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On Local Models with Special Parahoric Level Structure

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0. Introduction

Motivation and Main Results

For the study of arithmetic properties of a variety over an algebraic number field, it is of interest to have a model over the ring of integers. In the particular case of a Shimura variety, one likes to have a model over the ring of integers $\mathcal{O}_E$, where E is the completion of the reflex field at a finite prime of residue characteristic p. It should be flat and have only mild singularities. If the Shimura variety is the moduli space over Spec E of abelian varieties with additional polarization, endomorphisms, and level structure (a Shimura variety of PEL type), then it is natural to define a model by posing the moduli problem over $\mathcal{O}_E$. In the case of a parahoric level structure at p with the parahoric defined in an elementary way as the stabilizer of a self-dual periodic lattice chain, such a model has been given by Rapoport and Zink [RZ].

Although in special cases this model is shown to be flat with reduced special fiber and with irreducible components that are normal and have only rational singularities [Gö1; Gö2], in general it is not flat, as has been pointed out by Pappas [P]. In a series of papers, Pappas and Rapoport [PR1; PR2; PR4] examine how to define closed subschemes of this *naive model* that are more likely to be flat. Flatness can be brutally enforced by taking the (reduced) Zariski closure of the generic fiber in the naive model. Aside from that, by adding further conditions one can attempt to cut out this closed subscheme, or at least give a better approximation. If the parahoric subgroup is the stabilizer of a self-dual periodic lattice chain, then these questions can be reduced to problems of the corresponding *local* models [RZ]. Locally for the étale topology around each point of the special fiber, these models coincide with the corresponding moduli schemes. This approach has the advantage of leading to varieties that can be defined in terms of linear algebra and thus can be handled more easily. In the setting of unitary groups considered here, Pappas [P] defines in this way the *wedge local model,* a closed subscheme of the naive local model. The *local model* is defined to be the closure of the generic fiber in the naive local model; it is also a closed subscheme of the wedge local model.

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In one of their recent papers, Pappas and Rapoport [PR4] study the case where the group defining the Shimura variety is the group of unitary similitudes corresponding to a quadratic extension of $\mathbb{Q}$ that is ramified at p. Assuming the so-called coherence conjecture, the reducedness of the geometric special fiber of the local model is proved, and it is shown that its irreducible components are normal and with only rational singularities [PR4, Thm. 4.1]. Some special cases, however, can be treated without relying on this conjecture. We will prove the following theorem.

THEOREM 0.1 (cf. Theorem 2.1). *Let the level structure at p be given by a parahoric that is defined as the stabilizer of a self-dual periodic lattice chain (see Section 1.2 for details) and that is "special" in the sense of Bruhat–Tits theory* [T]. *Then the special fiber of the local model is irreducible and reduced; furthermore, the special fiber is normal, Frobenius split, and with only rational singularities.*

REMARK 0.2. As a consequence of this theorem, the corresponding Shimura variety has a p-adic model that is normal and has only rational singularities. Moreover, the special fiber of this model is irreducible and reduced.

The proof of the theorem is divided into two major steps, in which we prove the following results.

THEOREM 0.3 (cf. Theorem 3.1). *Let the assumptions be the same as in Theorem 0.1. Then the special fiber of the local model contains a nonempty open subset that is reduced.*

THEOREM 0.4 (cf. Theorem 4.1). *Under the assumptions of Theorem 0.1, the special fiber of the local model is irreducible.*

Once it is shown that the special fiber of the local model is irreducible and generically reduced, the other properties stated in Theorem 0.1 follow by standard methods given in the paper by Pappas and Rapoport [PR4, Proof of Thm. 5.1].

It is shown in [PR4, Sec. 1.2] that there are exactly three cases in which the stabilizer subgroup is a special parahoric. Two of these cases have been treated by Pappas and Rapoport [PR4, Sec. 5], where proofs of the theorems in these cases are given. The focus of this paper is on the proof of the third case, which has not yet been treated (in full generality) in the literature (see Remark 2.3). Moreover, we obtain the following result.

THEOREM 0.5 (cf. Theorem 3.1 and Theorem 5.1). *Let the same assumptions hold true as in Theorem 0.1. Then the local model contains a nonempty open subset that is isomorphic to affine space.*

All of the results mentioned previously are achieved by first evaluating the conditions of the wedge local model for open neighborhoods of certain special points (the "best point" and the "worst point"; see Section 3.1 and Section 4.1) and then passing to the actual local model using dimension arguments.

More precisely, the conditions of the wedge local model translate into several matrix identities, and we examine the schemes defined in this way. In the cases of Theorem 0.3 and Theorem 0.5, this leads to affine spaces described by simple matrix equations. In the case of Theorem 0.4, we must deal with a more complicated matrix scheme. We exploit that the symplectic group acts thereon, and by considering an equivariant projection morphism, we can confine ourselves to the study of certain fibers. These can be described using results of Ohta [O] and of Kostant and Rallis [KRa] on the structure of nilpotent orbits in the classical symmetric pair $(\mathfrak{gl}_n, \mathfrak{sp}_n)$, as in [PR4, Sec. 5.5].

By definition, the local model is flat; hence, its special fiber is equidimensional and has the same dimension as the generic fiber. The aforementioned matrix schemes are seen either to be irreducible of that dimension or to contain a single irreducible component of that dimension with all other irreducible components having smaller dimension. Since the local model is a closed subscheme of the wedge local model, this allows us to deduce our results on the local models.

The paper is divided into five sections. In the first section we recall the construction of the local model for the situation considered above. In Section 2 we formulate our main theorem (Theorem 0.1), whose two-part proof ranges over Sections 3 and 4 (where we establish Theorem 0.3 and Theorem 0.4, respectively). As mentioned previously, a slightly stronger result (Theorem 0.5) is obtained during the proof of Theorem 0.3; its validity in the cases treated by Pappas and Rapoport is shown in Section 5.

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1. Definition of the Local Model

We recall the construction of the local model for the ramified unitary group as given by Pappas and Rapoport [PR4]. We first introduce the basic notions and then define the naive local model. This is followed by a short discussion of the wedge local model, which provides a closed subscheme of the naive local model. Finally, we give the definition of the local model.

1.1. Standard Lattices

We use the notation of [PR4]. Let F_0 be a complete discretely valued field with ring of integers $\mathcal{O}_{F_0}$, perfect residue field k of characteristic $\neq 2$, and uniformizer π_0. Let F/F_0 be a ramified quadratic extension and $\pi \in F$ a uniformizer with $\pi^2 = \pi_0$. Let V be an F-vector space of dimension $n \geq 3$ with an (F/F_0)-hermitian form

$$\phi\colon V \times V \to F,$$

which we assume to be split. This means that there exists a basis $e_1, \dots, e_n$ of V such that

$$\phi(e_i, e_{n+1-j}) = \delta_{i,j} \quad \text{for all } \ i, j = 1, \dots, n.$$

We have two associated F_0-bilinear forms,

$$\langle x, y\rangle := \frac{1}{2}\operatorname{Tr}_{F/F_0}(\pi^{-1}\phi(x,y)) \quad \text{and}$$
$$(x, y) := \frac{1}{2}\operatorname{Tr}_{F/F_0}(\phi(x,y)).$$

The form $\langle\cdot,\cdot\rangle$ is alternating, and $(\cdot,\cdot)$ is symmetric. For any $\mathcal{O}_F$-lattice Λ in V, we denote by

$$\hat{\Lambda} := \{v \in V \mid \phi(v, \Lambda) \subset \mathcal{O}_F\} = \{v \in V \mid \langle v, \Lambda\rangle \subset \mathcal{O}_{F_0}\}$$

the dual lattice with respect to the alternating form and by

$$\hat{\Lambda}^{\mathrm{s}} := \{v \in V \mid (v, \Lambda) \subset \mathcal{O}_{F_0}\}$$

the dual lattice with respect to the symmetric form. We have $\hat{\Lambda}^{\mathrm{s}} = \pi^{-1}\hat{\Lambda}$.

For $i = 0, \dots, n-1$, we define the standard lattices

$$\Lambda_i := \operatorname{span}_{\mathcal{O}_F}\{\pi^{-1}e_1, \dots, \pi^{-1}e_i, e_{i+1}, \dots, e_n\}.$$

1.2. Self-Dual Periodic Lattice Chain

Write $n = 2m$ if n is even and $n = 2m+1$ if n is odd. We consider nonempty subsets $I \subset \{0, \dots, m\}$ with the requirement that for $n = 2m$ even, if $m-1$ is in I then also m is in I. We complete the Λ_i with $i \in I$ to a self-dual periodic lattice chain by first including the duals $\Lambda_{n-i} := \hat{\Lambda}_i^{\mathrm{s}}$ for $i \in I \setminus \{0\}$ and then all the π-multiples: for $j \in \mathbb{Z}$ of the form $j = kn + i$ with $k \in \mathbb{Z}$ and $i \in I$ or $n - i \in I$, we set $\Lambda_j := \pi^{-k}\Lambda_i$. Then the Λ_j form a periodic lattice chain Λ_I, which satisfies $\hat{\Lambda}_j = \Lambda_{-j}$.

The index sets I of the form just described are in one-to-one correspondence with the parahoric subgroups of the unitary similitude group

$$\mathrm{GU}(V,\phi) = \{g \in \mathrm{GL}_F(V) \mid \phi(gx, gy) = c(g)\phi(x,y),\ c(g) \in F_0^\times\}$$

of the vector space V and the form ϕ, as shown in [PR4, Sec. 1.2.3]. If $n = 2m+1$ is odd, then the correspondence is given by assigning the stabilizer subgroup

$$P_I := \{g \in \mathrm{GU}(V,\phi) \mid g\Lambda_i = \Lambda_i \text{ for all } i \in I\} \subset \mathrm{GU}(V,\phi)$$

to the lattice chain Λ_I. If $n = 2m$ is even, the situation is slightly more complicated. One must consider a certain subgroup of P_I (the kernel of the Kottwitz homomorphism), which gives a proper subgroup (of index 2) exactly when I does not contain m.

