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A Counterexample to Uniform Approximation on Totally Real Manifolds in $\mathbb{C}^3$

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1. Introduction and Main Result

Let X be a closed subset of $\mathbb{C}^n$. We say that X admits *uniform approximation* if for any continuous function $f \in \mathcal{C}(X)$ and for all $\varepsilon > 0$ there exists a holomorphic function $g \in \mathcal{O}(\mathbb{C}^n)$ such that $\sup_{x\in X}|f(x) - g(x)| < \varepsilon$. We will be concerned with the case where X is a totally real manifold: a smooth manifold whose tangent space at no point contains a nontrivial complex subspace (in which case there are no Cauchy–Riemann equations on X!).

One of the first results in approximation theory in complex analysis was the well-known theorem of Weierstrass [15]: If $X \subset \mathbb{C}$ is an in interval of the real line, then X admits uniform approximation. This result was generalized by Carleman [4] to the effect that X could be taken to be the entire real line in the complex plane. A complete characterization of the subsets of $\mathbb{C}$ that admit uniform approximation now exists: X admits uniform approximation if and only if (i) $\mathbb{C} \setminus X$ has no relatively compact components, (ii) X has no interior, and (iii) $\mathbb{C} \setminus X$ is locally connected at infinity (see e.g. [14]). In particular we have that a closed smooth 1-dimensional submanifold of the complex plane admits uniform approximation.

When considering the state of affairs in several complex variables it is natural to consider the compact and noncompact cases separately. Hörmander and Wermer showed that if X is a polynomially convex compact totally real manifold then X admits uniform approximation [7]. It is also possible to get $\mathcal{C}^k$-approximation on X—depending on the smoothness of X and the function f [11].

In the noncompact case the situation is best understood if the "size" of X is small. Following the work of Alexander [1], Gauthier and Zeron [5] have shown that uniform approximation is possible if $X \subset \mathbb{C}^n$ is a locally rectifiable dendrite—that is, if it is closed and connected and has locally finite 1-dimensional measure and if $\check{H}^1(X,\mathbb{Z}) = 0$. In the case of "bigger" sets it is known that one can take $X = \mathbb{R}^n \subset \mathbb{C}^n$ [6; 12]. Manne [9] has shown that if X is the union of two totally real planes such that X is polynomially convex, then X admits uniform approximation.

The purpose of this paper is to produce an example that demonstrates the following theorem.

Theorem 1.1. *There exists a proper $\mathcal{C}^\infty$-smooth embedding $\phi\colon \mathbb{R}^2 \to \mathbb{C}^3$ such that the following statements hold for $M := \phi(\mathbb{R}^2)$:*

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(A) *M is totally real and*
(B) *M is polynomially convex, yet*
(C) *M does not admit uniform approximation.*

That the set M is polynomially convex means that M has an exhaustion of polynomially convex compact sets.

We note that even though uniform approximation *by entire functions* fails, Manne [8] has shown that there exists a neighborhood Ω of M such that uniform approximation is possible by functions holomorphic on Ω. Approximation by functions on varying neighborhoods was shown by Nunemacher in [10].

2. Sketch of the Construction

Before plunging into the details we shall try to explain the main idea of the construction. We will choose a smoothly bounded (topological) disk D_0 contained in the unit disk in the complex plane (D_0 will agree with the unit disk on the right half-plane). For each $j \in \mathbb{N}$ we define the shifted disk $D_j = D_0 + j \cdot \sqrt{2}$. Then $\partial D_j \cap \partial D_{j+1}$ will consist of exactly two points and $\partial D_j \cap \partial D_k$ will be empty when $k \neq j \pm 1$. For a specific choice of functions $f_j \in \mathcal{O}(D_j) \cap \mathcal{C}^\infty(\bar{D}_j)$ and a sequence of positive real numbers ε_j, we define the sets

(a) $A_j := \{(z, w) \in \mathbb{C}^2 \mid z \in D_j,\, w = f_j(z) + t \cdot i,\, -\varepsilon_j \leq t \leq \varepsilon_j\}$,
(b) $X_j := \{(z, w) \in \mathbb{C}^2 \mid z \in \partial D_j,\, w = f_j(z) + t \cdot i,\, -\varepsilon_j \leq t \leq \varepsilon_j\}$.

