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Chow Motive of Fulton–MacPherson Configuration Spaces and Wonderful Compactifications

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1. Introduction

The purpose of this paper is to study the Chow groups and Chow motives of the so-called wonderful compactifications of an arrangement of subvarieties, in particular the Fulton–MacPherson configuration spaces.

All the varieties in the paper are over an algebraically closed field. Let Y be a nonsingular quasi-projective variety. Let $\mathcal{S}$ be an arrangement of subvarieties of Y (cf. Definition 2.2). Let $\mathcal{G}$ be a building set of $\mathcal{S}$, that is, a finite set of nonsingular subvarieties in $\mathcal{S}$ satisfying Definition 2.3. The wonderful compactification $Y_{\mathcal{G}}$ is constructed by blowing up Y along subvarieties in $\mathcal{G}$ successively (cf. Definition 2.5). There are different orders in which the blow-ups can be carried out; for example, we can blow up along the centers in any order that is compatible with the inclusion relation. There are many important examples of such compactifications: De Concini and Procesi's wonderful model of a subspace arrangement, the Fulton–MacPherson configuration spaces, the moduli space $\overline{\mathcal{M}}_{0,n}$ of stable rational curves with n marked points, and others. These spaces have many properties in common. Studying them with a uniform method gives us a better understanding of these spaces. In this paper, we study their Chow groups and Chow motives.

If we assume that Y is projective, then the Chow motive of $Y_{\mathcal{G}}$, denoted by $h(Y_{\mathcal{G}})$, can be decomposed canonically into a direct sum of the motive of Y and the twisted motives of the subvarieties in the arrangement (see Section 2.1 for a review of Chow motives). We will prove the following theorem, where the precise definitions of the set $M_{\mathcal{T}}$ and the subvarieties $Y_0\mathcal{T}$ of Y are given in Section 3.

Main Theorem (Theorems 3.1 and 3.2). *Let Y be a nonsingular quasi-projective variety, let $\mathcal{G}$ be a building set, and let $Y_{\mathcal{G}}$ be the wonderful compactification $Y_{\mathcal{G}}$. Then we have the Chow group decomposition*

$$A^*Y_{\mathcal{G}} = A^*Y \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu} \in M_{\mathcal{T}}} A^{*-\|\underline{\mu}\|}(Y_0\mathcal{T}),$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests. Moreover, when Y is projective we have the Chow motive decomposition

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$$h(Y_{\mathcal{G}}) \cong h(Y) \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu} \in M_{\mathcal{T}}} h(Y_0\mathcal{T})(\|\underline{\mu}\|),$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests. In this case the correspondences giving the previous isomorphism are canonical in the following sense: there is no canonical order of blow-ups (in general) to construct $Y_{\mathcal{G}}$, and the correspondences turn out to be independent of the order we choose.

The Fulton–MacPherson configuration space $X[n]$ is one of the most interesting examples of the wonderful compactification $Y_{\mathcal{G}}$, where $Y = X^n$ and $\mathcal{G}$ is the the set of all the diagonals in X^n (see Section 4.1). Applying the main theorem to $X[n]$, we obtain the following theorem, where the precise definition of the nests $\mathcal{S}$, the polydiagonals $\Delta_{\mathcal{S}}$, the integers $c(\mathcal{S})$, the sets of lattice points $M_{\mathcal{S}}$, and the correspondences $\alpha_{\mathcal{S},\underline{\mu}}$ and $\beta_{\mathcal{S},\underline{\mu}}$ are given in Section 4.1.

THEOREM 4.2. *Let X be a nonsingular projective variety. Then there is a canonical isomorphism of Chow motives*

$$\bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} \alpha_{\mathcal{S},\underline{\mu}} \colon h(X[n]) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} h(\Delta_{\mathcal{S}})(\|\underline{\mu}\|)$$

with the inverse $\sum_{\mathcal{S}} \sum_{\underline{\mu} \in \mathcal{S}} \beta_{\mathcal{S},\underline{\mu}}$. Equivalently, we have the following decomposition of the Chow motive of $X[n]$:

$$h(X[n]) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} h(X^{c(\mathcal{S})})(\|\underline{\mu}\|).$$

The first consequence of this theorem is that we can easily express the decomposition of $h(X[n])$ using a generating function $N(x,t)$, as follows.

THEOREM 4.3. *Define $f_i(x)$ to be the polynomials whose exponential generating function $N(x,t) = \sum_{i \geq 1} f_i(x)\frac{t^i}{i!}$ satisfies the identity*

$$(1-x)x^d t + (1 - x^{d+1}) = \exp(x^d N) - x^{d+1}\exp(N),$$

where $d = \dim X$. Then

$$h(X[n]) = \bigoplus_{\substack{1 \leq k \leq n \\ i \geq 0}} (h(X^k)(i))^{\oplus [x^i t^n/n!](N^k/k!)}.$$

The second consequence is a decomposition of the Chow motive of the quotient variety $X[n]/\mathfrak{S}_n$ obtained from the natural symmetric group $\mathfrak{S}_n$ action on $X[n]$. To make sense of the motive of a quotient variety, we assume the base field is of characteristic 0. The correspondences appearing in Theorem 4.2 are canonical and therefore symmetric with respect to the symmetric group $\mathfrak{S}_n$. It is then possible to compute the $\mathfrak{S}_n$-invariant part of $h(X[n])$, which is the Chow motive of $X[n]/\mathfrak{S}_n$. As pointed out by [FM], unlike the isotropy groups of a point in X^n, the isotropy group of any point in $X[n]$ is always solvable; therefore, the singularity of $X[n]/\mathfrak{S}_n$ is “better” than the singularity of the symmetric product $X^{(n)} := X^n/\mathfrak{S}_n$. It would be interesting to see how different the Chow motive $h(X[n]/\mathfrak{S}_n)$

is from $h(X^{(n)})$. In the following theorem, an unlabeled weighted forest is a forest whose nodes are not labeled and such that each nonleaf node is attached by a positive integer called a *weight*. We call an unlabeled weighted forest *of type* $\nu := \{n_1, \dots, n_r\}$ if the forest is of the form $n_1T_1 + \dots + n_rT_r$, where the T_i are mutually distinct unlabeled weighted trees.

THEOREM 5.3. *For any unordered set of positive integers* $\nu = \{n_1, \dots, n_r\}$ *and any nonnegative integer* m, *let* $\lambda(\nu, m)$ *be the number of unlabeled weighted forests with* n *leaves, of type* ν, *of total weight* m, *and such that—at each nonleaf* v *with* c_v *children—the weight* m_v *satisfies* $1 \le m_v \le (c_v - 1)\dim X - 1$. *Then*

$$h(X[n]/\mathfrak{S}_n) = \bigoplus_{\nu, m} [h(X^{(n_1)} \times \dots \times X^{(n_r)})(m)]^{\oplus\lambda(\nu,m)}.$$

The importance of all the preceding results on Chow motives can be seen through a working principle.

PRINCIPLE. *A result proved for Chow motives is valid if we replace them by homological/numerical motives, Chow groups* $A^*_{\mathbb{Q}}$, *cohomology groups* $H^*_{\mathbb{Q}}$, *Grothendieck groups* (*the aforementioned groups are taken with* $\mathbb{Q}$*-coefficients*), *Hodge structures, and so forth.*

Thus, for example, we have a decomposition for the $\mathbb{Q}$-coefficient singular cohomology of $Y_{\mathcal{G}}$, $X[n]$, and $X[n]/\mathfrak{S}_n$.

The paper is organized as follows. Section 2 contains a review of motives and the wonderful compactifications of arrangement of subvarieties. In Section 3, a motivic decomposition for the wonderful compactifications is proved. In Section 4 we give a motivic decomposition for the Fulton–MacPherson configuration spaces. Finally, Section 5 gives a motivic decomposition for the quotient variety $X[n]/\mathfrak{S}_n$.

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2. Preliminaries

2.1. Motives

Given an algebraic variety X of dimension d, let $A^iX = A_{d-i}X$ be the Chow group of codimension i (i.e., the group of algebraic cycles of codimension i in X modulo rational equivalence). Define $A^i_{\mathbb{Q}}X = A^iX \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let X and Y be two nonsingular projective varieties. The group of *correspondences of degree* r from X to Y is defined as

$$\mathrm{Corr}^r(X,Y) := A^{\dim X+r}(X\times Y).$$

The group $\mathrm{Corr}^r_{\mathbb{Q}}(X,Y)$ denotes the tensor of $\mathrm{Corr}^r(X,Y)$ with $\mathbb{Q}$.

The *composition* of two correspondences

$$f\in \mathrm{Corr}^r(X_1,X_2) \quad \text{and} \quad g\in \mathrm{Corr}^s(X_2,X_3)$$

is a correspondence in $\mathrm{Corr}^{r+s}(X_1,X_3)$ defined as

$$g\circ f := \pi_{13*}(\pi_{12}^* f\cdot \pi_{23}^* g),$$

where π_{ij} is the projection from $X_1\times X_2\times X_3$ to $X_i\times X_j$.

A correspondence $p\in\mathrm{Corr}^0(X,X)$ is called a *projector* of X if $p^2=p$ (where $p^2 := p\circ p$). Let $\mathcal{V}$ denote the category of (not necessarily connected) nonsingular projective varieties over a field k.

DEFINITION 2.1 [CH]. The *category of Chow motives over* k, denoted by $CH\mathcal{M}$, is defined as follows: an object of $CH\mathcal{M}$, called a *Chow motive,* is a triple (X,p,r) for X a nonsingular projective variety, p a projector of X, and r an integer. The morphisms in $CH\mathcal{M}$ are defined as

$$\mathrm{Hom}_{CH\mathcal{M}}((X,p,r),(Y,q,s)) := q\circ \mathrm{Corr}^{s-r}(X,Y)\circ p.$$

The composition of morphisms is defined as the composition of correspondences.

For a Chow motive $M=(X,p,r)$ and an integer ℓ, we define

$$M(\ell) := (X,p,r+\ell).$$

There is a natural contravariant functor h from $\mathcal{V}$ to $CH\mathcal{M}$ that sends X to $(X,\mathrm{id}_X,0)$ and also sends a morphism $f\colon X\to Y$ to $\Gamma_f^t\colon h(Y)\to h(X)$, the transpose of the graph of f. Naturally, $h(X)(\ell)$ stands for the Chow motive (X,id_X,ℓ).

According to [dBN], we can generalize the theory of Chow motives on nonsingular projective varieties to the one on varieties that are quotients of smooth projective varieties by finite group actions. To be more precise, let $\mathcal{V}'$ be the category of varieties of type X/G with $X\in Ob\mathcal{V}$ and G a finite group. We can define the group of correspondences $\mathrm{Corr}^r_{\mathbb{Q}}(X',Y')$ for $X',Y'\in\mathcal{V}'$ and the category of Chow motives $CH\mathcal{M}'$ in a manner that is similar to the nonsingular case (the difference is that we must use $\mathbb{Q}$-coefficients). There is a natural contravariant functor $h\colon \mathcal{V}'\to CH\mathcal{M}'$.

Define the G-average correspondence ave_G as

$$\mathrm{ave}_G := \frac{1}{|G|}\sum [g]\in \mathrm{Corr}^0_{\mathbb{Q}}(X,X),$$

where $[g]$ is given by the graph of g in $X\times X$. By [dBN, Prop. 1.2], there is an isomorphism

$$h(X/G)\cong (X,\mathrm{ave}\,\Delta)\cong h(X)^G.$$

Such a definition is consistent with the realization functors and $\mathbb{Q}$-coefficient Chow groups.

2.2. Wonderful Compactification of an Arrangement of Subvarieties

The wonderful compactification of an arrangement of subvarieties is introduced in [L] as a generalization of De Concini and Procesi's wonderful model of subspace arrangements. We briefly review the definition and some properties of such compactifications. For details we refer to [L].

DEFINITION 2.2. A (simple) *arrangement* of subvarieties of Y is a finite set $\mathcal{S} = \{S_i\}$ of nonsingular closed subvarieties of Y satisfying the following conditions:

(1) S_i and S_j intersect cleanly (in other words, their intersection is nonsingular and $T(S_i \cap S_j) = T(S_i)|_{(S_i \cap S_j)} \cap T(S_j)|_{(S_i \cap S_j)}$); and
(2) $S_i \cap S_j$ is either empty or equal to some $S_k \in \mathcal{S}$.

DEFINITION 2.3. Let $\mathcal{S}$ be an arrangement of subvarieties of Y. A subset $\mathcal{G} \subseteq \mathcal{S}$ is called a *building set* of $\mathcal{S}$ if, for all $S \in \mathcal{S}$, the minimal elements in the $\mathcal{G}$ that contains S intersect transversally and their intersection is S (this condition is always satisfied if $S \in \mathcal{G}$). These minimal elements are called the $\mathcal{G}$*-factors* of S. We call a finite set $\mathcal{G}$ of subvarieties a building set if the set

$$\mathcal{S} := \left\{ \bigcap_{V \in \mathcal{F}} V \right\}_{\mathcal{F}} \quad \text{(where } \mathcal{F} \text{ runs through all subsets of } \mathcal{G}\text{)}$$

is an arrangement and $\mathcal{G}$ is a building set of $\mathcal{S}$ $\left(\text{for } \mathcal{F} = \emptyset \text{ we set } \bigcap_{V \in \mathcal{F}} V = Y\right)$. In this case we call $\mathcal{S}$ the *induced arrangement* of $\mathcal{G}$.

DEFINITION 2.4. Let $\mathcal{G}$ be a building set. A subset $\mathcal{T} \subseteq \mathcal{G}$ is called $\mathcal{G}$*-nested* (or a $\mathcal{G}$*-nest*) if it satisfies one of the following equivalent relations.

(1) There is a flag of elements in $\mathcal{S}$, $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$, such that

$$\mathcal{T} = \bigcup_{i=1}^{k} \{A : A \text{ is a } \mathcal{G}\text{-factor of } S_i\}.$$

(We say that $\mathcal{T}$ is *induced* by the flag $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$.)
(2) Let $A_1, \dots, A_k$ be the minimal elements of $\mathcal{T}$; then they are all the $\mathcal{G}$-factors of a certain element in $\mathcal{S}$ and, for each $1 \le i \le k$, the set $\{A \in \mathcal{T} : A \supsetneq A_i\}$ is also $\mathcal{G}$-nested as defined by induction.

The wonderful compactification is defined as follows.

DEFINITION 2.5. Denote $Y^\circ = Y \setminus \bigcup_{G \in \mathcal{G}} G$. There is a natural locally closed embedding

$$Y^\circ \hookrightarrow Y \times \prod_{G \in \mathcal{G}} Bl_G Y.$$

The closure of this embedding, denoted by $Y_{\mathcal{G}}$, is called the *wonderful compactification* of $\mathcal{G}$.

The wonderful compactification $Y_{\mathcal{G}}$ of $\mathcal{G}$ has the properties described in Theorem 2.6, where parts (i) and (ii) are in [L, Thm. 1.2] and (iii) is clear from the proof there.

THEOREM 2.6. *The variety $Y_{\mathcal{G}}$ is nonsingular. For each $G \in \mathcal{G}$, there is a nonsingular divisor D_G on $Y_{\mathcal{G}}$ such that the following statements hold.*

(i) *The union of the divisors D_G is $Y_{\mathcal{G}} \setminus Y^\circ$.*

(ii) *Any collection of the divisors D_G intersects transversally. An intersection of divisors $D_{T_1} \cap \cdots \cap D_{T_r}$ is nonempty exactly when $\{T_1, \dots, T_r\}$ forms a $\mathcal{G}$-nest.*

(iii) *Each D_G is the unique connected component of $\pi^{-1}(G)$ that maps surjectively to the subvariety G, where π is the natural morphism $Y_{\mathcal{G}} \to Y$. (This D_G is called the* dominant transform *of G and is denoted by $\widetilde{G}$ in* [L].)

