solution of order 1:

$$f(t, z) = e^{t+z}.$$

Let's compare f with the following entire function of order 1:

$$g(t, z) = e^{-t-z}.$$

Obviously, f and g share $0, 1, -1, \infty$ counting multiplicity, but $g \neq f, gf = 1$. Now the differential equation

$$Lu + Du + a_6 = u_{tt} + u_{zz} - u_t + u_t^2 + u_z^2 - 1 = 0$$

has the nonconstant polynomial solution

$$u(t, z) = t + z.$$

Condition (A) is meaningful. For example, Theorem 1.1 shows that the differential equation (2) admits many entire solutions of finite order. Furthermore, we can prove that condition (A) when associated with the differential equation (2) holds, thereby obtaining the following result.

COROLLARY 1.4. *Let $f(t, z)$ be a nonconstant meromorphic solution of (2) such that $\text{ord}(f) < \infty$, and let g be a nonconstant meromorphic function of finite order on \mathbb{C}^2 . If f and g share $0, 1, \infty$ counting multiplicity, then we have $g = f$ or $gf = 1$ or $f^2g^2 = 3fg - f - g$.*

Case (b) in Theorem 1.3 may actually occur for $a_6 = 0$. For example, consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial z} = 0, \tag{7}$$

which has an entire solution $f(t, z) = e^{t+z}$ of order 1 such that assumption (A) clearly holds. The entire solution f and the function $g = e^{-t-z}$ share $0, 1, \infty$ counting multiplicity and also satisfy $gf = 1$; this is case (b) in Theorem 1.3.

For a real number x , let $[x]$ denote the maximal integer $\leq x$. We give the following result, which is an analogue of Anastassiadis's theorem [1] on uniqueness of entire functions of one variable.

THEOREM 1.5. *Let $f(t, z)$ and $g(t, z)$ be transcendental entire solutions of (2) such that $\text{ord}(f) < \infty, \text{ord}(g) < \infty$, and*

$$\frac{\partial^{2j} f}{\partial t^j \partial z^j}(0, 0) = \frac{\partial^{2j} g}{\partial t^j \partial z^j}(0, 0), \quad j = 0, 1, \dots, q,$$

where

$$q = \max\{[\text{ord}(f)], [\text{ord}(g)]\}.$$

If there exists a complex number a with $(a, f(0, 0)) \neq (0, 0)$ such that f and g share a counting multiplicity, then $f = g$.

Theorem 1.3 shows that, when $a_6 = 0$, global solutions of the equation (1) can be quite complicated; however, when $a_6 \neq 0$, these solutions have normal properties. Our next result also supports this view. Theorem 1.6 extends a theorem [6, Thm. 5.8] on meromorphic solutions of linear ordinary differential equations.

THEOREM 1.6. *Assume that all a_k in (1) are entire functions on \mathbb{C}^2 that grow more slowly than a meromorphic solution of equations (1) on \mathbb{C}^2 . If $a_6 \neq 0$, then the deficiency of the solution for each nonzero complex number is zero.*

Some notation and remarks related to Theorem 1.6 will be given in Section 7. For example, the so-called telegraph equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial z^2} + 2\alpha \frac{\partial u}{\partial t} + \alpha^2 u = 0$$

has entire solutions

$$u(t, z) = e^{-\alpha t} \{f(z + ct) + g(z - ct)\},$$

where f and g are entire functions on \mathbb{C} . If $\alpha \neq 0$, Theorem 1.6 shows that the deficiency of a nonconstant $u(t, z)$ for each nonzero complex number a is zero, which means that the equation

$$f(z + ct) + g(z - ct) - ae^{\alpha t} = 0$$

has zeros.

2. Proof of Theorem 1.1

First of all, we assume that $u = f(t, z)$ is an entire function on \mathbb{C}^2 satisfying (2). Then we have the Taylor expansion

$$f(t, z) = \sum_{n=0}^{\infty} w_n(t) z^n,$$

where

$$w_n(t) = \frac{1}{n!} \frac{\partial^n f}{\partial z^n}(t, 0).$$

Since $f(t, z)$ satisfies equation (2), we find

$$\begin{aligned} t^2 \frac{\partial^2 f}{\partial t^2} - z^2 \frac{\partial^2 f}{\partial z^2} + t \frac{\partial f}{\partial t} - z \frac{\partial f}{\partial z} + t^2 u \\ = \sum_{n=0}^{\infty} \left(t^2 \frac{d^2 w_n}{dt^2} + t \frac{dw_n}{dt} + (t^2 - n^2) w_n \right) z^n = 0; \end{aligned}$$

that is, the $w_n(t)$ are entire solutions of Bessel's differential equations

$$t^2 \frac{d^2 w}{dt^2} + t \frac{dw}{dt} + (t^2 - n^2) w = 0, \quad n = 0, 1, \dots \quad (8)$$

Because the first kind of Bessel function of order n ,

$$J_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\theta - t \sin \theta) d\theta,$$

and the second kind of Bessel function (Neumann function) of order n ,

$$\begin{aligned} N_n(t) = \frac{2}{\pi} J_n(t) \left(C + \log \frac{t}{2} \right) - \frac{1}{\pi} \left(\frac{t}{2} \right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{t}{2} \right)^{2k} \\ - \frac{1}{\pi} \left(\frac{t}{2} \right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (n+k)!} \left(\sum_{m=1}^k \frac{1}{m} + \sum_{m=1}^{n+k} \frac{1}{m} \right) \left(\frac{t}{2} \right)^{2k}, \end{aligned}$$

are linearly independent solutions of (8), there must exist constants c_n and d_n such that

$$w_n(t) = n! c_n J_n(t) + d_n N_n(t).$$

This equation easily yields $d_n = 0$ if we examine the singularity at $t = 0$. Thus we obtain the expansion (3).

To prove (4), we next study the limit

$$\lambda(t) = \limsup_{n \rightarrow \infty} (n! |c_n J_n(t)|)^{1/n}.$$

By using the Poisson formula

$$J_n(t) = \frac{2}{\sqrt{\pi} \Gamma(n + \frac{1}{2})} \left(\frac{t}{2} \right)^n \int_0^{\pi/2} \cos(t \cos x) \sin^{2n} x dx$$

and the Stirling formula

$$\Gamma(z) = e^{-z} z^{z-1/2} \sqrt{2\pi} \left\{ 1 + O\left(\frac{1}{z}\right) \right\}, \quad |\arg z| < \pi,$$

it is easy to show

$$\lambda(t) \leq \frac{\kappa}{2}|t|,$$

where

$$\kappa = \limsup_{n \rightarrow \infty} |c_n|^{1/n}.$$

Note that, when $t = 1$,

$$\int_0^{\pi/2} \cos(t \cos x) \sin^{2n} x \, dx \geq \cos 1 \int_0^{\pi/2} \sin^{2n} x \, dx = \frac{\pi(2n)!}{2(2^n n!)^2} \cos 1.$$

We obtain easily

$$\lambda(1) \geq \frac{\kappa}{2}.$$

Therefore,

$$\limsup_{n \rightarrow \infty} (n! |c_n J_n(1) z^n|)^{1/n} = \lambda(1) |z| = \frac{\kappa}{2} |z|.$$

Since $f(t, z)$ is an entire function (i.e., the series (3) converges for all $(t, z) \in \mathbb{C}^2$), it follows that $\kappa = 0$ and hence (4) is proved.

Conversely, if $f(t, z)$ is an entire function of the form (3) satisfying (4) then it is trivial to check that it also satisfies the partial differential equation (2), since

$$\begin{aligned} t^2 \frac{\partial^2 f}{\partial t^2} - z^2 \frac{\partial^2 f}{\partial z^2} + t \frac{\partial f}{\partial t} - z \frac{\partial f}{\partial z} + t^2 f \\ = \sum_{n=0}^{\infty} n! c_n \left(t^2 \frac{d^2 J_n}{dt^2} + t \frac{dJ_n}{dt} + (t^2 - n^2) J_n \right) z^n = 0 \end{aligned}$$

if we recall that J_n is a solution of (8).