Then A_j is a family of holomorphic disks and X_j, which is a totally real annulus, is the union of their boundaries. Now the main point is to choose the domain D_0 and the functions f_j such that (i) $X_i \cap X_j = \emptyset$ for all $i \neq j$ and (ii) for each $j \geq 1$ there is a piece E_j of X_j such that $M_j := \overline{X_j \setminus E_j}$ is a polynomially convex disk and $E_j \subset A_{j-1}$. If we choose a point $q_j \in A_j$ for each $j \in \mathbb{N}$ then it is not hard to see that uniform approximation cannot be possible on the union $\tilde{M} := \bigcup_{j=0}^{\infty}(M_j \cup \{q_j\})$; this is demonstrated in Lemma 3.3.

In Lemma 3.1 we construct a function $f \in \mathcal{O}(\triangle) \cap \mathcal{C}^\infty(\bar{\triangle})$ with very specific information about the values of f over certain parts of $\bar{\triangle}$. We then use this f to define the domain D_0 and the functions f_j, and we proceed to show that we can likewise define sets A_j, M_j, E_j with the desired properties (Lemma 3.2). Finally, in Lemma 3.4 we show that in $\mathbb{C}^3$ we can find a totally real embedded $\mathbb{R}^2$ that is polynomially convex and contains $\tilde{M}$.

3. The Construction

Let $\triangle$ denote the open unit disk in $\mathbb{C}$ and let $\triangle(\sqrt{2})$ denote the open disk of radius 1 centered at the point $\sqrt{2}$. We choose two subsets of $\triangle$: $A := \triangle \setminus \overline{\triangle(\sqrt{2})}$ and $B := \triangle \cap \triangle(\sqrt{2})$. Let $\{z_1, z_2\} = \partial\triangle \cap \partial\triangle(\sqrt{2})$ (with z_1 having positive imaginary part). Note that $\partial\triangle(\sqrt{2}) \cap \triangle \cap \mathbb{R} = \{\sqrt{2} - 1\}$. Let Q be the interior of a closed (topological) disk obtained by smoothing the corners of the square $[-1,1] \times [-1,1]$ in such a way that Q is invariant under reflections with respect to both axes. Choose Q such that $\{-1\} \times \left[-\frac{1}{2}, \frac{1}{2}\right] \subset \partial Q$ (the precise number $\frac{1}{2}$ is irrelevant to the construction) and such that ∂Q is $\mathcal{C}^\infty$-smooth. Let $Q^L = Q \cap \{\mathrm{Re}(z) < 0\}$ and $Q^R = Q \cap \{\mathrm{Re}(z) > 0\}$. We define $\triangle^{L/R}$ similarly.

LEMMA 3.1. *There exists a $\mathcal{C}^\infty$-smooth homemorphism $f : \overline{\triangle} \to \overline{Q}$, conformal on $\triangle$, such that*:

(1) $f(A) = Q^L$;
(2) $f(B) = Q^R$;
(3) $f(-1) = -1$, $f(1) = 1$, $f(\sqrt{2} - 1) = 0$, $f(z_1) = i$, *and* $f(z_2) = -i$; *and*
(4) $f(\bar{z}) = \overline{f(z)}$ *for all* $z \in \triangle$.