The dominant transform can also be defined as follows. Let $\pi : \widetilde{Y} \to Y$ be the blow-up along a nonsingular subvariety $G \subsetneq Y$. For any irreducible subvariety V in Y, we define the dominant transform of V, denoted by $\widetilde{V}$ or $V^{\sim}$, to be the strict transform of V when $V \nsubseteq G$ and to be $\pi^{-1}(V)$ when $V \subseteq G$. For a sequence of N blow-ups $Y_N \to Y_{N-1} \to \cdots \to Y_1 \to Y_0$ and a subvariety $V \subseteq Y_0$, we define the dominant transform $\widetilde{V} \subseteq Y_N$ to be the Nth iterated dominant transform $(\cdots((V^{\sim})^{\sim})\cdots)^{\sim}$.

It is known (see [L]) that $Y_{\mathcal{G}}$ can be constructed by a sequence of blow-ups as follows. Let Y be a nonsingular variety, let $\mathcal{S}$ be an arrangement of subvarieties, and let

$$\mathcal{G} = \{G_1, \dots, G_N\}$$

be a building set with respect to $\mathcal{S}$. Suppose the subvarieties in $\mathcal{G} = \{G_1, \dots, G_N\}$ are indexed in an order that is compatible with inclusion relations (i.e., $i \le j$ if $G_i \subseteq G_j$). We define the triple $(Y_k, \mathcal{S}^{(k)}, \mathcal{G}^{(k)})$ inductively with respect to k, where Y_k is a nonsingular variety, $\mathcal{S}^{(k)}$ is an arrangement of subvarieties of Y_k, and $\mathcal{G}^{(k)}$ is a building set with respect to $\mathcal{S}^{(k)}$.

(1) For $k = 0$, define $Y_0 = Y$, $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{G}^{(0)} = \mathcal{G} = \{G_1, \dots, G_N\}$, and $G_i^{(0)} = G_i$ for $1 \le i \le N$.

(2) Assume the triple $(Y_k, \mathcal{S}^{(k)}, \mathcal{G}^{(k)})$ has been constructed. Define Y_k to be the blow-up of Y_{k-1} along the nonsingular subvariety $G_k^{(k-1)}$, and define $G^{(k)}$ to be the dominant transform $(G^{(k-1)})^{\sim}$ for all $G \in \mathcal{G}$. Then $\mathcal{G}^{(k)} := \{G^{(k)}\}_{G \in \mathcal{G}}$ is a building set (by [L, Prop. 2.8]). We denote the induced arrangement by $\mathcal{S}^{(k)}$.

(3) Continue the inductive construction until $k = N$. We obtain a nonsingular variety Y_N and that all elements in the building set $\mathcal{G}^{(N)}$ are divisors. The resulting variety is isomorphic to $Y_{\mathcal{G}}$.

For any $\mathcal{G}$-nest $\mathcal{T}$, define

$$Y_k \mathcal{T} = \bigcap_{G \in \mathcal{T}} G^{(k)}.$$

The following property of $Y_k \mathcal{T}$ is used often throughout the paper.

PROPOSITION 2.7. *Let $0 \le k \le N-2$ and let $\mathcal{T} \subseteq \{G_{k+2}, G_{k+3}, \dots, G_N\}$ be a $\mathcal{G}$-nest. Then $Y_{k+1}\mathcal{T}$ is an irreducible nonsingular subvariety of Y_{k+1} with the following property.*

If $\{G_{k+1}\} \cup \mathcal{T}$ is not a $\mathcal{G}$-nest, then $G_{k+1}^{(k)} \cap Y_k\mathcal{T} = \emptyset$ and $Y_{k+1}\mathcal{T} \cong Y_k\mathcal{T}$; otherwise, the intersection $G_{k+1}^{(k)} \cap Y_k\mathcal{T}$ is clean, $Y_{k+1}\mathcal{T}$ is isomorphic to the blow-up of $Y_k\mathcal{T}$ along $G_{k+1}^{(k)} \cap Y_k\mathcal{T}$ with exceptional divisor $G_{k+1}^{(k+1)} \cap Y_{k+1}\mathcal{T}$ (where the intersection is transverse), and the codimension of $G_{k+1}^{(k)} \cap Y_k\mathcal{T}$ in $Y_k\mathcal{T}$ is equal to

$$\begin{cases} \dim \bigcap_{G_{k+1} \subsetneq G \in \mathcal{T}} G - \dim G_{k+1} & \text{if } \{G : G_{k+1} \subsetneq G \in \mathcal{T}\} \neq \emptyset, \\ \dim Y - \dim G_{k+1} & \text{otherwise.} \end{cases}$$

Proof. We prove the statement by induction on k. The case $k = 0$ is obvious. Now assume that the statement is true for k.

(i) Suppose that $\{G_{k+1}\} \cup \mathcal{T}$ is not a $\mathcal{G}$-nest. We will show that $G_{k+1}^{(k)} \cap Y_k\mathcal{T} = \emptyset$. As a consequence we have $Y_{k+1}\mathcal{T} \cong Y_k\mathcal{T}$, since the center of the blow-up is away from $Y_k\mathcal{T}$.

We prove by way of contradiction. Assume that $G_{k+1}^{(k)} \cap Y_k\mathcal{T} \neq \emptyset$. Since $\mathcal{T}$ is a $\mathcal{G}$-nest, $\{G^{(k)}\}_{G \in \mathcal{T}}$ is a $\mathcal{G}^{(k)}$-nest by [L, Prop. 2.8(3)]. By Definition 2.4(1), the nest $\{G^{(k)}\}_{G \in \mathcal{T}}$ is induced by a flag

$$S_1' \subseteq S_2' \subseteq \cdots \subseteq S_l',$$

where $S_i' \in \mathcal{S}^{(k)}$. We claim that $\{G_{k+1}^{(k)}\} \cup \{G^{(k)}\}_{G \in \mathcal{T}} \subseteq \mathcal{G}^{(k)}$ is a $\mathcal{G}^{(k)}$-nest induced by the flag

$$(G_{k+1}^{(k)} \cap S_1') \subseteq S_1' \subseteq S_2' \subseteq \cdots \subseteq S_l'.$$

Indeed, since $Y_k\mathcal{T} = S_1'$, we know that $G_{k+1}^{(k)} \cap S_1' \neq \emptyset$. By [L, Lemma 2.4(ii)], the $\mathcal{G}^{(k)}$-factors of $G_{k+1}^{(k)} \cap S_1'$ are $G^{(k)}$ and some $\mathcal{G}^{(k)}$-factors of S_1'; hence our claim follows. Then [L, Prop. 2.8(3)] asserts that, since $\{G_{k+1}^{(k)}\} \cup \{G^{(k)}\}_{G \in \mathcal{T}}$ is a $\mathcal{G}^{(k)}$-nest, $\{G_{k+1}\} \cup \mathcal{T}$ must be a $\mathcal{G}$-nest. But by assumption $\{G_{k+1}\} \cup \mathcal{T}$ is not a $\mathcal{G}$-nest—contradiction.

(ii) Suppose that $\mathcal{T} \cup \{G_{k+1}\}$ is a $\mathcal{G}$-nest. Let the $\mathcal{G}^{(k)}$-factors of $Y_k\mathcal{T}$ be $G_1', \dots, G_r'$. Then they are minimal elements in the $\mathcal{G}^{(k)}$-nest $\{G^{(k)}\}_{G \in \mathcal{T}}$ by the definition of nest. Assume without loss of generality that the first m subvarieties $G_1', \dots, G_m'$ contain $G_{k+1}^{(k)}$. Define $A = \bigcap_{i=1}^m G_i'$ and $B = \bigcap_{i=m+1}^r G_i'$; then $Y_k\mathcal{T} = A \cap B$ is the $G_{k+1}^{(k)}$-factorization of $Y_k\mathcal{T}$ by [L, Lemma 2.6.]

Observe that for $p, q \ge k+2$ and $G_p^{(k)} \subseteq G_q^{(k)}$ we have $G_p^{(k+1)} \subseteq G_q^{(k+1)}$, because strict transforms keep the inclusion relation. Moreover, since $G_1', \dots, G_r'$ are the minimal elements in $\mathcal{G}^{(k)}$ that contain $Y_k\mathcal{T}$, the subvariety $Y_{k+1}\mathcal{T}$ is the intersection $\bigcap_{i=1}^r \widetilde{G}_i'$. Then

$$\widetilde{A} = \bigcap_{i=1}^m \widetilde{G}_i', \quad \widetilde{B} = \bigcap_{i=m+1}^r \widetilde{G}_i', \qquad (A \cap B)^{\sim} = \widetilde{A} \cap \widetilde{B} = \bigcap_{i=1}^m \widetilde{G}_i'$$

by [L, Lemma 2.9]. Thus $Y_{k+1}\mathcal{T} = (Y_k\mathcal{T})^{\sim}$. By the definition of arrangement we know that $Y_k\mathcal{T}$ and $G_{k+1}^{(k)}$ intersect cleanly, so $Y_{k+1}\mathcal{T}$ is the blow-up of $Y_k\mathcal{T}$ along the center $G_{k+1}^{(k)} \cap Y_k\mathcal{T}$. The exceptional divisor is the preimage of the center and hence is $G_{k+1}^{(k+1)} \cap Y_{k+1}\mathcal{T}$. Since $G_{k+1}^{(k+1)}$ and $Y_{k+1}\mathcal{T}$ intersect cleanly and since the divisor $G_{k+1}^{(k+1)}$ does not contain $Y_{k+1}\mathcal{T}$, it follows that the intersection $G_{k+1}^{(k)} \cap Y_k\mathcal{T}$ is actually transversal.

The codimension of the center $Y_k\mathcal{T} \cap G_{k+1}^{(k)}$ in $Y_k\mathcal{T}$ equals

$$\operatorname{codim}(A \cap B \cap G_{k+1}^{(k)}, A \cap B) = \operatorname{codim}(G_{k+1}^{(k)} \cap B, A \cap B) = \operatorname{codim}(G_{k+1}^{(k)}, A),$$

where the second equality follows from the transversality of the intersection $G_{k+1}^{(k)} \cap B$. If no elements in $\mathcal{T}$ contain G_{k+1}, then $A = Y$ and

$$\operatorname{codim}(G_{k+1}^{(k)}, A) = \dim Y - \dim G_{k+1};$$

otherwise,

$$\operatorname{codim}(G_{k+1}^{(k)}, A) = \operatorname{codim}\left(G_{k+1}^{(k)}, \bigcap_{i=1}^{m} G_i'\right) = \operatorname{codim}\left(G_{k+1}, \bigcap_{G_{k+1}\subsetneq G\in\mathcal{T}} G\right)$$
$$= \dim \bigcap_{G_{k+1}\subsetneq G\in\mathcal{T}} G - \dim G_{k+1}.$$

Thus the proof is complete. □

3. The Motive of Wonderful Compactifications

NOTATION.

- Let Y be a nonsingular quasi-projective variety with an arrangement of subvarieties $\mathcal{S}$, and let $\mathcal{G}$ be a building set with respect to $\mathcal{S}$. Let $Y_{\mathcal{G}}$ be the wonderful compactification. Let $\mathcal{T}$ be a $\mathcal{G}$-nest.
- For $T \in \mathcal{G}$, define D_T to be the divisor $T^{(N)}$ in $Y_{\mathcal{G}}$. When no confusion can arise, we use the same notation D_T for its restriction to a subvariety of $Y_{\mathcal{G}}$.
- Denote by $j_{\mathcal{T}} \colon Y_{\mathcal{G}}\mathcal{T} \to Y_{\mathcal{G}}$ the natural imbedding; denote by $g_{\mathcal{T}} \colon Y_{\mathcal{G}}\mathcal{T} \to Y_0\mathcal{T}$ the restriction of the natural morphism $Y_{\mathcal{G}} \to Y$.
- Suppose $j \colon B \to C$ and $g \colon B \to D$ are two morphisms of varieties. Denote by $(j, g) \colon B \to C \times D$ the composition of the diagonal map Δ with $f \times g$,

$$(j, g) \colon B \xrightarrow{\Delta} B \times B \xrightarrow{f\times g} C \times D.$$

- Given $a \in A(P)$, denote by $\{a\}_i$ the image of the projection $A(P) \to A^i(P)$ of the Chow ring to its degree-i direct summand (i.e., taking the codimension-i part of a).
- We set $\bigcap_{G\subsetneq T\in\mathcal{T}} T = Y$ if no T satisfies $G \subsetneq T \in \mathcal{T}$. Define

$$r_G := \dim\left(\bigcap_{G\subsetneq T\in\mathcal{T}} T\right) - \dim G$$

and define

$$N_G := N_G\left(\bigcap_{G\subsetneq T\in\mathcal{T}} T\right)\Big|_{Y_0\mathcal{T}},$$

which is the restriction to $Y_0\mathcal{T}$ of the normal bundle of G in the ambient space $\left(\bigcap_{G\subsetneq T\in\mathcal{T}} T\right)$. Define

$$M_{\mathcal{T}} := \{\underline{\mu} = \{\mu_G\}_{G\in\mathcal{G}} : 1 \le \mu_G \le r_G - 1,\ \mu_G \in \mathbb{Z}\},$$

and define $\|\underline{\mu}\| := \sum_{G\in\mathcal{G}} \mu_G$ for $\underline{\mu} \in M_{\mathcal{T}}$.

THEOREM 3.1. *We have the Chow group decomposition*

$$A^*Y_{\mathcal{G}} = A^*Y \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu}\in M_{\mathcal{T}}} A^{*-\|\underline{\mu}\|}(Y_0\mathcal{T}),$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests.

Moreover, when Y is complete, we have the Chow motive decomposition

$$h(Y_{\mathcal{G}}) = h(Y) \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu}\in M_{\mathcal{T}}} h(Y_0\mathcal{T})(\|\underline{\mu}\|),$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests.

THEOREM 3.2. *The correspondence that gives each of the preceding direct summands can be explicitly expressed as*

$$\alpha\colon h(Y_{\mathcal{G}}) \to h(Y_0\mathcal{T})(\|\underline{\mu}\|),$$
$$\alpha = (j_{\mathcal{T}}, g_{\mathcal{T}})_* \prod_{G\in\mathcal{T}} \left\{ c\left(g_{\mathcal{T}}^*(N_G) \otimes \mathcal{O}\left(-\sum_{(\bigstar)} D_{G'}\right)\right) \frac{1}{1+D_G} \right\}_{r_G-1-\mu_G}.$$

Here c is total Chern class, the subscript $r_G - 1 - \mu_G$ means the codimension-$(r_G - 1 - \mu_G)$ part, and condition $(\bigstar)$ is: $G' \subsetneq G$ and $\mathcal{T} \cup \{G'\}$ is a $\mathcal{G}$-nest.