Finally, we prove $\rho \leq \text{ord}(f) \leq \max\{1, \rho\}$. We first show $\rho \leq \text{ord}(f)$. Without loss of generality, we may assume $0 < \rho \leq \infty$. Take ε with $0 < \varepsilon < \rho$, and set

$$k = \begin{cases} \rho - \varepsilon & \text{if } \rho < \infty, \\ 1/\varepsilon & \text{if } \rho = \infty. \end{cases}$$

Then there exists a sequence $n_j \rightarrow \infty$ such that

$$2n_j \log n_j \geq k \log |c_{n_j}|^{-1}.$$

Note that

$$J_n(t) = \left(\frac{t}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!} \left(\frac{t}{2}\right)^{2m}$$

and so

$$\frac{\partial^{2n} f}{\partial t^n \partial z^n}(0, 0) = c_n (n!)^2 J_n^{(n)}(0) = \frac{(n!)^2}{2^n} c_n.$$

By using Cauchy's inequality

$$\left| \frac{\partial^{2n} f}{\partial t^n \partial z^n}(0, 0) \right| \leq (n!)^2 r^{-2n} M(r, f),$$

which is equivalent to

$$2^{-n} |c_n| \leq r^{-2n} M(r, f), \tag{9}$$

we find that

$$\log M(r, f) \geq \log|c_n| + 2n \log r - n \log 2.$$

Then, when j is large, we have

$$\begin{aligned} \log M(r, f) &\geq \log|c_{n_j}| + 2n_j \log r - n_j \log 2 \\ &> 2n_j \left(\log r - \frac{1}{k} \log n_j - \frac{1}{2} \log 2 \right). \end{aligned}$$

Now take

$$r_j = \sqrt{2}(en_j)^{1/k}.$$

If j is large, then

$$\log M(r_j, f) > \frac{2^{1-k/2}}{ek} r_j^k,$$

which means that

$$\text{ord}(f) \geq \limsup_{j \rightarrow \infty} \frac{\log^+ \log^+ M(r_j, f)}{\log r_j} \geq k;$$

hence, if we let $\varepsilon \rightarrow 0$ then $\text{ord}(f) \geq \rho$ follows.

Next we show another inequality, $\text{ord}(f) \leq \max\{1, \rho\}$. Toward this end, we may assume $\rho < \infty$. For any $\varepsilon > 0$ there exists an $n_0 > 1$ such that, when $n \geq n_0$,

$$0 \leq \frac{2 \log n}{\log(1/|c_n|^{1/n})} < \rho + \varepsilon;$$

that is,

$$|c_n| < n^{-2n/(\rho+\varepsilon)}.$$

By using the Poisson formula and noting that

$$|\cos(t \cos x)| \leq e^r,$$

where $r = |t|$, we can obtain the inequality

$$n! |J_n(t)| \leq \frac{\sqrt{\pi}(2n)!}{2^{2n}\Gamma(n + \frac{1}{2})n!} \left(\frac{r}{2}\right)^n e^r. \tag{10}$$

Since the Stirling formula easily implies that

$$\lim_{n \rightarrow \infty} \frac{\sqrt{\pi}(2n)!}{2^{2n}\Gamma(n + \frac{1}{2})n!} = 1,$$

we can choose n_0 sufficiently large so that when $n \geq n_0$ we have

$$n! |J_n(t)| \leq 2 \left(\frac{r}{2}\right)^n e^r, \tag{11}$$

where $r = |t|$. However, when $|t| = r$,

$$\begin{aligned} M(r, f) &\leq \sum_{n=0}^{n_0-1} n! |c_n J_n(t)| r^n + 2e^r \sum_{n=n_0}^{\infty} 2^{-n} n^{-2n/(\rho+\varepsilon)} r^{2n} \\ &\leq 2e^r \left\{ Ar^{2n_0-2} + \sum_{n=n_0}^{\infty} n^{-2n/(\rho+\varepsilon)} \left(\frac{r^2}{2}\right)^n \right\}. \end{aligned}$$

Put $m(r) = (2r)^{\rho+\varepsilon}$. Then

$$\sum_{n \geq m(r)} n^{-2n/(\rho+\varepsilon)} \left(\frac{r^2}{2}\right)^n \leq \sum_{n=0}^{\infty} \frac{1}{8^n} = \frac{8}{7}.$$

We also have

$$\sum_{n_0 \leq n < m(r)} n^{-2n/(\rho+\varepsilon)} \left(\frac{r^2}{2}\right)^n \leq \left(\frac{r^2}{2}\right)^{m(r)} \sum_{n=1}^{\infty} n^{-2n/(\rho+\varepsilon)} = B \left(\frac{r^2}{2}\right)^{m(r)}.$$

Therefore,

$$M(r, f) \leq 2e^r \left\{ Ar^{2n_0-2} + B \left(\frac{r^2}{2}\right)^{m(r)} + \frac{8}{7} \right\}, \tag{12}$$

which means that

$$\text{ord}(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ M(r, f)}{\log r} \leq \max\{1, \rho\} + \varepsilon.$$

Hence, if we let $\varepsilon \rightarrow 0$ then $\text{ord}(f) \leq \max\{1, \rho\}$ follows.

3. Proof of Theorem 1.2

Set

$$\varrho = 2^{-\lambda/2} \limsup_{n \rightarrow \infty} 2n |c_n|^{\lambda/(2n)}.$$

We first show $\varrho \geq e\lambda\sigma$. To do this, we may assume that $\varrho < \infty$. Then for any $\varepsilon > 0$ there exists an $n_0 > 1$ such that, when $n \geq n_0$,

$$2^{-\lambda/2} 2n |c_n|^{\lambda/(2n)} < \varrho + \varepsilon$$

or, equivalently,

$$2^{-n} |c_n| < \left(\frac{\varrho + \varepsilon}{2n}\right)^{2n/\lambda}.$$

Therefore, when $|t| = r$,

$$M(r, f) \leq 2e^r \left\{ Ar^{2n_0-2} + \sum_{n \geq n_0} \left(\frac{\varrho + \varepsilon}{2n}\right)^{2n/\lambda} r^{2n} \right\}.$$

Note that, for $a > 0$ and $b > 0$,

$$\max_{x>0} \left(\frac{a}{x}\right)^{x/b} = \exp\left(\frac{a}{eb}\right).$$

Then

$$\sum_{n_0 \leq n < (\varrho+\varepsilon)r^\lambda} \left(\frac{\varrho + \varepsilon}{2n}\right)^{2n/\lambda} r^{2n} \leq (\varrho + \varepsilon)r^\lambda \exp\left(\frac{(\varrho + \varepsilon)r^\lambda}{e\lambda}\right),$$

whereas

$$\sum_{n \geq (\varrho+\varepsilon)r^\lambda} \left(\frac{\varrho + \varepsilon}{2n}\right)^{2n/\lambda} r^{2n} \leq \sum_{n=0}^{\infty} \frac{1}{4^{n/\lambda}} = \frac{4^{1/\lambda}}{4^{1/\lambda} - 1} = C.$$

Therefore,

$$M(r, f) \leq 2e^r \left\{ Ar^{2n_0-2} + (\varrho + \varepsilon)r^\lambda \exp\left(\frac{(\varrho + \varepsilon)r^\lambda}{e\lambda}\right) + C \right\},$$

which easily yields

$$\sigma \leq \frac{\varrho + \varepsilon}{e\lambda};$$

that is, $e\lambda\sigma \leq \varrho$ if we let $\varepsilon \rightarrow 0$.

