Michigan Math. J. 58 (2009)

On the Weak-type Constant of the Beurling–Ahlfors Transform

Rodrigo Bañuelos & Prabhu Janakiraman

1. Introduction

The Beurling–Ahlfors operator B defined on $L^p(\mathbb{C})$ by

$$Bf(z) = \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\,dm(w) \tag{1.1}$$

is a well-known example of a Calderón–Zygmund singular integral operator of convolution type. The operator arises naturally in the study of the regularity of solutions of the Beltrami equation and thus has applications to quasiconformal mapping theory and partial differential equations (see [1; 9; 12; 14; 15]). In particular, the Iwaniec conjecture [12] that the L^p norm

$$\|B\|_p = p^* - 1, \tag{1.2}$$

where $1 < p < \infty$ and $p^* = \max\{p, \frac{p}{p-1}\}$, is partly motivated by its relation to the Gehring–Reich conjecture proved by Astala in [1]. (The lower bound of p^*-1 was obtained by Lehto [16] in 1965.) Recent work has revealed B as an exemplary junction between Fourier analysis and probability, and martingale methods established by Burkholder have led to the present best known estimates on $\|B\|_p$; see [3; 4; 10].

In this paper we investigate the action of B on the radial function subspaces (for m a nonnegative integer)

$$\mathcal{R}_m^p = \{f \in L^p(\mathbb{C}) : f(re^{i\theta}) = H(r)e^{-im\theta}\}, \tag{1.3}$$

and we arrive at a corresponding family of one-dimensional operators $\{\Lambda_m\}_{m\ge 0}$ on $L^p([0,\infty))$ defined by

$$\begin{aligned}\Lambda_m g(u) = (\mathcal{H}_m - I)g(u) &= \frac{2m+2}{m+2}\int_0^1 g(uv^{2/(m+2)})\,dv - g(u)\\ &= \int_0^1 g(uv)(m+1)v^{m/2}\,dv - g(u).\end{aligned} \tag{1.4}$$

Received January 10, 2008. Revision received August 24, 2008.
The first author is supported in part by NSF grant no. 0603701-DMS-DMS. The second author thanks the Mathematics Department at the University of Illinois Urbana-Champaign, where most of this work was done while he was partially supported by NSF VIGRE grant no. 0555905.

In his attempts to simplify the proof of Hilbert's double series theorem, Hardy showed in 1920 (see [11, Thm. 327; 18, Sec. 2.4]) that

$$\int_0^\infty |\mathcal{H}_0 g(u)|^p \, du < \left(\frac{p}{p-1}\right)^p \int_0^\infty |g(u)|^p \, du.$$

This is Hardy's famous inequality with best constant, which has led to wide research in basic inequalities and is at the heart of much of modern harmonic analysis. In view of this fact, $\mathcal{H}_m$ may be referred as the *Hardy* average operator of order m, and $\mathcal{H} := \mathcal{H}_0$ as simply the Hardy operator.

The next objective is naturally to exploit the connection between B and the Λ family. The following is an immediate but nontrivial fact that follows from B being an L^2 isometry.

THEOREM 1.1. *For each $m \geq 0$, Λ_m is an $L^2([0,\infty))$ isometry.*

Similarly, since Λ_m is a one-dimensional operator, it is likely easier to derive information on its norm; whatever is gained then automatically gives parallel information on $B|_{\mathcal{R}_m^p}$. Indeed, better norm estimates are obtained on these spaces. In the special case $m = 0$ and $1 < p \leq 2$, the L^p norm $\|\Lambda_0\|_p$ equals $p^* - 1$, which is a direct consequence of results in [2]. These results are discussed in Sections 4 and 5. The main result of this paper is computation of the weak-type constant of Λ_0 and thereby a new lower bound for the weak-type constant of B. Recall that the general Calderón–Zygmund theory shows that B is L^p bounded for $1 < p < \infty$ but not for $p = 1$. Instead there exists a universal constant $C_1 > 0$ such that, for all $f \in L^1(\mathbb{C})$ and $\lambda > 0$,

$$m\{z \in \mathbb{C} : |Bf(z)| > \lambda\} \leq \frac{C_1}{\lambda}\|f\|_1.$$

This is the weak-type $(1,1)$ inequality, and the minimal C_1 is the weak-type $(1,1)$ norm of B denoted in this paper by $\|B\|_{w(1)}$.

THEOREM 1.2.

$$\|B\|_{w(1)} \geq \|\Lambda_0\|_{w(1)} = \frac{1}{\log 2}. \tag{1.5}$$

REMARK 1.1. In the first version of this paper, the authors had made the assumption in the proof of Theorem 3.2 (without actually calculating (3.4)) that $\mu[0,\beta) = 0$ is an optimizing condition. This required that we fix $\beta = \frac{e}{\lambda}$ in (3.3) and gave the smaller bound of $\frac{e}{2}$. James Gill (personal communication) pointed out that this condition is not optimizing and that the constant is actually $\frac{1}{\log 2}$. Except for the change in (3.3) and (3.4), the arguments here are exactly as in that first version.

The rest of the paper is organized as follows. In Section 2 we prove the isometry of Λ_m on L^2 and, along the way, derive various properties of B acting on $\mathcal{R}_m^p$; in this section we also prove Theorem 1.1. In Section 3, we prove the weak-type estimates for Λ_0 and derive the consequences for B. In Sections 4 and 5, we derive L^p estimates for Λ_0 and Λ_m, respectively. We end the paper with a conjecture on the possible best weak-type $(1,1)$ constant for B restricted to real-valued functions.

2. Λ_m is an L^2 Isometry

Iwaniec shows in [13] that the BMO_2 norm of B is exactly 3. In Section 1 he writes (independent of his main result): "The following formulas are worth recording." For each integer $m \geq 0$, let

$$\rho_{m,r}(z) = \bar{z}^m \chi_{B(0,r)}(z). \tag{2.1}$$

Then

$$B\rho_{m,r}(z) = \frac{r^{2m+2}}{z^{m+2}} \chi_{B(0,r)^c}(z). \tag{2.2}$$

Here $B(0,r) = \{z \in \mathbb{C} : |z| < r\}$ is the disk of radius r and $B(0,r)^c = \mathbb{C} \setminus B(0,r)$ is the complement. Iwaniec actually proves this formula for the special case when $r = 1$. Equation (2.2) follows from this because the Beurling–Ahlfors commutes with dilations, and it is used in what follows to find the formula that relates Λ_m and B.

REMARK 2.1. (1) Iwaniec also gives $B\rho(z) = (1 + \log|z|^2)\chi_B(z)$, where $\rho(z) = \left(\frac{z}{|z|}\right)^2 \chi_B(z)$. Because $\int B\rho \cdot \bar{g} = \int \rho B\bar{g}$ for g a radial complex-valued function (see Theorem 2.1), the action of B on ρ corresponds to the action of the adjoint of Λ_0. Hence, for such a class of functions we expect parallel L^p estimates in the range $2 < p < \infty$.

(2) James Gill (personal communication) has computed that the adjoint also has the weak-type constant $\frac{1}{\log 2}$. This is work in preparation.

(3) We may also question whether the nonnegative integer m can be replaced by nonnegative reals and then similar formulas obtained.

2.1. Λ_m and B

Let $\mathcal{S}_m^p$ be the subspace of $\mathcal{R}_m^p$ containing functions of the form

$$\bar{z}^m \sum_k a_k \chi_{B(0,r_k)}(z),$$

where $a_k \in \mathbb{C}$ for each k and where $0 < r_1 < r_2 < \cdots$.