The inverse correspondence is

$$\beta\colon h(Y_0\mathcal{T})(\|\underline{\mu}\|) \to h(Y_{\mathcal{G}}),$$
$$\beta = (g_{\mathcal{T}}, j_{\mathcal{T}})_* \prod_{G\in\mathcal{T}} (-D_G)^{\mu_G-1}.$$

3.1. Proof of Theorem 3.1

LEMMA 3.3. *Suppose we have a $\mathcal{G}$-nest $\mathcal{T} \subseteq \{G_{k+2}, \dots, G_N\}$, and suppose that $\mathcal{T}' := \mathcal{T} \cup \{G_{k+1}\}$ is also a $\mathcal{G}$-nest. Define $r = r_{k,\mathcal{T}}$ to be*

$$\begin{cases} \dim \bigcap_{G_{k+1}\subsetneq G\in\mathcal{T}} G - \dim G_{k+1} & \text{if } \{G : G_{k+1} \subsetneq G \in \mathcal{T}\} \ne \emptyset, \\ \dim Y - \dim G_{k+1} & \text{otherwise.} \end{cases}$$

Then the following Chow group decomposition holds:

$$A^*(Y_{k+1}\mathcal{T}) = A^*(Y_k\mathcal{T}) \oplus \bigoplus_{t=1}^{r-1} A^{*-t}(Y_k\mathcal{T}').$$

When Y is complete, we also have the motivic decomposition

$$h(Y_{k+1}\mathcal{T}) = h(Y_k\mathcal{T}) \oplus \bigoplus_{t=1}^{r-1} h(Y_k\mathcal{T}')(t).$$

Proof. Applying the well-known blow-up formula for the Chow group and for the Chow motive (Theorem A.2) to Proposition 2.7 immediately gives the conclusion. □

Iteratively applying Lemma 3.3 yields the proof of Theorem 3.1, as follows.

Proof of Theorem 3.1. Define

$$M_{\mathcal{T}}^{(k)} = \big\{\underline{\mu} = \{\mu_G\}_{G\in\mathcal{G}} : 1 \le \mu_G \le \dim\big(\textstyle\bigcap_T T^{(k)}\big) - \dim G^{(k)} - 1,\ \mu_G \in \mathbb{Z}\big\},$$

where T runs through the subvarieties in $\mathcal{T}$ such that $G^{(k)} \subsetneq T^{(k)}$. Define $\|\underline{\mu}\| := \sum_{G\in\mathcal{G}} \mu_G$ for $\underline{\mu} \in M_{\mathcal{T}}^{(k)}$.

We prove the following statement using a downward induction on k:

$$A^*Y_{\mathcal{G}} = A^*Y_k \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu}\in M_{\mathcal{T}}^{(k)}} A^{*-\|\underline{\mu}\|}(Y_k\mathcal{T}), \tag{3.1}$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests such that $\mathcal{T} \subseteq \{G_{k+1}, G_{k+2}, \dots, G_N\}$.

The assertion for $k = N$ is trivial because all $G^{(N)}$ are divisors in $Y_{\mathcal{G}}$ (and hence of codimension 1) and $M_{\mathcal{T}}^{(k)} = \emptyset$.

Assume (3.1) has been proved for $k + 1$. In other words, assume that

$$A^*Y_{\mathcal{G}} = A^*Y_{k+1} \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu}\in M_{\mathcal{T}}^{(k+1)}} A^{*-\|\mu\|}(Y_{k+1}\mathcal{T}),$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests such that $\mathcal{T} \subseteq \{G_{k+2}, G_{k+3}, \dots, G_N\}$. Applying Lemma 3.3, we have

$$\begin{aligned} A^*Y_{\mathcal{G}} = A^*Y_k &\oplus \Bigg(\bigoplus_{t=1}^{\operatorname{codim}(G_{k+1},Y)-1} A^{*-t}(G_{k+1}^{(k)})\Bigg) \\ &\oplus \Bigg(\bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu}\in M_{\mathcal{T}}^{(k+1)}} A^{*-\|\mu\|}(Y_k\mathcal{T})\Bigg) \\ &\oplus \Bigg(\bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu}\in M_{\mathcal{T}}^{(k+1)}} \bigoplus_{t=1}^{r_{k+1,\mathcal{T}}-1} A^{*-\|\mu\|-t}(Y_k(\{G_{k+1}\}\cup\mathcal{T}))\Bigg). \end{aligned} \tag{3.2}$$

This immediately gives the Chow group decomposition (3.1) for k. Indeed, any $\mathcal{G}$-nest contained in $\{G_{k+1}, G_{k+2}, \dots, G_N\}$ must be one of $\{G_{k+1}\}$, a $\mathcal{G}$-nest $\mathcal{T}$ contained in $\{G_{k+2}, G_{k+3}, \dots, G_N\}$, or $\{G_{k+1}\} \cup \mathcal{T}$. These possibilities correspond (respectively) to the second, third, and last summands in (3.2). (Notice that, by Proposition 2.7, $Y_k(\{G_{k+1}\} \cup \mathcal{T}) = \emptyset$ if $\{G_{k+1}\} \cup \mathcal{T}$ is not a $\mathcal{G}$-nest.)

Therefore, the Chow group decomposition (3.1) holds for all k; in particular, the case $k = 0$ gives the desired Chow group decomposition. For the proof of the Chow motive decomposition, we can either repeat the preceding proof nearly verbatim or, as the referee pointed out, simply observe that the Chow motive decomposition follows from the result on the Chow groups and Manin's identity principle. □

3.2. Proof of Theorem 3.2

First, we introduce some notation for a given $\mathcal{G}$-nest $\mathcal{T}$.

- Define $\mathcal{T}_k := \mathcal{T} \cap \{G_{k+1}, G_{k+2}, \dots, G_N\}$ for $0 \le k \le N$. Then we have a chain of $\mathcal{G}$-nests $\mathcal{T}_0 \supseteq \mathcal{T}_1 \supseteq \cdots \supseteq \mathcal{T}_N$, where $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{T}_N = \emptyset$.
- For $\underline{\mu} \in \mathcal{M}_{\mathcal{T}}$ and $1 \le i \le N$, define

$$\mu_i := \begin{cases} \mu_{G_i} & \text{if } G_i \in \mathcal{T}, \\ 0 & \text{otherwise.} \end{cases}$$

- j_{kl} and g_{kl} $(N \ge k > l \ge 0)$ are the natural morphisms, as seen in the following diagram.

$$\begin{tikzcd}
Y_N\mathcal{T}_0 \arrow[r, "j_{N0}"] \arrow[d, "g_{N0}"'] \arrow[rrrr, bend left, "j_{\mathcal{T}}"] \arrow[dddd, bend right, "g_{\mathcal{T}}"'] & Y_N\mathcal{T}_1 \arrow[r, "j_{N1}"] \arrow[d, "g_{N1}"'] & \cdots \arrow[r] & Y_N\mathcal{T}_{N-1} \arrow[r, "j_{N,N-1}"] \arrow[d, "g_{N,N-1}"'] & Y_N\mathcal{T}_N \arrow[dl, dotted, bend left, "\beta_N"] \\
Y_{N-1}\mathcal{T}_0 \arrow[r, "j_{N-1,0}"] \arrow[d, "g_{N-1,0}"'] & Y_{N-1}\mathcal{T}_1 \arrow[r, "j_{N-1,1}"] \arrow[d, "g_{N-1,1}"'] & \cdots \arrow[r] & Y_{N-1}\mathcal{T}_{N-1} \arrow[ur, dotted, bend left, "\alpha_N"] & \\
\cdots \arrow[d, "g_{20}"'] & \cdots \arrow[d, "g_{21}"'] & \cdots & & \\
Y_1\mathcal{T}_0 \arrow[r, "j_{10}"] \arrow[d, "g_{10}"'] & Y_1\mathcal{T}_1 \arrow[dl, dotted, bend left, "\beta_1"] & & & \\
Y_0\mathcal{T}_0 \arrow[ur, dotted, bend left, "\alpha_1"] & & & &
\end{tikzcd} \tag{3.3}$$

LEMMA 3.4. *Denote by $g\colon Y_k \to Y_{k-1}$ the natural morphism. Then, for $l \le k-1$,*

$$g^{-1}(G_l^{(k-1)}) = G_l^{(k)}.$$

Proof. First we claim that $G_l^{(k-1)} \not\supseteq G_k^{(k-1)}$. Otherwise, $G_l \supseteq G_k$ because they are the respective images of $G_l^{(k-1)}$ and $G_k^{(k-1)}$ under $Y_k \to Y_0$. But then, given our assumption that the order of $\{G_i\}$ is compatible with inclusion relations, we obtain the contradiction $l \ge k$.

Next, it is easy to see that $G_l^{(k-1)} \not\subseteq G_k^{(k-1)}$ because $G_l^{(k-1)}$ is a divisor. Now we know that the two nonsingular subvarieties $G_l^{(k-1)}$ and $G_k^{(k-1)}$ intersect cleanly and that neither one contains the other; hence they must intersect transversally. Then it is standard to show by calculation of local coordinates that the following isomorphism between ideal sheaves holds:

$$g^{-1}\mathcal{I}(G_l^{(k-1)}) \cdot \mathcal{O}_{Y_k} \cong \mathcal{I}(G_l^{(k)}).$$

The desired conclusion follows from this isomorphism. $\square$

LEMMA 3.5. *In* (3.3), *all squares are fiber squares. Moreover, for any* $N \geq k > l \geq 0$ *we have*:

(i) j_{kl} *is injective*;

(ii) *if* $G_k \in \mathcal{T}$, *then* g_{kl} *is the projection of a projective bundle with fiber isomorphic to a projective space of dimension* $r_{k,\mathcal{T}} - 1$;

(iii) *if* $G_k \notin \mathcal{T}$ *but* $\{G_k\} \cup \mathcal{T}_l$ *is a* $\mathcal{G}$-*nest, then* g_{kl} *is the blow-up of* $Y_{k-1}\mathcal{T}_l$ *along the center* $G_k^{(k-1)} \cap Y_{k-1}\mathcal{T}_l$;

(iv) *if* $\{G_k\} \cup \mathcal{T}_l$ *is not a* $\mathcal{G}$-*nest, then* g_{kl} *is an isomorphism.*

Proof. It is obvious that j_{kl} is injective.

Now we show that g_{kl} is the projection of a projective bundle if $G_k \in \mathcal{T}$. By Proposition 2.7, the variety $Y_k\mathcal{T}_k$ is the blow-up of $Y_{k-1}\mathcal{T}_k$ along the center $Y_{k-1}\mathcal{T}_{k-1}$, and the exceptional divisor is $Y_k\mathcal{T}_{k-1}$ (note that $Y_{k-1}\mathcal{T}_k \cap G_k^{(k-1)} = Y_{k-1}\mathcal{T}_{k-1}$ and $Y_k\mathcal{T}_k \cap G_k^{(k)} = Y_k\mathcal{T}_{k-1}$). Therefore, $g_{k,k-1}\colon Y_k\mathcal{T}_{k-1} \to Y_{k-1}\mathcal{T}_{k-1}$ is a projective bundle, and the dimension of a fiber is $r_{k,\mathcal{T}} - 1$. Next we show that, for any $l \leq k-1$, g_{kl} is the restriction of $g_{k,k-1}$ to a smaller base $Y_{k-1}\mathcal{T}_l$; this, in turn, will show that g_{kl} is also a projective bundle with fiber of the same dimension $r_{k,\mathcal{T}} - 1$. Fix k and use downward induction on l. By inductive assumption, $g_{k,l+1}$ is a restriction of $g_{k,k-1}$. Since

$$g_{k,l+1}^{-1}(G_{l+1}^{(k-1)} \cap Y_{k-1}\mathcal{T}_l) = G_{l+1}^{(k)} \cap Y_k\mathcal{T}_l$$

by Lemma 3.4, the restriction of the projective bundle $g_{k,l+1}$ to a smaller base space $Y_{k-1}\mathcal{T}_l = Y_{k-1}\mathcal{T}_{l+1} \cap G_{l+1}^{(k-1)}$ is exactly g_{kl}.

We now show that g_{kl} is birational if $G_k \notin \mathcal{T}$. This is again implied by Proposition 2.7. Observe that $G_k^{(k-1)}$ is minimal in

$$\mathcal{T}' := \{G_k^{(k-1)}\} \cup \{G^{(k-1)}\}_{G \in \mathcal{T}_l}.$$

If $\mathcal{T}'$ is a $\mathcal{G}^{(k-1)}$-nest, then $g_{kl}\colon Y_k\mathcal{T}_l \to Y_{k-1}\mathcal{T}_l$ is a blow-up along the center $G_k^{(k-1)} \cap Y_{k-1}\mathcal{T}_l$; otherwise, g_{kl} is an isomorphism. In both cases, g_{kl} is birational.

Finally, all squares in (3.3) are fiber squares because g_{kl} is a restriction of $g_{k,l+1}$ for all $l \leq k-2$. The proof is complete. □

The following lemma computes the composition of correspondences in certain diagrams. Thanks to the referee for suggesting a proof much simpler than the author's original.

LEMMA 3.6. *Let* W, U, V, X, Y, Z *be nonsingular quasi-projective varieties. Suppose the square in the following diagram is a fiber square.*

$$\begin{tikzcd}
W \arrow[r, "j_3"] \arrow[d, "g_3"'] \arrow[dr, phantom, "\square"] & U \arrow[r, "j_2"] \arrow[d, "g_2"] & X \arrow[dl, dotted, bend right, "\alpha_2"'] \\
V \arrow[r, "j_1"] \arrow[d, "g_1"'] & Y \arrow[ur, dotted, bend right, "\beta_2"'] \arrow[dl, dotted, bend right, "\alpha_1"'] & \\
Z \arrow[ur, dotted, bend right, "\beta_1"'] & &
\end{tikzcd}$$

Suppose also that $\dim W - \dim V = \dim U - \dim Y$ *and that* j_k, g_k $(1 \le k \le 3)$ *are proper. Take* $\gamma_1, \gamma_1' \in A(V)$ *and* $\gamma_2, \gamma_2' \in A(U)$ *and define correspondences*

$$\alpha_k = (j_k, g_k)_* \gamma_k \quad \textit{and} \quad \beta_k = (g_k, j_k)_* \gamma_k' \quad \textit{for } k = 1, 2.$$

Then

$$\alpha_1 \alpha_2 = (j_2 j_3, g_1 g_3)_* (j_3^* \gamma_2 \cdot g_3^* \gamma_1), \tag{3.4}$$

$$\beta_2 \beta_1 = (g_1 g_3, j_2 j_3)_* (g_3^* \gamma_1' \cdot j_3^* \gamma_2'). \tag{3.5}$$

Proof. By abuse of notation, for $\gamma \in A(V)$ we use the same γ to denote the correspondence $(\Delta_V)_*(\gamma) \in A(V \times V)$, where $\Delta_V : V \to V \times V$ is the diagonal embedding. For a map $j : U \to X$, we denote by j_* the correspondence Γ_j (i.e., the graph of j) and by j^* the correspondence Γ_j' (i.e., the transpose of Γ_j).

First observe that $\alpha_k = g_{k*} \circ \gamma \circ j_k^*$ for $k = 1, 2$. Indeed, by properties of correspondences (see e.g. [F, Prop. 16.1.1(c)]), we have $\Gamma_j \circ \gamma = (1_U \times j)_* \gamma$ and $\gamma \circ \Gamma_g' = (g \times 1_U)_* \gamma$, so

$$g_{k*} \circ \gamma_k \circ j_k^* = \Gamma_{g_k} \circ \gamma_k \circ \Gamma_{j_k}^* = (g_k \times j_k)_* \gamma_k = (g_k, j_k)_* \gamma_k = \alpha_k \quad \text{for } k = 1, 2.$$

Given this observation, (3.4) is equivalent to

$$g_{1*} \gamma_1 j_1^* g_{2*} \gamma_2 j_2^* = (g_1 g_3)_* (j_3^* \gamma_2 \cdot g_3^* \gamma_1)(j_2 j_3)^*;$$

hence it suffices to prove that

$$\gamma_1 j_1^* g_{2*} \gamma_2 = g_{3*} (j_3^* \gamma_2 \cdot g_3^* \gamma_1) j_3^*. \tag{3.6}$$

For any $u \in A(U)$, we have

$$\gamma_1 j_1^* g_{2*} \gamma_2(u) = \gamma_1 g_{3*} j_3^* \gamma_2(u) = g_{3*}(g_3^* \gamma_1 \cdot j_3^*(\gamma_2 u)) = g_{3*}(g_3^* \gamma_1 \cdot j_3^* \gamma_2) j_3^*(u),$$

where the first equality follows because $\dim W - \dim V = \dim U - \dim Y$ and the second because of the projection formula. Then we apply Manin's identity principle to obtain (3.6) and hence (3.4). The identity (3.5) can be obtained by transposing (3.4). $\square$

Now we state a simple lemma, omitting the proof.