Finally, we show the converse inequality $e\lambda\sigma \geq \varrho$. Now we may assume $\varrho > 0$. Take ε with $0 < \varepsilon < \varrho$ and set

$$\kappa = \begin{cases} \varrho - \varepsilon & \text{if } \varrho < \infty, \\ 1/\varepsilon & \text{if } \varrho = \infty. \end{cases}$$

Then there exists a sequence $n_j \rightarrow \infty$ satisfying

$$2^{-\lambda/2} 2n_j |c_{n_j}|^{\lambda/(2n_j)} > \kappa$$

or, equivalently,

$$2^{-n_j} |c_{n_j}| > \left(\frac{\kappa}{2n_j}\right)^{2n_j/\lambda}.$$

Using Cauchy's inequality (9),

$$2^{-n} |c_n| \leq r^{-2n} M(r, f),$$

for large j we find that

$$M(r, f) \geq \left(\frac{\kappa}{2n_j} r^\lambda\right)^{2n_j/\lambda}.$$

Take

$$r_j^\lambda = \frac{2en_j}{\kappa}.$$

We obtain

$$M(r_j, f) \geq \exp\left(\frac{\kappa}{e\lambda} r_j^\lambda\right),$$

which yields

$$\sigma \geq \limsup_{j \rightarrow \infty} \frac{\log^+ M(r_j, f)}{r_j^\lambda} \geq \frac{\kappa}{e\lambda}.$$

Hence $e\lambda\sigma \geq \varrho$ follows if we let $\varepsilon \rightarrow 0$.

4. Proof of Theorem 1.3

By [16, Prop. 6.2], there exist two polynomials α, β satisfying

$$\frac{f}{g} = e^\alpha, \quad \frac{f-1}{g-1} = e^\beta. \tag{13}$$

If $e^\alpha = e^\beta$, then

$$\frac{f}{g} = \frac{f-1}{g-1}$$

and so $g = f$ follows. Conversely, if $g = f$, then $e^\alpha = 1 = e^\beta$ follows from (13).

Next we study the case

$$e^\alpha \neq e^\beta, \quad g \neq f. \tag{14}$$

It is easy to obtain the expressions

$$f = \frac{e^\beta - 1}{e^\gamma - 1} \quad \text{and} \quad g = \frac{e^\gamma}{e^\gamma - 1}(1 - e^{-\beta}), \tag{15}$$

where $\gamma = \beta - \alpha$. Note that

$$\begin{aligned} f_t &= (e^{-\alpha} - e^{-\beta})^{-2} \{ \alpha_t e^{-\alpha} - \beta_t e^{-\beta} + \gamma_t e^{-\alpha-\beta} \}, \\ f_{tt} &= (e^{-\alpha} - e^{-\beta})^{-3} \{ (\alpha_{tt} + \alpha_t^2) e^{-2\alpha} + (\beta_{tt} + \beta_t^2) e^{-2\beta} \\ &\quad + (\gamma_{tt} - \gamma_t^2) e^{-2\alpha-\beta} - (\gamma_{tt} + \gamma_t^2) e^{-\alpha-2\beta} \\ &\quad - (\alpha_{tt} + \beta_{tt} + 2\alpha_t \beta_t - \gamma_t^2) e^{-\alpha-\beta} \}, \\ f_{tz} &= (e^{-\alpha} - e^{-\beta})^{-3} \{ (\alpha_{tz} + \alpha_t \alpha_z) e^{-2\alpha} + (\beta_{tz} + \beta_t \beta_z) e^{-2\beta} \\ &\quad + (\gamma_{tz} - \gamma_t \gamma_z) e^{-2\alpha-\beta} - (\gamma_{tz} + \gamma_t \gamma_z) e^{-\alpha-2\beta} \\ &\quad - (\alpha_{tz} + \beta_{tz} + \alpha_t \beta_z + \alpha_z \beta_t - \gamma_t \gamma_z) e^{-\alpha-\beta} \}. \end{aligned}$$

Symmetrically, we can obtain derivatives of f with respect to z . Substituting into the differential equation (1) yields

$$\begin{aligned} 0 &= (L\alpha + D\alpha + a_6) e^{-2\alpha} + (L\beta + D\beta + a_6) e^{-2\beta} \\ &\quad + (L\gamma - D\gamma - a_6) e^{-2\alpha-\beta} - (L\gamma + D\gamma - 2a_6) e^{-\alpha-2\beta} \\ &\quad - \{ L\alpha + L\beta + 2D(\alpha, \beta) - D\gamma + 2a_6 \} e^{-\alpha-\beta} - a_6 e^{-3\beta}, \end{aligned} \tag{16}$$

where

$$D(\alpha, \beta) = a_0 \alpha_t \beta_t + a_1 (\alpha_t \beta_z + \alpha_z \beta_t) + a_2 \alpha_z \beta_z.$$

We further distinguish several cases in our study of (14).

Case 1: We claim that the polynomial α is not constant. Otherwise, if α is a constant c , then equation (16) becomes

$$\begin{aligned} 0 &= -a_6 e^{-3\beta} + \{ L_1 \beta - e^{-c} (L_1 \beta - 3a_6) \} e^{-2\beta} \\ &\quad - e^{-c} \{ L_2 \beta + 5a_6 - e^{-c} (L_2 \beta + 2a_6) \} e^{-\beta} + a_6 e^{-2c}, \end{aligned} \tag{17}$$

where

$$L_1 \beta = L\beta + D\beta + a_6, \quad L_2 \beta = L\beta - D\beta - 3a_6.$$

Observe that β is not a constant in this case, for otherwise we could deduce from (15) that f is a constant. If we apply a generalized Borel–Nevanlinna theorem to (17) (cf. [8, Thm. 3.4]), then it follows that the coefficients of exponential functions in (17) all are zero; that is,

$$a_6 = 0, \quad (1 - e^{-c}) L_1 \beta = 0, \quad (1 - e^{-c}) L_2 \beta = 0.$$

Note that $e^c \neq 1$; otherwise, $g = f$ follows from (13), which contradicts (14). Hence we must have

$$L_1 \beta = 0 \quad \text{and} \quad L_2 \beta = 0,$$

which means that

$$L\beta - a_6 = 0 \quad \text{and} \quad D\beta + 2a_6 = 0. \quad (18)$$

This is a contradiction, so the claim is proved.

Case 2: We claim that the polynomial β is not constant. Otherwise, if β is a constant c , then equation (16) becomes

$$(1 - e^{-c})(L_1\alpha)e^{-2\alpha} - e^{-c}(1 - e^{-c})(L_2\alpha + 5a_6)e^{-\alpha} + e^{-2c}(1 - e^{-c})a_6 = 0, \quad (19)$$

where

$$L_1\alpha = L\alpha + D\alpha + a_6, \quad L_2\alpha = L\alpha - D\alpha - 3a_6.$$

Observe that α is not a constant in this case, for otherwise we could deduce from (15) that f is a constant. If we apply a generalized Borel–Nevanlinna theorem to (19) (cf. [8, Thm. 3.4]), then it follows that the coefficients of exponential functions in (19) all are zero; that is,

$$(1 - e^{-c})a_6 = 0, \quad (1 - e^{-c})L_1\alpha = 0, \quad (1 - e^{-c})(L_2\alpha + 5a_6) = 0.$$

Note that $e^c \neq 1$; otherwise, $g = f$ follows from (13), which contradicts (14). Hence we must have

$$a_6 = 0, \quad L_1\alpha = 0, \quad L_2\alpha = 0,$$

which means that

$$L\alpha - a_6 = 0 \quad \text{and} \quad D\alpha + 2a_6 = 0. \quad (20)$$

This is a contradiction, so the claim is proved.