1.3. Reflex Field

Let F_0^{sep} be a fixed separable closure of F_0. For each of the two embeddings $\varphi\colon F \to F_0^{\mathrm{sep}}$, we fix an integer r_φ with $0 \le r_\varphi \le n$. The reflex field E associated to these data is the finite field extension of F_0 contained in F_0^{sep} with

$$\mathrm{Gal}(F_0^{\mathrm{sep}}/E) = \{\tau \in \mathrm{Gal}(F_0^{\mathrm{sep}}/F_0) \mid r_{\tau\varphi} = r_\varphi \text{ for all } \varphi\}.$$

1.4. Naive Local Model

We fix nonnegative integers r and s with $n = r + s$. In the theory of Shimura varieties, these integers correspond to the signature of the algebraic group associated to the Shimura variety (after base change to the real numbers). Replacing ϕ by $-\phi$ if necessary, we may assume $s \le r$. We further assume $s > 0$ (otherwise, the corresponding Shimura variety is 0-dimensional). With r and s taken for r_φ in Section 1.3, the reflex field E equals F if $r \ne s$ or F_0 if $r = s$.

For ease of notation, we denote the tensor product over $\mathcal{O}_{F_0}$ just by $\otimes$. We formulate a moduli problem M_I^{naive} on the category of $\mathcal{O}_E$-schemes: A point of M_I^{naive} with values in an $\mathcal{O}_E$-scheme S is given by $(\mathcal{O}_F \otimes \mathcal{O}_S)$-submodules

$$\mathcal{F}_j \subset \Lambda_j \otimes \mathcal{O}_S$$

for each $j \in \mathbb{Z}$ of the form $j = kn \pm i$ with $k \in \mathbb{Z}$ and $i \in I$. For each such j, the following conditions must be satisfied.

(N1) As an $\mathcal{O}_S$-module, $\mathcal{F}_j$ is locally on S, a direct summand of rank n.

(N2) For each $j < j'$, there is a commutative diagram

$$\begin{array}{ccc} \Lambda_j \otimes \mathcal{O}_S & \longrightarrow & \Lambda_{j'} \otimes \mathcal{O}_S \\ \cup & & \cup \\ \mathcal{F}_j & \longrightarrow & \mathcal{F}_{j'}, \end{array}$$

where the top horizontal map is induced by the lattice inclusion $\Lambda_j \subset \Lambda_{j'}$ and where, for each j, the isomorphism $\pi : \Lambda_j \to \Lambda_{j-n}$ induces an isomorphism of $\mathcal{F}_j$ with $\mathcal{F}_{j-n}$.

(N3) $\mathcal{F}_{-j} = \mathcal{F}_j^{\perp}$, with $\mathcal{F}_j^{\perp}$ denoting the orthogonal complement of $\mathcal{F}_j$ under the natural perfect pairing

$$\langle \cdot,\cdot \rangle \otimes \mathcal{O}_S\colon (\Lambda_{-j} \otimes \mathcal{O}_S) \times (\Lambda_j \otimes \mathcal{O}_S) \to \mathcal{O}_S.$$

(N4) We denote by Π the respective action on $\Lambda_j \otimes \mathcal{O}_S$ given by multiplication with $\pi \otimes 1$. Since $\mathcal{F}_j$ is required to be an $\mathcal{O}_F \otimes \mathcal{O}_S$-module, Π restricts to an action on $\mathcal{F}_j$. The characteristic polynomial equals

$$\det(T\,\mathrm{id} - \Pi|_{\mathcal{F}_j}) = (T - \pi)^s (T + \pi)^r \in \mathcal{O}_S[T].$$

The moduli problem formulated in this way is representable by a projective scheme over $\mathrm{Spec}\,\mathcal{O}_E$, since conditions (N1)–(N4) define a closed subfunctor of a product of Grassmann functors. We call M_I^{naive} the *naive local model* associated to the group $\mathrm{GU}(V,\phi)$, the signature type (r,s), and the self-dual periodic lattice chain Λ_I.

1.5. Wedge Local Model

As mentioned in the Introduction, the naive local model is almost never flat over $\mathcal{O}_E$. Pappas [P] defines a closed subscheme of M_I^{naive} by imposing an additional (wedge) condition:

(W) If $r \neq s$ then, for each j,

$$\wedge^{r+1}\big(\Pi - \sqrt{\pi_0}\,|_{\mathcal{F}_j}\big) = 0,$$
$$\wedge^{s+1}\big(\Pi + \sqrt{\pi_0}\,|_{\mathcal{F}_j}\big) = 0.$$

Here we have written $\sqrt{\pi_0}$ for the action on $\Lambda_j \otimes \mathcal{O}_S$ given by multiplication with $1 \otimes \pi$. Note that the assumption $r \neq s$ implies $\pi \in \mathcal{O}_S$.

We denote the corresponding closed subscheme by $M_I^{\wedge}$. It is called the *wedge local model.*

LEMMA 1.1. *The wedge local model has the same generic fiber as the naive local model.*

Proof. We may assume that $r \neq s$ because otherwise the wedge condition is trivial. In order to examine the generic fiber of the naive local model, we must consider A-valued points with A an arbitrary E-algebra. These points are given by subspaces $\mathcal{F}_j \subset \Lambda_j \otimes A$ subject to conditions (N1)–(N4). We fix an $\mathcal{O}_F$-basis $f_1, \dots, f_n$ of Λ_j. This induces an A-basis $f_1, \pi f_1, \dots, f_n, \pi f_n$ of $\Lambda_j \otimes A$ via the identification $\mathcal{O}_F \cong \mathcal{O}_{F_0} \cdot 1 + \mathcal{O}_{F_0} \cdot \pi$. Then Π is represented by the diagonal block matrix $\mathrm{diag}(B, \dots, B)$ of size $2n$, where the square matrix B of size 2 is given by $\left(\begin{smallmatrix} & \pi_0 \\ 1 & \end{smallmatrix}\right)$.

Since the characteristic polynomial of B is $T^2 - \pi_0 = (T - \pi)(T + \pi)$, it follows that the endomorphism Π is diagonalizable over A, and so is the restriction to the Π-stable subspace $\mathcal{F}_j$. By (N4), the corresponding characteristic polynomial equals $(T - \pi)^s (T + \pi)^r$; hence, we can choose a basis such that $\Pi|_{\mathcal{F}_j}$ is represented by the diagonal matrix $\mathrm{diag}(\pi, \dots, \pi, -\pi, \dots, -\pi)$, with π occurring s times and $-\pi$ occurring r times. Now it is obvious that (W) is automatically satisfied in the situation considered. Therefore, the wedge condition does not alter the generic fiber. $\square$

1.6. Local Model

The *local model* M_I^{loc} is defined to be the scheme-theoretic closure of the generic fiber in the naive local model M_I^{naive}. In particular, their generic fibers coincide. The following result will be used later on.

LEMMA 1.2. *The generic fiber of the local model is irreducible and of dimension rs.*

Proof. The generic fiber can be identified with the Grassmannian of r-dimensional subspaces of $r + s = n$-dimensional space; see [PR4, Sec. 1.5.3]. □

By Lemma 1.1, the local model is also a closed subscheme of the wedge local model. Pappas and Rapoport [PR4, Rem. 7.4] give examples showing that, in general, the wedge condition is not sufficient to cut out the local model; they also propose one further condition (the so-called Spin condition) that should take care of this. Nevertheless, in some of the special cases we consider here, the local model should already be given by the wedge local model (see Remark 2.2 for a precise statement).

2. Special Parahoric Level Structures

We examine the local model M_I^{loc} for special choices of the index set I. If $n = 2m+1$ is odd, we consider the cases $I = \{0\}$ and $I = \{m\}$; if $n = 2m$ is even, we consider the case $I = \{m\}$. In [PR4, Sec. 1.2.3] it is shown that these are exactly the index sets for which the parahoric subgroups P_I preserving the lattice sets Λ_i with $i \in I$ are "special" in the sense of Bruhat–Tits theory [T]. The following theorem describes the special fibers of the corresponding local models.

THEOREM 2.1. *Let $I = \{0\}$ or $I = \{m\}$ if $n = 2m + 1$ is odd, and let $I = \{m\}$ if $n = 2m$ is even. Then the special fiber of the local model M_I^{loc} is irreducible and reduced; furthermore, the special fiber is normal, is Frobenius split, and has only rational singularities.*

REMARK 2.2. Pappas and Rapoport conjecture that, under the assumptions of Theorem 2.1, the wedge local model $M_I^{\wedge}$ is flat—provided that s is even if n is even (cf. [PR4, Rem. 5.3] and Proposition 4.16).

REMARK 2.3. The cases $n = 2m + 1$ odd, $I = \{0\}$, and $n = 2m$ even, $I = \{m\}$, have been treated in [PR4, Thm. 5.1]. Calculations for the low-dimensional case $n = 3$ odd, $I = \{1\}$, have been given in [PR4, Prop. 6.2]. However, the arguments cannot be generalized directly to the case of general $n = 2m + 1$ odd, $I = \{m\}$.

By Remark 2.3, to prove Theorem 2.1 we must deal with the case of general $n = 2m+1$ odd, $I = \{m\}$. As explained in the Introduction, essentially two results are required for the proof. These results are obtained in the next two sections, where we first show that the special fiber of the local model contains a nonempty open subset that is reduced (Theorem 3.1) and then show that the special fiber of the local model is irreducible (Theorem 4.1).

3. Open Reduced Subset of the Special Fiber

Recall from our definition of the naive local model that we have fixed the signature type (r, s) of the unitary group. The first result required in the proof of Theorem 2.1 is the next statement.

THEOREM 3.1. *Let $n = 2m + 1$ be odd and $I = \{m\}$. Then the local model M_I^{loc} contains an affine space of dimension rs as an open subset. In particular, the special fiber of the local model contains a nonempty open subset that is reduced.*

We will first prove a corresponding statement for the wedge local model; from that, the theorem will be derived.

PROPOSITION 3.2. *Let $n = 2m + 1$ be odd and $I = \{m\}$. Then the wedge local model $M_I^{\wedge}$ contains an affine space of dimension rs as an open subset.*

Proof. Before starting the actual proof, which ranges over Sections 3.1–3.10, we introduce some matrices that will occur frequently from now on.