LEMMA 3.7. *If* A, B_i, *and* C_{ij} *are motives such that*

(i) $\bigoplus_i \alpha_i : A \cong \bigoplus_i B_i$ *is an isomorphism with inverse* $\sum_i \beta_i$ *and*
(ii) $\bigoplus_j \alpha_{ij} : B_i \cong \bigoplus_j C_{ij}$ *is an isomorphism with inverse* $\sum_j \beta_{ij}$,

then the correspondence $\bigoplus_{i,j} \alpha_{ij} \circ \alpha_i$ *gives an isomorphism* $A \cong \bigoplus_{i,j} C_{ij}$ *with inverse* $\sum_{i,j} \beta_i \circ \beta_{ij}$.

For $G_k \in \mathcal{T}$, define $h_k \in A^1(Y_k \mathcal{T}_{k-1})$ to be first Chern class of the invertible sheaf $\mathcal{O}(1)$ of the projective bundle $g_{k,k-1}$. Define

$$\alpha_k = \begin{cases} (j_{k,k-1}, g_{k,k-1})_* 1 & \text{if } G_k \notin \mathcal{T}, \\ (j_{k,k-1}, g_{k,k-1})_* \left(\left\{ g_{k,k-1}^* c(N_k) \dfrac{1}{1 - h_k} \right\}_{r_k - 1 - \mu_k} \right) & \text{if } G_k \in \mathcal{T}, \end{cases}$$

where $N_k := N_{Y_{k-1} \mathcal{T}_{k-1}} Y_{k-1} \mathcal{T}_k$. Define

$$\beta_k = \begin{cases} (g_{k,k-1}, j_{k,k-1})_* 1 & \text{if } G_k \notin \mathcal{T}, \\ (g_{k,k-1}, j_{k,k-1})_* h_k^{\mu_k - 1} & \text{if } G_k \in \mathcal{T}. \end{cases}$$

By the blow-up formula of motives (Theorem A.2), the correspondence

$$a_k\colon h(Y_k\mathcal{T}_k)\Big(\sum_{i=k+1}^{N} \mu_i\Big) \to h(Y_{k-1}\mathcal{T}_{k-1})\Big(\sum_{i=k}^{N} \mu_i\Big)$$

expresses $h(Y_{k-1}\mathcal{T}_{k-1})\big(\sum_k^N \mu_i\big)$ as a direct summand of $h(Y_k\mathcal{T}_k)\big(\sum_{k+1}^N \mu_i\big)$ with right inverse β_k.

By Lemma 3.7, the correspondence

$$\alpha_{\mathcal{T},\underline{\mu}}\colon h(Y_{\mathcal{G}}) \to h(Y_0\mathcal{T})(\|\underline{\mu}\|)$$

that gives the direct summand $h(Y_0\mathcal{T})(\|\underline{\mu}\|)$ in Theorem 3.1 can be expressed as the composition $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_N$ with right inverse $\beta_N \circ \cdots \circ \beta_1$. Therefore we have the following result.

PROPOSITION 3.8. *Denote by $f_k\colon Y_N\mathcal{T}_0 \to Y_k\mathcal{T}_{k-1}$ the natural map in* (3.3) (*i.e.,* $g_{k+1,k-1} \circ \cdots \circ g_{N,k-1} \circ j_{N,k-2} \circ \cdots \circ j_{N0}$). *Then*

$$\alpha_1 \circ \cdots \circ \alpha_N = (j_{\mathcal{T}}, g_{\mathcal{T}})_* \prod_{G_k \in \mathcal{T}} \left\{ f_k^* g_{k,k-1}^* c(N_k) \frac{1}{1 - f_k^* h_k} \right\}_{r_k - 1 - \mu_k},$$

$$\beta_N \circ \cdots \circ \beta_1 = (g_{\mathcal{T}}, j_{\mathcal{T}})_* \prod_{G_k \in \mathcal{T}} f_k^* h_k^{\mu_k - 1}.$$

Proof. Combine Lemma 3.5 and Lemma 3.6 with the previous discussion. □

The following two standard facts about normal bundles of subvarieties are used in the proof of Theorem 3.2.

FACT 3.9. *Let Z be a nonsingular variety. Let Y and W be nonsingular proper subvarieties of Z and assume that Y intersects transversally with W. Let $\pi\colon \widetilde{Z} \to Z$ be the blow-up of Z along W, and let $\widetilde{Y}$ be the strict transform of Y. Then*

$$N_{\widetilde{Y}}\widetilde{Z} \simeq \pi^* N_Y Z.$$

FACT 3.10. *Let $W \subsetneq Y \subsetneq Z$ be nonsingular varieties and let $\pi\colon \widetilde{Z} \to Z$ be the blow-up of Z along W. Denote by $\widetilde{Y}$ the strict transform of Y, and denote by E the exceptional divisor on $\widetilde{Y}$. Then*

$$N_{\widetilde{Y}}\widetilde{Z} \simeq \pi^* N_Y Z \otimes \mathcal{O}(-E).$$

Proof of Fact 3.9 and Fact 3.10. Use local coordinates or see [F]. □

Proof of Theorem 3.2. To deduce Theorem 3.2 from Proposition 3.8, we proceed in three steps.

Step 1. Show $f_k^* h_k = -D_{G_k}|_{Y_N\mathcal{T}_0}$. Recall that, for $G_k \in \mathcal{T}$, h_k is first Chern class of the invertible sheaf $\mathcal{O}(1)$ of the projective bundle $g_{k,k-1}$.

Consider the following diagram (not necessarily a fiber square), where π and j are the natural morphisms.

$$\begin{tikzcd} Y_N\mathcal{T}_0 \arrow[r, "j_{\mathcal{T}}"] \arrow[d, "f_k"] & Y_N \arrow[d, "\pi"] \\ Y_k\mathcal{T}_{k-1} \arrow[r, "j"] & Y_k\,. \end{tikzcd}$$

By Proposition 2.7, we have that $Y_k\mathcal{T}_{k-1}$ is the exceptional divisor of the blow-up $g_{k,k-1}\colon Y_k\mathcal{T}_{k-1} \to Y_{k-1}\mathcal{T}_{k-1}$, so $h_k = -j^*_{k,k-1}[Y_k\mathcal{T}_{k-1}]$. Since $Y_k\mathcal{T}_{k-1}$ is the transversal intersection $Y_k\mathcal{T}_k \cap G_k^{(k)}$, it follows that $h_k = -j^*[G_k^{(k)}]$. Then

$$f_k^* h_k = -f_k^* j^*[G_k^{(k)}] = -j_{\mathcal{T}}^*\pi^*[G_k^{(k)}] = -j_{\mathcal{T}}^* D_{G_k} = -D_{G_k}|_{Y_N\mathcal{T}_0},$$

where the third equality can be proved by successively applying Lemma 3.4.

Step 2. Let $0 \le s < k \le N$. Let $g_{sk}\colon Y_s\mathcal{T}_k \to Y_{s-1}\mathcal{T}_k$ denote the natural map induced from Y_s to Y_{s-1}. We make the following claim. If $G_k \in \mathcal{T}$ (and hence $\mathcal{T}_{k-1} = \mathcal{T}_k \cup \{G_k\}$), then the normal bundle $N_{Y_s\mathcal{T}_{k-1}}Y_s\mathcal{T}_k$ is isomorphic to

$$\begin{cases} g^*_{s,k-1}(N_{Y_{s-1}\mathcal{T}_{k-1}}Y_{s-1}\mathcal{T}_k) \otimes (-[G_s^{(s)}]|_{Y_s\mathcal{T}_{k-1}}) & \text{if } (**) \text{ holds,} \\ g^*_{s,k-1}(N_{Y_{s-1}\mathcal{T}_{k-1}}Y_{s-1}\mathcal{T}_k) & \text{otherwise,} \end{cases}$$

where condition $(**)$ is: $G_s \subsetneq G_k$ and $\mathcal{T}_k \cup \{G_s\}$ is a $\mathcal{G}$-nest. For the proof, we discuss three cases.

Case (i): condition $(**)$ *holds.* This is a direct conclusion of Fact 3.10. Indeed, to apply Fact 3.10 we need to show that

$$Y_{s-1}\mathcal{T}_k \cap G_s^{(s-1)} \subsetneq Y_{s-1}\mathcal{T}_k \cap G_k^{(s-1)} \subsetneq Y_{s-1}\mathcal{T}_k.$$

The second inequality is obvious. The first inclusion is strict for the following reason: $G_s^{(s-1)}$ is a $\mathcal{G}^{(s-1)}$-factor of $Y_{s-1}\mathcal{T}_k \cap G_s^{(s-1)}$; therefore, $G_k^{(s-1)}$ is not a $\mathcal{G}^{(s-1)}$-factor because it strictly contains $G_s^{(s-1)}$. On the other hand, $G_k^{(s-1)}$ is a $\mathcal{G}^{(s-1)}$-factor of $Y_{s-1}\mathcal{T}_k \cap G_k^{(s-1)}$ and so the first inclusion is strict.

Case (ii): $\mathcal{T}_k \cup \{G_s\}$ *is not* $\mathcal{G}$*-nested.* In this case, $G_s^{(s-1)} \cap Y_{s-1}\mathcal{T}_k = \emptyset$ by Proposition 2.7. Hence no twisting is needed for the normal bundle.

Case (iii): $\mathcal{T}_k \cup \{G_s\}$ *is* $\mathcal{G}$*-nested but* G_s *is not strictly contained in* G_k. If $\mathcal{T}_{k-1} \cup \{G_s\}$ is not a $\mathcal{G}$-nest, then $G_s^{(s-1)} \cap Y_{s-1}\mathcal{T}_{k-1} = \emptyset$ by Proposition 2.7. Hence blowing up along $G_s^{(s-1)}$ will not affect the normal bundle of $Y_{s-1}\mathcal{T}_{k-1}$, so no twisting is needed. Otherwise, assume $\mathcal{T}_{k-1} \cup \{G_s\}$ is a $\mathcal{G}$-nest. Both G_s and G_k are minimal in the $\mathcal{G}$-nest $\mathcal{T}_{k-1} \cup \{G_s\}$. Then $G_s^{(s-1)}$ and $G_k^{(s-1)}$ are minimal in a nest, and neither one contains the other; hence they intersect transversally by the definition of nest. Thus $Y_{s-1}\mathcal{T}_k \cap G_k^{(s-1)}$ and $Y_{s-1}\mathcal{T}_k \cap G_s^{(s-1)}$, regarded as subvarieties of the ambient space $Y_{s-1}\mathcal{T}_k$, intersect transversally. Therefore Fact 3.9 applies, and no twisting is needed for the normal bundle.

Step 3. Apply the result of Step 2 successively for $s = 1, 2, \ldots, k-1$. The normal bundle $N_{Y_{k-1}\mathcal{T}_{k-1}}Y_{k-1}\mathcal{T}_k$ is isomorphic to

$$(g_{k-1,k-1}^* \cdots g_{1,k-1}^*(N_{Y_0\mathcal{T}_{k-1}} Y_0\mathcal{T}_k)) \otimes \Big(-\sum_{(**)} [G_s^{(k-1)}]|_{Y_{k-1}\mathcal{T}_{k-1}}\Big),$$

where the sum is over all s that satisfy condition $(**)$. (Here we have used Lemma 3.4.) Therefore,

$$\begin{aligned} &f_k^* g_{k,k-1}^* c(N_{Y_{k-1}\mathcal{T}_{k-1}} Y_{k-1}\mathcal{T}_k) \\ &\qquad = c\Big(g_{\mathcal{T}}^*(N_{Y_0\mathcal{T}_{k-1}} Y_0\mathcal{T}_k|_{Y_0\mathcal{T}}) \otimes \mathcal{O}\Big(-\sum_{(**)} [D_{G_s}]|_{Y_N\mathcal{T}_{k-1}}\Big)\Big). \end{aligned}$$

Notice that

$$(N_{Y_0\mathcal{T}_{k-1}} Y_0\mathcal{T}_k)|_{Y_0\mathcal{T}} = N_{G_k}\Big(\bigcap_{G_k \subsetneq G \in \mathcal{T}} G\Big)\Big|_{Y_0\mathcal{T}},$$

which we denote by N_{G_k}. The proof is as follows. Suppose $T_1, \ldots, T_m, T_{m+1}, \ldots, T_r$ are the minimal elements of the nest $\mathcal{T}_k$, where the first m elements contain G_k. Then the minimal elements of the nest $\mathcal{T}_{k-1}$ are $G_k, T_{m+1}, \ldots, T_r$. By the definition of nest, $Y_0\mathcal{T}_k$ is the transversal intersection $T_1 \cap \cdots \cap T_m \cap T_{m+1} \cap \cdots \cap T_r$ and $Y_0\mathcal{T}_{k-1}$ is the transversal intersection $G_k \cap T_{m+1} \cap \cdots \cap T_r$. Therefore,

$$N_{Y_0\mathcal{T}_{k-1}} Y_0\mathcal{T}_k = N_{G_k}(T_1 \cap \cdots \cap T_m)|_{Y_0\mathcal{T}_{k-1}}.$$

Since $T_1 \cap \cdots \cap T_m = \bigcap_{G_k \subsetneq G \in \mathcal{T}} G$, the conclusion follows immediately.

Now substituting everything into Corollary 3.8, we have:

$$\begin{aligned} &\alpha_1 \circ \cdots \circ \alpha_N \\ &\quad = (j_{\mathcal{T}}, g_{\mathcal{T}})_* \prod_{G_k \in \mathcal{T}} \Big\{ c(g_{\mathcal{T}}^*(N_{G_k}) \otimes \mathcal{O}\Big(-\sum_{(**)} [D_{G_s}]|_{Y_N\mathcal{T}}\Big) \frac{1}{1 + D_{G_k}|_{Y_N\mathcal{T}}} \Big\}_{r_k - 1 - \mu_k}, \\ &\beta_N \circ \cdots \circ \beta_1 = (g_{\mathcal{T}}, j_{\mathcal{T}})_* \prod_{G_k \in \mathcal{T}} (-D_{G_k})^{\mu_k - 1}|_{Y_N\mathcal{T}}. \end{aligned}$$

Finally, we show that $(**)$ can be replaced by the following condition:

$$G_s \subsetneq G_k \text{ and } \mathcal{T} \cup \{G_s\} \text{ is a } \mathcal{G}\text{-nest}. \tag{$\bigstar$}$$

Indeed, $(\bigstar)$ is stronger than $(**)$. However, for those G_s satisfying $(**)$ but not $(\bigstar)$, the divisor $[D_{G_s}]|_{Y_N\mathcal{T}}$ would be trivial because $D_{G_s} \cap Y_N\mathcal{T} = \emptyset$. Therefore, replacing $(**)$ by $(\bigstar)$ will not affect the resulting correspondence.

The proof is now complete. □

We write the following direct conclusion from Step 3 for later use.