PROPOSITION 2.1. *$\mathcal{S}_m^p$ is an L^p dense subset of $\mathcal{R}_m^p$. That is, given $\varepsilon > 0$ and $f \in \mathcal{R}_m^p$, there exists a $g \in \mathcal{S}_m^p$ such that $\|f - g\|_p < \varepsilon$.*

Proof. Let $f(z) = \frac{\bar{z}^m}{|z|^m} h(z)$ where, without loss of generality, $h \in \mathcal{R}_0^p(\mathbb{C}) \cap C_c^\infty(\mathbb{C})$. Assume that f is supported in $B(0,R)$. For M a positive integer and N such that $\lceil \frac{N}{M} \rceil = \lceil R \rceil$, define

$$f_M(z) = \frac{\bar{z}^m}{|z|^m} \sum_{k=1}^{N} h\left(\frac{k-1}{M}\right) \chi_{A_k}(z),$$

where $A_k = B\big(0, \frac{k}{M}\big) \setminus B\big(0, \frac{k-1}{M}\big)$. Choose M large enough that $\|f - f_M\|_\infty < \varepsilon$. Then $\|f - f_M\|_p < \varepsilon |B(0,R)|$. In particular, $f_M \to f$ in L^p. Hence we can assume without loss of generality that

$$f(z) = \frac{\bar{z}^m}{|z|^m} \sum_{k=1}^{N} a_k \chi_{B(0,k/M)}(z) = \frac{\bar{z}^m}{|z|^m} \sum_{k=1}^{N} \tilde{a}_k \chi_{A_k}(z).$$

For $\delta > 0$, define also

$$\tilde{f}_\delta(z) = \frac{\bar{z}^m}{|z|^m} \sum_{k=2}^{N} \frac{\tilde{a}_k}{r_{k-1}^m} |z|^m \chi_{A_k}(z) \chi_{B(0,\delta)^c}(z).$$

Then

$$\begin{aligned} |\tilde{f}_\delta - f| &\le |f\chi_{B(0,\delta)}| + \sum_{k=\lfloor \delta M \rfloor}^{N} |\tilde{a}_k| \left| \frac{|z|^m}{r_{k-1}^m} - 1 \right| \chi_{A_k}(z) \\ &\le |f\chi_{B(0,\delta)}| + \left(\frac{|z|^m}{r_{k-1}^m} - 1 \right) |f\chi_{B(0,(\lfloor \delta M \rfloor - 1)/M)^c}|. \end{aligned}$$

It is clear that $\|\tilde{f}_\delta - f\|_p < \varepsilon$ for δ small and M large. Since $g = \tilde{f}_\delta$ is in $\mathcal{S}_m^p$, the proof is complete. □

Given $f(z) = \bar{z}^m \sum_k a_k \chi_{B(0,r_k)}(z)$, it follows from (2.2) that

$$Bf(z) = \frac{\sum_{k=1}^{j} a_k r_k^{2m+2}}{z^{m+2}}, \quad r_j \le |z| < r_{j+1}. \tag{2.3}$$

The fundamental representation theorem may be stated as follows.

THEOREM 2.1. *Let $f \in \mathcal{R}_m^p$. Then $Bf \in \mathcal{R}_{m+2}^p$ and*

$$Bf(z) = \frac{\bar{z}^2}{|z|^2} \left[\frac{2m+2}{m+2} \int_0^1 f(v^{1/(m+2)} z)\, dv - f(z) \right]. \tag{2.4}$$

Proof. The formula clearly shows that $B\colon \mathcal{R}_m^p \to \mathcal{R}_{m+2}^p$. It suffices to prove the theorem for $f(z) = \bar{z}^m \sum_k a_k \chi_{B_k}(z)$; the general case then follows by approximation. Let $r_n \le |z| < r_{n+1}$ and define $\sigma(k,z) = \left(\frac{r_k}{|z|}\right)^{m+2}$. We have

$$\begin{aligned} \int_0^1 f(v^{1/(m+2)} z)\, dv &= \bar{z}^m \int_0^1 v^{m/(m+2)} \sum_k a_k \chi_{B_k}(v^{1/(m+2)} z)\, dv \\ &= \bar{z}^m \int_0^1 v^{m/(m+2)} \sum_k a_k \chi_{[0,\sigma(k,z))}(v)\, dv \\ &= \sum_{k=1}^{n} \left(\int_{\sigma(k-1,z)}^{\sigma(k,z)} v^{m/(m+2)} \left(\sum_{l=k}^{\infty} a_l \right) dv \right) \bar{z}^m \\ &\quad + \int_{\sigma(n,z)}^{1} v^{m/(m+2)}\, dv \left(\sum_{l=n+1}^{\infty} a_l \right) \bar{z}^m \\ &= \frac{m+2}{2m+2} \frac{\bar{z}^m}{|z|^{2m+2}} \sum_{k=1}^{n} \left(\sum_{l=k}^{\infty} a_l \right) (r_k^{2m+2} - r_{k-1}^{2m+2}) \\ &\quad + \frac{m+2}{2m+2} \bar{z}^m \left(1 - \left(\frac{r_n}{|z|} \right)^{2m+2} \right) \left(\sum_{l=n+1}^{\infty} a_l \right). \end{aligned}$$

Since $f(z) = \sum_{l=n+1}^{\infty} a_l$, it follows that

$$\frac{\bar{z}^2}{|z|^2}\left(\frac{2m+2}{m+2}\int_0^1 f(v^{1/(m+2)}z)\,dv - f(z)\right) = \frac{1}{z^{m+2}}\sum_{l=1}^{n} a_l r_l^{2m+2}.$$

By (2.3), this equals $Bf(z)$. □

Given $f \in L^2(0,\infty)$, let $\tilde{f}(z) = f(|z|^2)$ be the associated *radial* function and let $F(z) = \frac{\bar{z}^m}{|z|^m}\tilde{f}(z)$. Define the operator $\tilde{\Lambda}_m$ by

$$\tilde{\Lambda}_m g(z) = \frac{2m+2}{m+2}\int_0^1 g(v^{1/(m+2)}z)\,dv - g(z),$$

so that $BF(z) = \left(\frac{\bar{z}}{|z|}\right)^2 \tilde{\Lambda}_m F(z)$ by Theorem 2.1. Observe finally that

$$BF(z) = \frac{\bar{z}^{m+2}}{|z|^{m+2}}\Lambda_m f(|z|^2). \tag{2.5}$$

2.2. *Proof of Theorem 1.1*

The first step is to prove that Λ_m preserves the L^2-inner product.

THEOREM 2.2. *Let $f, g \in L^2(0,\infty)$. Then*

$$\int_0^\infty \Lambda_m f(u)\overline{\Lambda_m g}(u)\,du = \int_0^\infty f(u)\bar{g}(u)\,du.$$

Proof.

$$\begin{aligned}
\int_0^\infty \Lambda_m f(u)\Lambda_m \bar{g}(u)\,du &= \int_0^\infty \Lambda_m f(|z|^2)\overline{\Lambda_m g}(|z|^2)\,d(|z|^2)\\
&= \int_0^\infty \tilde{\Lambda}_m \tilde{f}(re^{i\theta})\overline{\tilde{\Lambda}_m \tilde{g}}(re^{i\theta})\,d(r^2)\\
&= \frac{1}{\pi}\int_0^\infty \int_{\partial B(0,r)} \tilde{\Lambda}_m \tilde{f}(\eta)\overline{\tilde{\Lambda}_m \tilde{g}}(\eta)\,d\sigma(\eta)\,dr\\
&= \frac{1}{\pi}\int_{\mathbb{R}^2} \tilde{\Lambda}_m F(z)\overline{\tilde{\Lambda}_m G}(z)\,dz.
\end{aligned}$$

In the last line, $F(z) = \frac{\bar{z}^m}{|z|^m}\tilde{f}(z)$ and $G(z) = \frac{\bar{z}^m}{|z|^m}\tilde{g}(z)$. Because of the conjugation of G, the two unimodular terms cancel out and the integral value does not change. It is important to note that $F, G \in \mathcal{R}_m^2$; hence, for instance,

$$BF(z) = \frac{\bar{z}^2}{|z|^2}\tilde{\Lambda}_m F(z).$$

As a consequence,

$$\begin{aligned}\int_0^\infty \Lambda_m f(u)\overline{\Lambda_m g}(u)\,du &= \frac{1}{\pi}\int_{\mathbb{R}^2} BF(z)\overline{BG}(z)\,dz\\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} F(z)\bar{G}(z)\,dz\\ &\ \vdots\\ &= \int_0^\infty f(u)\bar{g}(u)\,du.\end{aligned}$$

This completes the proof of the theorem. □

Proof. An alternate direct proof may also be obtained using the integration-by-parts technique employed by Hardy. We outline such a proof.