Case 3: The polynomial $\gamma = \beta - \alpha$ is a constant c . Now (16) becomes

$$0 = \{L_1\alpha + e^{-2c}L_1\alpha - 2e^{-c}L_1\alpha\}e^{-2\alpha} - e^{-c}(1 - 2e^{-c} + e^{-2c})a_6e^{-3\alpha}. \quad (21)$$

Note that α is not a constant in this case, for otherwise we could deduce from (15) that f is a constant. It follows from (21) that

$$(1 - e^{-c})^2a_6 = 0 \quad \text{and} \quad (1 - e^{-c})^2L_1\alpha = 0.$$

Since $e^\alpha \neq e^\beta$, we have $b = e^c \neq 1$ and hence

$$a_6 = 0 \quad \text{and} \quad L\alpha + D\alpha = 0.$$

Thus Theorem 1.3(d) follows from (15).

Case 4: The polynomial $\alpha - 2\beta$ is a constant c . Now equation (16) becomes

$$0 = (L_1\beta)e^{-2\beta} - \{e^{-c}(3L_1\beta - a_6) + a_6\}e^{-3\beta} - e^{-2c}(L_1\beta)e^{-5\beta} + e^{-c}\{(2L_1\beta + 2D\beta - a_6)e^{-c} + L_2\beta + 5a_6\}e^{-4\beta}. \quad (22)$$

Note that β is not a constant in this case, for otherwise we could deduce from (15) that f is a constant. If we apply a generalized Borel–Nevanlinna theorem to

equation (22) (cf. [8, Thm. 3.4]), then it follows that the coefficients of exponential functions in (22) all are zero; that is,

$$L_1\beta = 0, \quad (1 - e^{-c})a_6 = 0, \quad (1 - e^{-c})(2D\beta - a_6) = 0. \quad (23)$$

If $e^c \neq 1$, then (23) implies

$$L_1\beta = 0, \quad a_6 = 0, \quad 2D\beta - a_6 = 0;$$

hence

$$L\beta = 0, \quad D\beta = 0.$$

This is in contradiction to (A). It follows that $e^c = 1$; that is, $e^{\alpha-2\beta} = 1$. Thus we obtain

$$\frac{f}{g} \left(\frac{g-1}{f-1} \right)^2 = 1$$

or, equivalently,

$$gf(g-f) = g-f;$$

hence $gf = 1$ follows since $g \neq f$. This is Theorem 1.3(b).

Case 5: The polynomial $\beta - 2\alpha$ is a constant c . Now (16) becomes

$$0 = (L_1\alpha)e^{-2\alpha} - e^{-c}(3L_1\alpha - a_6)e^{-3\alpha} - e^{-2c}(L_1\alpha - 3a_6)e^{-5\alpha} + e^{-c}\{(2L_1\alpha + 2D\alpha - a_6)e^{-c} + L_2\alpha + 2a_6\}e^{-4\alpha} - e^{-3c}a_6e^{-6\alpha}. \quad (24)$$

Note that α is not a constant in this case, for otherwise we could deduce from (15) that f is a constant. If we apply a generalized Borel–Nevanlinna theorem to (24) (cf. [8, Thm. 3.4]), then it follows that the coefficients of exponential functions in (24) all are zero; that is,

$$a_6 = 0, \quad L_1\alpha = 0, \quad (1 - e^{-c})D\alpha = 0.$$

As in Case 4, we can show that $e^{\beta-2\alpha} = e^c = 1$. Therefore,

$$\left(\frac{g}{f} \right)^2 \frac{f-1}{g-1} = e^{\beta-2\alpha} = 1$$

or, equivalently,

$$gf(g-f) = (g-f)(g+f);$$

hence $gf = f + g$ follows since $g \neq f$. This is Theorem 1.3(c).

Case 6: The polynomial $3\beta - 2\alpha$ is a constant c . Now equation (16) becomes

$$0 = e^{-3c}(L\gamma - D\gamma - a_6)e^{8\gamma} - e^{-3c}(L\gamma + D\gamma - 2a_6)e^{7\gamma} - e^{-2c}\{3L\gamma - 9D\gamma - a_6 + e^{-c}a_6\}e^{6\gamma} + e^{-2c}(5L\gamma - 11D\gamma - 2a_6)e^{5\gamma} - e^{-2c}(2L\gamma - 4D\gamma - a_6)e^{4\gamma}. \quad (25)$$

Observe that γ is not a constant in this case, for otherwise we could deduce from (15) that f is a constant. If we apply a generalized Borel–Nevanlinna theorem to equation (25) (cf. [8, Thm. 3.4]), then it follows that the coefficients of exponential functions in (25) all are zero; that is,

$$L\gamma + D\gamma - 2a_6 = 0, \quad L\gamma - D\gamma - a_6 = 0, \quad 2L\gamma - 4D\gamma - a_6 = 0,$$

$$5L\gamma - 11D\gamma - 2a_6 = 0, \quad 3L\gamma - 9D\gamma + (e^{-c} - 1)a_6 = 0.$$

This means that

$$a_6 = 0, \quad L\gamma = 0, \quad D\gamma = 0$$

when $e^c \neq 1$ or $a_6 = 0$, which is a contradiction.

We thus have $e^c = 1, a_6 = 0$, and hence

$$\left(\frac{f-1}{g-1}\right)^3 \left(\frac{g}{f}\right)^2 = 1,$$

which yields Theorem 1.3(e) because $f \neq g$.

Case 7: We rule out the case that $\alpha, \beta, \beta - \alpha, \alpha - 2\beta, 2\alpha - \beta$, and $3\beta - 2\alpha$ all are not constant. Otherwise, if we apply a generalized Borel–Nevanlinna theorem to equation (16) (cf. [8, Thm. 3.4]), then it follows that the coefficients of exponential functions in (16) all are zero. In particular, we have

$$a_6 = 0, \quad L\gamma + D\gamma = 0, \quad L\gamma - D\gamma = 0,$$

which implies that

$$a_6 = 0, \quad L\gamma = 0, \quad D\gamma = 0. \tag{26}$$

This is a contradiction.

5. Proof of Theorem 1.5

Let f be defined by (3) and write

$$g(t, z) = \sum_{n=0}^{\infty} n! b_n J_n(t) z^n. \tag{27}$$

Note that

$$J_0(0) = 1, \quad J_n^{(n)}(0) = 2^{-n}.$$

By the assumptions of Theorem 1.5, we have

$$c_j = b_j, \quad j = 0, 1, \dots, q; \quad (a, c_0) \neq (0, 0). \tag{28}$$

Since f and g share a counting multiplicity (i.e., the zero divisors of both $f - a$ and $g - a$ are the same), it follows that [16, Prop. 6.2] implies the existence of a polynomial $P(t, z)$ on \mathbb{C}^2 satisfying

$$\frac{f(t, z) - a}{g(t, z) - a} = e^{P(t, z)}. \tag{29}$$

Moreover, either $P(t, z) \equiv 0$ or

$$\deg(P) \leq \max\{\text{ord}(f), \text{ord}(g)\}.$$

If $P = 0$, then $f = g$ follows immediately.