COROLLARY 3.11. *Denote* $\pi : G_{k+1}^{(k)} \to G_{k+1}$. *Then*

$$c(N_{G_{k+1}^{(k)}} Y_k) = c\Big(\pi^* N_{(G_{k+1})} Y \otimes \sum_{G_{k+1} \supsetneq G \in \mathcal{T}} (-[D_G])|_{G_{k+1}^{(k)}}\Big).$$

Proof. Apply Step 3 to the nest $\mathcal{T} = \{G_{k+1}\}$. □

4. Fulton–MacPherson Configuration Spaces

Fix a nonsingular variety X of dimension d. The configuration space of n distinct ordered points on X, denoted by $F(X,n)$, can be naturally identified with an open subvariety of the Cartesian product X^n:

$$F(X,n) := \{(x_1, x_2, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}.$$

In their celebrated paper [FM], Fulton and MacPherson discovered an interesting compactification $X[n]$ of the configuration space $F(X,n)$. The compactification is obtained by replacing the diagonals of X^n by a simple normal crossing divisor. It has many attractive properties—for example, the geometry when n points collide (i.e., the degenerate configuration) can be explicitly described using $X[n]$. Also, $X[n]$ is closely related to the well-known compactification $\overline{\mathcal{M}}_{0,n}$ of the moduli space of stable rational curves with n marked points. The reader is referred to the beautiful paper [FM] for the original construction and various applications of the Fulton–MacPherson configuration space.

The Fulton–MacPherson configuration space $X[n]$ can be realized as a wonderful compactification of an arrangement of subvarieties by taking Y to be X^n and $\mathcal{G}$ to be the collection of all diagonals of X^n; hence, the induced arrangement is the set of intersections of diagonals, which are called *polydiagonals* (see [L]).

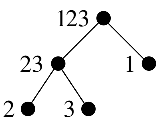
4.1. Main Theorems

First we fix some additional notation.

(i) Denote $[n] := \{1, 2, \ldots, n\}$. We call two subsets $I, J \subseteq [n]$ *overlapped* if $I \cap J$ is a nonempty proper subset of I and J. For a set $\mathcal{S}$ of subsets of $[n]$, we say that I is *compatible* with $\mathcal{S}$ (denoted by $I \sim \mathcal{S}$) if I does not overlap any element in $\mathcal{S}$.

A *nest* $\mathcal{S}$ is a set of subsets of $[n]$ such that any two elements $I \neq J \in \mathcal{S}$ are not overlapped and all singletons $\{1\}, \ldots, \{n\}$ are in $\mathcal{S}$. Notice that the nest defined here, unlike the one defined in [FM], is allowed to contain singletons. Given a nest $\mathcal{S}$, define $\mathcal{S}^\circ = \mathcal{S} \setminus \{\{1\}, \ldots, \{n\}\}$. In the description of nests by forests to follow, $\mathcal{S}^\circ$ corresponds to the forest $\mathcal{S}$ cutting of all leaves.

A nest $\mathcal{S}$ naturally corresponds to a forest (i.e., to a not necessarily connected tree) each node of which is labeled by an element in $\mathcal{S}$. For example, the following forest corresponds to a nest $\mathcal{S} = \{1, 2, 3, 23, 123\}$.



Denote by $c(\mathcal{S})$ the number of connected components of the forest—that is, the number of maximal elements of $\mathcal{S}$. Denote by $c_I(\mathcal{S})$ (or c_I if no ambiguity will arise) the number of maximal elements of the set $\{J \in \mathcal{S} \mid J \subsetneq I\}$ (i.e., the number of "children" of the node I). In the preceding example, $c(\mathcal{S}) = 1$ and $c_{123} = c_{23} = 2$.

(ii) For a subset $I \subseteq [n]$ consisting of at least two elements, define the diagonal

$$\Delta_I := \{(x_1, \dots, x_n) \in X^n : x_i = x_j \text{ if } i, j \in I\}.$$

It is shown in [FM] that a complement of $F(x, n)$ in the Fulton–MacPherson compactification $X[n]$ is a union of normal crossing nonsingular divisors D_I, indexed by subsets $I \subseteq [n]$ with at least two elements. More precisely, D_I is the dominant transform $\widetilde{\Delta}_I$ under the natural morphism $X[n] \to X^n$.

For every nest $\mathcal{S}$, $X(\mathcal{S}) := \bigcap_{I \in \mathcal{S}} D_I$ is a nonsingular subvariety of $X[n]$. Define $j_{\mathcal{S}} \colon X(\mathcal{S}) \hookrightarrow X[n]$ to be the natural inclusion.

Define $\Delta_{\mathcal{S}} := \bigcap_{I \in \mathcal{S}} \Delta_I$. Define $g_{\mathcal{S}} \colon X(\mathcal{S}) \to \Delta_{\mathcal{S}}$ to be the restriction of the morphism $\pi \colon X[n] \to X^n$ to the subvariety $X(\mathcal{S})$.

(iii) Let $p_I \colon X[n] \to X$ be the composition of $\pi \colon X[n] \to X^n$ with the projection $X^n \to X$ to the ith factor for an arbitrary $i \in I$. The choice of $i \in I$ is not essential: indeed, the only place we need p_I is in the formulation of $\alpha_{\mathcal{S},\mu}$ that follows, where we need the composition $j_{\mathcal{S}}^* p_I^*$. By the diagram

$$\begin{array}{ccc} X(\mathcal{S}) & \xrightarrow{j_{\mathcal{S}}} & X[n] \\ \Big\downarrow{\scriptstyle g_{\mathcal{S}}} & & \Big\downarrow{\scriptstyle p_i} \\ \Delta_{\mathcal{S}} & \xrightarrow{q_i} & X \end{array}$$

with $i \in I$, we have $j_{\mathcal{S}}^* p_i^* = g_{\mathcal{S}}^* q_i^*$. But q_i is independent of the choice of $i \in I$ since $\Delta_{\mathcal{S}} \subseteq \Delta_I$, so $j_{\mathcal{S}}^* p_I^*$ is independent of the choice of $i \in I$ for p_I.

(iv) For a nest $\mathcal{S} \neq \{\{1\}, \dots, \{n\}\}$ (i.e., $\mathcal{S}^\circ \neq \emptyset$), define

$$M_{\mathcal{S}} := \{\underline{\mu} = \{\mu_I\}_{I \in \mathcal{S}^\circ} : 1 \le \mu_I \le d(c_I - 1) - 1,\ \mu_I \in \mathbb{Z}\}$$

(recall that $d = \dim X$ and that $c_I = c_I(\mathcal{S})$ is defined in (i)) and define

$$\|\underline{\mu}\| := \sum_{I \in \mathcal{S}^\circ} \mu_I \quad \forall \underline{\mu} \in M_{\mathcal{S}}.$$

For $\mathcal{S} = \{\{1\}, \dots, \{n\}\}$, assume that $M_{\mathcal{S}} = \{\underline{\mu}\}$ with $\|\underline{\mu}\| = 0$.

We will show in the proof of Theorem 4.1 that $M_{\mathcal{S}}$ is the special case of $M_{\mathcal{T}}$ defined in Section 3 where Y is X^n, $\mathcal{G}$ is the set of diagonals of X^n, and $\mathcal{T}$ is the set of $\mathcal{G}$-nests.

Define the function

$$\zeta(x) := \sum_{i=0}^{d} (1 + x)^{d-i} c_i(T_X).$$

Also define $\alpha_{\mathcal{S},\underline{\mu}} \in \operatorname{Corr}^{-\|\underline{\mu}\|}(X[n], \Delta_{\mathcal{S}})$, $\beta_{\mathcal{S},\underline{\mu}} \in \operatorname{Corr}^{\|\underline{\mu}\|}(\Delta_{\mathcal{S}}, X[n])$, and $p_{\mathcal{S},\underline{\mu}} \in \operatorname{Corr}^0(X[n], X[n])$ as follows:

$$\alpha_{\mathcal{S},\underline{\mu}} = (j_{\mathcal{S}}, g_{\mathcal{S}})_* j_{\mathcal{S}}^* \Bigg(\prod_{I \in \mathcal{S}^\circ} \Bigg\{ -p_I^* \zeta \Big(-\sum_{\substack{J \sim \mathcal{S} \\ J \supsetneq I}} D_J \Big)^{c_I - 1} \frac{1}{1 + D_I} \Bigg\}_{d(c_I - 1) - 1 - \mu_I} \Bigg);$$

$$\beta_{\mathcal{S},\underline{\mu}} = (g_{\mathcal{S}}, j_{\mathcal{S}})_* j_{\mathcal{S}}^* \Bigg(\prod_{I \in \mathcal{S}^\circ} D_I^{\mu_I - 1} \Bigg);$$

$$p_{\mathcal{S},\underline{\mu}} = \beta_{\mathcal{S},\underline{\mu}} \circ \alpha_{\mathcal{S},\underline{\mu}}.$$

In these definitions of $\alpha_{\mathcal{S},\underline{\mu}}$ and $\beta_{\mathcal{S},\underline{\mu}}$, the products are set to be $1_{X(\mathcal{S})} \in A^0(X(\mathcal{S}))$ if $\mathcal{S}^\circ = \emptyset$.

The following are the main theorems on the Chow groups and Chow motives of Fulton–MacPherson configuration spaces.

THEOREM 4.1. *Let X be a nonsingular quasi-projective variety. Then there is an isomorphism of Chow groups,*

$$A^*(X[n]) = \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} A^{*-\|\underline{\mu}\|}(X^{c(\mathcal{S})}),$$

where $\mathcal{S}$ runs through all nests of $[n]$.

THEOREM 4.2. *Let X be a nonsingular projective variety. Then there is a canonical isomorphism of Chow motives*

$$\bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} \alpha_{\mathcal{S},\underline{\mu}} \colon h(X[n]) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} h(\Delta_{\mathcal{S}})(\|\underline{\mu}\|)$$

with the inverse $\sum_{\mathcal{S}} \sum_{\underline{\mu} \in \mathcal{S}} \beta_{\mathcal{S},\underline{\mu}}$. Equivalently, we have

$$h(X[n]) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} h(X^{c(\mathcal{S})})(\|\underline{\mu}\|).$$

REMARK. Observe that the two sets of correspondences $\{\alpha_{\mathcal{S},\underline{\mu}}\}$ and $\{\beta_{\mathcal{S},\underline{\mu}}\}$ are $\mathfrak{S}_n$-symmetric in the sense that the following diagram commutes for any $\sigma \in \mathfrak{S}_n$.

$$\begin{tikzcd}
h(X[n]) \arrow[r, "\alpha_{\mathcal{S},\underline{\mu}}"] \arrow[d, "\sigma"] & h(\Delta_{\mathcal{S}})(\|\underline{\mu}\|) \arrow[r, "\beta_{\mathcal{S},\underline{\mu}}"] \arrow[d, "\sigma"] & h(X[n]) \arrow[d, "\sigma"] \\
h(X[n]) \arrow[r, "\alpha_{\sigma(\mathcal{S},\underline{\mu})}"] & h(\Delta_{\mathcal{S}})(\|\underline{\mu}\|) \arrow[r, "\beta_{\sigma(\mathcal{S},\underline{\mu})}"] & h(X[n])
\end{tikzcd}$$

Proof of Theorem 4.1. Apply Theorem 3.1 with the ambient space $Y = X^n$ and the building set

$$\mathcal{G} = \{\Delta_I\}_{I \subseteq [n], |I| \geq 2}.$$

First notice that a nest $\mathcal{S}$ of $[n]$ gives a $\mathcal{G}$-nest $\mathcal{T} = \{\Delta_I\}_{I \in \mathcal{S}^\circ}$. Moreover, the inverse is also true: a $\mathcal{G}$-nest will give a nest of $[n]$. Indeed, given a partition $\Pi = (I_1, \dots, I_t)$ of $[n]$, a $\mathcal{G}$-factor of Δ_Π by definition is a minimal element in $\{G \in \mathcal{G} : G \supseteq \Delta_\Pi\}$. So $\{\Delta_{I_1}, \dots, \Delta_{I_t}\}$ are all the $\mathcal{G}$-factors of Δ_Π. By the definition of $\mathcal{G}$-nest, $\mathcal{T}$ is induced from a flag of strata

$$\Delta_{\Pi_1} \supseteq \Delta_{\Pi_2} \supseteq \cdots \supseteq \Delta_{\Pi_t}.$$

Then

$$\Pi_1 \geq \Pi_2 \geq \cdots \geq \Pi_k.$$

(Here $\Pi \geq \Pi'$ means that Π is a finer partition than Π'; e.g., $(12, 3, 4) \geq (123, 4)$.) The nest $\mathcal{T}$ is induced by "taking the union of all factors of each Δ_Π", which corresponds to "taking all Is that appear in any of the partitions Π_i". Since the partitions are totally ordered, the set of Is forms a nest of $[n]$.

Next we prove that the range of $\underline{\mu}$ is as stated. Theorem 3.1 asserts that

$$1 \leq \mu_G \leq r_G - 1.$$

Now $G = \Delta_I$ is a diagonal, so by definition we have

$$\begin{aligned} r_G &:= \dim\Big(\bigcap_{G \subsetneq T \in \mathcal{T}} T\Big) - \dim G \\ &= \dim\Big(\bigcap_{I \supsetneq I' \in \mathcal{S}} \Delta_{I'}\Big) - \dim \Delta_I \\ &= d(c_I - 1). \end{aligned}$$

Finally, observe that

$$Y_0\mathcal{T} = \bigcap_{G \in \mathcal{T}} G = \bigcap_{I \in \mathcal{S}} \Delta_I = \Delta_{\mathcal{S}} \cong X^{c(\mathcal{S})}.$$

Therefore, the expected conclusion is implied by Theorem 3.1. □

Proof of Theorem 4.2. The statement of the motive decomposition is proved exactly as in the proof of Theorem 4.1.

The correspondences are induced from Theorem 3.2. The improvement of this theorem over Theorem 3.2 is that here we can say more about the Chern classes appearing in the correspondence $\alpha_{\mathcal{S},\underline{\mu}}$.

First, given $G = \Delta_I$, let $\Pi = (I_1, \ldots, I_{c_I})$ be the partition containing all children of I in $\mathcal{S}$. We compute the normal bundle $N_G := N_{\Delta_I}\Delta_\Pi$. Without loss of generality, assume that $I = (1, 2, \ldots, m)$ where $m \leq n$.

Denote by $p_i \colon \Delta_I \to X$ and $q_i \colon \Delta_\Pi \to X$ the projections induced from the projection of X^n to the ith factor. For each $1 \leq i \leq c_I$, pick an $a_i \in I_i$. Then

$$\begin{aligned} T_{\Delta_I} &= p_1^* T_X \oplus p_{m+1}^* T_X \oplus \cdots \oplus p_n^* T_X, \\ T_{\Delta_\Pi} &= q_{a_1}^* T_X \oplus \cdots \oplus q_{a_{c_I}}^* T_X \oplus q_{m+1}^* T_X \oplus \cdots \oplus q_n^* T_X, \\ T_{\Delta_\Pi}|_{\Delta_I} &= p_1^* T_X \oplus \cdots \oplus p_1^* T_X \oplus q_{m+1}^* T_X \oplus \cdots \oplus q_n^* T_X. \end{aligned}$$

Therefore, $c(N_G) = p_1^* c(T_X)^{c_I - 1}$.