Let $f, g \in C_c^\infty([0,\infty))$. A change of variables reveals that

$$\mathcal{H}_m f(u) = \frac{1}{u^{(m+2)/2}}\int_0^u f(v)(m+1)v^{m/2}\,dv.$$

Let $F(u) = \int_0^u f(v)(m+1)v^{m/2}\,dv$ and $G(u) = \int_0^u g(v)(m+1)v^{m/2}\,dv$. Then

$$\begin{aligned}&\int \Lambda_m f\cdot\Lambda_m\bar{g}\\ &\quad= \int\left(\frac{F-u^{(m+2)/2}f}{u^{(m+2)/2}}\right)\cdot\left(\frac{\bar{G}-u^{(m+2)/2}\bar{g}}{u^{(m+2)/2}}\right)\\ &\quad= \frac{1}{m+1}\int\bigg[(\bar{G}-u^{(m+2)/2}\bar{g})\left(\frac{m}{2}f-uf'\right)\\ &\qquad\qquad + (F-u^{(m+2)/2}f)\left(\frac{m}{2}\bar{g}-u\bar{g}'\right)\bigg]u^{-(m+2)/2}\,du\\ &\quad= -\frac{m}{m+1}\int f\bar{g} + \frac{m}{2(m+1)}\int u^{-(m+2)/2}(\bar{G}f+F\bar{g})\\ &\qquad + \frac{1}{m+1}\int[(\bar{G}-u^{(m+2)/2}\bar{g})(-u^{-m/2}f') + (F-u^{(m+2)/2}f)(-u^{-m/2}\bar{g}')].\end{aligned}$$

Applying integration by parts to the last term shows that it equals

$$\frac{-m}{2(m+1)}\int u^{-(m+2)/2}(\bar{G}f+F\bar{g}) + \frac{2m+1}{m+1}\int f\bar{g}.$$

Hence the sum equals $\int f\bar{g}$ as required. □

Although this proof is straightforward, it would not be intuitive to guess that Λ_m preserves inner product without knowledge of its relationship with B. Furthermore, our proof that Λ_m is a surjection, and hence an isometry on $L^2([0,\infty))$, depends on the corresponding property of B.

THEOREM 2.3. $\Lambda_m\colon L^2([0,\infty)) \to L^2([0,\infty))$ *is a surjection.*

Proof. Let $h \in L^2([0,\infty))$ and let $\tilde{h} \in L^2(\mathbb{C})$ be the associated radial function: $\tilde{h}(z) = h(|z|^2)$. Define $H(z) = \frac{z^{m+2}}{|z|^{m+2}}\tilde{h}(z)$. Then it is easy to check that $\tilde{H} = \bar{B}H \in \mathcal{R}_m^2$. To see this, write out the singular integral and observe how the argument of $\tilde{H}$ changes as that of z is varied. It follows that $H(z) = \frac{\bar{z}^2}{|z|^2}\tilde{\Lambda}_m \tilde{H}(z)$ and $\tilde{h}(z) = \tilde{\Lambda}_m\big(\frac{|z|^m}{\bar{z}^m}\tilde{H}\big)(z)$. It follows that h is in the range of Λ_m. $\square$

Thus Λ_m preserves inner product and is onto; hence it is an L^2 isometry. This completes the proof of Theorem 1.1.

3. The Weak-type Estimates for Λ_0 and B

$\Lambda := \Lambda_0$ is defined on $L^1([0,\infty))$ by

$$\begin{aligned}\Lambda f(u) &= \mathcal{H}f(u) - f(u)\\ &= \frac{1}{u}\int_0^u f(v)\,dv - f(u)\\ &= \int_0^1 f(uv)\,dv - f(u).\end{aligned}$$

Because the delta measure plays a crucial role in the optimization procedures of the proofs, it is useful to generalize the definition for arbitrary finite measures as follows. Let $d\mu(v) = f(v)\,dv + d\nu(v)$, where ν is mutually singular with respect to the Lebesgue measure. Define

$$\begin{aligned}\Lambda\mu(u) &= \mathcal{H}\mu(u) - f(u)\\ &= \frac{\mu([0,u])}{u} - f(u).\end{aligned}$$

Define the weak-type "norm" of a function F as

$$\|F\|_{w(1)} = \sup_{\lambda>0} \lambda m\{u \in [0,\infty) : |F(u)| > \lambda\};$$

here, as always, $dm(u)$ denotes the Lebesgue measure.

THEOREM 3.1. *Let $d\mu = f dv + d\nu$ be a positive finite measure* $[0,\infty)$. *Then*

$$\|\Lambda\mu\|_{w(1)} \leq \|\mu\|_1.$$

This is best possible.

Proof. Without loss of generality, assume $\|\mu\|_1 = 1$. Observe that:

(1) $\mathcal{H}\delta_0(x) = \frac{1}{x}$;
(2) since $\mu \geq 0$, $|\Lambda\mu| \leq \max(f, \mathcal{H}\mu)$;
(3) $\{\max(f, \mathcal{H}\mu) \geq \lambda\} = \{\max(f \wedge \lambda, \mathcal{H}\mu) \geq \lambda\}$; and
(4) if $\int_0^x f \leq \int_0^x g$ for all $x > 0$, then $\mathcal{H}f \leq \mathcal{H}g$.

In light of (3) and (4), it is optimizing to assume that $\mu = a\delta_0 + \lambda\chi_E$, where $m(E) = \frac{1-a}{\lambda}$. (In regions where $f < \lambda$, $|\Lambda\mu| \geq \lambda$ only if $\mathcal{H}\mu \geq \lambda$, and $\mathcal{H}\mu$ would be higher if we took the f values here—along with the singular ν measure—and stored them at 0.)

So assume that μ has this form with $f = \lambda\chi_E$. Then

$$\mathcal{H}\mu(u) = \frac{a}{u} + \frac{\lambda|E \cap [0,u)|}{u},$$

and $\mathcal{H}\mu(u) \geq \lambda$ if and only if $|E \cap [0,u)| \geq \frac{\lambda u - a}{\lambda}$. Since $u - |E \cap [0,u)| = |[0,u) \setminus E|$, it follows that $\mathcal{H}\mu(u) \geq \lambda$ if and only if $|[0,u) \setminus E| \leq \frac{a}{\lambda}$. Let u^* be the maximal such u. Note that $u^* \leq \frac{1}{\lambda}$.

The set $\{\max(f, \mathcal{H}\mu) \geq \lambda\}$ has an intersection within $[0,u^*]$ and also one outside of it. The outside part reduces to $E \setminus [0,u^*]$; the inside part is the union of $\{\mathcal{H}\mu \geq \lambda\}$ and $E \cap [0,u^*]$. This is the same as

$$(\{\mathcal{H}\mu \geq \lambda\} \setminus E) \cup (E \cap [0,u^*]) \subset ([0,u^*] \setminus E) \cup (E \cap [0,u^*]).$$

Therefore, joining the outside and inside shows that

$$\{\max(f, \mathcal{H}\mu) \geq \lambda\} \subset ([0,u^*] \setminus E) \cup E.$$

We thus conclude that

$$\begin{aligned} m(|\Lambda\mu| \geq \lambda) &\leq m(\max(f, \mathcal{H}\mu) \geq \lambda) \\ &\leq |[0,u^*] \setminus E| + |E| \\ &\leq \frac{a}{\lambda} + \frac{1-a}{\lambda} = 1. \end{aligned}$$

Since $\|\mu\|_1 = 1$, this completes the proof of the theorem. To show that this is optimal, just take μ to be the delta measure δ_0. □

Alternatively, we can argue as follows.