We write I_l for the unit matrix of size l,

$$I_l := \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix},$$

and H_l for the unit antidiagonal matrix of size l,

$$H_l := \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

The matrix $J_{k,l}$ is given by the antidiagonal matrix of size $k + l$,

$$J_{k,l} := \begin{pmatrix} & H_l \\ -H_k & \end{pmatrix}.$$

The special case $k = l$ is abbreviated to $J_{2k} := J_{k,k}$.

3.1. Best Point

Recall from Section 1.2 the notion of the parahoric subgroup P_I: in the current situation of odd n, it is the stabilizer subgroup preserving the lattice chain Λ_I. The corresponding group scheme acts on the special fibers of the models M_I^{naive}, $M_I^{\wedge}$, and M_I^{loc}. In [PR4, Sec. 3.3], Pappas and Rapoport construct an embedding of the geometric special fiber of the naive local model into a partial affine flag variety (associated to the unitary similitude group). This closed immersion is equivariant for the action of the parahoric and so its image is a union of Schubert varieties, which are enumerated by certain elements of the corresponding affine Weyl group.

In [PR4, Prop. 3.1] it is shown that the union of Schubert varieties over elements of the so-called μ-admissible set is contained in the geometric special fiber of the local model. This union is denoted by $\mathcal{A}^I(\mu)$; it is closed because the μ-admissible set is closed under the Bruhat order. In [PR4, Sec. 3.4], points of the local model are constructed that reduce to points lying in the Schubert varieties corresponding to the extreme elements of the μ-admissible set. The open subset of the local model we are about to construct will contain one of these "best points". (A posteriori we can see that, in the situation under consideration, there is only a single extreme orbit; see Remark 4.15.)

3.2. Conditions of the Wedge Local Model

We specialize the definition of the wedge local model to the case $n = 2m+1$ odd, $I = \{m\}$. The essential part of the periodic lattice chain is given by

$$\cdots \to \Lambda_m \to \Lambda_{m+1} \to \cdots,$$

where Λ_m and Λ_{m+1} are the standard lattices

$$\Lambda_m = \operatorname{span}_{\mathcal{O}_F}\{\pi^{-1}e_1, \ldots, \pi^{-1}e_m, e_{m+1}, \ldots, e_n\} \quad \text{and}$$
$$\Lambda_{m+1} = \operatorname{span}_{\mathcal{O}_F}\{\pi^{-1}e_1, \ldots, \pi^{-1}e_{m+1}, e_{m+2}, \ldots, e_n\}.$$

Denoting the above basis of Λ_m by $f_1, \ldots, f_n$ and that of Λ_{m+1} by $g_1, \ldots, g_n$, we have corresponding $\mathcal{O}_{F_0}$-bases $f_1, \ldots, f_n, \pi f_1, \ldots, \pi f_n$ and $g_1, \ldots, g_n, \pi g_1, \ldots, \pi g_n$, respectively.

We have to examine A-valued points of $M_I^\wedge$, with A an arbitrary $\mathcal{O}_E$-algebra. This means considering $(\mathcal{O}_F \otimes A)$-submodules

$$\mathcal{F} \subset \Lambda_m \otimes A \quad \text{and}$$
$$\mathcal{G} \subset \Lambda_{m+1} \otimes A$$

subject to the conditions of the wedge local model. These conditions translate into the following.

(N1) As A-modules, $\mathcal{F}$ and $\mathcal{G}$ are locally direct summands of rank n. Identifying $\Lambda_m \otimes A$ and $\Lambda_{m+1} \otimes A$ with A^{2n} via the preceding $\mathcal{O}_{F_0}$-bases, we can consider $\mathcal{F}$ and $\mathcal{G}$ as A-valued points of the Grassmannian $\mathrm{Grass}_{n,2n}$.

(N2) The maps induced by the inclusions $\Lambda_m \subset \Lambda_{m+1}$ and $\Lambda_{m+1} \subset \pi^{-1}\Lambda_m$ restrict to maps

$$\mathcal{F} \to \mathcal{G} \to \pi^{-1}\mathcal{F}.$$

Here $\pi^{-1}\mathcal{F}$ is the image of $\mathcal{F}$ under the map induced by the isomorphism $\pi^{-1}\colon \Lambda_m \to \pi^{-1}\Lambda_m$.

(N3) $\mathcal{G} = \mathcal{F}^\perp$, where $\mathcal{F}^\perp$ denotes the orthogonal complement of $\mathcal{F}$ under the natural perfect pairing

$$(\cdot,\cdot) \otimes A\colon (\Lambda_{m+1} \otimes A) \times (\Lambda_m \otimes A) \to A. \tag{3.1}$$

With respect to the chosen bases, the form is represented by the $2n \times 2n$ matrix

$$M := \begin{pmatrix} & -J_{m,m+1} \\ J_{m,m+1} & \end{pmatrix}.$$

(N4) The characteristic polynomial of $\Pi|_{\mathcal{F}}$ is given by

$$\det(T\,\mathrm{id} - \Pi|_{\mathcal{F}}) = (T-\pi)^s(T+\pi)^r \in A[T],$$

and the analogous statement holds true for $\mathcal{G}$.

(W) We have

$$\wedge^{r+1}\big(\Pi - \sqrt{\pi_0}|_{\mathcal{F}}\big) = 0,$$
$$\wedge^{s+1}\big(\Pi + \sqrt{\pi_0}|_{\mathcal{F}}\big) = 0,$$

and the analogous statement holds true for $\mathcal{G}$.

If we view $\mathcal{F}$ and $\mathcal{G}$ as A-modules, then the requirement that they be modules over $\mathcal{O}_F \otimes A$ translates into an additional condition:

(Pi) $\mathcal{F}$ and $\mathcal{G}$ are Π-stable.

3.3. Orthogonal Complement

Condition (N3) implies that the subspace $\mathcal{G}$ is determined by $\mathcal{F}$ as its orthogonal complement. We denote by W the corresponding subfunctor of $\mathrm{Grass}_{n,2n} \times \mathrm{Grass}_{n,2n}$ that satisfies (N3). Then the projection onto the first factor,

$$\mathrm{pr}_{\mathcal{F}} \colon \mathrm{Grass}_{n,2n} \times \mathrm{Grass}_{n,2n} \to \mathrm{Grass}_{n,2n},$$

restricts to an isomorphism of functors:

$$\mathrm{pr}_{\mathcal{F}}|_W \colon W \xrightarrow{\sim} \mathrm{Grass}_{n,2n}.$$

This is because the assignment $\mathcal{F} \mapsto (\mathcal{F}, \mathcal{F}^{\perp})$ on A-valued points induces an inverse morphism, as can be seen from the explicit determination of the orthogonal complement in Lemma 3.3 (to follow). Because our objective is to construct an open subset of the wedge local model, we may restrict ourselves to considering subfunctors of W that are induced via the isomorphism $\mathrm{pr}_{\mathcal{F}}|_W$ by open subfunctors of $\mathrm{Grass}_{n,2n}$.

Recall that the Grassmann functor is covered by the open subfunctors

$$\begin{aligned}&\mathrm{Grass}^J_{n,2n}(A)\\ &\quad := \{\mathcal{U} \in \mathrm{Grass}_{n,2n}(A) \mid \mathcal{O}_A^J \hookrightarrow \mathcal{O}_A^{2n} \twoheadrightarrow \mathcal{O}_A^{2n}/\mathcal{U} \text{ is an isomorphism}\},\end{aligned}$$

where $J \subset \{1, \dots, 2n\}$ is a subset consisting of n elements and the arrows denote the obvious homomorphisms. These functors are represented by affine space of dimension n^2.

We consider the complement J of the index set that corresponds to the basis elements $f_1, \dots, f_s, \pi f_1, \dots, \pi f_r$ (see Remark 3.4 concerning a motivation for this choice). The elements of $\mathrm{Grass}^J_{n,2n}(A)$ can be described as the column span of $2n \times n$ matrices $\mathcal{F}$ with entries in A and of the following form:

$$\mathcal{F} = \begin{pmatrix} I_s & \\ a & b \\ & I_r \\ c & d \end{pmatrix}. \tag{3.2}$$

Here the submatrix a has r rows and s columns, d has s rows and r columns, and (as usual) I_s and I_r are the unit matrices of sizes s and r, respectively; see Figure 3.1. We denote the subspace $\mathcal{F}$ and the matrix representing it (as a column span) by the same symbol. This should not lead to any confusion, since the intended meaning will be clear from the context.

To describe the orthogonal complement of $\mathcal{F}$ in a clear way, we introduce further notation. For the moment, let B be an arbitrary matrix with k rows and l columns. We define the involution ι as follows:

$$\iota(B) := H_l B^{\mathrm{t}} H_k.$$

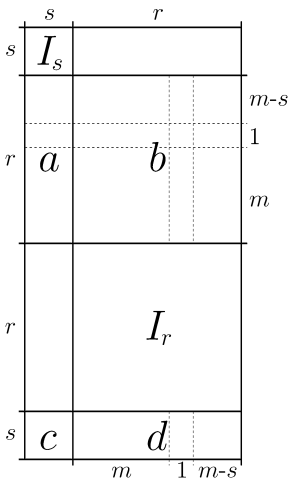

Figure 3.1 Typical form of the matrix $\mathcal{F}$ (for $n = 2m + 1$ odd). The partitioning shown corresponds to $n = 9$ and $s = 2$ (then $m = 4$ and $r = 5$). The solid lines separate the main blocks; the dotted lines indicate a finer subdivision helpful for the upcoming calculations. The labels outside denote the sizes of the blocks.

This is the matrix obtained from B by reflection at the first angle bisector going through the lower left matrix entry (which is precisely the antidiagonal in the case of a square matrix). Assuming $i \leq k$, we denote by $B^{[i]}$ the matrix consisting of the first i rows of B and by $B_{[i]}$ the matrix consisting of the last i rows. The ith row is denoted by $B^{(i)}$. Likewise, assuming $j \leq l$, we write ${}^{[j]}B$, ${}_{[j]}B$, and ${}^{(j)}B$ for the first j columns of B, the last j columns, and the jth column, respectively. We use $B_{i,j}$ to denote the single matrix entry in the ith row and jth column.