To compute the Chern classes of N_G twisted by a line bundle L, we use the Chern root technique. For any vector bundle N on X, define the Chern polynomial as

$$c_y(N) := c_0(N) + c_1(N)y + c_2(N)y^2 + \cdots.$$

Define $x = c_1(L)$. Recall that the rank of N_G is $r_G = d(c_I - 1)$. Now

$$\begin{aligned} c(N_G \otimes L) &= c_{r_G}(N_G) + c_{r_G - 1}(N_G)(1 + x) + \cdots + c_0(N_G)(1 + x)^{r_G} \\ &= (x + 1)^{r_G} c_{1/(x+1)}(N_G) \\ &= (x + 1)^{d(c_I - 1)} p_1^* c_{1/(x+1)}(T_X)^{c_I - 1} \\ &= p_1^*[(x + 1)^d c_{1/(x+1)}(T_X)]^{c_I - 1} = p_1^* \zeta(x)^{c_I - 1}. \end{aligned}$$

Finally, by restricting to $\Delta_{\mathcal{S}}$ and pulling back to $X(\mathcal{S})$ we get the expected formula for correspondences $\alpha_{\mathcal{S},\underline{\mu}}$. □

4.2. *A Formula for the Generating Function of Chow Groups and Chow Motive of $X[n]$*

In this section, we express decompositions of the Chow groups (Theorem 4.1) and the Chow motive (Theorem 4.2) in terms of exponential generating functions.

Define $[x^i]$ to be the function that picks up the coefficient of x^i from a power series. Define $\left[\frac{x^i t^n}{n!}\right]$ to be the function that picks up the coefficient of $\frac{x^i t^n}{n!}$ from a power series with two variables x and t:

$$\left[\frac{x^i t^n}{n!}\right] \sum_{j,m} a_{jm} \frac{x^j t^m}{m!} := a_{in}.$$

The main theorem of this section is as follows.

THEOREM 4.3. *Define $f_i(x)$ to be the polynomials whose exponential generating function $N(x,t) = \sum_{i\geq 1} f_i(x) \frac{t^i}{i!}$ satisfies the identity*

$$(1-x)x^d t + (1 - x^{d+1}) = \exp(x^d N) - x^{d+1} \exp(N).$$

Then, for a nonsingular d-dimensional quasi-projective variety X,

$$A^*(X[n]) = \bigoplus_{\substack{1\leq k\leq n \\ i\geq 0}} A^{*-i}(X^k)^{\oplus [x^i t^n/n!](N^k/k!)}.$$

Moreover, if X is projective then we have the motive decomposition

$$\begin{aligned} h(X[n]) &= \bigoplus_{\Pi=(I_1,\dots,I_k)} (h(\Delta_\Pi)(i))^{\oplus [x^i](f_{|I_1|}(x)\cdots f_{|I_k|}(x))} \\ &= \bigoplus_{\substack{1\leq k\leq n \\ i\geq 0}} (h(X^k)(i))^{\oplus [x^i t^n/n!](N^k/k!)}, \end{aligned}$$

where Π is a partition of $[n]$.

REMARK. One can write down by hand the first several terms of N. Define $\sigma_j = \sum_{i=1}^{dj-1} x^i$ (when $d = 1$, define $\sigma_1 = 0$). Then

$$\begin{aligned} N = t + \sigma_1 \frac{t^2}{2!} + (\sigma_2 + 3\sigma_1^2)\frac{t^3}{3!} + (\sigma_3 + 10\sigma_1\sigma_2 + 15\sigma_1^3)\frac{t^4}{4!} \\ + (\sigma_4 + 15\sigma_1\sigma_3 + 10\sigma_2^2 + 105\sigma_1^2\sigma_2 + 105\sigma_1^4)\frac{t^5}{5!} + \cdots. \end{aligned}$$

Proof of Theorem 4.3. We prove only the statement for motives, since the statement for Chow groups can be proved by exactly the same method.

By Theorem 4.2, for any given i and k we want to count how many possible $\mathcal{S}$ and $\underline{\mu} \in \mathcal{S}$ satisfy $c(\mathcal{S}) = k$ and $\|\underline{\mu}\| = i$. First, consider the case when $c(\mathcal{S}) = 1$ (i.e., when $\mathcal{S}$ is a connected forest).

Define

$$f_n(x) := \sum_{\mathcal{S}:c(\mathcal{S})=1} \sum_{\underline{\mu}\in M_{\mathcal{S}}} x^{\|\underline{\mu}\|},$$

and define $f_1(x) = 1$.

For a nest $\mathcal{S}$ of $[n]$ with $c(\mathcal{S}) = 1$, we have

$$\sum_{\underline{\mu} \in M_{\mathcal{S}}} x^{\|\underline{\mu}\|} = \prod_{I \in \mathcal{S}^{\circ}} \sigma_{(c_I - 1)};$$

that is, I goes through all nonleaves of $\mathcal{S}$ (if $n = 1$, then the sum is set to be 1). Since the children of the root of $\mathcal{S}$ correspond to a partition $\{I_1, \dots, I_k\}$ of $[n]$, we have following formula for $n \geq 2$:

$$f_n(x) = \sum_{\Pi} f_{|I_1|} f_{|I_2|} \cdots f_{|I_k|} \sigma_{k-1},$$

where $\Pi = \{I_1, \dots, I_k\}$ is (as before) a partition of $[n]$, $\sigma_k = \sum_{i=1}^{dk-1} x^i$ for $k > 0$, and $\sigma_0 = 0$. The equality does not hold for $n = 1$ when $f_1(x) = 1$ and the right side is 0, so one can define

$$\tilde{f}_n(x) = \begin{cases} f_n(x) & \text{if } n > 1, \\ 0 & \text{if } n = 1. \end{cases}$$

Then the following holds for any $n \geq 1$:

$$\tilde{f}_n(x) = \sum_{\Pi} f_{|I_1|} f_{|I_2|} \cdots f_{|I_k|} \sigma_{k-1}.$$

Recall the compositional formula of exponential generating functions (cf. [S, Thm. 5.1.4]), which asserts that if an equation like the previous one holds then

$$E_{\tilde{f}}(t) = E_{\sigma}(E_f(t)),$$

where

$$\begin{aligned} E_{\tilde{f}}(t) &= 1 + \tilde{f}_1 t + \tilde{f}_2 t^2/2! + \tilde{f}_3 t^3/3! + \cdots, \\ E_{\sigma}(t) &= 1 + \sigma_0 t + \sigma_1 t^2/2! + \sigma_2 t^3/3! + \cdots, \\ E_f(t) &= f_1 t + f_2 t^2/2! + f_3 t^3/3! + \cdots. \end{aligned}$$

By the definition of $\tilde{f}$, we have $E_{\tilde{f}} = E_f - t + 1$. If we denote $N = E_f$ then

$$N - t + 1 = E_g(N),$$

and standard computation shows that

$$E_g(N) = 1 + N + \frac{1}{x-1}\left[\frac{1}{x^d}(e^{x^d N} - 1) - xe^N + x\right].$$

Therefore,

$$(1-x)x^d t + (1 - x^{d+1}) = \exp(x^d N) - x^{d+1}\exp(N).$$

Now consider the case when $c(\mathcal{S})$ is not necessarily 1—that is, the forest $\mathcal{S}$ is not necessarily connected. For a partition $\Pi = \{I_1, \dots, I_k\}$ of $[n]$, the number of times that $h(\Delta_{\Pi})(i)$ appears in the decomposition of $h(X[n])$ is equal to $[x^k](f_{|I_1|}(x) \cdots f_{|I_k|}(x))$, the coefficient of x^k in the product. Denote by $a_{k,i}$ the sum of these numbers for all partitions with k blocks. Then $a_{k,i}$ is the number of times that $h(X^k)(i)$ appears in the decomposition of $H(X[n])$.

Define

$$F_n(y) = \sum_{\Pi} f_{|I_1|} f_{|I_2|} \cdots f_{|I_k|} y^k;$$

then the coefficient $[y^k]F_n(y) = \sum a_{k,i} x^i$. Using the compositional formula again, we have

$$F_n = \left[\frac{t^n}{n!}\right] \exp(yN).$$

Therefore,

$$\begin{aligned} [y^k]F_n(y) &= [y^k]\left[\frac{t^n}{n!}\right] \exp(yN) \\ &= \left[\frac{t^n}{n!}\right][y^k] \exp(yN) \\ &= \left[\frac{t^n}{n!}\right]\frac{N^k}{k!}. \end{aligned}$$

This yields the formula for the decomposition of the Chow motive $h(X[n])$. □

4.3. *Description of $X[n]$ for Small n*

In this section we explain the previous theorems (Theorems 4.1, 4.2, and 4.3) about Fulton–MacPherson configuration space $X[n]$ for $n = 2, 3, 4$.

For unification of expression, assume $d > 1$ in the following examples. (The case $d = 1$ is simpler but the expression would need to be modified.)

EXAMPLE FOR $n = 2$. The morphism $\pi : X[2] \to X^2$ is a blow-up along the diagonal Δ_{12}. Theorem 4.3 asserts that

$$h(X[2]) \cong h(X^2) \oplus \bigoplus_{i=1}^{d-1} h(\Delta_{12})(i) \cong h(X^2) \oplus \bigoplus_{i=1}^{d-1} h(X)(i). \tag{4.1}$$

There are two possible nests, $\mathcal{S} = \{1, 2\}$ and $\mathcal{S} = \{1, 2, 12\}$. Theorem 4.2 asserts the following.

For the first nest, $M_{\mathcal{S}}$ contains only one element $\underline{\mu}$ with $\|\underline{\mu}\| = 0$. Therefore $\alpha = \Gamma_\pi$, $\beta = \Gamma_\pi^t$, and $p = \Gamma_\pi^t \circ \Gamma_\pi$, which together give the first direct summand in the decomposition (4.1).

For the second nest we have $\mathcal{S}^\circ = \{12\}$ and $1 \le \mu_{12} \le d - 1$, so there are $d - 1$ direct summands for this nest. Denoting $j : D_{12} \hookrightarrow X[2]$ and $g : D_{12} \to \Delta_{12}$ as the natural map, we have

$$\begin{aligned} \alpha_{\mathcal{S},\underline{\mu}} &= -(j, g)_* j^* \left(\sum_{i=0}^{d-1-\mu_{12}} p_1^* c_i(T_X)(-D_{12})^{d-1-\mu_{12}-i} \right), \\ \beta_{\mathcal{S},\underline{\mu}} &= (g, j)_* j^*(D^{\mu_{12}-1}), \\ p_{\mathcal{S},\underline{\mu}} &= \beta_{\mathcal{S},\underline{\mu}} \circ \alpha_{\mathcal{S},\underline{\mu}}. \end{aligned}$$

These terms give the direct summand $h(\Delta_{12})(\mu_{12})$ in the decomposition (4.1).

EXAMPLE FOR $n = 3$. Applying Theorem 4.3, we have

$$\begin{aligned} h(X[3]) &\cong h(X^3) \oplus \bigoplus_{i=1}^{d-1} h(\Delta_{12})(i) \oplus \bigoplus_{i=1}^{d-1} h(\Delta_{13})(i) \oplus \bigoplus_{i=1}^{d-1} h(\Delta_{23})(i) \\ &\quad \oplus \bigoplus_{i=1}^{2d-1} (h(\Delta_{123})(i))^{\oplus \min\{3i-2, 6d-3i-2\}} \\ &\cong h(X^3) \oplus \bigoplus_{i=1}^{d-1} (h(X^2)(i))^{\oplus 3} \oplus \bigoplus_{i=1}^{2d-1} (h(X)(i))^{\oplus \min\{3i-2, 6d-3i-2\}}. \end{aligned}$$

Now we write out all the correspondences that give the decomposition of motives. There are eight possible nests, corresponding to eight trees (see the right side of Figure 1).

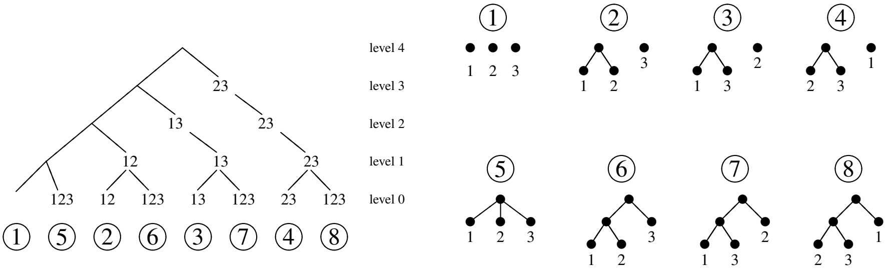


Figure 1 $X[3]$ by the symmetric construction

The tree on the left side of Figure 1 helps us understand the relation between subvarieties of different Y_i (i.e., at different levels): each node with label I at level k corresponds to the subvariety $Y_k I := (\Delta_I)^{(k)}$ in Y_k. The node at level k without a label corresponds to Y_k. For example, the root at level 4 corresponds to Y_4, its two successors correspond to Y_3 and $Y_3(23)$, and the relation is that Y_4 is the blow-up of Y_3 along $Y_3(23)$.

Here is a list of those correspondences α, β, p for the eight trees.

① gives $\alpha = \Gamma_\pi$, $\beta = \Gamma_\pi^t$, $p = \Gamma_\pi^t \circ \Gamma_\pi$.

② (③ and ④ are similar) gives

$$\alpha_{\mathcal{S},\underline{\mu}} = (j_{\mathcal{S}}, g_{\mathcal{S}})_* j_{\mathcal{S}}^* \left(\left\{ -p_1^* \zeta(-D_{123}) \frac{1}{1+D_{12}} \right\}_{d-1-\mu_{12}} \right),$$

$$\beta_{\mathcal{S},\underline{\mu}} = (g_{\mathcal{S}}, j_{\mathcal{S}})_* j_{\mathcal{S}}^* (D_{12}^{\mu_{12}-1});$$

here $X(\mathcal{S}) = D_{12}$ and $1 \le \mu_{12} \le d-1$.

⑤ gives

$$\alpha_{\mathcal{S},\underline{\mu}} = (j_{\mathcal{S}}, g_{\mathcal{S}})_* j_{\mathcal{S}}^* \left(\left\{ -p_1^* \zeta(\mathcal{O})^2 \frac{1}{1+D_{123}} \right\}_{2d-1-\mu_{123}} \right),$$

$$\beta_{\mathcal{S},\underline{\mu}} = (g_{\mathcal{S}}, j_{\mathcal{S}})_* j_{\mathcal{S}}^* (D_{123}^{\mu_{123}-1});$$

here $X(\mathcal{S}) = D_{123}$ and $1 \le \mu_{123} \le 2d-1$.

⑥ (⑦ and ⑧ are similar) gives

$$\alpha_{\mathcal{S},\underline{\mu}} = (j_{\mathcal{S}}, g_{\mathcal{S}})_* j_{\mathcal{S}}^* \biggl(\biggl\{ p_1^* \zeta(-D_{123}) \frac{1}{1+D_{12}} \biggr\}_{d-1-\mu_{12}} \cdot \biggl\{ p_1^* \zeta(\mathcal{O}) \frac{1}{1+D_{123}} \biggr\}_{d-1-\mu_{123}} \biggr),$$

$$\beta_{\mathcal{S},\underline{\mu}} = (g_{\mathcal{S}}, j_{\mathcal{S}})_* j_{\mathcal{S}}^* (D_{12}^{\mu_{12}-1} D_{123}^{\mu_{123}-1});$$

here $X(\mathcal{S}) = D_{12} \cap D_{123}$ and $1 \le \mu_{12}, \mu_{123} \le d-1$.