Proof. Let $a_1 = \sup\{x > 0 : \mathcal{H}\mu(x) \geq \lambda\}$. Then $\mathcal{H}\mu(a_1) = \lambda$ and $\mathcal{H}\mu(x) < \lambda$ for $x > a_1$. Store all of $\mu([0,a_1))$ in 0: that is, replace μ by $\mu([0,a_1])\delta_0 + \mu|_{(a_1,\infty)}$. Observe that the choice of a_1 implies $\mu([0,a_1]) = \lambda a_1$. In particular,

$$m\{x < a_1 : |\Lambda\mu(x)| \geq \lambda\} = a_1 = \frac{\mu([0,a_1))}{\lambda}. \tag{3.1}$$

Notice that $\{x > a_1 : |\Lambda\mu(x)| \geq \lambda\}$ is a subset of $\{x > a_1 : f(x) \geq \lambda\}$ because $|\Lambda f| \leq \max(f, \mathcal{H}\mu)$ and $\mathcal{H}\mu(x) < \lambda$ here. Therefore,

$$m\{x > a_1 : |\Lambda\mu(x)| \geq \lambda\} \leq \frac{1}{\lambda}\int_{a_1}^{\infty} f \leq \frac{\mu((a_1,\infty))}{\lambda}. \tag{3.2}$$

Combining (3.1) and (3.2) finishes the proof. □

Thus the positive measure case follows from a rather trivial-looking proof. The behavior of Λ on positive measures is controlled by that of the maximum of the Hardy operator and the identity, which fortunately is easy to analyze in the weak-type case. What happens, then, when the domain is the general finite measure space? It is easy to see that, since $|\mathcal{H}\mu| \leq \mathcal{H}|\mu|$, the weak-type norm of the Hardy operator remains the same (one) with the delta measure δ_0 as extremal. The Λ operator, on the other hand, appears much more formidable, and the subtracted $\mathcal{H} - I$ actually turns out to have a higher weak-type constant $\frac{1}{\log 2}$ as stated in Theorem 1.2. Fortunately again, the operator allows for a constructive optimization procedure (in the weak-type case) that can be taken to the end. We begin with obtaining upper estimates by finding the weak-type constants of a couple of bigger operators.

Consider Θ defined by

$$\Theta f(u) = f(u) + \frac{\|f\|_1}{u}.$$

It is clear that $|\Lambda f| \le \mathcal{H}|f| + |f| \le \Theta|f|$ and hence $\|\Lambda\|_{w(1)} \le \|\mathcal{H} + I\|_{w(1)} \le \|\Theta\|_{w(1)}$.

PROPOSITION 3.1. $\|\Theta\|_{w(1)}$ *is the unique solution of the equation*

$$\alpha - e^{\alpha-2} = 0.$$

Proof. Without loss of generality, assume that $\|f\|_1 = 1$ and is nonnegative. Note that $\Theta f(u) = f(u) + \frac{1}{u}$ is bigger than λ when $u \le \frac{1}{\lambda}$. To obtain the upper bound, consider the graph of $\frac{1}{u}$ and find α such that

$$\int_{1/\lambda}^{\alpha} \left(\lambda - \frac{1}{u}\right) du = 1.$$

Thus the contribution of

$$f(u) = \left(\lambda - \frac{1}{u}\right) \chi_{(1/\lambda,\alpha)}(u)$$

is kept completely separate from the place where $\frac{1}{u}$ exceeds λ. Moreover, this is the optimal way to choose f because, at points further away, more height is required to make $f(u) + \frac{1}{u} = \lambda$.

It is a simple exercise to show that α solves the equation $\alpha\lambda - e^{\alpha\lambda-2} = 0$. Since this choice of f is optimizing, the weak-type constant of Θ is as required. By a numerical calculation we see that the value is slightly smaller than 3.15. $\square$

A better estimate is obtained by considering $\mathcal{H} + I$.

PROPOSITION 3.2. $\|\Lambda\|_{w(1)} \le \|\mathcal{H} + I\|_{w(1)} = 2$.

Proof. It suffices to work with nonnegative functions. Observe that $\mathcal{H}f + f \le 2\max\{\mathcal{H}f, f\}$, and hence 2 is an upper bound (following the proof of Theorem 3.1). To see that it is also a lower bound, consider $f(u) = \frac{\lambda}{2}\chi_{[0,1]}$. $\square$

The next theorem estimates $\|\Lambda\|_{w(1)}$ from below.

THEOREM 3.2. $\|\Lambda\|_{w(1)} \ge \frac{1}{\log 2}$.

Proof. In Theorem 3.1 it is shown that the positive function case is optimized by the delta measure. Then $\Lambda\delta_0(u) = \mathcal{H}\delta_0(u) = \frac{1}{u}$. Starting at $u = \frac{1}{\lambda}$, a negative function $f = f\chi_{(1/\lambda,\beta)}$ can be added so that $\Lambda\mu = \mathcal{H}\mu - f = \lambda$ in $\left(\frac{1}{\lambda}, \beta\right)$. This addition increases both the measure in consideration and the total integral. It is shown that the optimal $\beta = \frac{2}{\lambda}$.

Let $d\mu(u) = \delta_0(u) + f(u)\,du$ satisfy the following conditions:

(1) for $0 < t < \beta - \frac{1}{\lambda}$,

$$\frac{1 + \int_{1/\lambda}^{1/\lambda + t} f(v)\,dv}{1/\lambda + t} - f\left(\frac{1}{\lambda} + t\right) = \lambda;$$

(2) $f(u) = 0$ for $u > \beta$.

The unique solution is

$$d\mu(u) = \delta_0(u) - \lambda \log(\lambda u)\chi_{(1/\lambda,\beta)}(u)\,du. \tag{3.3}$$

It can be verified that

$$\frac{\lambda m\{|\Lambda\mu| \geq \lambda\}}{\|\mu\|} = \frac{\beta\lambda}{2 - \beta\lambda + \beta\lambda\log\beta\lambda}, \tag{3.4}$$

which acquires maximum value $\frac{1}{\log 2}$ at $\beta = \frac{2}{\lambda}$. Hence $\frac{1}{\log 2}$ is a lower bound for the weak-type constant. □

The next result is an immediate consequence of Theorem 3.2.

COROLLARY 3.1. $\|B\|_{w(1)} \geq \frac{1}{\log 2}$.

Proof. Let $\tilde{f}(z) = f(|z|^2)$ be radial. We have

$$\begin{aligned}
m(|B\tilde{f}| \geq \lambda) &= m(|\tilde{\Lambda}\tilde{f}| \geq \lambda) \\
&= \int_0^\infty \chi_{\{r:|\tilde{\Lambda}\tilde{f}(re^{i0})| \geq \lambda\}}(s) 2\pi s\,ds \\
&= \int_0^\infty \chi_{\{r:|\Lambda f(r^2)| \geq \lambda\}}(s) 2\pi s\,ds \\
&= \pi \int_0^\infty \chi_{\{r:|\Lambda f(r^2)| \geq \lambda\}}(\sqrt{s}\,)\,ds \\
&= \pi \int_0^\infty \chi_{\{r:|\Lambda f(r)| \geq \lambda\}}(s)\,ds \\
&= \pi |\{|\Lambda f| \geq \lambda\}|.
\end{aligned}$$

Now observe that $\|\tilde{f}\|_{L^1(\mathbb{R}^2)} = \pi \|f\|_{L^1(0,\infty)}$. The proof follows from the corresponding lower-bound theorem for Λ. □

This also proves the first inequality of Theorem 1.5. The most difficult work is in showing that $\frac{1}{\log 2}$ is also the upper bound for the weak-type constant of Λ. Luckily the most straightforward approach works. We are able to establish an optimization process that starts with an arbitrary function and ends up with a measure that has a form analogous to (3.3). That is, the optimizing representative will have a delta measure followed by a logarithmic negative function. Then Theorem 3.2 shows that the function in (3.3) is the weak-type extremal, and hence $\frac{1}{\log 2}$ is the upper bound.