Proof. The map f will be the composition of two maps. First, the map

$$\varphi(z) = \frac{z - (\sqrt{2} - 1)}{1 - (\sqrt{2} - 1)z}$$

is a holomorphic automorphism of $\triangle$ that maps B to $\triangle^R$ with $\varphi(\sqrt{2} - 1) = 0$, $\varphi(z_1) = i$, $\varphi(z_2) = -i$, $\varphi(-1) = -1$, and $\varphi(1) = 1$. To see this, let $\tilde{\varphi}(z) := \frac{z+1}{1-z}$, a Möbius transformation that sends $\triangle$ onto the right half-plane with $\tilde{\varphi}(-1) = 0$ and $\tilde{\varphi}(1) = \infty$. It sends the piece of $\partial\triangle$ with strictly positive imaginary part to the positive imaginary axis. The map $\tilde{\varphi}(z - \sqrt{2})$ has the corresponding properties for $\triangle(\sqrt{2})$, and $\tilde{\varphi}(z - \sqrt{2}) = 1/(\sqrt{2} + 1) \cdot \varphi(z)$. It is well known that $\varphi \in \mathrm{Aut}_{\mathrm{hol}}(\triangle)$, and it follows that φ maps B onto $\triangle^R$. Since the piece of $\partial B \cap \triangle$ with positive imaginary part is sent to the positive imaginary axis, we have $\phi(z_1) = i$ and consequently $\varphi(z_2) = -i$. Clearly $\varphi(\bar{z}) = \overline{\varphi(z)}$ for all $z \in \triangle$.

To create the second map, let ψ be a conformal map that maps the part of $\triangle^L$ with positive imaginary part onto the part of Q^L with positive imaginary part. The existence of ψ follows from the Riemann mapping theorem, and—since it is possible to interpolate over three points in $\partial\triangle$ by elements of $\mathrm{Aut}_{\mathrm{hol}}(\triangle)$—we may assume that $\psi(-1) = -1$, $\psi(0) = 0$, and $\psi(i) = i$. By the Schwarz reflection principle, ψ extends to a map $\psi : \triangle \to Q$. Then $\psi(\bar{z}) = \overline{\psi(z)}$ for all $z \in \triangle$ and so $f := \psi \circ \varphi$ satisfies (1)–(4). See [3] for the fact that f is $\mathcal{C}^\infty$-smooth up to the boundary. □

Pick two points $y_1^0 \in \partial B \cap \triangle(\sqrt{2})$ and $y_2^0 = \bar{y}_1^0$ such that $\mathrm{Re}(f(y_i^0)) = 1$ for $i = 1, 2$ (see Figure 3.1). Let $x_1^0 := y_1^0 - \sqrt{2}$ and $x_2^0 := y_2^0 - \sqrt{2}$. Since $x_1^0, x_2^0 \in A$ it follows that $\mathrm{Re}(f(x_i^0)) = \alpha$ for some $-1 < \alpha < 0$.

For any $0 < \delta \leq \frac{1}{2}$, let $I_0^1(\delta)$ denote the line segment $I_0^1(\delta) := \{-1\} \times [-\delta, \delta]$ and let $I_0^2(\delta) := \{0\} \times [-\delta, \delta]$. Let $\gamma_0^1(\delta) := f^{-1}(I_0^1(\delta))$ and $\gamma_0^2(\delta) := f^{-1}(I_0^2(\delta))$. Pick a $\delta^0 \leq \frac{1}{2}$ and choose a smoothly bounded domain $D_0 \subset \triangle$ as in the figure. The domain D_0 should have the following properties:

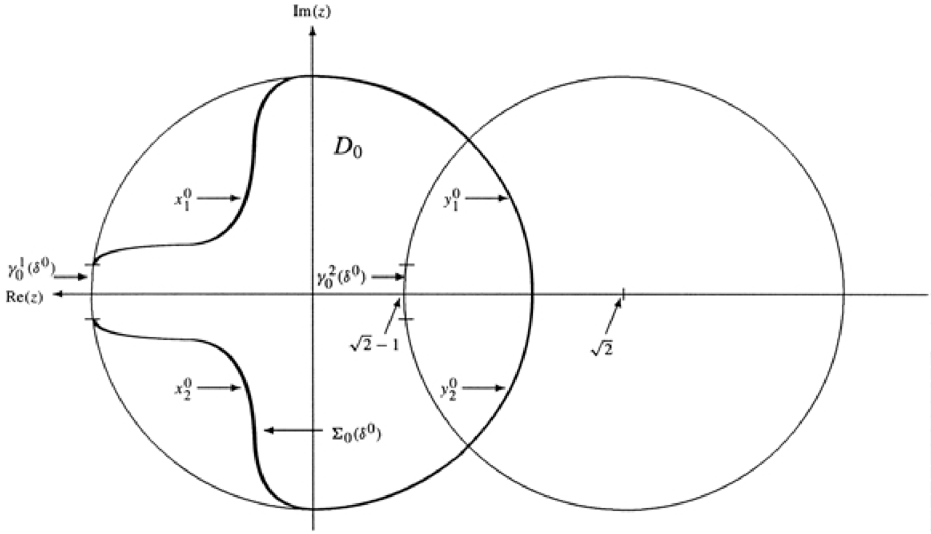

Figure 3.1

(i) $D_0 \subset \triangle$, and D_0 agrees with $\triangle$ on the right half-plane;
(ii) $[\overline{D_0} + \sqrt{2}\,] \cap \overline{D_0} = \{y_1^0, y_2^0\}$;
(iii) $\gamma_0^1(\delta^0) \subset \partial D_0$; and
(iv) $\gamma_0^2(\delta^0) \subset D_0$.

Define disks $D_j = D_0 + j \cdot \sqrt{2}$ for all integers $j \geq 0$, and let $f_j \colon \overline{D_j} \to \mathbb{C}$ be defined inductively by $f_0 := f$; then $f_j(z) := f_{j-1}(z - \sqrt{2}) + 1$. Let $\{x_1^j, x_2^j, y_1^j, y_2^j\} = \{x_1^0, x_2^0, y_1^0, y_2^0\} + j \cdot \sqrt{2}$, and for all $\delta \leq \delta^0$ let $\gamma_j^i(\delta) = \gamma_0^i(\delta) + j \cdot \sqrt{2}$, $I_j^i(\delta) = I_0^i(\delta) + j \cdot \sqrt{2}$, and $\Sigma_j(\delta) := \overline{\partial D_j \setminus \gamma_j^1(\delta)}$.

The following properties are easily confirmed for all j and all $\delta \leq \delta^0$:

(v) $\mathrm{Re}(f_j(z)) = -1 + j$ for all $z \in \gamma_j^1(\delta)$;
(vi) $\mathrm{Re}(f_j(z)) = j$ for all $z \in \gamma_j^2(\delta)$;
(vii) $\mathrm{Re}(f_j(x_i^j)) = \alpha + j$, where $-1 < \alpha < 0$;
(viii) $\mathrm{Re}(f_j(y_i^j)) = 1 + j$;
(ix) $\partial D_j \cap \partial D_{j+1} = \{y_1^j, y_2^j\} = \{x_1^{j+1}, x_2^{j+1}\}$;
(x) $\partial D_j \cap \partial D_k = \emptyset$ for $k \neq j \pm 1$; and
(xi) $f_j(j \cdot \sqrt{2} + (\sqrt{2} - 1)) = j = f_{j+1}(j \cdot \sqrt{2} + (\sqrt{2} - 1))$.

For $0 \leq \delta \leq \delta^0$ and $\varepsilon > 0$ we define the following sets:

(xii) $M_j(\delta, \varepsilon) := \{(z, w) \in \mathbb{C}^2 \mid z \in \Sigma_j(\delta), w = f_j(z) + t \cdot i, t \in [-\varepsilon, \varepsilon]\}$;
(xiii) $E_j(\delta, \varepsilon) := \{(z, w) \in \mathbb{C}^2 \mid z \in \gamma_j^1(\delta), w = f_j(z) + t \cdot i, t \in [-\varepsilon, \varepsilon]\}$;
(xiv) $A_j^t := \{(z, w) \in \mathbb{C}^2 \mid z \in D_j, w = f_j(z) + t \cdot i\}$; and
(xv) $A_j(\varepsilon) = \bigcup_{t \in [-\varepsilon, \varepsilon]} A_j^t$.