LEMMA 3.3. *With respect to the perfect pairing* (3.1), *the orthogonal complement of $\mathcal{F}$ is given by the column span of the matrix*

$$\mathcal{G} = \begin{pmatrix} I_s & \\ \tilde{a} & \tilde{b} \\ & I_r \\ \tilde{c} & \tilde{d} \end{pmatrix},$$

where

$$\tilde{a} = \begin{pmatrix} -\iota({}_{[r-m]}d) \\ \iota({}^{[m]}d) \end{pmatrix},$$

$$\tilde{b} = \begin{pmatrix} \iota({}_{[r-m]}b_{[m+1]}) & -\iota({}_{[r-m]}b^{[m-s]}) \\ -\iota({}^{[m]}b_{[m+1]}) & \iota({}^{[m]}b^{[m-s]}) \end{pmatrix},$$

$$\tilde{c} = -\iota(c),$$

$$\tilde{d} = (\iota(a_{[m+1]}) \quad -\iota(a^{[m-s]})).$$

Proof. $\mathcal{G}$ is a subspace of rank n, and one calculates $\mathcal{G}^{\mathrm{t}} M \mathcal{F} = 0$ (recall that M is the matrix representing the perfect pairing). □

REMARK 3.4. It can be easily checked that $(\mathcal{F}_1, \mathcal{G}_1) \in \mathrm{Grass}^J_{n,2n}(k) \times \mathrm{Grass}^J_{n,2n}(k)$, given by the k-subspaces

$$\mathcal{F}_1 := \mathrm{span}_k\{f_1, \dots, f_s, \pi f_1, \dots, \pi f_r\},$$
$$\mathcal{G}_1 := \mathrm{span}_k\{g_1, \dots, g_s, \pi g_1, \dots, \pi g_r\},$$

satisfies the conditions of the wedge local model and thus represents a point of the special fiber of the wedge local model (in the preceding notation, $\mathcal{F}_1$ corresponds to $a = b = c = d = 0$ and $\mathcal{G}_1$ corresponds to $\tilde{a} = \tilde{b} = \tilde{c} = \tilde{d} = 0$). More precisely, this is one of the special points mentioned in Section 3.1, as follows from [PR4, Sec. 3.4] by considering (in their notation) the subset $S = [n+1-s, n]$. Therefore, $(\mathcal{F}_1, \mathcal{G}_1)$ lies in the special fiber of the local model.

3.4. Pi-Stability

We continue to evaluate the conditions of the wedge local model. We are given pairs of subspaces $(\mathcal{F}, \mathcal{F}^\perp)$. Condition (Pi), concerning the stability of $\mathcal{F}$ under the action of Π, translates into the equation

$$\Pi\mathcal{F} = \mathcal{F}R. \tag{3.3}$$

Here R is a square matrix of size n, which we subdivide into four blocks as follows:

$$R = \begin{pmatrix} S & T \\ U & V \end{pmatrix},$$

with S a square matrix of size s and V a square matrix of size r. With respect to the chosen basis, the operator Π is given by the matrix

$$\Pi = \begin{pmatrix} & \pi_0 I_n \\ I_n & \end{pmatrix}.$$

Then (3.3) becomes

$$\begin{pmatrix} 0 & \pi_0 I_r \\ \pi_0 c & \pi_0 d \\ I_s & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} S & T \\ aS + bU & aT + bV \\ U & V \\ cS + dU & cT + dV \end{pmatrix}. \tag{3.4}$$

Comparing the matrices yields several identities involving the a-, b-, c-, and d-variables. This must be done carefully, since the blocks of the matrices that seem to correspond are of different sizes.

To begin with, we obtain from (3.4) the following identities concerning the blocks of the matrix R:

$$S = 0, \quad T = \pi_0 {I_r}^{[s]}, \quad U = \begin{pmatrix} I_s \\ a^{[r-s]} \end{pmatrix}, \quad V = \begin{pmatrix} 0 \\ b^{[r-s]} \end{pmatrix}. \tag{3.5}$$

Thus, the matrix R takes the form

$$R = \begin{pmatrix} & \pi_0 I_s & \\ I_s & & \\ a^{[r-s]} & {}^{[s]}b^{[r-s]} & {}_{[r-s]}b^{[r-s]} \end{pmatrix}. \tag{3.6}$$

3.5. Wedge Condition

Before examining the remaining blocks of (3.4), we take a look at the wedge condition (W).

Since $\Pi|_{\mathcal{F}}$ is given by the matrix R, all minors of size $r+1$ of

$$R - \pi I_n = \begin{pmatrix} -\pi I_s & \pi_0 I_s & \\ I_s & -\pi I_s & \\ a^{[r-s]} & {}^{[s]}b^{[r-s]} & {}_{[r-s]}b^{[r-s]} - \pi I_{r-s} \end{pmatrix} \tag{3.7}$$

must be zero. Note that the first s rows are multiples of the following s rows. Since any minor of size $r+1$ includes at least one pair of such corresponding rows, all these minors are zero.

All minors of size $s+1$ of

$$R + \pi I_n = \begin{pmatrix} \pi I_s & \pi_0 I_s & \\ I_s & \pi I_s & \\ a^{[r-s]} & {}^{[s]}b^{[r-s]} & {}_{[r-s]}b^{[r-s]} + \pi I_{r-s} \end{pmatrix} \tag{3.8}$$

must also be zero. First, we consider the minors of size $s+1$ obtained by keeping only the rows with row number in $\{s+1,\dots,2s,2s+i\}$ and the columns with column number in $\{1,\dots,s,s+j\}$. Here i and j denote integers with $1 \le i \le r-s$ and $1 \le j \le s$. We use Laplace expansion along the last column and calculate

$$\det\begin{pmatrix} I_s & \pi^{(j)} I_s \\ a^{(i)} & b_{i,j} \end{pmatrix} = (-1)^{2(s+1)} b_{i,j} + (-1)^{s+1+j}(-1)^{s-j}\pi a_{i,j}.$$

Since these minors are zero, we get

$${}^{[s]}b^{[r-s]} = \pi a^{[r-s]}. \tag{3.9}$$

Next, we consider the minors obtained by keeping the rows $\{s+1,\dots,2s,2s+i\}$ and the columns $\{1,\dots,s,2s+j\}$, with $1 \le i, j \le r-s$:

$$\det\begin{pmatrix} I_s & 0 \\ a^{(i)} & ({}_{[r-s]}b^{[r-s]} + \pi I_{r-s})_{i,j} \end{pmatrix} = ({}_{[r-s]}b^{[r-s]} + \pi I_{r-s})_{i,j}.$$

Since these minors are zero, we obtain

$${}_{[r-s]}b^{[r-s]} = -\pi I_{r-s}. \tag{3.10}$$

Finally, all remaining minors of size $s+1$ are now automatically zero.

3.6. Characteristic Polynomial

The characteristic polynomial of $\Pi|_{\mathcal{F}}$ is given by $\det(TI_n - R)$, with R as in (3.6). Making use of (3.10), we calculate

$$\det\begin{pmatrix} TI_s & -\pi_0 I_s & \\ -I_s & TI_s & \\ -a^{[r-s]} & -{}^{[s]}b^{[r-s]} & (T+\pi)I_{r-s} \end{pmatrix} = (T-\pi)^s(T+\pi)^r \in A[T],$$

which is in accordance with (N4).

3.7. *Pi-Stability* (*continued*)

We show that the b-variables are determined by the a-, c-, and d-variables. For this purpose, we consider the matrix equation $cT + dV = b_{[s]}$, which is obtained from the lower right blocks of the matrices in (3.4). Using (3.5), (3.9), and (3.10), the first s columns give

$$ {}^{[s]}b_{[s]} = \pi_0 c + {}_{[r-s]}d\, {}^{[s]}b^{[r-s]} = \pi_0 c + \pi\, {}_{[r-s]}da^{[r-s]}, \tag{3.11}$$

and the last $r - s$ columns give

$$ {}_{[r-s]}b_{[s]} = {}_{[r-s]}d\, {}_{[r-s]}b^{[r-s]} = -\pi\, {}_{[r-s]}d. \tag{3.12}$$

Combining (3.9)–(3.12) yields the following description of the submatrix b:

$$b = \begin{pmatrix} \pi a^{[r-s]} & -\pi I_{r-s} \\ \pi_0 c + \pi\, {}_{[r-s]}da^{[r-s]} & -\pi\, {}_{[r-s]}d \end{pmatrix}. \tag{3.13}$$

Hence, the b-variables are determined by the other variables.

With (3.5), the lower left blocks of the matrices in (3.4) give the identity

$$ {}^{[s]}d = a_{[s]} - {}_{[r-s]}da^{[r-s]}, \tag{3.14}$$

to which we shall return later.

The remaining blocks of the matrices in (3.4) give nothing new.

3.8. *Lattice Inclusion Map*

We can deduce further constraints on the a-, c-, and d-variables from (N2), which demands that the maps induced by the lattice inclusions restrict to the considered subspaces.