NOTATION. For convenience, f is used to denote the entire measure μ and not just the absolutely continuous part.

THEOREM 3.3. *Let f be real-valued and integrable. Then*

$$\|\Lambda f\|_{w(1)} \leq \frac{1}{\log 2}\|f\|_1.$$

This is best possible.

Proof. Start, without loss of generality, with a continuous function f on $[0,\infty)$. The proof proceeds in fourteen steps as follows.

1. A slight modification will ensure that $\mathcal{H}f$ is nonnegative. Just change the sign of f, in the intervals where it is negative, in a sequential manner. (Changing f at x does not affect values at $y < x$ but could affect values at $y > x$; this is why the procedure should be done from left to right.) The resulting function is piecewise continuous, but neither $|f|$ nor $|\Lambda f|$ has been changed. *In all the modifications that follow, $\mathcal{H}f$ remains nonzero and generally increases.* This is reiterated at various points hereafter.

2. *Type-1 optimization.* Identify the maximal a_1 satisfying $\mathcal{H}f(a_1) = \lambda$. Without loss of generality, assume $a_1 > 0$. (Otherwise add an infinitesimal value at 0.) Observe that by storing $f\chi_{(0,a_1)}$ at 0, as $\left(\int_0^{a_1} f\right)\delta_0$, we have $\mathcal{H}f(x) > \lambda$ for $0 < x < a_1$ and values after a_1 remain unchanged.

3. Partition as $0 < a_1 < a_2 < \cdots \to \infty$, where $f < 0$ in (a_1, a_2), $f \geq 0$ in (a_2, a_3), and so on in an alternating manner. Without loss of generality, assume that $f < 0$ immediately after a_1; otherwise, an infinitesimal modification will obtain this.

4. *Type-2 optimization.* It is shown next that f may be assumed to equal either λ or 0 wherever it is nonnegative. Suppose $f \geq 0$ in $I_i = (a_i, a_{i+1})$ for some $i \neq 0$. Then $\mathcal{H}f(u) < \lambda$ in this region (since $u > a_1$), so if $|\Lambda f| \geq \lambda$ then $f \geq \lambda$. The upper estimate will be obtained by considering only f; hence a modification that increases $\mathcal{H}f$ without changing where $f \geq \lambda$ (or properly compensating for any changes) will not adversely affect the net measure.

5. For this purpose, replace f here with $f \wedge \lambda\chi_{f\geq\lambda}$ and store all of the removed integral at 0. The storage at 0 *increases $\mathcal{H}f$* everywhere. The estimation in regions where $f < 0$ will be improved because $|\Lambda f|$ increases there. Where $f \geq 0$, as said before, since only f is considered, it follows that the maximal region where the original function was $\geq \lambda$ will still remain the same (with $= \lambda$). As for the possibility that in these regions more points occur (owing to modification) where $\mathcal{H}f > \lambda$, the issue is avoided by repeating Type-1 optimization with this new function. So a new a_1 starts off the show, with all of $(0, a_1)$ being included in the measure count and after which $\mathcal{H}f$ may play a role only where $f < 0$. In the end, f starts as a delta measure and then, after a_1 switches between $-$ and $+$, f takes as its value either 0 or λ in $+$ regions. Note that $\mathcal{H}f$ remains nonnegative throughout and is less than λ after a_1.

6. The objective is to initiate a reduction process that eliminates all but $[a_0, a_1)$, $[a_1, a_2)$, and $[a_2, \infty)$. Then a simple optimization is done to prove the theorem.

7. MOD 1. First remove f in (a_2, a_3) (where $f \geq 0$) and make it 0 there. The integral loss would be $\beta = \int_{a_2}^{a_3} f$: store this instead at 0. Observe that $\frac{\beta}{\lambda} \leq a_3 - a_2$ since $f \leq \lambda$ in this region by Type-2 optimization. The values after a_3 are unchanged for f and $\mathcal{H}f$; however, the $\mathcal{H}f$ has *increased* in $(0, a_3)$.

8. Two things must be considered. First, there may have been points in (a_2, a_3) where $f = \lambda$, and these must be compensated. Second, the modification has made $\mathcal{H}f(a_1) > \lambda$. These are fixed in the next step.

9. *Compensation for potential loss of* $\{f = \lambda\}$. We remark that $\frac{\beta}{\lambda} = \frac{1}{\lambda}\int_{a_2}^{a_3} f \leq a_3 - a_2$ is the maximal measure in the region of removal where f could have equaled λ; we would obtain this measure if and only if $f = \lambda$ wherever it is nonzero. Now make the second modification, MOD 2: shift the negative f in (a_1, a_2) to $\left(a_1 + \frac{\beta}{\lambda}, a_2 + \frac{\beta}{\lambda}\right)$ without any changes (just translation). Since we have stored at 0 the measure $\left(\lambda a_1 + \frac{\beta}{\lambda}\right)\delta_0$ and since $\mathcal{H}\left(\left(\lambda a_1 + \frac{\beta}{\lambda}\right)\delta_0\right)\left(a_1 + \frac{\beta}{\lambda}\right) = \lambda$, the MOD 2 acquires the entire set $\left(a_1, a_1 + \frac{\beta}{\lambda}\right)$ of measure $\frac{\beta}{\lambda}$ within $\{|\Lambda f_{\text{new}}| \geq \lambda\}$, where f_{new} is the modified f. Thus the potential loss of $\{f = \lambda\} \cap (a_2, a_3)$ that occurred from MOD 1 is optimally compensated by MOD 2.

10. It is clear that the "new a_1" should equal the present $a_1 + \frac{\beta}{\lambda}$. This change in notation is made in step 12. Next it is verified that MOD 2 did not reduce the subset in (a_1, a_2) where $\Lambda f \geq \lambda$. That is, it is verified that

$$|\{|\Lambda f| \geq \lambda\} \cap (a_1, a_2)| \leq \left|\{|\Lambda f_{\text{new}}| \geq \lambda\} \cap \left(a_1 + \frac{\beta}{\lambda}, a_2 + \frac{\beta}{\lambda}\right)\right|.$$

As $f_{\text{new}}\left(x + \frac{\beta}{\lambda}\right) = f(x)$ for $a_1 < x < a_2$, we must check that $\mathcal{H}f_{\text{new}}\left(a_1 + \frac{\beta}{\lambda} + t\right) \geq \mathcal{H}f(a_1 + t)$ for $0 < t < a_2 - a_1$. In other words, is

$$\frac{(\lambda a_1 + \beta) + \gamma}{a_1 + \beta/\lambda + t} \geq \frac{\lambda a_1 + \gamma}{a_1 + t}?$$

Here λa_1 is the original measure stored at 0 and hence integral of f up to a_1 (recall that $\mathcal{H}f(a_1) = \lambda$; the new a_1 after verifications are done will be $a_1 + \frac{\beta}{\lambda}$, but not yet). Observe that γ is the integral of f in $(a_1, a_1 + t)$ and of f_{new} in $\left(a_1 + \frac{\beta}{\lambda}, a_1 + \frac{\beta}{\lambda} + t\right)$, and γ is negative in value. Thus:

$$\begin{aligned}
((\lambda a_1 + \beta) + \gamma)(a_1 + t) \quad &? \quad (\lambda a_1 + \gamma)\left(a_1 + \frac{\beta}{\lambda} + t\right), \\
\beta(a_1 + t), \quad &? \quad \frac{\beta}{\lambda}(\lambda a_1 + \gamma), \\
a_1 + t \quad &? \quad a_1 + \frac{\gamma}{\lambda}, \\
t \quad &? \quad \frac{\gamma}{\lambda}.
\end{aligned}$$

Since γ is negative, each ? mark may be replaced with a $\geq$ sign. In other words, the shifting of f is beneficial and increases the measure of set where $\Lambda f \geq \lambda$. Hence f_{new} is indeed a good optimization.