Remark. If we used Fulton and MacPherson's nonsymmetric construction of $X[3]$, then we would obtain another set of correspondences that also gives a decomposition of the motive $h(X[n])$. This set of correspondences turns out to be different than what we have already given: a straightforward calculation shows that, by the nonsymmetric construction of $X[3]$, the correspondence that gives the direct summand $h(\Delta_{12})(\mu_{12})$ is

$$\alpha\colon h(X[3]) \to h(\Delta_{12})(\mu_{12}),$$

$$\alpha = (j_{12}, g_{12})_* j_{12}^* \biggl(\biggl\{ p_1^* \zeta(\mathcal{O}) \frac{1}{1+D_{12}} \biggr\}_{d-1-\mu_{12}} \biggr);$$

here $j_{12}\colon D_{12} \hookrightarrow X[3]$ and $g_{12}\colon D_{12} \to \Delta_{12}$ are the natural morphisms. However, the correspondence giving the direct summand $h(\Delta_{13})(\mu_{13})$ is

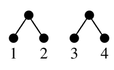
$$\alpha'\colon h(X[3]) \to h(\Delta_{13}) \otimes \mathbb{L}^{\mu_{13}},$$

$$\alpha' = (j_{13}, g_{13})_* j_{13}^* \biggl(\biggl\{ p_1^* \zeta(-D_{123}) \frac{1}{1+D_{13}} \biggr\}_{d-1-\mu_{13}} \biggr);$$

here $j_{13}\colon D_{13} \hookrightarrow X[3]$ and $g_{13}\colon D_{13} \to \Delta_{13}$ are the natural morphisms. Notice that α and α' are not of similar form (compare $\zeta(\mathcal{O})$ with $\zeta(-D_{123})$). Hence the nonsymmetry of the construction of $X[3]$ induces the nonsymmetry of correspondences. Actually, this is one reason why we choose the symmetric construction of $X[n]$ (cf. Remark 4.1).

Example for $n = 4$. We only look at one nest $\mathcal{S}$:

$\mathcal{S}$



We have $X(\mathcal{S}) = D_{12} \cap D_{34}$, $1 \le \mu_{12}, \mu_{34} \le d-1$, and

$$\alpha_{\mathcal{S},\underline{\mu}} = (j_{\mathcal{S}}, g_{\mathcal{S}})_* j_{\mathcal{S}}^* \biggl(\biggl\{ p_1^* \zeta(-D_{1234}) \frac{1}{1+D_{12}} \biggr\}_{d-1-\mu_{12}} \cdot \biggl\{ p_3^* \zeta(-D_{1234}) \frac{1}{1+D_{34}} \biggr\}_{d-1-\mu_{34}} \biggr),$$

$$\beta_{\mathcal{S},\underline{\mu}} = (g_{\mathcal{S}}, j_{\mathcal{S}})_* j_{\mathcal{S}}^* (D_{12}^{\mu_{12}-1} D_{34}^{\mu_{34}-1}).$$

Because Δ_{12} and Δ_{34} would not be disjoint in the procedure of blow-ups, we must a priori decide whether to blow up along (the strict transform of) Δ_{12} first or rather along (the strict transform of) Δ_{34} first. Although we must choose (noncanonically) an order so that we can compute the correspondences, it turns out that the correspondences (hence projectors) that give the motive decomposition in Theorem 4.2 are actually independent of the choice and thus "canonical". This independence is a special case of Remark 4.1: for $\sigma = (13)(24) \in \mathbb{S}_4$, the preceding correspondences are invariant under the action induced by σ.

One application of Theorem 4.3 is to computing the rank of $A(X[n])$ as an abelian group; we need only the ranks of $A(X^k)$ for all $1 \le k \le n$ (assuming that the ranks of the $A(X^k)$ are finite). Let us take $\mathbb{P}^d[5]$ as an example. Since the rank of $A((\mathbb{P}^d)^k)$ is $(d+1)^k$, Theorem 4.3 implies that the rank of $A(\mathbb{P}^d[5])$ is

$$\sum_{1\le k\le 5}(d+1)^k\left(\left[\frac{t^5}{t!}\right]\left(\frac{N^k}{k!}\Big|_{x=1}\right)\right).$$

By Remark 4.2, we can compute

$$\frac{N^2}{2!}=\frac{t^2}{2!}+3\sigma_1\frac{t^3}{3!}+(15\sigma_1^2+4\sigma_2)\frac{t^4}{4!}+(105\sigma_1^3+60\sigma_1\sigma_2+5\sigma_3)\frac{t^5}{5!}+\cdots,$$
$$\frac{N^3}{3!}=\frac{t^3}{3!}+6\sigma_1\frac{t^4}{4!}+(45\sigma_1^2+10\sigma_2)\frac{t^5}{5!}+\cdots,$$
$$\frac{N^4}{4!}=\frac{t^4}{4!}+10\sigma_1\frac{t^5}{5!}+\cdots,$$
$$\frac{N^5}{5!}=\frac{t^5}{5!}+\cdots.$$

Now plug in $x = 1$ to obtain $\sigma_j = dj - 1$. The sum just displayed is a polynomial of d as follows:

$$\begin{aligned}(d+1)^5&+(d+1)^4 10\sigma_1+(d+1)^3(45\sigma_1^2+10\sigma_2)\\&+(d+1)^2(105\sigma_1^3+60\sigma_1\sigma_2+5\sigma_3)\\&+(d+1)(\sigma_4+15\sigma_1\sigma_3+10\sigma_2^2+105\sigma_1^2\sigma_2+105\sigma_1^4).\end{aligned}$$

In particular, the rank of $A(\mathbb{P}^1[5])$ is 178 and the rank of $A(\mathbb{P}^2[5])$ is 7644.

REMARK. For the example $X = \mathbb{P}^d$, since $X[n]$ has an affine cell decomposition it follows that the rank of the Chow group $A_k(X[n])$ coincides with the $2k$th Betti number of $X[n]$. Hence we could also derive the ranks reported here by using the Poincaré polynomial of $X[n]$ computed in [FM]. However, the rank of $A(X[n])$ for a general variety X is not implied by the Poincaré polynomial of $X[n]$.

5. Chow Motives of $X[n]/\mathfrak{S}_n$

It is proved in [FM] that the isotropy group of any point in $X[n]$ is a solvable group. It is natural to consider the quotient space $X[n]/\mathfrak{S}_n$. In this section, we compute its Chow motive in terms of the Chow motives of the Cartesian products of symmetric products of X.

The base field is of characteristic 0 throughout this section.

LEMMA 5.1. *Suppose a finite group G acts on a nonsingular projective variety Y. Suppose $p_1, \dots, p_k$ are orthogonal projectors of Y such that*:

(i) $\sigma p_i = p_i \sigma$ *for all* $1 \le i \le k$ *and for all* $\sigma \in G$; *and*
(ii) $p_1 + p_2 + \cdots + p_k = \Delta_Y$.

Then $\operatorname{ave} \Delta_Y = \sum \operatorname{ave} \circ p_i$, *where* $\operatorname{ave} \circ p_1, \dots, \operatorname{ave} \circ p_k$ *are orthogonal projectors. Consequently,* $h(Y) = \bigoplus (Y, \operatorname{ave} \circ p_i)$.

Proof. We have

$$\begin{aligned}(\operatorname{ave} p_i)(\operatorname{ave} p_j) &= \left(\frac{1}{|G|}\sum_\sigma \sigma p_i\right)\left(\frac{1}{|G|}\sum_\tau \tau p_j\right)\\ &= \frac{1}{|G|^2}\sum_{\sigma,\tau} \sigma\tau p_i p_j = \frac{1}{|G|}\sum_\sigma \sigma \delta_{ij} p_i = \delta_{ij}(\operatorname{ave} p_j).\end{aligned}$$

Hence the lemma follows. □

LEMMA 5.2. *Suppose Y and Z are nonsingular (not necessarily connected) projective varieties with finite group G actions. Suppose that* $\alpha \in \operatorname{Corr}^{-m}(Y, Z)$ *has an inverse* $\beta \in \operatorname{Corr}^m(Z, Y)$ *and that* α *gives an isomorphism of Chow motives*

$$(Y, p) \cong h(Z)(m),$$

where $p = \beta\alpha$ *and where* $\alpha\sigma = \sigma\alpha$ *and* $\beta\sigma = \sigma\beta$ *for all* $\sigma \in G$. *Then*

$$(Y, \operatorname{ave} \circ p) \cong h(Z/G)(m).$$

Proof. Much as in the proof of Lemma 5.1, we have $(\operatorname{ave} p)^2 = \operatorname{ave} p$ and the following commutative diagram.

$$\begin{tikzcd} Y \arrow[r, "\operatorname{ave}\alpha"] \arrow[d, "\operatorname{ave} p"'] & Z \arrow[r, "\operatorname{ave}\beta"] \arrow[d, "\operatorname{ave}\Delta_Z"] & Y \arrow[d, "\operatorname{ave} p"] \\ Y \arrow[r, "\operatorname{ave}\alpha"'] & Z \arrow[r, "\operatorname{ave}\beta"'] & Y \end{tikzcd}$$

Therefore, $(Y, \operatorname{ave} p) \cong (Z, \operatorname{ave} \Delta_Z)(m) \cong h(Z/G)(m)$. □

Now we consider the quotient variety $X[n]/\mathfrak{S}_n$. For convenience, define $G := \mathfrak{S}_n$. There is a natural action of G on the set $\{(\mathcal{S}, \underline{\mu})\}$, where $\mathcal{S}$ are nests and $\underline{\mu} \in M_{\mathcal{S}}$. Define the subgroup $G_{\mathcal{S},\underline{\mu}}$ of G as

$$G_{\mathcal{S},\underline{\mu}} = \{\sigma \in \mathfrak{S}_n : \sigma(\mathcal{S}, \underline{\mu}) = (\mathcal{S}, \underline{\mu})\}.$$

Define $\overline{(\mathcal{S}, \underline{\mu})}$ to be the class of G-orbit $G \cdot (\mathcal{S}, \underline{\mu})$. Then

$$\Delta_Y = \sum_{\mathcal{S},\underline{\mu}} p_{\mathcal{S},\underline{\mu}} = \sum_{\overline{(\mathcal{S},\underline{\mu})}} \sum_{\bar\sigma\in G/G_{\mathcal{S},\underline{\mu}}} p_{\sigma(\mathcal{S},\underline{\mu})}.$$

Since both $\{\alpha_{\mathcal{S},\underline{\mu}}\}$ and $\{\beta_{\mathcal{S},\underline{\mu}}\}$ are $\mathfrak{S}_n$-symmetric (see the Remark following Theorem 4.2), it is easy to check that $\sum_{\bar\sigma\in G/G_{\mathcal{S},\underline{\mu}}} p_{\sigma(\mathcal{S},\underline{\mu})}$ commutes with every $\tau\in G$. By Lemma 5.1,

$$h(X[n]/G) \cong (Y, \text{ave}\circ\Delta_Y) \cong \bigoplus_{\overline{(\mathcal{S},\underline{\mu})}} \Bigl(Y, \text{ave}\circ \sum_{\bar\sigma\in G/G_{\mathcal{S},\underline{\mu}}} p_{\sigma(\mathcal{S},\underline{\mu})}\Bigr).$$

Since

$$\Bigl(Y, \sum_{\bar\sigma\in G/G_{\mathcal{S},\underline{\mu}}} p_{\sigma(\mathcal{S},\underline{\mu})}\Bigr) \cong \Bigl(\coprod_{\bar\sigma\in G/G_{\mathcal{S},\underline{\mu}}} \Delta_{\sigma(\mathcal{S})}\Bigr)(\|\underline{\mu}\|),$$

by Lemma 5.2 we have

$$\begin{aligned}\Bigl(Y, \text{ave}\circ \sum_{\bar\sigma\in G/G_{\mathcal{S},\underline{\mu}}} p_{\sigma(\mathcal{S},\underline{\mu})}\Bigr) &\cong h\Bigl(\Bigl(\coprod_{\bar\sigma\in G/G_{\mathcal{S},\underline{\mu}}} \Delta_{\sigma(\mathcal{S})}\Bigr)\Big/G\Bigr)(\|\underline{\mu}\|)\\ &\cong h(\Delta_{\mathcal{S}}/G_{\mathcal{S},\mu})(\|\underline{\mu}\|).\end{aligned}$$

The space $\Delta_{\mathcal{S}}/G_{\mathcal{S},\underline{\mu}}$ can be described as follows. Each $(\mathcal{S},\underline{\mu})$ corresponds to a labeled "weighted" forest, with the correspondence given by attaching an integer μ_I to each nonleaf node I of the labeled forest $\mathcal{S}$. Forgetting all the labels on the nodes of $\mathcal{S}$, we obtain an unlabeled weighted forest of the form $n_1T_1+\cdots+n_rT_r$, where the T_i are mutually distinct unlabeled weighted trees (we say that such a tree is of type $\{n_1,\dots,n_r\}$). Then

$$\Delta_{\mathcal{S}}/G_{\mathcal{S},\underline{\mu}} \cong X^{(n_1)}\times\cdots\times X^{(n_r)}.$$

Figure 2 gives an example of a labeled weighted forest and the corresponding unlabeled weighted forest. The weights a, b are integers.

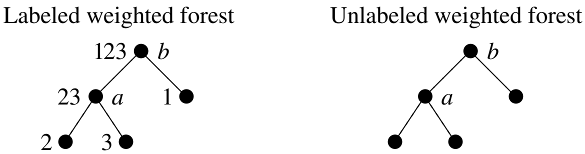


Figure 2 Labeled and unlabeled weighted forests

We have therefore proved the following decomposition of the Chow motive of $X[n]/\mathfrak{S}_n$.

THEOREM 5.3. *For any unordered set of integers $\nu=\{n_1,\dots,n_r\}$ and any nonnegative integer m, let $\lambda(\nu,m)$ to be the number of unlabeled weighted forests with n leaves, of type ν, of total weight m, and such that—at each nonleaf v with c_v children—the weight m_v satisfies $1\le m_v\le (c_v-1)\dim X-1$. Then*

$$h(X[n]/\mathfrak{S}_n) = \bigoplus_{\nu,m} [h(X^{(n_1)}\times\cdots\times X^{(n_r)})(m)]^{\oplus\lambda(\nu,m)}.$$

REMARK. We offer the following application of this theorem. MacDonald proved a formula that relates the Betti number of X and its symmetric powers:

$$\sum_{n=0}^{\infty} P_t X^{(n)} \cdot T^n = \frac{(1+tT)^{b_1}(1+t^3T)^{b_3}\cdots}{(1-T)^{b_0}(1-t^2T)^{b_2}\cdots},$$

where b_i is the ith Betti number of X. By the decomposition of the de Rham cohomology of $X[n]/\mathfrak{S}_n$ induced by the motivic decomposition formula in Theorem 5.3, we can compute the Betti number of $X[n]/\mathfrak{S}_n$ (modulo the combinatorial difficulty of calculating $\lambda(\nu, m)$).

EXAMPLES. Here are some examples of $h(X[n]/\mathfrak{S}_n)$ for small n. Let $d = \dim X$.

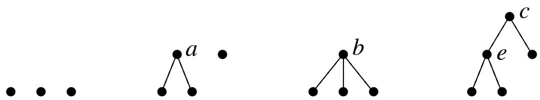
(i) $n = 2$. There are d different forests as follows, where each weight $a \in \mathbb{Z}$ $(1 \le a \le d-1)$ gives a forest.

$\nu = \{2\}$ $\qquad\qquad$ $\nu = \{1\}$

Therefore,

$$h(X[2]/\mathfrak{S}_2) \cong h(X^{(2)}) \oplus \bigoplus_{a=1}^{d-1} h(X)(a).$$

(ii) $n = 3$. The forests are as follows.