With respect to the chosen bases, the map corresponding to $\Lambda_m \subset \Lambda_{m+1}$ is given by the $2n \times 2n$ matrix

$$A := \begin{pmatrix} I_m & & & & & \\ & 0 & & & \pi_0 & \\ & & I_m & & & \\ & & & I_m & & \\ & 1 & & & 0 & \\ & & & & & I_m \end{pmatrix}.$$

Since this map is required to restrict to $\mathcal{F} \to \mathcal{F}^\perp$, we have to examine the conditions under which $A\mathcal{F}$ is perpendicular to $\mathcal{F}$. With M as in (N3), $C := \mathcal{F}^{\mathrm{t}}A^{\mathrm{t}}M\mathcal{F}$ must be the zero matrix of size n. We multiply the matrices on the right-hand side. Note the form of the matrix

$$
A^{\mathrm{t}}M = \begin{pmatrix}
 & & & & & -H_m \\
 & 1 & & & 0 & \\
 & & & H_m & & \\
 & & H_m & & & \\
 & 0 & & & -\pi_0 & \\
-H_m & & & & &
\end{pmatrix},
$$

which suggests that blockwise multiplying becomes easier when subdividing $\mathcal{F}$ into four groups of columns with the groups consisting of s, m, 1, and $m-s$ columns. This partitioning is shown in Figure 3.1. The symmetry of $A^{\mathrm{t}}M$ implies the symmetry of C, and we obtain ten conditions from the blocks of C:

(C1) $0 = -H_s c + a^{(m-s+1)\mathrm{t}} a^{(m-s+1)} - c^{\mathrm{t}} H_s$,
(C2) $0 = -H_s\, {}^{[m]}d + a^{(m-s+1)\mathrm{t}}\, {}^{[m]}b^{(m-s+1)} + a_{[m]}{}^{\mathrm{t}} H_m$,
(C3) $0 = -H_s\, {}^{(m+1)}d + a^{(m-s+1)\mathrm{t}} b_{m-s+1,m+1}$,
(C4) $0 = -H_s\, {}_{[m-s]}d - a^{[m-s]\mathrm{t}} H_{m-s} + a^{(m-s+1)\mathrm{t}}\, {}_{[m-s]}b^{(m-s+1)}$,
(C5) $0 = {}^{[m]}b^{(m-s+1)\mathrm{t}}\, {}^{[m]}b^{(m-s+1)} + {}^{[m]}b_{[m]}{}^{\mathrm{t}} H_m + H_m\, {}^{[m]}b_{[m]}$,
(C6) $0 = {}^{[m]}b^{(m-s+1)\mathrm{t}} b_{m-s+1,m+1} + H_m\, {}^{(m+1)}b_{[m]}$,
(C7) $0 = -{}^{[m]}b^{[m-s]\mathrm{t}} H_{m-s} + {}^{[m]}b^{(m-s+1)\mathrm{t}}\, {}_{[m-s]}b^{(m-s+1)} + H_m\, {}_{[m-s]}b_{[m]}$,
(C8) $0 = b_{m-s+1,m+1}{}^2 - \pi_0$,
(C9) $0 = -{}^{(m+1)}b^{[m-s]\mathrm{t}} H_{m-s} + b_{m-s+1,m+1}\, {}_{[m-s]}b^{(m-s+1)}$,
(C10) $0 = -{}_{[m-s]}b^{[m-s]\mathrm{t}} H_{m-s} + {}_{[m-s]}b^{(m-s+1)\mathrm{t}}\, {}_{[m-s]}b^{(m-s+1)} - H_{m-s}\, {}_{[m-s]}b^{[m-s]}$.

These conditions will now be evaluated, beginning with (C1). We collect the c-variables on the left-hand side and then left-multiply with H_s to obtain

(C1′) $c + \iota(c) = H_s a^{(m-s+1)\mathrm{t}} a^{(m-s+1)}$.

Both sides of this equation are symmetric with respect to reflection at the antidiagonal (that is, invariant under the involution ι). Therefore, it suffices to look at entries on or above the antidiagonal; these are the entries indexed by (i, j) with $1 \le i \le s$ and $1 \le j \le s+1-i$. Note that only a-variables occur on the right-hand side of (C1′), which we temporarily denote by B. We obtain equations of the form

$$c_{i,j} + c_{s+1-j,s+1-i} = B_{i,j}.$$

The entries on the antidiagonal give $c_{i,s+1-i} = B_{i,s+1-i}/2$ (by assumption, the characteristic $\neq 2$); those above the antidiagonal give $c_{s+1-j,s+1-i} = B_{i,j} - c_{i,j}$. Hence, we may keep the elements of the set

$$\{c_{i,j} \mid 1 \le i < s,\ 1 \le j < s+1-i\}$$

as free variables, determining (together with the a-variables) all remaining $c_{i,j}$ with $1 \le i \le s$, $s+1-i \le j \le s$. The free c-variables are $s(s-1)/2$ in number.

Analogously, we rearrange (C2)–(C4) to get

(C2′) ${}^{[m]}d = \iota(a_{[m]}) + H_s a^{(m-s+1)\mathrm{t}} (\pi a^{(m-s+1)}\ \ 0)$,
(C3′) ${}^{(m+1)}d = -\pi\iota(a^{(m-s+1)})$,
(C4′) ${}_{[m-s]}d = -\iota(a^{[m-s]})$.

Consequently, the d-variables are determined by the a-variables.

We split ${}_{[r-s]}da^{[r-s]}$ into three terms,

$$ {}_{[r-s]}da^{[r-s]} = {}_{[m-s]}{}^{[m]}da^{[m-s]} + {}^{(m+1)}da^{(m-s+1)} + {}_{[m-s]}da_{[m]}{}^{[m-s]}, \tag{3.15} $$

with which we substitute the corresponding term in equation (3.14) (this equation has not been considered yet). We then use (C2′)–(C4′) to replace the d-variables and obtain, after rearranging,

$$ a_{[s]} - \iota(a_{[s]}) = \iota(a_{[m]}{}^{[m-s]})a^{[m-s]} - \iota(a^{[m-s]})a_{[m]}{}^{[m-s]}. \tag{3.16} $$

All elements $a_{i,j}$ on the right-hand side have index (i, j) in the set

$$ \mathcal{I} := \{(i, j) \mid 1 \le i \le r - s, 1 \le j \le s\}, $$

whereas all elements on the left-hand side have index (i, j) in the complement

$$ \mathcal{Q} := \{(i, j) \mid r - s + 1 \le i \le r, 1 \le j \le s\}. $$

Both sides of (3.16) are antisymmetric with respect to reflection at the antidiagonal. We argue as before (in the case of the c-variables) and keep the elements of the set

$$ \{a_{i,j} \mid (i, j) \in \mathcal{I}\} \cup \{a_{i,j} \mid r - s + 1 \le i \le r, 1 \le j \le r + 1 - i\} $$

as free variables. These elements determine the remaining $a_{i,j}$ with $r - s + 1 < i \le r$ and $r+1-i < j \le s$. Hence, there are $s(r-s)+s(s+1)/2$ free a-variables.

Since the c-variables are independent of the a-variables, we conclude that the pairs $(\mathcal{F}, \mathcal{G})$ satisfying the conditions so far describe an affine space of dimension

$$ \frac{s(s-1)}{2} + s(r-s) + \frac{s(s+1)}{2} = rs, $$

which is in accordance with the assertion of the proposition.

3.9. Remaining Conditions

The remaining conditions can be verified by explicit calculations; this is done in the preprint [Ar]. Moreover, in the proof of Theorem 3.1 we give a dimension argument, which implies that the remaining conditions are automatically satisfied; see Remark 3.5.

3.10. Conclusion

This completes the proof of Proposition 3.2: we have shown that the wedge local model contains an open subset that is isomorphic to affine space of dimension rs. Moreover, the open subset is a neighborhood of the special point $(\mathcal{F}_1, \mathcal{G}_1)$. □

Now the assertions of the theorem can be deduced.

Proof of Theorem 3.1. We want to see that the local model contains an open subset that is isomorphic to affine space of dimension rs. For this, we show that the open subset constructed previously is actually lying in the local model.

We consider the closed subscheme of the product of Grassmannians that consists of pairs satisfying the conditions of the wedge local model treated in Sections 3.3–3.8:

$$Y := \{(\mathcal{F}, \mathcal{G}) \in \mathrm{Grass}_{n,2n} \times \mathrm{Grass}_{n,2n} \mid \text{conditions from Sections 3.3–3.8}\}.$$

The standard open subset $\mathrm{Grass}^J_{n,2n} \times \mathrm{Grass}^J_{n,2n}$ of the product of Grassmannians is abbreviated to U. We then have the following inclusions of closed subschemes:

$$M_I^{\mathrm{loc}} \cap U \subset M_I^{\wedge} \cap U \subset Y \cap U. \tag{3.17}$$

By Lemma 1.2, the generic fiber of the local model is irreducible and of dimension rs. As its closure (in the naive local model), the local model is also irreducible. The structure morphism to $\mathcal{O}_E$ is dominant, and since it is projective, the special fiber of the local model is nonempty. It follows from Chevalley's theorem [EGA IV$_3$, Thm. 13.1.3] that all irreducible components of the special fiber have dimension at least rs. By flatness, the special fiber of the local model is in fact equidimensional of dimension rs.

We have seen in the proof of Proposition 3.2 that the $\mathcal{O}_E$-scheme $Y \cap U$ is isomorphic to affine space of dimension rs; in particular, its special fiber and its generic fiber are both irreducible of dimension rs. Hence, on the level of reduced schemes, the inclusions in (3.17) are equalities. Since $Y \cap U$ is reduced, we obtain

$$Y \cap U = (Y \cap U)_{\mathrm{red}} = (M_I^{\mathrm{loc}} \cap U)_{\mathrm{red}} \subset M_I^{\mathrm{loc}} \cap U,$$

where the subscript "red" denotes the reduced structure. Together with (3.17), this implies that the affine space $Y \cap U$ coincides with the open subset $M_I^{\mathrm{loc}} \cap U$ of the local model. □

REMARK 3.5. The conditions of the wedge local model that were not explicitly verified during the calculations in the previous sections are, in fact, automatically satisfied: from the previous results we obtain the inclusion $Y \cap U \subset M_I^{\wedge} \cap U$, and the converse inclusion holds trivially.

4. Irreducibility of the Special Fiber

In this section, we will establish the second result required in the proof of Theorem 2.1.

THEOREM 4.1. *Let $n = 2m + 1$ be odd and let $I = \{m\}$. Then the special fiber of the local model M_I^{loc} is irreducible.*

Proof. The theorem is a consequence of an apparently weaker result, which is given in Proposition 4.3. Remark 4.2 ensures that this is actually sufficient. □

4.1. Worst Point

Recall from Section 3.1 that the special fiber of the local model is the union of Schubert varieties, enumerated by certain elements of the corresponding affine Weyl group. As in [PR4, Sec. 5.5], we can see that there is a unique closed orbit, which must be contained in the closed subset $\mathcal{A}^I(\mu)$. From [PR4, Sec. 2.4.2] it follows that, in the current situation, the closed orbit consists of the single point $(\mathcal{F}_0, \mathcal{G}_0)$ given by the subspaces

$$\mathcal{F}_0 := \Pi\Lambda_m \subset \Lambda_m,$$
$$\mathcal{G}_0 := \Pi\Lambda_{m+1} \subset \Lambda_{m+1}.$$

This point is, in some sense, at the opposite extreme of the previously considered best point $(\mathcal{F}_1, \mathcal{G}_1)$: it is contained in all irreducible components of the special fiber of the local model; the "worst singularities" occur at this point, so it is named "worst point".