11. After MOD 1 and MOD 2, we have $f_{\text{new}} < 0$ in $\left(a_1 + \frac{\beta}{\lambda}, a_2 + \frac{\beta}{\lambda}\right)$, $f_{\text{new}} = 0$ in $\left(a_2 + \frac{\beta}{\lambda}, a_3\right)$, and $f_{\text{new}} < 0$ in (a_3, a_4). MOD 3: Switch $\left(a_2 + \frac{\beta}{\lambda}, a_3\right)$ and

(a_3, a_4) (along with the corresponding function values). This joins the two alternate negative intervals into one: $\big(a_1 + \frac{\beta}{\lambda}, a_2 + \frac{\beta}{\lambda} + (a_4 - a_3)\big)$ and moves the intermediate 0 interval to $\big(a_2 + \frac{\beta}{\lambda} + (a_4 - a_3), a_4\big)$. Denote f_{new} as the function after MOD 1 and MOD 2 and denote $\tilde{f}_{\text{new}}$ as the function after MOD 3 as well. Then, for $x < a_4 - a_3$:

(a) $a_2 + \frac{\beta}{\lambda} \le a_3$;

(b) $\tilde{f}_{\text{new}}\big(a_2 + \frac{\beta}{\lambda} + x\big) = f_{\text{new}}(a_3 + x)$;

(c)

$$\begin{aligned}\mathcal{H}\tilde{f}_{\text{new}}\left(a_2 + \frac{\beta}{\lambda} + x\right) &= \frac{1}{a_2 + \beta/\lambda + x}\int_0^{a_2+\beta/\lambda+x} \tilde{f}_{\text{new}} \\ &= \frac{1}{a_2 + \beta/\lambda + x}\int_0^{a_3+x} f_{\text{new}} \\ &\ge \frac{1}{a_3 + x}\int_0^{a_3+x} f_{\text{new}} = \mathcal{H}f_{\text{new}}(a_3 + x);\end{aligned}$$

(d) both $\mathcal{H}\tilde{f}_{\text{new}}$ and $\mathcal{H}f_{\text{new}}$ are less than or equal to λ in their corresponding 0-intervals where $\tilde{f}_{\text{new}} = f_{\text{new}} = 0$.

These facts imply that

$$m\{|\Lambda\tilde{f}_{\text{new}}| \ge \lambda\} \ge m\{|\Lambda f_{\text{new}}| \ge \lambda\}.$$

Thus MOD 3 is also optimizing.

12. Relabel f as $\tilde{f}_{\text{new}}$, a_1 as $a_1 + \frac{\beta}{\lambda}$, a_2 as $a_2 + \frac{\beta}{\lambda} + (a_4 - a_3)$, and so on, following the rule described in step 3.

13. Repeat steps 3–12 indefinitely to obtain an optimizing f of the following form: it is $\lambda a_1 \delta_0$ at 0, has strictly negative values in some (a_1, a_2), and is 0 in (a_2, ∞). Since $\mathcal{H}f$ is nonnegative throughout, the absolute value of the integral in (a_1, a_2) is less than or equal to λa_1.

14. But at this stage the question remains: What is the optimal way to decrease f in (a_1, a_2)? The log decrease of Theorem 3.2 is the right answer.

(a) First, note that if $|\Lambda f| > \lambda$ anywhere in (a_1, a_2), then there is loss in measure of where $\Lambda f \ge \lambda$. (The excess function pulls down $\mathcal{H}f$ more quickly and dissipates more quickly its allowed integral value.) So f should satisfy $\Lambda f \le \lambda$. Now suppose in a subinterval $(x, x + \varepsilon)$ that $\Lambda f < \lambda$.

(b) Set $f = 0$ in $(x, x + \varepsilon)$. Observe that this modification (i) does not change the values of f, $\mathcal{H}f$, and Λf to the left of x, (ii) increases $\mathcal{H}f$ and so $\Lambda f = \mathcal{H}f - f$ to the right of $x + \varepsilon$, and (iii) leaves $\mathcal{H}f \ge 0$ everywhere. Although we have increased $\mathcal{H}f$ in $(x, x + \varepsilon)$ (by eliminating negative values of f), the values of $\mathcal{H}f$ remain less than or equal to λ. This is because $x > a_1$ and $\mathcal{H}f(a_1) = \lambda$; hence, for the modified f_{new},

$$\mathcal{H}f_{\text{new}}(x + \delta) = \frac{1}{x + \delta}\int_0^x f_{\text{new}} \le \mathcal{H}f(x) \le \lambda.$$

In conclusion, it is favorable to set $f = 0$ in $(x, x + \varepsilon)$, and we do so.

(c) Next, it is optimizing (as shown before) to push out the intervening 0 regions of f; so repeating the previous optimizing procedures shows that the best option is to have $|\Lambda f| = \lambda$ in (a_1, a_2). This is achieved by having f decrease logarithmically to an optimizing choice of a_2, which is precisely how the lower bound is computed in Theorem 3.2. Thus the lower-bound example is also an upper-bound extremal, and the weak-type constant $\|\Lambda\|_{w(1)} = \frac{1}{\log 2}$.

This completes the proof of Theorem 3.3. □

Theorem 3.3 and Corollary 3.1 together complete the proof of Theorem 1.5. An interesting problem is to extend this result for complex-valued radial functions.

4. $\|\Lambda_0\|_p$ for $1 < p < 2$

Baernstein and Smith introduce in [2] the family $\mathcal{S}$ of *stretch* functions of the form $g(r)e^{i\theta}$, where $g\colon [0,\infty) \to [0,\infty)$ is nonnegative and locally Lipschitz, $g(0) = g(0+) = 0$, and $\lim_{r\to\infty} g(r) = 0$. Consider the Cauchy–Riemann differential operators

$$\partial = \frac{\partial_x - i\partial_y}{2}, \qquad \bar{\partial} = \frac{\partial_x + i\partial_y}{2}.$$

Then, for a stretch function $h(re^{i\theta}) = g(r)e^{i\theta}$,

$$\tilde{f} = \partial h = \frac{1}{2}(g' + r^{-1}g) \quad \text{and} \quad B\tilde{f} = \bar{\partial} h = \frac{e^{i2\theta}}{2}(g' - r^{-1}g).$$

It is shown in [2] that $\|B\tilde{f}\|_p \le (p^* - 1)\|\tilde{f}\|_p$ for $1 < p < 2$. Hence Iwaniec's conjecture is verified (when $1 < p < 2$) for the subspace

$$\partial\mathcal{S} = \{\partial h \in L^p(\mathbb{C};\mathbb{R}) : h \in \mathcal{S}\} \subseteq \mathcal{R}_0^p(\mathbb{C}).$$

Define

$$\tilde{\mathcal{R}}_0^p(\mathbb{C}) = \{\tilde{f} \in \mathcal{R}_0^p(\mathbb{C}) : \mathcal{H}f \ge 0\}$$

and $\partial\mathcal{S}^\star = \partial\mathcal{S} \cap \tilde{\mathcal{R}}_0^p(\mathbb{C})$.