If $P \neq 0$, we claim that P is independent of z . In fact, if we write generally

$$\frac{\partial P}{\partial z} = \rho_0 + \rho_1 z + \dots + \rho_{\lambda-1} z^{\lambda-1},$$

where $\lambda = \deg(P) \leq q$ and $\rho_i = \rho_i(t)$ are polynomials of t but $\rho_{\lambda-1}$ is a constant, then by differentiating (29) on z we easily obtain

$$g \frac{\partial f}{\partial z} - f \frac{\partial g}{\partial z} - a \left(\frac{\partial f}{\partial z} - \frac{\partial g}{\partial z} \right) = \{fg - a(f + g) + a^2\} \frac{\partial P}{\partial z}.$$

Write

$$\begin{aligned} g \frac{\partial f}{\partial z} - f \frac{\partial g}{\partial z} &= \sum_{n=0}^{\infty} \alpha_n(t) z^n, \\ \frac{\partial f}{\partial z} - \frac{\partial g}{\partial z} &= \sum_{n=0}^{\infty} \beta_n(t) z^n, \\ f(t, z)g(t, z) &= \sum_{n=0}^{\infty} \gamma_n(t) z^n, \\ f(t, z) + g(t, z) &= \sum_{n=0}^{\infty} \delta_n(t) z^n; \end{aligned}$$

here $\alpha_n, \beta_n, \gamma_n, \delta_n$ are entire functions of t that satisfy the relations

$$\begin{aligned} \alpha_0 - a\beta_0 &= \rho_0(\gamma_0 - a\delta_0 + a^2), \\ \alpha_1 - a\beta_1 &= \rho_0(\gamma_1 - a\delta_1) + \rho_1(\gamma_0 - a\delta_0 + a^2), \\ &\vdots \end{aligned}$$

$$\alpha_{\lambda-1} - a\beta_{\lambda-1} = \rho_0(\gamma_{\lambda-1} - a\delta_{\lambda-1}) + \dots + \rho_{\lambda-1}(\gamma_0 - a\delta_0 + a^2),$$

and

$$\alpha_k - a\beta_k = \rho_0(\gamma_k - a\delta_k) + \dots + \rho_{\lambda-1}(\gamma_{k-\lambda+1} - a\delta_{k-\lambda+1}), \quad k \geq \lambda.$$

Combining these with (28), we obtain

$$\begin{aligned} \alpha_j &= 0, \quad j = 0, 1, \dots, q - 1, \\ \beta_j &= 0, \quad j = 0, 1, \dots, q - 1, \\ \gamma_0 &= c_0 b_0 J_0^2 = c_0^2 J_0^2, \\ \delta_0 &= c_0 J_0 + b_0 J_0 = 2c_0 J_0; \end{aligned}$$

hence

$$\gamma_0 - a\delta_0 + a^2 = (c_0 J_0 - a)^2 \neq 0.$$

Therefore,

$$\rho_0 = \rho_1 = \dots = \rho_{\lambda-1} = 0;$$

that is, $P(t, z) = P(t)$ is independent of z .

Setting $z = 0$ in (29), we find

$$e^{P(t)} = \frac{c_0 J_0(t) - a}{b_0 J_0(t) - a} = 1$$

and hence

$$\frac{f(t, z) - a}{g(t, z) - a} = 1.$$

Thus we obtain $f(t, z) \equiv g(t, z)$.

6. Proof of Theorem 1.6

We will use standard notation and terminologies in value distribution theory (see [8; 10]).

DEFINITION 6.1. A meromorphic function g is said to *grow more slowly* than another meromorphic function f if their Nevanlinna characteristic functions satisfy

$$\| T(r, g) = o(T(r, f)),$$

where the symbol $\|$ to the left of an expression denotes that the expression holds as $r \rightarrow \infty$ outside of a possible exceptional set E with $\int_E dr/r < \infty$.

Let's prove Theorem 1.6. Take a nonzero complex number a . Let f be a meromorphic solution of the equation (1) such that all a_k grow more slowly than f , and note that

$$\frac{1}{f - a} = -\frac{1}{a} - \frac{1}{aa_6(f - a)} \left(a_0 \frac{\partial^2 f}{\partial t^2} + 2a_1 \frac{\partial^2 f}{\partial t \partial z} + a_2 \frac{\partial^2 f}{\partial z^2} + a_3 \frac{\partial f}{\partial t} + a_4 \frac{\partial f}{\partial z} \right).$$

Then, using the lemma of logarithmic derivative (see [8; 10]), it is easy to show that the proximity function satisfies

$$\| m\left(r, \frac{1}{f - a}\right) = o(T(r, f)).$$

Therefore, the defect of f for a is just

$$\delta_f(a) = \liminf_{r \rightarrow \infty} \frac{m\left(r, \frac{1}{f - a}\right)}{T(r, f)} = 0.$$

This completes the proof of Theorem 1.6.

REMARK. Simple calculations show that a nonconstant meromorphic function f on \mathbb{C}^2 satisfies the estimates

$$\| T\left(r, \frac{\partial f}{\partial t}\right) \leq (2 + o(1))T(r, f) \quad (30)$$

and

$$\| T\left(r, \frac{\partial f}{\partial z}\right) \leq (2 + o(1))T(r, f). \quad (31)$$

Conversely, we can prove the following statement.

THEOREM 6.2. Assume that all a_k in equation (1) are entire functions on \mathbb{C}^2 . If $a_6 \neq 0$, then each nonconstant meromorphic solution f of (1) satisfies

$$\| T(r, f) \leq T\left(r, \frac{\partial f}{\partial t}\right) + T\left(r, \frac{\partial f}{\partial z}\right) + \sum_k T(r, a_k) + o(T(r, f)).$$

Proof. Without loss of generality, we may assume that

$$\frac{\partial f}{\partial t} \neq 0 \quad \text{and} \quad \frac{\partial f}{\partial z} \neq 0.$$

Now f satisfies

$$f = -a_6^{-1} \frac{\partial f}{\partial t} \left(a_0 \frac{\partial^2 f}{\partial t^2} \Big/ \frac{\partial f}{\partial t} + 2a_1 \frac{\partial^2 f}{\partial t \partial z} \Big/ \frac{\partial f}{\partial t} + a_3 \right) - a_6^{-1} \frac{\partial f}{\partial z} \left(a_2 \frac{\partial^2 f}{\partial z^2} \Big/ \frac{\partial f}{\partial z} + a_4 \right).$$

By using the lemma of logarithmic derivative, we easily obtain

$$\begin{aligned} \| m(r, f) \leq m\left(r, \frac{\partial f}{\partial t}\right) + m\left(r, \frac{\partial f}{\partial z}\right) + \sum_k T(r, a_k) \\ + o\left\{T\left(r, \frac{\partial f}{\partial t}\right)\right\} + o\left\{T\left(r, \frac{\partial f}{\partial z}\right)\right\} + O(1). \end{aligned}$$

Combining this with (30) and (31) yields

$$\| m(r, f) \leq m\left(r, \frac{\partial f}{\partial t}\right) + m\left(r, \frac{\partial f}{\partial z}\right) + \sum_k T(r, a_k) + o(T(r, f)).$$

Since

$$N(r, f) \leq N\left(r, \frac{\partial f}{\partial t}\right) \quad \text{and} \quad N(r, f) \leq N\left(r, \frac{\partial f}{\partial z}\right),$$

the proof of Theorem 6.2 is complete. □

7. Factorization of Meromorphic Solutions

Given the results due to Valiron [17] and Brownawell [5], we can derive the following theorem.

THEOREM 7.1. *Assume that all a_k in equation (1) are polynomials on \mathbb{C}^2 . Let f be a nonconstant meromorphic function on \mathbb{C} , and let g be a nonconstant entire function on \mathbb{C}^2 such that $u = f \circ g$ is a solution of (1). Then g satisfies a nontrivial algebraic differential equation of the form*

$$\begin{aligned} 0 = a_0 P_{00}(g) \frac{\partial^2 g}{\partial t^2} + 2a_1 P_{01}(g) \frac{\partial^2 g}{\partial t \partial z} + a_2 P_{02}(g) \frac{\partial^2 g}{\partial z^2} \\ + a_0 P_{10}(g) \left(\frac{\partial g}{\partial t}\right)^2 + 2a_1 P_{11}(g) \frac{\partial g}{\partial t} \frac{\partial g}{\partial z} + a_2 P_{12}(g) \left(\frac{\partial g}{\partial z}\right)^2 \\ + a_3 P_{20}(g) \frac{\partial g}{\partial t} + a_4 P_{21}(g) \frac{\partial g}{\partial z} + a_6 P_{22}(g), \end{aligned} \tag{32}$$

where the P_{ij} are polynomials.