REMARK 4.2. To prove the irreducibility of the special fiber of the local model, it is sufficient to show that the worst point has an open neighborhood (in the special fiber of the local model) that is irreducible.

By this remark, the next proposition is enough to complete the proof of Theorem 4.1.

PROPOSITION 4.3. *Let $n = 2m+1$ be odd and let $I = \{m\}$. Then, in the special fiber of the local model, the point $(\mathcal{F}_0, \mathcal{G}_0)$ has an open neighborhood that is irreducible.*

Proof. We start with the description of an open neighborhood of the point $(\mathcal{F}_0, \mathcal{G}_0)$ in the special fiber of the wedge local model. Later we consider the intersection with the local model and deduce the statement of the proposition.

As in the previous section, we use matrices to describe an open subset. We consider points of the special fiber; therefore, unless explicitly mentioned otherwise, all schemes in this section are over the residue field k. Since we want to prove an irreducibility result, it is enough to consider the reduced scheme structures; therefore, unless otherwise specified, all schemes are equipped with the reduced structure. Moreover, the schemes involved in this section are all of finite type over k. Hence, it is enough to consider only geometric points—that is, $\bar{k}$-valued points, where $\bar{k}$ denotes a fixed algebraic closure of k.

4.2. Conditions of the Wedge Local Model

To simplify the upcoming calculations, we use rearranged bases of Λ_m and Λ_{m+1}:

$$\Lambda_m = \operatorname{span}_{\mathcal{O}_F}\{e_{m+2}, \dots, e_n, \pi^{-1}e_1, \dots, \pi^{-1}e_m, e_{m+1}\},$$
$$\Lambda_{m+1} = \operatorname{span}_{\mathcal{O}_F}\{e_{m+2}, \dots, e_n, \pi^{-1}e_1, \dots, \pi^{-1}e_m, \pi^{-1}e_{m+1}\}.$$

As usual, we obtain corresponding $\mathcal{O}_{F_0}$-bases by adding the π-multiples of the respective basis vectors displayed above (in the prescribed order; cf. Section 3.2).

Recall that the $\bar{k}$-valued points of the wedge local model are given by pairs of $(\mathcal{O}_F \otimes \bar{k})$-subspaces $(\mathcal{F}, \mathcal{G})$, with $\mathcal{F} \subset \Lambda_m \otimes \bar{k}$ and $\mathcal{G} \subset \Lambda_{m+1} \otimes \bar{k}$, subject to conditions (N1)–(N4), (W), and (Pi). In particular, $\mathcal{G}$ is determined by $\mathcal{F}$ as its orthogonal complement, and it suffices to consider $\bar{k}$-valued points $\mathcal{F}$ of some standard open subset $\mathrm{Grass}^J_{n,2n}$ of the Grassmannian. In order for the corresponding open subset of the product of Grassmannians to contain the special point $(\mathcal{F}_0, \mathcal{G}_0)$, the index set J must correspond to the first n elements of the previously chosen

$\mathcal{O}_{F_0}$-basis of Λ_m. Then the elements of $\mathrm{Grass}^J_{n,2n}(\bar{k})$ are represented by $2n \times n$ matrices

$$\mathcal{F} = \begin{pmatrix} X \\ I_n \end{pmatrix}, \tag{4.1}$$

with a square matrix X of size n having entries in $\bar{k}$. With respect to the upcoming calculations, we subdivide X into four smaller blocks,

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix},$$

where X_1 is a square matrix of size $n-1$ and X_4 a scalar (that is, a square matrix of size 1).

We evaluate the conditions of the wedge local model. By construction, (N1) and (N3) are satisfied. The remaining conditions translate into constraints on the matrix X.

4.3. Lattice Inclusion Map

Note that π is zero in $\bar{k}$. The map induced by the inclusion $\Lambda_m \subset \Lambda_{m+1}$ is described by the matrix

$$\bar{A} := \begin{pmatrix} I_n - K & \\ K & I_n - K \end{pmatrix},$$

where the $n \times n$ matrix K is defined as

$$K := \begin{pmatrix} 0_{2m} & \\ & 1 \end{pmatrix},$$

with 0_{2m} denoting the zero matrix of size $2m$. We introduce the square matrix J'_{2m} of size $2m+1$, following the notation of J_{2m}:

$$J'_{2m} := \begin{pmatrix} J_{2m} & \\ & 0 \end{pmatrix}.$$

The natural perfect pairing (3.1) is then represented by the matrix

$$M' := \begin{pmatrix} & J'_{2m} - K \\ -J'_{2m} + K & \end{pmatrix}.$$

Condition (N2) requires that the map $\bar{A}$ restricts to a map $\mathcal{F} \to \mathcal{F}^{\perp}$. The image of $\mathcal{F}$ lies in the orthogonal complement of $\mathcal{F}$ if $\mathcal{F}^{\mathrm{t}}\bar{A}^{\mathrm{t}}M'\mathcal{F} = 0$. We multiply these matrices; with

$$\bar{A}^{\mathrm{t}}M' = \begin{pmatrix} K & J'_{2m} \\ -J'_{2m} & \end{pmatrix},$$

we get the condition $X^{\mathrm{t}}KX + (X^{\mathrm{t}}J'_{2m} - J'_{2m}X) = 0$, which in block form is given by

$$\begin{pmatrix} X_3^{\mathrm{t}}X_3 & X_4X_3^{\mathrm{t}} \\ X_4X_3 & {X_4}^2 \end{pmatrix} + \begin{pmatrix} X_1^{\mathrm{t}}J_{2m} - J_{2m}X_1 & -J_{2m}X_2 \\ X_2^{\mathrm{t}}J_{2m} & 0 \end{pmatrix} = 0. \tag{4.2}$$

From the lower right blocks, we deduce ${X_4}^2 = 0$. Since we are looking at $\bar{k}$-valued points, it follows that $X_4 = 0$. Then the upper right blocks give $X_2 = 0$, and from the upper left blocks the identity $X_3^{\mathrm{t}} X_3 = J_{2m} X_1 - X_1^{\mathrm{t}} J_{2m}$ follows. By introducing an involution similar to ι, the latter equation can be expressed more clearly. For this, we multiply by $-J_{2m}$ from the left and obtain

$$-J_{2m} X_3^{\mathrm{t}} X_3 = X_1 + \sigma(X_1), \tag{4.3}$$

with the involution σ defined as follows. Let B be an arbitrary matrix with k rows and l columns. We write

$$\sigma(B) := D_l B^{\mathrm{t}} D_k,$$

where for an integer i the matrix D_i is defined to be J_i if i is even and H_i if i is odd. We calculate

$$\sigma(X_1) = J_{2m} X_1^{\mathrm{t}} J_{2m} = \begin{pmatrix} -\iota({}_{[m]}X_1{}_{[m]}) & \iota({}_{[m]}X_1{}^{[m]}) \\ \iota({}^{[m]}X_1{}_{[m]}) & -\iota({}^{[m]}X_1{}^{[m]}) \end{pmatrix}$$

and see that σ is a "signed reflection" at the antidiagonal. Therefore, (4.3) is to some extent a symmetry condition.

4.4. Pi-Stability

Over $\bar{k}$ and with respect to the chosen bases, the action induced by multiplication with $\pi \otimes 1$ is given by the matrix $\bar{\Pi} = \left(\begin{smallmatrix} & 0_n \\ I_n & \end{smallmatrix}\right)$. Condition (Pi) requires that $\mathcal{F}$ be $\bar{\Pi}$-stable. This holds true if there is an equation $\bar{\Pi}\mathcal{F} = \mathcal{F}R$ with a square matrix R of size n. We obtain

$$\begin{pmatrix} 0 \\ X \end{pmatrix} = \begin{pmatrix} XR \\ R \end{pmatrix}$$

and deduce that $R = X$ and $X^2 = 0$. The latter equation is given in block form by

$$\begin{pmatrix} X_1{}^2 & 0 \\ X_3 X_1 & 0 \end{pmatrix} = 0,$$

from which we deduce the identities

$$X_1{}^2 = 0, \tag{4.4}$$

$$X_3 X_1 = 0. \tag{4.5}$$

Here we have used that X_2 and X_4 are both zero.

4.5. Wedge Condition

Because the last column of X is identically zero, (W) translates into a wedge condition for the $(2m+1) \times 2m$ matrix composed of the blocks X_1 and X_3:

$$\wedge^{s+1} \begin{pmatrix} X_1 \\ X_3 \end{pmatrix} = 0 \tag{4.6}$$

(recall that $s < r$ and $\pi = 0 \in \bar{k}$).

4.6. Action of the Symplectic Group

We are left with pairs of matrices (X_1, X_3) subject to conditions (4.3)–(4.6). We denote this space of matrices by N.

Recall the definition of the symplectic group of size $2m$: it is the group of invertible $2m \times 2m$ matrices that preserve the antisymmetric form given by J_{2m},

$$\mathrm{Sp}_{2m} = \{g \in \mathrm{GL}_{2m} \mid g^{\mathrm{t}} J_{2m} g = J_{2m}\},$$

which we consider over k. It acts on N from the right:

$$N \times \mathrm{Sp}_{2m} \to N, \quad ((X_1, X_3), g) \mapsto (g^{-1} X_1 g, X_3 g). \tag{4.7}$$

Indeed, equations (4.3)–(4.5) are obviously invariant under this action. Since we are interested only in $\bar{k}$-valued points, the invariance of (4.6) follows from its interpretation as a rank condition.

We consider the projection morphism on the second factor,

$$\mathrm{pr}_{X_3} \colon N \to \mathbb{A}^{2m}, \quad (X_1, X_3) \mapsto X_3,$$

which is equivariant for the action of Sp_{2m} (with the action on $\mathbb{A}^{2m}$ given in the obvious way). By studying the fibers of pr_{X_3}, we expect to develop a better understanding of the whole space N.

We write $c_0 := (1 \ 0 \ \ldots \ 0)$ for the row vector of $\mathbb{A}^{2m}$ that has a 1 as first entry and a 0 in each of the remaining $2m - 1$ columns.