$\nu = \{3\}$ $\qquad$ $\nu = \{1,1\}$ $\qquad$ $\nu = \{1\}$ $\qquad$ $\nu = \{1\}$

Here the weights $a, b, c, e \in \mathbb{Z}$ satisfy $1 \le a, c, e \le d-1$ and $1 \le b \le 2d-1$. Then

$$h(X[3]/\mathfrak{S}_3) \cong h(X^{(3)}) \oplus \bigoplus_{i=1}^{d-1} (h(X^2)(i))^{\oplus 3} \oplus \bigoplus_{i=1}^{2d-1} (h(X)(i))^{\oplus \min\{i, 2d-i\}}.$$

(iii) $n = 4$. The varieties that appear in the decomposition of $h(X[4]/\mathfrak{S}_4)$ are

$$X^{(4)},\ X \times X^{(2)},\ X^2,\ X^{(2)},\ X.$$

The decomposition is a bit nasty to be written here, so we limit ourselves to pointing out one fact. Consider the forest in Figure 3, where $a, b \in \mathbb{Z}$ and $1 \le a, b \le d-1$. For any $a < b$, the weighted forest is of type $\nu = \{1,1\}$ and therefore gives a summand $h(X^2)(a+b)$. However, for $a = b$, this weighted forest has an automorphism exchanging the two trees; thus it is of type $\nu = \{2\}$ and gives a summand $h(X^{(2)})(2a)$. Because of this kind of automorphism of weighted forests, it seems difficult to compute $\lambda(\nu, m)$.

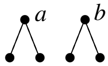


Figure 3 An unlabeled weighted forest when $n = 4$

QUESTION. Is there a clean formula for $\lambda(\nu, m)$? (Perhaps in terms of a generating function?)

6. Appendix: A Formula for the Motive of a Blow-up

Suppose $f\colon \widetilde{Y} \to Y$ is the blow-up of a nonsingular projective variety Y along a nonsingular closed subvariety V of Y, and denote by P the exceptional divisor. Denote by i, j, f, g the morphisms as in the following fiber square.

$$\begin{tikzcd} P \arrow[r, "j"] \arrow[d, "g"'] \arrow[dr, phantom, "\Box"] & \widetilde{Y} \arrow[d, "f"] \\ V \arrow[r, "i"'] & Y \end{tikzcd}$$

Denote by $N := N_V Y$ the normal bundle of V in Y. Let $h := c_1(\mathcal{O}_N(1)) \in A^1(P)$, and let $r := \operatorname{codim}_V Y$ be the codimension of V in Y.

For $1 \le k \le r-1$, define $\alpha_k \in \operatorname{Corr}^{-k}(\widetilde{Y}, V)$, $\beta_k \in \operatorname{Corr}^k(V, \widetilde{Y})$, $p_k \in \operatorname{Corr}^0(\widetilde{Y}, \widetilde{Y})$, $\alpha_0 \in \operatorname{Corr}^0(\widetilde{Y}, Y)$, $\beta_0 \in \operatorname{Corr}^0(Y, \widetilde{Y})$, and $p_0 \in \operatorname{Corr}^0(\widetilde{Y}, \widetilde{Y})$ as follows:

$$\begin{cases} \alpha_0 := \Gamma_f, \\ \beta_0 := \Gamma_f^t, \\ p_0 := \beta_0 \circ \alpha_0 = \Gamma_f^t \circ \Gamma_f = (f \times f)^* \Delta_Y, \\ \alpha_k := -(j, g)_* \Bigl(\displaystyle\sum_{l=0}^{r-1-k} g^* c_{r-1-k-l}(N) h^l \Bigr) \\ \qquad = -(j, g)_* \Bigl(\Bigl\{ g^* c(N) \dfrac{1}{1-h} \Bigr\}_{r-1-k} \Bigr), \\ \beta_k := (g, j)_* h^{k-1}, \\ p_k := \beta_k \circ \alpha_k; \end{cases} \tag{A.1}$$

here the subscript $r-1-k$ in the definition of α_k signifies taking the codimension-$(r-1-k)$ component. We will give the proof of the following proposition at the end of this section.

PROPOSITION A.1. *Define $\alpha_k, \beta_k, p_k, \alpha_0, \beta_0, p_0$ as before. Then the following statements hold.*

(i) $\alpha_0 \beta_0 = \Delta_Y$, $\alpha_k \beta_k = \Delta_V$ *for* $1 \le k \le r-1$, *and* $\alpha_i \beta_j = 0$ *for* $i \ne j$.

(ii) $p_0, p_1, p_2, \ldots, p_{r-1}$ *are mutually orthogonal projectors of* $\widetilde{Y}$, *and*

$$\sum_{i=0}^{r-1} p_i = \Delta_{\widetilde{Y}} \quad \text{in } A(\widetilde{Y} \times \widetilde{Y});$$

that is, equality holds up to rational equivalence.

(iii) *We have the following isomorphisms of motives*:

$\alpha_0\colon (\widetilde{Y}, p_0, 0) \simeq h(Y)$ *with inverse morphism* β_0;

$\alpha_k\colon (\widetilde{Y}, p_k, 0) \simeq h(V)(k)$ *with inverse morphism* β_k *for* $1 \le k \le r-1$.

Define $\Gamma := \bigoplus_{i=0}^{r-1} \alpha_i$ and $\Gamma' := \sum_{i=0}^{r-1} \beta_i$. Then Proposition A.1 can be conveniently reformulated as follows.

THEOREM A.2. *The correspondence* Γ *gives a canonical isomorphism in* $CH\mathcal{M}$,

$$\Gamma\colon h(\widetilde{Y}) \cong h(Y) \oplus \bigoplus_{k=1}^{r-1} h(V)(k),$$

with an inverse isomorphism given by Γ'.

REMARK. If the normal bundle N of V in Y is trivial (e.g., when V is a point), then P is isomorphic to a product space $V \times \mathbb{P}^{r-1}$ and $h = c_1(\mathcal{O}_P(1))$ can be represented (not canonically) by a product space $H = V \times \mathbb{P}^{r-2}$ in P. In this case, we have simple forms for the projectors:

$$p_k = -(j \times j)_*(H^{r-1-k} \times_V H^{k-1}) \quad \text{for } 1 \le k \le r-1;$$

$$p_0 = \Delta + \sum_{k=1}^{r-1} (j \times j)_*(H^{r-1-k} \times_V H^{k-1}).$$

In general, for a nontrivial normal bundle N, more terms involving the Chern classes of N are needed and the correspondences cannot be represented by explicit and natural algebraic cycles.

REMARK. The isomorphism of motives in Theorem A.2 is also a consequence of the "Theorem on the additive structure of the motif" of $\widetilde{Y}$ in [Man, Sec. 9]. In our notation, this theorem states that there is a split exact sequence

$$0 \longrightarrow h(V)(r) \xrightarrow{\ a\ } h(Y) \oplus h(P)(1) \xrightarrow{\ b\ } h(\widetilde{Y}) \longrightarrow 0.$$

The correspondences appearing in our theorem are not given, at least not explicitly, in Manin's paper.

In order to clarify this point, define:

$$\Phi = c_{r-1}(g^*N/\mathcal{O}_N(-1)) \in A^{r-1}(P),\ c_\Phi = \delta_{P*}(\Phi) \in \mathrm{Corr}(P, P);$$

$$a = (i_*, c_\Phi \circ g^*),\ a' = g_*;$$

$$b = f^* + j_*,\ b' \text{ its right inverse;}$$

$$d = \Delta_{Y \times P} - aa',\ d' = \Delta_Y \otimes (\Delta_P - p_0^P) \text{ (where } p_0^P = c_{h^{r-1}} \circ g^* \circ g_*).$$

Denote by $e\colon \bigoplus_{k=1}^{r-1} V(k) \to (P, \Delta_P - p_0^P)$ the isomorphism implicitly defined in [Man, Sec. 7], and denote by e' the inverse of e.

We have the isomorphisms

$$h(Y) \oplus \bigoplus_{k=1}^{r-1} h(V)(k) \underset{\Delta_Y \otimes e'}{\overset{\Delta_Y \otimes e}{\rightleftarrows}} (Y \sqcup P, (\Delta_Y, \Delta_P - p_0^P))$$

$$\underset{d'}{\overset{d}{\rightleftarrows}} (Y \sqcup P, \Delta_{Y \sqcup P} - aa') \underset{b'}{\overset{b}{\rightleftarrows}} (\widetilde{Y}, \Delta_{\widetilde{Y}}).$$

Hence the following is an isomorphism of Chow motives:

$$(\Delta_Y \otimes e') \circ d' \circ b' \colon h(\widetilde{Y}) \cong h(Y) \oplus \bigoplus_{k=1}^{r-1} h(V) \otimes \mathbb{L}^k$$

with inverse $b \circ d \circ (\Delta_Y \otimes e)$.

Therefore, to write down the correspondence $(\Delta_Y \otimes e') \circ d' \circ b'$, we must find explicitly the right inverse b' of b. However, in [Man] the construction of b' is based on the surjectivity of $\gamma \colon A(\widetilde{Y} \times (Y \sqcup P)) \to A(\widetilde{Y} \times \widetilde{Y})$ as follows. By the surjectivity of γ, there is a cycle class $c \in A(\widetilde{Y} \times (Y \sqcup P))$ (which is not explicitly given in [Man]) such that $\gamma(c) = \Delta_{\widetilde{Y}} \in A(\widetilde{Y} \times \widetilde{Y})$. Then b' is defined to be $(1 - aa')c$.

On the other hand, the correspondences Γ and Γ' that were constructed in Theorem A.2 yield an explicit construction of b'. Indeed, $b' = d \circ (\Delta_Y \otimes e) \circ \Gamma$.

Proof of Proposition A.1. In the proof, we assume that $1 \le k \le r-1$ and that $0 \le i, j \le r-1$.

The idea is as follows. We study the morphisms α_{i*}, β_{i*}, and p_{i*} of Chow groups induced by the correspondences α_i, β_i, and p_i. As a consequence, the identities of morphisms of Chow groups that are induced by the identities in parts (i) and (ii) of Proposition A.1 hold. On the other hand, Manin's identity principle asserts that the identities of morphisms of Chow groups imply the identities of correspondences—provided the correspondences are universal in certain sense.

By [V, Thm. 9.27], an element $\tilde{y} \in A(\widetilde{Y})$ can be expressed uniquely as

$$\tilde{y} = \sum_{i=1}^{r-1} j_*(g^* a_i \cdot h^{i-1}) + f^* y.$$

It is standard to verify the following statements.

(α_k) The morphism $\alpha_{k*} \colon A(\widetilde{Y}) \to A(V)$ maps $\tilde{y} \mapsto a_k$.
(β_k) The morphism $\beta_{k*} \colon A(V) \to A(\widetilde{Y})$ maps $x \mapsto j_*(g^* x \cdot h^{k-1})$.
(α_0) The morphism $\alpha_{0*} \colon A(\widetilde{Y}) \to A(Y)$ maps $\tilde{y} \mapsto y$.
(β_0) The morphism $\beta_{0*} \colon A(Y) \to A(\widetilde{Y})$ maps $y \mapsto f^* y$.

To give a flavor, we prove only the statement (α_k) (i.e., $\alpha_{k*}(\tilde{y}) = a_k$).

Define $a_0 = -i^* y$. Since $j^* j_* z = -h \cdot z$ for all $z \in A(P)$, we have

$$\begin{aligned} j^* \tilde{y} &= \sum_{i=1}^{r-1} j^* j_*(g^* a_i \cdot h^{i-1}) + j^* f^* y = -\sum_{i=1}^{r-1} g^* a_i \cdot h^i + g^* i^* y \\ &= -\sum_{i=0}^{r-1} g^* a_i \cdot h^i. \end{aligned}$$

By definition (see [F, Sec. 3]), the ith Segre class of N is

$$s_i(N) := g_*(h^{i+r-1}).$$

Therefore,

$$
\begin{aligned}
\alpha_{k*}(\tilde{y}) &= -g_*\Big(j^*\tilde{y}\cdot\sum_{l=0}^{r-1-k} g^*c_{r-1-k-l}(N)\cdot h^l\Big)\\
&= -g_*\Big(\Big(-\sum_{i=0}^{r-1} g^*a_i\cdot h^i\Big)\cdot\Big(\sum_{l=0}^{r-1-k} g^*c_{r-1-k-l}\cdot h^l\Big)\Big)\\
&= g_*\Big(\sum_{i=0}^{r-1}\sum_{l=0}^{r-1-k} g^*(a_i c_{r-1-k-l})h^{i+l}\Big)\\
&= \sum_{i=0}^{r-1} a_i\Big(\sum_{l=0}^{r-1-k} c_{r-1-k-l}s_{i+l+1-r}\Big).
\end{aligned}
$$

Because $c(N)s(N) = 1$, where $c(N) := \sum c_i(N)$ is the total Chern class and $s(N) := \sum s_i(N)$ is the total Segre class, we have

$$
\sum_{l=0}^{r-1-k} c_{r-1-k-l}s_{i+l+1-r} = \sum_{l=-\infty}^{+\infty} c_{r-1-k-l}s_{i+l+1-r} = \{c(N)s(N)\}_{i-k} = \delta_{ik}.
$$

The first equality holds because $s_{i+l+1-r} = 0$ for $l < 0$ and $c_{r-1-k-l} = 0$ for $l > r-1-k$. It follows that $\alpha_{k*}(\tilde{y}) = a_k$, as we claimed.

The statements (α_k), (β_k), (α_0), and (β_0) immediately imply the following identities:

$$
\alpha_{k*}\beta_{k*} = \mathrm{id}_{A(V)}, \qquad \alpha_{0*}\beta_{0*} = \mathrm{id}_{A(Y)},
$$
$$
\alpha_{i*}\beta_{j*} = 0 \quad \text{for } i \neq j,
$$
$$
(p_ip_j)_* = \delta_{ij}p_{i*}, \qquad \sum_{i=0}^{r-1} p_{i*} = \mathrm{id}_{A(\widetilde{Y})}.
$$

For any smooth scheme T, we write $T \times \widetilde{Y}$ to denote the blow-up of $T \times Y$ along the smooth subvariety $T \times V$. Denoting $j' = \mathrm{id}_T \times j$, $g' = \mathrm{id}_T \times g$, $f' = \mathrm{id}_T \times f$, and $i' = \mathrm{id}_T \times i$, we have the following fiber square.

$$
\begin{tikzcd}
T \times P \arrow[r, "j'"] \arrow[d, "g'"] \arrow[dr, phantom, "\square"] & T \times \widetilde{Y} \arrow[d, "f'"] \\
T \times V \arrow[r, "i'"] & T \times Y
\end{tikzcd}
$$

We can construct the correspondences $\alpha'_i, \beta'_i, p'_i$ for this fiber square as we did in (A.1); the result is

$$
\alpha'_i = \mathrm{id}_T \otimes \alpha_i, \quad \beta'_i = \mathrm{id}_T \otimes \beta_i, \quad p'_i = \mathrm{id}_T \otimes p_i.
$$

Then (i) and (ii) follow from Manin's identity principle.

For part (iii) of the proposition, in order to show that α_k gives an isomorphism $(\widetilde{Y}, p_k, 0) \simeq h(V) \otimes \mathbb{L}^k$ with inverse β_k, we must show that $p_k = p_k \circ \beta_k \circ \alpha_k$ and $\mathrm{id} = \mathrm{id} \circ \alpha_k \circ \beta_k$; but these equalities are direct consequences of $\alpha_k\beta_k = \Delta_V$ from part (i). The proof for $(\widetilde{Y}, p_0, 0) \simeq h(Y)$ is similar. $\square$

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