However, suppose one of the following conditions holds:

- (a) $a_6 \neq 0$;
- (b) g is not a solution of the equation

$$a_0 \frac{\partial^2 u}{\partial t^2} + 2a_1 \frac{\partial^2 u}{\partial t \partial z} + a_2 \frac{\partial^2 u}{\partial z^2} + a_3 \frac{\partial u}{\partial t} + a_4 \frac{\partial u}{\partial z} = 0;$$

(c) g is not a solution of the equation

$$a_0 \left(\frac{\partial u}{\partial t} \right)^2 + 2a_1 \frac{\partial u}{\partial t} \frac{\partial u}{\partial z} + a_2 \left(\frac{\partial u}{\partial z} \right)^2 = 0.$$

Then f satisfies a nontrivial differential equation of the form

$$Q_0(\zeta)f''(\zeta) + Q_1(\zeta)f'(\zeta) + Q_2(\zeta)f(\zeta) = 0, \quad (33)$$

where the $Q_k(\zeta)$ are polynomials on \mathbb{C} . In particular, f is of finite order. Further, if f is a transcendental entire function then the order of f is a positive rational number.

Proof. Because $u = f \circ g$ is a solution of (1), we have

$$B_0 f'' \circ g + B_1 f' \circ g + a_6 f \circ g = 0, \quad (34)$$

where

$$B_0 = a_0 \left(\frac{\partial g}{\partial t} \right)^2 + 2a_1 \frac{\partial g}{\partial t} \frac{\partial g}{\partial z} + a_2 \left(\frac{\partial g}{\partial z} \right)^2 \quad \text{and}$$

$$B_1 = a_0 \frac{\partial^2 g}{\partial t^2} + 2a_1 \frac{\partial^2 g}{\partial t \partial z} + a_2 \frac{\partial^2 g}{\partial z^2} + a_3 \frac{\partial g}{\partial t} + a_4 \frac{\partial g}{\partial z}.$$

The assumptions of Theorem 7.1 make equation (34) nontrivial. By [5, Thm. 1], g and f satisfy a nontrivial differential equation of the form (32) and (33), respectively. Then, according to a result due to Von Koch-Perron (or see [6]), f is of finite order. Furthermore, if f is a transcendental entire function, then it follows from Valiron's theorem [17] that the order of f is a positive rational number. This completes the proof of Theorem 7.1. \square

A result due to Steinmetz [15] claims that each meromorphic solution of a linear ordinary differential equation with rational coefficients is pseudo-prime—that is, one of factors in each factorization of the function is not transcendental. However, the functions f and g in Theorem 7.1 may be transcendental. For example, the function $u(t, z) = \exp(2te^z - e^{2z})$ with $f(\zeta) = e^\zeta$ and $g(t, z) = 2te^z - e^{2z}$ satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} - 2t \frac{\partial u}{\partial t} + 2 \frac{\partial u}{\partial z} = 0, \quad (35)$$

but $u(t, z)$ is not pseudo-prime. Hence Steinmetz's theorem cannot be directly extended to the partial differential equations (1). Observe that $g(t, z)$ satisfies the nonlinear differential equation

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{\partial u}{\partial t} \right)^2 - 2t \frac{\partial u}{\partial t} + 2 \frac{\partial u}{\partial z} = 0.$$

On the other hand, $g(t, z)$ satisfies condition (b) in Theorem 7.1; that is, g is not a solution of (35), and f is of order 1. Therefore, $u(t, z)$ satisfies the conclusions of Theorem 7.1.

8. Existence of Entire Solutions

Generally, by Cauchy's existence theorem (or the Cauchy–Kowalewski theorem) we can find local holomorphic solutions of equation (1) (see Section 9), but this theorem does not tell us how to confirm completely the domains of holomorphic solutions. If Σ is of the form $D \times \mathbb{C}$ for a domain $D \subseteq \mathbb{C}$, then the following result gives definite domains of holomorphic solutions for a class of differential equations.

THEOREM 8.1. *Let D be a domain in \mathbb{C} , and take $t_0 \in D$. Assume that $a_0 = a_0(t)$ is independent of z and has no zeros in a disc*

$$\Delta = \Delta_\rho = \{t \in \mathbb{C} \mid |t - t_0| < \rho\} \subset D.$$

Suppose one of the following conditions holds:

(i) *equation (1) has the form*

$$a_0 \frac{\partial^2 u}{\partial t^2} + 2A_1 z \frac{\partial^2 u}{\partial t \partial z} + A_2 z^2 \frac{\partial^2 u}{\partial z^2} + A_3 \frac{\partial u}{\partial t} + A_4 z \frac{\partial u}{\partial z} + A_6 u = 0, \quad (36)$$

where all $A_k = A_k(t)$ are holomorphic functions for one variable $t \in D$;

(ii) *the holomorphic functions a_k on $\Sigma = D \times \mathbb{C}$ are independent of the variable z .*

Then equation (1) has nonconstant holomorphic solutions on $\Delta \times \mathbb{C}$ that satisfy

$$\dim \mathcal{S}_{\text{hol}}(\Delta \times \mathbb{C}) = \infty.$$

In particular, we have the following statement.

COROLLARY 8.2. *Under the conditions of Theorem 8.1 with $D = \mathbb{C}$ and $a_0 = 1$, equation (1) has nonconstant entire solutions $u(t, z)$ on \mathbb{C}^2 .*

For example, Laplace's equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

Helmholtz's equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0,$$

the telegraph equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial z^2} + 2\alpha \frac{\partial u}{\partial t} + 2\beta \frac{\partial u}{\partial z} + \gamma u = 0,$$

and Čaplygin's equation

$$\frac{\partial^2 u}{\partial t^2} - K(t) \frac{\partial^2 u}{\partial z^2} = 0$$

with entire $K(t)$ all have nonconstant entire solutions $u(t, z)$ on \mathbb{C}^2 .

We will first consider the condition (i) in Theorem 8.1. We shall seek analytic solutions to (36) of the form $w(t)z^n$ for each integer $n \in \mathbb{Z}$. In other words, we wish to solve $w(t)$ from the associated differential equations

$$a_0 \frac{d^2 w}{dt^2} + \{2nA_1 + A_3\} \frac{dw}{dt} + \{n(n-1)A_2 + nA_4 + A_6\} w = 0, \quad n \in \mathbb{Z}, \quad (37)$$

or (equivalently)

$$\frac{d^2 w}{dt^2} + p_n \frac{dw}{dt} + q_n w = 0, \quad n \in \mathbb{Z}, \quad (38)$$

where

$$p_n = \frac{1}{a_0} \{2nA_1 + A_3\}, \quad q_n = \frac{1}{a_0} \{n(n-1)A_2 + nA_4 + A_6\}$$

are meromorphic functions on D but are holomorphic on Δ . We have the following basic fact.

LEMMA 8.3 (cf. [18]). *Equation (38) has a unique solution $w_n(t)$ inside the disc Δ . This solution satisfies the initial conditions*

$$w_n(t_0) = b_n \quad \text{and} \quad \frac{dw_n}{dt}(t_0) = b'_n,$$

where b_n, b'_n are arbitrary constants and $w_n(t)$ is single-valued and holomorphic in the disc Δ .