LEMMA 4.1. *$\partial\mathcal{S}^\star$ is L^p dense in $\tilde{\mathcal{R}}_0^p(\mathbb{C})$.*

Proof. Let $\tilde{f} \in \tilde{\mathcal{R}}_0^p(\mathbb{C}) \cap C_c^\infty(\mathbb{C})$. As before, let $f\colon [0,\infty) \to \mathbb{R}$ satisfy $\tilde{f}(z) = f(|z|^2)$; then $\mathcal{H}f \ge 0$. Define $g(r) = r\mathcal{H}f(r^2)$. Then

$$\frac{1}{2}(g'(r) + r^{-1}g(r)) = f(r^2) = \tilde{f}(z).$$

Thus $\partial\mathcal{S}^\star \supset \tilde{\mathcal{R}}_0^p(\mathbb{C}) \cap C_c^\infty(\mathbb{C})$ and hence is dense in $\tilde{\mathcal{R}}_0^p(\mathbb{C})$. □

THEOREM 4.1. *For $1 < p < 2$, $\|\Lambda_0\|_{L^p([0,\infty),\mathbb{R})} = p^* - 1 = \frac{1}{p-1}$.*

Proof. It suffices to consider the action of Λ_0 on functions satisfying $\mathcal{H}f \ge 0$ or, equivalently, the action of B on $\tilde{\mathcal{R}}_0^p(\mathbb{C})$. In particular, $\|B\|_{\mathcal{R}_0^p} = \|B\|_{\tilde{\mathcal{R}}_0^p} = \|\Lambda_0\|_p$. Refer to step 1 of the proof of Theorem 3.3.

By Lemma 4.1, $\partial\mathcal{S}^\star$ is dense in $\tilde{\mathcal{R}}_0^p(\mathbb{C})$. Therefore, since Iwaniec's conjecture holds on $\partial\mathcal{S}^\star$, it follows that $\|B\|_{\tilde{\mathcal{R}}_0^p} = \|\Lambda_0\|_p \leq p^* - 1$. The extremal family $f_\varepsilon(u) = u^{\varepsilon-1/p}\chi_{(0,1)}(u)$ verifies that $p^* - 1$ is also the lower bound. This completes the proof of the theorem. □

REMARK 4.1. (1) Theorems 3.3 and 4.1 also hold for the Beurling–Ahlfors transform acting on functions of the form $\sum_k f_k \chi_{B_k}$, where the B_k are disjoint disks and each f_k is radially supported in B_k and with integral 0. This is because Bf_k is supported in B_k for each k.

(2) Instead of relying on the Baerstein–Smith result for stretch functions, it may be possible to apply parallel techniques (i.e., to prove Sverák's conjecture; see [2]) directly to Λ_0. The authors were able to verify this for f smooth and decreasing in $[0, \infty)$ and believe that some standard analysis should lead to the general result.

4.1. An Estimate for $\|\Lambda_0\|_{L^p([0,\infty),\mathbb{C})}$

PROPOSITION 4.1. *Given $F : [0, a) \to \mathbb{C}$, smooth and compactly supported, there exists a $G = g_1 + ig_2$ such that $\mathcal{H}g_2(t) < \varepsilon$,*

$$|\Lambda_0 F(t)| = |\Lambda_0 G(t)|, \quad \textit{and} \quad |F(t)| = |G(t)|$$

for each $t \in [0, a)$.

Proof. Let $F = f_1 + if_2$. First change the sign of f_2 in all intervals where $\mathcal{H}f_2 < 0$. Then change the sign of f_1 in all intervals where $\mathcal{H}f_1 < 0$. Observe that the modifications ensure that $\mathcal{H}f_1$ and $\mathcal{H}f_2$ are nonnegative, so $\mathcal{H}F$ maps into the first quadrant of the plane. Moreover, $|\Lambda_0 F|$ and $|F|$ remain unchanged because the signs of $\mathcal{H}f_i$ and f_i are modified simultaneously.

Let k initially equal 2. Let $F_1 = F$.

(1) Multiply F_{k-1} by $e^{i(\pi/2^k)}$. Then the range of the function is contained within the arguments $-\pi/2^k$ and $\pi/2^k$.
(2) If $F_{k-1} = h_1 + ih_2$, then change the sign of h_2 wherever $\mathcal{H}h_2 < 0$. Let this modified function be F_k. Note that $\mathcal{H}F_k$ has arguments between 0 and $\pi/2^k$.
(3) Add 1 to k.

Repeat these three steps until $\|\mathcal{H}h_2\|_\infty < \varepsilon$ (where $F_k = h_1 + ih_2$). Let $G = F_k$. Note that $|\Lambda_0 G| = |\Lambda_0 F|$ and similarly $|G| = |F|$. □

Observe that, since G is bounded and compactly supported, $\|\mathcal{H}g_2\|_p$ may be taken arbitrarily small by choosing ε small enough. Therefore,

$$\|\Lambda_0 F\|_p = \|\Lambda_0 G\|_p \approx \|\Lambda_0 g_1 + ig_2\|_p.$$

Hence the norm of Λ_0 on complex functions over $[0, \infty)$ is no greater than the norm of the operator L defined by $LF = \Lambda_0 f_1 + if_2$. In fact, it is equal because for any real-valued function f there exists an $\tilde{f}$ with $|\mathcal{H}\tilde{f}|$ small. This is obtained by multiplying f by a suitable sharply oscillating function with absolute value 1.

THEOREM 4.2. *Let $f = f_1 + if_2 \in L^p([0,\infty))$ for $1 < p < 2$. Then*

$$\|Lf\|_p = \|\Lambda_0 f_1 + if_2\|_p \le \frac{C_p}{p-1}\|f\|_p, \tag{4.1}$$

where

$$C_p = \frac{(1+(p-1)^{2p/(2-p)})^{1/p}}{\sqrt{1+(p-1)^{2p/(2-p)}}}. \tag{4.2}$$

This C_p is slightly larger than the expected value of 1. The maximum is approximately 1.003074795 at $p = 1.3224$ (checked for various p-values using Maple). The main estimate in the proof is not sharp enough to give 1.

Proof. The key estimate is

$$|\Lambda_0 f_1 + if_2|^p \le |\Lambda_0 f_1|^p + |f_2|^p.$$

Since $\int |\Lambda_0 f_1|^p \le (p^*-1)^p \int |f_1|^p$, the goal becomes to show that

$$(p^*-1)^p \int |f_1|^p + \int |f_2|^p \le C_p (p^*-1)^p \int |f_1 + if_2|^p$$

or, equivalently,

$$\int \left(|f_1|^p + (p-1)^p |f_2|^p - C_p\left(\sqrt{f_1^2+f_2^2}\,\right)^p\right) \le 0.$$

By maximizing the function $s^p + (p-1)^p\left(\sqrt{1-s^2}\,\right)^p$ on $(0,1)$, it can be deduced that C_p equals the value in (4.2). (Note that the functions can be chosen so that this is best possible.) $\square$

5. The L^p Norm Estimates for Λ_m, $m \ge 1$

In this section, we give upper and lower estimates for the L^p norm of $\mathcal{H}_m$, of Λ_m, and hence of B restricted to $\mathcal{R}_m^p$ spaces. As for the special case $m = 0$, the norm of the general Hardy average operator $\mathcal{H}_m$ is amenable to computation via the standard Minkowski integral inequality. And the same extremal class works for all m.

PROPOSITION 5.1. $\|\mathcal{H}_m\|_p = \frac{p(2m+2)}{p(m+2)-2}$.