LEMMA 4.4. *The orbit of c_0 under the action of the symplectic group consists of all nonzero row vectors of $\mathbb{A}^{2m}$; that is, we have a surjection*

$$\{c_0\} \times \mathrm{Sp}_{2m} \twoheadrightarrow \mathbb{A}^{2m} \setminus \{0\}, \quad (c_0, g) \mapsto c_0 g.$$

Proof. This is clear: for a given row vector $c_1 \neq 0 \in \mathbb{A}^{2m}(\bar{k})$, one may construct a symplectic matrix $g \in \mathrm{Sp}_{2m}(\bar{k})$ that has c_1 as its first row. □

Because of this transitivity result, there are essentially two fibers to examine. On the one hand, we must look at the fiber over the zero vector; on the other hand, we must determine the fiber over c_0.

4.7. Zero Fiber

The next lemma describes the fiber over the zero vector.

LEMMA 4.5. *The fiber $\mathrm{pr}_{X_3}^{-1}(0)$ is given by the k-scheme of $2m \times 2m$ matrices X_1 that satisfy the conditions*

$$X_1^2 = 0, \quad X_1 + \sigma(X_1) = 0, \quad \wedge^{s+1} X_1 = 0.$$

This scheme is irreducible. It has dimension $(2m - s)s$ if s is even or dimension $(2m - s + 1)(s - 1)$ if s is odd. In both cases, the dimension is smaller than rs.

Proof. The description of the fiber is obvious from (4.3)–(4.6); in particular, because $X_3 = 0$, the wedge condition (4.6) translates into the wedge condition involving just X_1.

The stabilizer of the zero vector is the whole symplectic group, $\mathrm{Stab}_0 = \mathrm{Sp}_{2m}$. It acts by conjugation on the elements X_1 contained in the zero fiber. Pappas and Rapoport have considered this matrix scheme [PR4, Sec. 5.5]. In their notation, it coincides with the special fiber of the matrix scheme $U^{\wedge}_{r',s}$, where we have set $r' := 2m - s$. It is shown in [PR4] that the special fiber is irreducible and of dimension $r's$ if s is even or of dimension $(r'+1)(s-1)$ if s is odd. Since $r' = r - 1$, the lemma is proved.

The argument of [PR4] is as follows. Consider the matrix scheme $V^{\wedge}_{r',s}$ of $2m \times 2m$ matrices X_1 over k that satisfy the conditions

$$X_1{}^2 = 0, \qquad \wedge^{s+1} X_1 = 0.$$

This scheme is the union of the nilpotent GL_{2m}-conjugation orbits $\mathcal{O}_{2^i,1^{2m-i}}$ with $i \leq s$, which contain the Jordan matrices with exactly i nilpotent Jordan blocks of size 2 while all other blocks are zero. The orbits have dimension $2(2m-i)i$, and the following closure relation holds true:

$$\mathcal{O}_{2^i,1^{2m-i}} \subset \overline{\mathcal{O}_{2^j,1^{2m-j}}} \quad \text{if and only if} \quad i \leq j \tag{4.8}$$

[PR1, Rem. 4.2]. We denote by $U^{\wedge}_{r',s}$ the fixed point scheme of $V^{\wedge}_{r',s}$ under the involution $-\sigma$. The symplectic group acts on this scheme by conjugation; slightly abusing notation, we denote the corresponding nilpotent conjugation orbits by the same symbols. It follows from [O, Prop. 1] that $U^{\wedge}_{r',s}$ is the union of the orbits $\mathcal{O}_{2^i,1^{2m-i}}$ with even $i \leq s$. By [O, Thm. 1], a closure relation as in (4.8) also holds true in this context. We conclude that $U^{\wedge}_{r',s}$ is either the closure of $\mathcal{O}_{2^s,1^{r'}}$ if s is even or the closure of $\mathcal{O}_{2^{s-1},1^{r'+1}}$ if s is odd. The irreducibility of the symplectic group implies the irreducibility of its orbits and their closures. The dimension of these Sp_{2m}-orbits is half the dimension of the corresponding GL_{2m}-orbits [KRa, Prop. 5 and its proof]. □

4.8. Nonzero Fiber

The following lemma gives a description of the fiber over c_0.

LEMMA 4.6. *The fiber* $\mathrm{pr}_{X_3}{}^{-1}(c_0)$ *is given by the k-scheme N' of pairs of matrices* (Y_1, Y_2), *subject to the following conditions*:

$$Y_1{}^2 = 0, \quad Y_1 + \sigma(Y_1) = 0, \quad \wedge^s \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = 0, \quad Y_2 Y_1 = 0.$$

Here Y_1 denotes a square matrix of size $2m-2$ and Y_2 a row vector of size $2m-2$.

Proof. We describe the matrices X_1 lying over c_0 by evaluating (4.3)–(4.6).

Equation (4.5) applied with $X_3 = c_0$ implies that the first row of X_1 is zero.

Since $-J_{2m} c_0^{\mathrm{t}} c_0 = K H_{2m}$, it follows that the left-hand side of (4.3) is the square matrix with all entries 0 except the lower left, which is 1. As noted before, σ is a signed reflection at the antidiagonal; hence (4.3) implies that X_1 has the following form:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ \sigma(Y_2) & Y_1 & 0 \\ 1/2 & Y_2 & 0 \end{pmatrix}. \tag{4.9}$$

Here Y_1 is a square matrix of size $2m-2$ that satisfies the symmetry condition

$$Y_1 + \sigma(Y_1) = 0 \tag{4.10}$$

and Y_2 is a row vector with $2m-2$ columns.

Because c_0 has a unit in the first entry and zeros everywhere else, (4.6) translates via Laplace expansion along c_0 into a wedge condition for the $(2m-1)\times(2m-2)$ matrix consisting of Y_1 and Y_2:

$$\wedge^s \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = 0. \tag{4.11}$$

By (4.4), the square of X_1 must be zero. Using (4.9), this results in

$$Y_1{}^2 = 0, \tag{4.12}$$

$$Y_2 Y_1 = 0. \tag{4.13}$$

As previously asserted, equations (4.10)–(4.13) describe the fiber over c_0. □

Next we determine the stabilizer of $c_0 \in \mathbb{A}^{2m}$ and its action on the fiber over c_0.

LEMMA 4.7. *The stabilizer* $\mathrm{Stab}_{c_0} \subset \mathrm{Sp}_{2m}$ *of* c_0 *is given by symplectic matrices* g *of the form*

$$g = \begin{pmatrix} 1 & & \\ -g_1\sigma(g_2) & g_1 & \\ g_3 & g_2 & 1 \end{pmatrix},$$

with a symplectic matrix g_1 *of size* $2m-2$, *a row vector* g_2 *of corresponding size, and a scalar* g_3. *Referring to these matrices by giving the essential data in the form of a triple* (g_1, g_2, g_3), *the induced action on* N' *can be described as follows*:

$$N' \times \mathrm{Stab}_{c_0} \to N',$$
$$((Y_1, Y_2), (g_1, g_2, g_3)) \mapsto (g_1{}^{-1} Y_1 g_1, Y_2 g_1 - g_2 g_1{}^{-1} Y_1 g_1). \tag{4.14}$$

Proof. Let $g \in \mathrm{Sp}_{2m}$ stabilize c_0. Then the first row of g must be c_0. We subdivide g into blocks,

$$g = \begin{pmatrix} 1 & 0 & 0 \\ g_4 & g_1 & g_5 \\ g_3 & g_2 & g_6 \end{pmatrix},$$

with a square matrix g_1 of size $2m-2$, a row vector g_2 with $2m-2$ columns, and a scalar g_3. Once we evaluate the condition of g as being symplectic, the description of the stabilizer follows. □

REMARK 4.8. The entry g_3 of an element $(g_1, g_2, g_3) \in \mathrm{Stab}_{c_0}$ does not occur on the right-hand side of (4.14); hence, it has no effect on the induced action on N'.

REMARK 4.9. The symplectic group of size $2m-2$ can be regarded as a subgroup of the stabilizer of c_0: we have the inclusion morphism

$$\mathrm{Sp}_{2m-2} \hookrightarrow \mathrm{Stab}_{c_0}, \quad g_1 \mapsto (g_1, 0, 0).$$

The corresponding action on N' is given by

$$N' \times \mathrm{Sp}_{2m-2} \to N', \quad ((Y_1, Y_2), g_1) \mapsto ({g_1}^{-1} Y_1 g_1, Y_2 g_1),$$

which is completely analogous to (4.7).

Recall that we have set $r' = 2m - s$. We consider the k-scheme $U^{\wedge}_{r'-1,s-1}$ defined analogously to the matrix scheme $U^{\wedge}_{r',s}$ from the proof of Lemma 4.5: it is given by square matrices Y_1 of size $2m - 2$ satisfying ${Y_1}^2 = 0$, $Y_1 + \sigma(Y_1) = 0$, and $\wedge^s Y_1 = 0$. The symplectic group acts on this scheme by conjugation, and $U^{\wedge}_{r'-1,s-1}$ is the union of the finitely many Sp_{2m-2}-orbits $\mathcal{O}_{2^i,1^{2m-2-i}}$ with even $i \leq s - 1$. The orbits are irreducible and have dimension $(2m - 2 - i)i$, and a closure relation analogous to (4.8) holds true. Hence, there is an open dense orbit; it is $\mathcal{O}_{2^{s-1},1^{r'-1}}$ if $s - 1$ is even or $\mathcal{O}_{2^{s-2},1^{r'}}$ if $s - 1$ is odd.

The first component Y_1 of a point $(Y_1, Y_2) \in N'$ gives a point in $U^{\wedge}_{r'-1,s-1}$. This is true because (4.11) implies in particular that $\wedge^s Y_1 = 0$. We study the projection morphism on the first factor,

$$\mathrm{pr}_{Y_1} \colon N' \to U^{\wedge}_{r'-1,s-1}, \quad (Y_1, Y_2) \mapsto Y_1,$$

which is equivariant for the action of Sp_{2m-2}.