In particular, for any nonnegative integer n taking nonzero constants b_n and b'_n , we obtain nonconstant holomorphic solutions $w_n(t)z^n$ on $\Delta \times \mathbb{C}$. Obviously, the family $\{w_n(t)z^n\}_{n \geq 0}$ is linearly independent, so

$$\dim \mathcal{S}_{\text{hol}}(\Delta \times \mathbb{C}) = \infty.$$

For condition (ii) of Theorem 8.1, we may try to find analytic solutions to (36) of the form $w(t)e^{nz}$ for each integer $n \in \mathbb{Z}$; that is, to solve $w(t)$ from the associated differential equations

$$a_0 \frac{d^2 w}{dt^2} + \{2na_1 + a_3\} \frac{dw}{dt} + \{n^2 a_2 + na_4 + a_6\} w = 0, \quad n \in \mathbb{Z}. \quad (39)$$

Using a similar argument as before, we can obtain nonconstant holomorphic solutions $w_n(t)e^{nz}$ on $\Delta \times \mathbb{C}$. The proof of Theorem 8.1 is now complete.

According to the proof of Theorem 8.1, for each $n \in \mathbb{Z}$ equation (36) has solutions $w_n(t)z^n$, where $w_n(t)$ are analytic solutions of equation (37) on Δ . Take a real number r with $0 < r < \rho$. Obviously, we may choose constants $c_n = c_n(r)$ such that, for any compact set $E \subset \mathbb{C}_* = \mathbb{C} - \{0\}$, the series

$$u(t, z) = \sum_{n=-\infty}^{\infty} c_n w_n(t) z^n \quad (40)$$

is uniformly convergent for $t \in \Delta_r$ and $z \in E$, so it expresses a holomorphic function on $\Delta_r \times \mathbb{C}_*$. Now it is trivial to check that $u(t, z)$ satisfies equation (36).

Near a zero of a_0 , we may construct two analytic solutions $w_{n1}(t)z^n$ and $w_{n2}(t)z^n$ of (36) by using the following fact.

LEMMA 8.4 (cf. [18]). *If t_0 is a singular point of equation (38) (i.e., if t_0 is a zero of a_0), then the two linearly independent solutions of (38) in $0 < |t - t_0| < \rho$ (where ρ is sufficiently small so that there are no zeros of a_0 in the annular domain) are*

$$w_{n1}(t) = (t - t_0)^{\tau_1} \sum_{k=-\infty}^{\infty} \alpha_k (t - t_0)^k \quad \text{and}$$

$$w_{n2}(t) = \beta w_{n1}(t) \log(t - t_0) + (t - t_0)^{\tau_2} \sum_{k=-\infty}^{\infty} \beta_k (t - t_0)^k,$$

where $\tau_1 - \tau_2 \in \mathbb{Z}$.

If we make a transform $z = e^w$, then equation (36) becomes

$$a_0 \frac{\partial^2 u}{\partial t^2} + 2A_1 \frac{\partial^2 u}{\partial t \partial w} + A_2 \left(\frac{\partial^2 u}{\partial w^2} - \frac{\partial u}{\partial w} \right) + A_3 \frac{\partial u}{\partial t} + A_4 \frac{\partial u}{\partial w} + A_6 u = 0. \quad (41)$$

Thus a solution $w_n(t)z^n$ of (36) becomes a solution $w_n(t)e^{nw}$ of (41).

Assume in general that all A_k in (36) are entire functions on \mathbb{C}^2 with $a_0 = 1$, so we may write

$$A_k(t, z) = \sum_{n=0}^{\infty} A_{kn}(t)z^n.$$

Next we try to find solutions to (36) of the form

$$u(t, z) = \sum_{n=0}^{\infty} w_n(t)z^n.$$

Recall that equation (36) has the following form:

$$0 = \sum_{n=0}^{\infty} w_n'' z^n + 2 \sum_{n=0}^{\infty} \left(\sum_{k=0}^n k A_{1,n-k} w_k' \right) z^n + \sum_{n=0}^{\infty} \left(\sum_{k=0}^n k(k-1) A_{2,n-k} w_k \right) z^n$$

$$+ \sum_{n=0}^{\infty} \left(\sum_{k=0}^n A_{3,n-k} w_k' \right) z^n + \sum_{n=0}^{\infty} \left(\sum_{k=0}^n k A_{4,n-k} w_k \right) z^n$$

$$+ \sum_{n=0}^{\infty} \left(\sum_{k=0}^n A_{6,n-k} w_k \right) z^n;$$

hence, it is enough to solve the equations

$$w_n'' + \sum_{k=0}^n (2k A_{1,n-k} + A_{3,n-k}) w_k'$$

$$+ \sum_{k=0}^n (k(k-1) A_{2,n-k} + k A_{4,n-k} + A_{6,n-k}) w_k = 0$$

for all $n \geq 0$. Inductively, we can find entire solutions $w_n(t)$ satisfying conditions

$$w_n(0) = b_n \quad \text{and} \quad \frac{dw_n}{dt}(0) = b'_n,$$

where b_n and b'_n are arbitrary constants (see Lemma 9.4). Moreover, by choosing proper constants b_n and b'_n , we may obtain holomorphic solutions $u(t, z)$.

9. Final Notes

Generally, we consider partial differential equations of order n in two independent complex variables

$$F(t, z, u, u_{10}, u_{01}, \dots, u_{n0}, \dots, u_{0n}) = 0, \tag{42}$$

where

$$u_{jk} = \frac{\partial^{j+k} u}{\partial t^j \partial z^k} \quad \text{and} \quad u_{00} = u.$$

With regard to this equation, Cauchy’s problem may be stated as follows: Find a solution $u = u(t, z)$ of (43) such that, for $t = t(\lambda)$ and $z = z(\lambda)$, one has (depending on a parameter λ)

$$\frac{\partial^{j+k} u(t, z)}{\partial t^j \partial z^k} = u_{jk}(\lambda), \quad j + k \leq n - 1.$$

THEOREM 9.1 (Cauchy’s existence theorem). *Take a point*

$$Z_0 = (t^0, z^0, u^0, u_{10}^0, u_{01}^0, \dots, u_{n0}^0, \dots, u_{0n}^0) \in \mathbb{C}^{2+(n+1)(n+2)/2}.$$

Assume that F is holomorphic at the point Z_0 with $F(Z_0) = 0$, and assume that $t(\lambda)$, $z(\lambda)$, and $u_{jk}(\lambda)$ are holomorphic at $\lambda = 0$ and satisfy

$$t(0) = t^0, \quad z(0) = z^0, \quad u_{jk}(0) = u_{jk}^0.$$

If the function F satisfies

$$\left(\frac{\partial F}{\partial u_{n0}} dz^n - \frac{\partial F}{\partial u_{n-1,1}} dt dz^{n-1} + \dots + (-1)^n \frac{\partial F}{\partial u_{0n}} dt^n \right)_{Z_0} \neq 0, \tag{43}$$

then Cauchy’s problem has a unique holomorphic solution near (t^0, z^0) .

If equation (42) can be written in the form

$$\frac{\partial^n u}{\partial t^n} = F(t, z, u, u_{10}, u_{01}, \dots, u_{n0}, \dots, u_{0n}) \tag{44}$$

such that the function F is independent of u_{n0} , then Cauchy’s problem consists of finding the solution of equation (44) that for $t = 0$ satisfies the conditions

$$\frac{\partial^j u}{\partial t^j} = u_j(z), \quad j = 0, 1, \dots, n - 1. \tag{45}$$

The following theorem of Kowalewski is fundamental in this connection.