Proof.

$$\begin{aligned}
\|\mathcal{H}_m f\|_p &= \left(\int_0^\infty \left|\int_0^1 f(uv)(m+1)v^{m/2}\,dv\right|^p du\right)^{1/p} \\
&\le \int_0^1 \left(\int_0^\infty |f(uv)|^p\,du\right)^{1/p} (m+1)v^{m/2}\,dv \\
&= \int_0^1 (m+1)v^{m/2-1/p}\,dv \|f\|_p \\
&= \frac{p(2m+2)}{p(m+2)-2}\|f\|_p.
\end{aligned}$$

To show that the same constant is also a lower bound, verify that

$$f_\varepsilon(x) = x^{-1/p+\varepsilon}\chi_{(0,1)} \tag{5.1}$$

extremizes the norm as $\varepsilon \to 0$. □

An immediate corollary is as follows.

COROLLARY 5.1.

$$\|B\|_{\mathcal{R}^p_m} = \|\Lambda_m\|_p \le \frac{(3m+4)p-2}{(m+2)p-2}. \tag{5.2}$$

The first equality follows from the representation Theorem 2.1 and the second inequality from the fact that $\|\Lambda_m\| = \|\mathcal{H}_m - I\| \le \|\mathcal{H}_m\| + 1$. It is evident that this upper bound is not the correct constant, since for $p = 2$ the value does not equal 1 (though Λ_m is an L^2 isometry). A lower bound

$$\|\Lambda_m\|_p \ge \frac{mp+2}{p(m+2)-2}, \quad 1 < p < \infty, \tag{5.3}$$

may be obtained by considering the extremal family in (5.1) for $\mathcal{H}_m$. Recall that Iwaniec's conjecture predicts $\|B\|_p = \frac{1}{p-1}$ for $1 < p < 2$, and observe that $\|\Lambda_0\|_p = \frac{1}{p-1} = \frac{p}{p-1} - 1$ for $1 < p < 2$. Since $\|B\|_{\mathcal{R}^p_0} = \|\Lambda_0\|_p \le \|B\|_p$, we have the following reasonable conjecture.

CONJECTURE 1. *For* $1 < p < 2$,

$$\|\Lambda_m\|_p = \|\mathcal{H}_m - I\|_p = \|\mathcal{H}_m\|_p - 1 = \frac{mp+2}{p(m+2)-2} \tag{5.4}$$

with the extremal family as given in (5.1).

Naturally one might consider generalizing the techniques for the case $m = 0$ (in [2]) to verify Conjecture 1. The conjecture is probably false for $p > 2$. In fact, it is easy to see that

$$\|\Lambda_m\|_\infty = \frac{2(m+1)}{m+2} + 1. \tag{5.5}$$

An extremal is the function $\chi_{[0,1)} - \chi_{[1,2)}$. At present, we do not have a conjecture for $p > 2$.

REMARK 5.1. For the operator Λ_m (and Λ_0), we may conjecture that the main results of the paper hold when Λ_m acts on complex-valued functions. This is true for the Hardy operators, but the subtraction by identity makes the problem nontrivial.

6. Concluding Remarks

Observe from Theorem 4.1 that the extremals for $\|B\|_p$, $1 < p < 2$, are expected to be radial, nonnegative, and decreasing. For $p = 1$, we have shown that the extremals for $\|B\|_{w(1)}$ are definitely *not* nonnegative and radially decreasing. However, it may be expected that—at least in the real-valued setting—the extremals remain real-valued and radial. Therefore, a conjecture based on Theorem 1.2 could be as follows.

CONJECTURE 2. *For $f \in L^1(\mathbb{C}; \mathbb{R})$ and $\lambda > 0$,*

$$\lambda m\{z \in \mathbb{C} : |Bf(z)| > \lambda\} \leq \frac{1}{\log 2}\|f\|_1.$$

This is best possible.

A natural question is whether the radial result can lead to a proof for Iwaniec's conjecture. It suffices to show that the conjecture is true for the class of simple functions of the form $\sum_i a_i \chi_{B_i}$, where $a_i \in \mathbb{C}$ and $\{B_i\}$ is a finite collection of disjoint unit disks. Given any fixed function f in this family, Bf can be explicitly computed; however, even the case $f = a_1\chi_{B_1} + a_2\chi_{B_2}$ appears impossible to verify because integration techniques are lacking. Still, owing to the explicit nature of f and Bf, this class may be amenable to computer evaluations similar to those performed in [2].

ACKNOWLEDGMENTS. We are very grateful to an anonymous referee for a careful reading of this paper and for many comments that improved its readability. We also thank James Gill for his important observation mentioned in Remark 1.1.

References

[1] K. Astala, *Area distortion of quasiconformal mappings,* Acta Math. 173 (1994), 37–60.

[2] A. Baernstein II and S. Montgomery-Smith, *Some conjectures about integral means of ∂f and $\bar{\partial} f$,* Complex analysis and differential equations (Uppsala, 1997), Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., 64, pp. 92–109, Uppsala Univ., Uppsala, 1999.

[3] R. Bañuelos and P. Janakiraman, *L^p-bounds for the Beurling–Ahlfors transform,* Trans. Amer. Math. Soc. 360 (2008), 3603–3612.

[4] R. Bañuelos and P. Méndez-Hernández, *Space-time Brownian motion and the Beurling–Ahlfors transform,* Indiana Univ. Math. J. 52 (2003), 981–990.

[5] R. Bañuelos and G. Wang, *Sharp inequalities for martingales with applications to the Beurling–Ahlfors and Riesz transforms,* Duke Math. J. 80 (1995), 575–600.

[6] D. L. Burkholder, *Boundary value problems and sharp inequalities for Martingale transforms,* Ann. Probab. 12 (1984), 647–702.

[7] ———, *Sharp inequalities for martingales and stochastic integrals,* Colloque Paul Lévy sur les processus stochastiques (Palaiseau, 1987), Astérisque 157–158 (1988), 75–94.

[8] ———, *Martingales and singular integrals in Banach spaces,* Handbook of the geometry of Banach spaces (W. B. Johnson, J. Lindenstrauss, eds.), vol. 1, pp. 233–269, Elsevier, Amsterdam, 2001.

[9] S. Donaldson and D. Sullivan, *Quasiconformal 4-manifolds,* Acta Math. 163 (1989), 181–252.

[10] O. Dragičević and A. Volberg, *Bellman functions, Littlewood–Paley estimates and asymptotics for the Ahlfors–Beurling operator in $L^p(\mathbb{C})$*, Indiana Univ. Math. J. 54 (2005), 971–995.

[11] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities,* Cambridge Univ. Press, Cambridge, 1934.
[12] T. Iwaniec, *Extremal inequalities in Sobolev spaces and quasiconformal mappings,* Z. Anal. Anwendungen 1 (1982), 1–16.
[13] ———, *The best constant in a BMO-inequality for the Beurling–Ahlfors transform,* Michigan Math. J. 33 (1986), 387–394.
[14] ———, *L^p-theory of quasiregular mappings. Quasiconformal space mappings,* Lecture Notes in Math., 1508, Springer-Verlag, Berlin, 1992.
[15] T. Iwaniec and G. Martin, *Quasiregular mappings in even dimensions,* Acta Math. 170 (1993), 29–81.
[16] O. Lehto, *Remarks on the integrability of the derivatives of quasiconformal mappings,* Ann. Acad. Sci. Fenn. Ser. A I Math. 371 (1965), 3–8.
[17] F. Nazarov and A. Volberg, *Heat extension of the Beurling operator and estimates for its norm,* St. Petersburg Math. J. 15 (2004), 563–573.
[18] B. G. Pachpatte, *Mathematical inequalities,* North-Holland Math. Library, 67, Elsevier, Amsterdam, 2005.

R. Bañuelos
Department of Mathematics
Purdue University
West Lafayette, IN 47907

banuelos@math.purdue.edu

P. Janakiraman
Department of Mathematics
Purdue University
West Lafayette, IN 47907

pjanakir@math.purdue.edu