LEMMA 4.10. *Over each orbit $\mathcal{O}_{2^i,1^{2m-2-i}}$ with even $i \leq s-1$, the projection morphism* $\mathrm{pr}_{Y_1} \colon N' \to U^{\wedge}_{r'-1,s-1}$ *is a fibration into affine spaces. The inverse images of these orbits are irreducible subsets that partition N'. The inverse image of the open dense orbit has dimension* $(2m - s)(s - 1)$; *the inverse images of the other orbits have smaller dimension.*

Proof. We fix an orbit $\mathcal{O}_{2^i,1^{2m-2-i}}$ with even $i \leq s - 1$ and consider an arbitrary point Y_1 thereof. We determine the points of N' lying above Y_1; that is, we identify the vectors Y_2 giving elements $(Y_1, Y_2) \in N'$. We distinguish the cases $i = s - 1$ and $i < s - 1$.

In the former case, the rank of the matrix Y_1 equals $s - 1$; thus, (4.11) implies that Y_2 belongs to the image of Y_1, and we can write $Y_2 = aY_1$ with a row vector a of size $2m - 2$. Then (4.13), which is the second condition mixing Y_1 and Y_2, automatically holds true: $Y_2 Y_1 = a{Y_1}^2 = 0$. It follows that exactly those elements in the image of Y_1, which is an $(s - 1)$-dimensional vector space, correspond to points $(Y_1, Y_2) \in N'$. Locally on $\mathcal{O}_{2^i,1^{2m-2-i}}$, this gives trivializations with linear isomorphisms as transition maps; in other words, we get a vector bundle over the orbit $\mathcal{O}_{2^i,1^{2m-2-i}}$.

If $i < s - 1$, then (4.11) is automatically satisfied because the rank i of Y_1 is smaller than $s - 1$. Hence, Y_2 determines a point $(Y_1, Y_2) \in N'$ if and only if (4.13) is satisfied—that is, if and only if Y_2 lies in the kernel of Y_1. It follows that every fiber is a vector space of dimension $2m - 2 - i$. Again, we get a vector bundle over the orbit $\mathcal{O}_{2^i,1^{2m-2-i}}$.

The total space of a vector bundle over an irreducible base is irreducible, and its dimension is the sum of the base dimension and the typical fiber dimension. Hence, the dimension of the inverse image of the open dense orbit is calculated to be $(2m-2-(s-1))(s-1)+(s-1)$ if $s-1$ is even or $(2m-2-(s-2))(s-2)+(2m-2-(s-2))$ if $s-1$ is odd. In both cases, this equals $(2m-s)(s-1)$. The inverse images of the other orbits (corresponding to even $i < s-2$) have smaller dimension $(2m-2-i)(i+1)$: note that $i+1 < s-1 \leq m-1$. □

REMARK 4.11. The action of Stab_{c_0} on the inverse image of the open dense orbit is transitive if $s-1$ is even but is not transitive if $s-1$ is odd. The actions on the inverse images of the other orbits are not transitive.

Taking the respective closures in N' of the inverse images of the orbits and omitting redundant terms then yields the decomposition of the fiber over c_0 into irreducible components, as follows.

COROLLARY 4.12. *The fiber* $\mathrm{pr}_{X_3}^{-1}(c_0)$ *contains an irreducible component* $Z_{\max}$ *of dimension* $(2m-s)(s-1)$. *All other irreducible components* Z_γ *with* $\gamma \in \Gamma$ (*and* Γ *a finite, possibly empty index set*) *have smaller dimension.*

4.9. Action of the Symplectic Group (*continued*)

The action (4.7) of the symplectic group Sp_{2m} on N gives rise to the surjective morphism

$$\phi\colon \mathrm{pr}_{X_3}^{-1}(c_0) \times \mathrm{Sp}_{2m} \to \mathrm{pr}_{X_3}^{-1}(X_3 \neq 0),$$
$$((X_1, c_0), g) \mapsto (g^{-1}X_1 g, c_0 g).$$

We consider the images under ϕ of the sets $Z_{\max} \times \mathrm{Sp}_{2m}$ and $Z_\gamma \times \mathrm{Sp}_{2m}$ with $\gamma \in \Gamma$, and we denote their respective closures in N by $Z'_{\max}$ and Z'_γ with $\gamma \in \Gamma$.

LEMMA 4.13. *The sets* $Z'_{\max}$ *and* Z'_γ *with* $\gamma \in \Gamma$ *are irreducible subsets of* N. *The dimension of* $Z'_{\max}$ *equals* rs. *Any* Z'_γ *with* $\gamma \in \Gamma$ *has smaller dimension.*

Proof. The irreducibility is obvious since images of irreducible subsets under morphisms are irreducible, and so are their closures. As for the dimension assertion, we consider the restriction of the projection morphism pr_{X_3} to $\phi(Z_{\max} \times \mathrm{Sp}_{2m})$:

$$\mathrm{pr}_{X_3}|_{\phi(Z_{\max}\times \mathrm{Sp}_{2m})}\colon \phi(Z_{\max} \times \mathrm{Sp}_{2m}) \to \mathbb{A}^{2m} \setminus \{0\}, \quad (X_1, X_3) \mapsto X_3.$$

This is a surjective morphism between irreducible schemes of finite type over k with all fibers isomorphic to $Z_{\max}$. Since $\phi(Z_{\max} \times \mathrm{Sp}_{2m})$ has the same dimension as its closure, we calculate $\dim Z'_{\max} = 2m + (2m-s)(s-1) = rs$. Analogous reasoning shows that the dimensions of the other subsets is smaller. □

By Lemma 4.5, the subset $\mathrm{pr}_{X_3}^{-1}(0)$ is irreducible of dimension smaller than rs. Together with $Z'_{\max}$ and Z'_γ for $\gamma \in \Gamma$, we get a finite covering of N by irreducible subsets. By omitting redundant terms, we obtain the decomposition of N into irreducible components, as follows.

COROLLARY 4.14. *The scheme N contains the irreducible component $Z'_{\max}$, which has dimension rs. All other irreducible components of N (if there are any at all) are of smaller dimension.* □

4.10. Intersection with the Local Model

We will now pass to the local model, finishing therewith the proof of Proposition 4.3. The arguments resemble those used in the proof of Theorem 3.1.

Recall that, in this Section 4, all schemes are over k and equipped with the reduced structure (unless explicitly stated otherwise). The standard open subset $\mathrm{Grass}^J_{n,2n} \times \mathrm{Grass}^J_{n,2n}$ of the product of Grassmannians is abbreviated to U. As usual, $\bar{M}^{\mathrm{loc}}_I$ denotes the special fiber of the local model and $\bar{M}^{\wedge}_I$ the special fiber of the wedge local model. We have closed immersions

$$\bar{M}^{\mathrm{loc}}_I \cap U \subset \bar{M}^{\wedge}_I \cap U \subset N.$$

Following the same arguments as given in the proof of Theorem 3.1, we deduce that the open subset $\bar{M}^{\mathrm{loc}}_I \cap U$ of the special fiber of the local model coincides with the irreducible component $Z'_{\max}$ of N. On the one hand, $\bar{M}^{\mathrm{loc}}_I \cap U$ is nonempty (it contains the special point $(\mathcal{F}_0, \mathcal{G}_0)$; see Section 4.1) and equidimensional of dimension rs. On the other hand, by Corollary 4.14 the decomposition of N into irreducible components is given by $Z'_{\max}$ (which has dimension rs) and irreducible components of smaller dimension (if there are any at all).

We conclude that $\bar{M}^{\mathrm{loc}}_I \cap U$ is an irreducible open neighborhood of the point $(\mathcal{F}_0, \mathcal{G}_0)$. This completes the proof of Proposition 4.3 and therefore also of Theorem 4.1. □

REMARK 4.15. In the case considered, the set $\mathcal{A}^I(\mu)$ (which was mentioned in Section 3.1) is the closure of a single extreme orbit and coincides with the underlying reduced scheme of the geometric special fiber of the local model. This follows from dimension arguments in the same manner as before. Observe that the open subset constructed in Section 3 is a neighborhood of one of the best points and has dimension rs; and by the results of this section, the (geometric) special fiber of the local model is irreducible of dimension rs.

We are now in a position to prove the main theorem.

Proof of Theorem 2.1. As we have shown, the special fiber of the local model is irreducible and generically reduced. Now the remaining assertions follow by arguments given by Pappas and Rapoport [PR4, Proof of Thm. 5.1]. The main result [PR3, Thm. 8.4] in one of their previous papers allows us to deduce the three properties "normal, Frobenius split, and having only rational singularities". Finally, an application of Hironaka's lemma [EGA IV$_2$, Prop. 5.12.8] yields that the special fiber of the local model is reduced. □

T. Richarz has pointed out that, with the same methods used here, one can prove the following statement.

PROPOSITION 4.16 (T. Richarz). *In the case of signature* $(n-1,1)$ (*i.e., when* $s=1$), *the local model* M_I^{loc} *is smooth. Furthermore, in this case* $\bar{M}_I^{\mathrm{loc}} = (\bar{M}_I^{\wedge})_{\mathrm{red}}$.

Proof. In this situation and with notation as before, Lemmas 4.4–4.6 show that the morphism $\mathrm{pr}_{X_3}\colon N_{\mathrm{red}} \to \mathbb{A}^{2m}$ is bijective on $\bar{k}$-valued points and also that it is birational. By Zariski's main theorem, it is an isomorphism. Now the previous reasoning implies the assertions of the proposition. □

5. Other Special Parahoric Level Structures

In this section, we take a look at the cases treated by Pappas and Rapoport (see Remark 2.3). Transferring our methods from Section 3 to this situation, we obtain analogues of Theorem 3.1 and Proposition 3.2. In this way, we can strengthen some of Pappas and Rapoport's results.

THEOREM 5.1. *Let* $I=\{0\}$ *if* $n=2m+1$ *is odd and* $I=\{m\}$ *if* $n=2m$ *is even. Then the local model* M_I^{loc} *contains an affine space of dimension* rs *as an open subset.*

Proof. Arguing as in the proof of Theorem 3.1, this is a consequence of the following proposition. □

PROPOSITION 5.2. *Let* $I=\{0\}$ *if* $n=2m+1$ *is odd and* $I=\{m\}$ *if* $n=2m$ *is even. Then the wedge local model* $M_I^{\wedge}$ *contains an affine space of dimension* rs *as an open subset.*

Proof. We proceed as in the proof of Proposition 3.2 and specialize the definition of the wedge local model to the current situation. The evaluation of the corresponding conditions leads to an open subset isomorphic to an affine space of the desired dimension. (See [Ar] for details of the calculations.) □

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