THEOREM 9.2 (Cauchy–Kowalewski theorem). *Assume that F in equation (44) is independent of u_{n0} , and assume that F is holomorphic with respect to all its arguments in a neighborhood of some fixed values (which, for the sake of simplicity, we shall assume to be zero). Let the functions u_j as well as all their derivatives up to the n th order vanish when $z = 0$. Then Cauchy’s problem (44) with (45) has one and only one holomorphic solution in a certain neighborhood of the origin.*

By using Cauchy’s existence theorem (or the Cauchy–Kowalewski theorem), we may study local holomorphic solutions of the homogeneous linear partial differential equations of order n in two independent complex variables

$$\sum_{j+k \leq n} a_{jk} \frac{\partial^{j+k} u}{\partial t^j \partial z^k} = 0, \tag{46}$$

where $a_{jk} = a_{jk}(t, z)$ are holomorphic functions for $t \in D$ and $z \in \Omega$ and where $D \subseteq \mathbb{C}$ and $\Omega \subseteq \mathbb{C}$ are domains. For this case, condition (43) becomes

$$(a_{n0} dz^n - a_{n-1,1} dt dz^{n-1} + \dots + (-1)^n a_{0n} dt^n)_{(t^0, z^0)} \neq 0. \tag{47}$$

It is evident that, by the methods and arguments used previously, one can use (46) to derive results similar to Theorems 8.1, 7.1, and 1.6. Their analogues can be stated as follows.

THEOREM 9.3. *Take $t_0 \in D$. Assume that $a_{n0} = a_{n0}(t)$ is independent of z and has no zeros in a disc*

$$\Delta = \Delta_\rho = \{t \in \mathbb{C} \mid |t - t_0| < \rho\} \subset D.$$

Suppose one of the following conditions holds:

- (i) *the coefficients of equation (46) have the form*

$$a_{jk}(t, z) = A_{jk}(t)z^k,$$

where all $A_{jk} = A_{jk}(t)$ are holomorphic functions for one variable $t \in D$;

- (ii) *all a_{jk} are independent of the variable z .*

Then equation (46) has nonconstant holomorphic solutions $u(t, z)$ on $\Delta \times \mathbb{C}$.

Theorem 9.3 can be proved similarly by using the following result for ordinary differential equations of higher orders.

LEMMA 9.4 (cf. [6]). *Let $p_1(t), \dots, p_n(t)$ and $q(t)$ be holomorphic functions on a domain $D \subset \mathbb{C}$. Then, for each $t_0 \in D$ and $(b, b', \dots, b^{(n-1)}) \in \mathbb{C}^n$, the ordinary differential equation of order n ,*

$$\frac{d^n w}{dt^n} + p_1(t) \frac{d^{n-1} w}{dt^{n-1}} + \dots + p_n(t) w = q(t),$$

has a unique holomorphic solution $w = w(t)$ on D satisfying

$$w(t_0) = b, w'(t_0) = b', \dots, w^{(n-1)}(t_0) = b^{(n-1)}.$$

THEOREM 9.5. *Assume that all a_{jk} in (46) are polynomials on \mathbb{C}^2 . Let f be a nonconstant meromorphic function on \mathbb{C} , and let g be a nonconstant entire function on \mathbb{C}^2 such that $u = f \circ g$ is a solution of (46). Suppose one of the following conditions holds:*

- (a) $a_{00} \neq 0$;
 (b) g is not a solution of the equation

$$\sum_{j+k \leq n} a_{jk} \frac{\partial^{j+k} u}{\partial t^j \partial z^k} = a_{00} u;$$

- (c) g is not a solution of the equation

$$\sum_{j+k=n} a_{jk} \left(\frac{\partial u}{\partial t} \right)^j \left(\frac{\partial u}{\partial z} \right)^k = 0.$$

Then f satisfies a nontrivial differential equation of the form

$$Q_0(\zeta) f^{(n)}(\zeta) + Q_1(\zeta) f^{(n-1)}(\zeta) + \cdots + Q_n(\zeta) f(\zeta) = 0, \quad (48)$$

where the $Q_k(\zeta)$ are polynomials on \mathbb{C} . In particular, f is of finite order. Furthermore, if f is a transcendental entire function, then the order of f is a positive rational number.

Moreover, under the conditions of Theorem 9.5, one can prove that g satisfies a rather complicated algebraic differential equation (see e.g. [5]). Finally, we would like to point out that the arguments used in this paper's proofs can easily lead to the following result.

THEOREM 9.6. *Assume that all a_{jk} in (46) are entire functions on \mathbb{C}^2 that grow more slowly than a meromorphic solution of equations (46) on \mathbb{C}^2 . If $a_{00} \neq 0$, then the deficiency of the solution for each nonzero complex number is zero.*

ACKNOWLEDGMENT. The authors wish to thank the referee for several helpful comments.

References

- [1] J. Anastassiadis, *Recherches algébriques sur le théorème de Picard-Montel*, Exposés sur la théorie des fonctions XVIII, Hermann, Paris, 1959.
- [2] C. A. Berenstein and B. Q. Li, *On certain first-order partial differential equations in \mathbb{C}^n* , Harmonic analysis, signal processing, and complexity, Progr. Math., 238, pp. 29–36, Birkhäuser, Boston, 2005.
- [3] S. N. Bernšteĭn, *Sur la nature analytique des solutions de certaines équations aux dérivées partielles du second order*, C. R. Acad. Sci. Paris 137 (1903), 778–781.
- [4] G. Brosch, *Eindeutigkeitssätze für meromorphe Funktionen*, Ph.D. thesis, Technical University of Aachen, 1989.
- [5] W. D. Brownawell, *On the factorization of partial differential equations*, Canad. J. Math. 39 (1987), 825–834.

- [6] Y. Z. He and X. Z. Xiao, *Algebroid functions and ordinary differential equations*, Science Press, Beijing, 1988.
- [7] E. W. Hobson, *Spherical and ellipsoidal harmonics*, Cambridge Univ. Press, Cambridge, 1931.
- [8] P. C. Hu, P. Li, and C. C. Yang, *Unicity of meromorphic mappings*, Adv. Complex Anal. Appl., 1, Kluwer, Dordrecht, 2003.
- [9] P. C. Hu and C. C. Yang, *Malmquist type theorem and factorization of meromorphic solutions of partial differential equations*, Complex Variables Theory Appl. 27 (1995), 269–285.
- [10] ———, *Value distribution theory related to number theory*, Birkhäuser, Basel, 2006.
- [11] H. Lewy, *Neuer Beweis des analytischen Charakters der Lösungen elliptische Differentialgleichungen*, Math. Ann. 101 (1929), 609–619.
- [12] B. Q. Li, *Entire solutions of certain partial differential equations and factorization of partial derivatives*, Trans. Amer. Math. Soc. 357 (2005), 3169–3177.
- [13] B. Q. Li and E. G. Saleeby, *Entire solutions of first-order partial differential equations*, Complex Variables Theory Appl. 48 (2003), 657–661.
- [14] I. G. Petrovskiĭ, Dokl. Akad. Nauk SSSR 17 (1937), 343–346; Mat. Sb. (N.S.) 5(47) (1939), 3–70.
- [15] N. Steinmetz, *Über die faktorisierten Lösungen gewöhnlicher Differentialgleichungen*, Math. Z. 170 (1980), 169–180.
- [16] W. Stoll, *Holomorphic functions of finite orders in several complex variables*, CBMS Regional Conf. Ser. in Math., 21, Amer. Math. Soc., Providence, RI, 1974.
- [17] G. Valiron, *Lectures on the general theory of integral functions*, Édouard Privat, Toulouse, 1923.
- [18] Z. X. Wang and D. R. Guo, *Special functions*, World Scientific, Teaneck, NJ, 1989.

P.-C. Hu
Department of Mathematics
Shandong University
Jinan 250100
Shandong
P. R. China
pchu@sdu.edu.cn

C.-C. Yang
Department of Mathematics
Hong Kong University
of Science & Technology
Clear Water Bay
Kowloon
Hong Kong
chungchun.yang@gmail.com