The sets $M_j(\delta,\varepsilon)$ are totally real disks, and the union $M_j(\delta,\varepsilon)\cup E_j(\delta,\varepsilon)$ bounds the family of analytic disks $A_j(\varepsilon)$. Moreover, if $j\neq k$ then $M_j(\delta,\varepsilon)\cap M_k(\delta',\varepsilon')=\emptyset$ for all $\delta,\delta'\leq\delta^0$ and all $\varepsilon,\varepsilon'>0$. Hence $\Sigma_j(\delta)\cap\Sigma_{j+1}(\delta')=\{y_1^j,y_2^j\}=\{x_1^{j+1},x_2^{j+1}\}$, so in this case (xii) follows from (vii) and (viii). If $k\neq j\pm 1$ then $\Sigma_j(\delta)\cap\Sigma_k(\delta')=\emptyset$ by (x).

For $j\geq 0$ let $q_j\in\mathbb{C}^2$ denote the point $q_j=(j\cdot\sqrt{2},f(0)+j)$. Then $q_j\in A_j^0$ for all j. For a positive number μ_j we will now let $B_{\mu_j}(q_j)$ denote the ball of radius μ_j centered at the point q_j.

Lemma 3.2. *There are sequences of strictly positive real numbers $\delta_j'<\delta_j\leq\delta^0$, $\varepsilon_j<\varepsilon_j'$, and $\mu_j<\sqrt{2}-1$ such that:*

(1) *$E_{j+1}(\delta_{j+1},\varepsilon_{j+1})\subset A_j(\varepsilon_j)$ for $j=0,1,2,\ldots$; and*

(2) *any finite union $\bigcup_{j=0}^m(M_j(\delta_j',\varepsilon_j')\cup\overline{B_{\mu_j}(q_j)})$ is polynomially convex.*

Proof. The proof proceeds by induction on j. Let $\delta_0:=\delta^0$, choose any $\varepsilon_0>0$, and let $\delta_0'<\delta_0$ and $\varepsilon_0'>\varepsilon_0$. We have that $M_0(\varepsilon_0',\delta_0')$ is polynomially convex because it projects onto a polynomially convex smooth arc in the plane and the fibers are straight lines. If μ_0 is chosen small enough, then (2) holds with $m=0$ and (1) is void at this step.

Assume now that we have chosen ε_j, ε_j', δ_j, δ_j', and μ_j for $0\leq j\leq N$ and that (1) and (2) hold for $0\leq j,m\leq N$. If t_1 is small enough, then from (vi), (xi), and the definition of $A_N(\varepsilon_N)$ it follows that

$$\gamma_N^2(t_1)\times I_N^2(t_1)\subset A_N(\varepsilon_N).$$

If $t_1'<t_1$ and if s_1 and s_2 are small enough, it follows from (v), (xi), and the definition of $E_{N+1}(\varepsilon_{N+1},\delta_{N+1})$ that

$$E_{N+1}(\varepsilon_{N+1},\delta_{N+1})\subset\gamma_{N+1}^1(t_1')\times I_{N+1}^1(t_1)\subset\gamma_N^2(t_1)\times I_N^2(t_1)$$

for all $\varepsilon_{N+1}\leq s_1$ and $\delta_{N+1}\leq s_2$. Choose $\delta_{N+1}'<s_2$. If $\varepsilon_{N+1}'<s_1$ is chosen small enough then $\bigcup_{j=0}^N(M_j(\varepsilon_j',\delta_j')\cup\overline{B_{\mu_j}(q_j)})\cup M_{N+1}(\varepsilon_{N+1}',\delta_{N+1}')$ is polynomially convex: By assumption, $K:=\bigcup_{j=0}^N(M_j(\varepsilon_j',\delta_j')\cup\overline{B_{\mu_j}(q_j)})$ is polynomially convex, and we have that $K\cup M_{N+1}(0,\delta_{N+1}')$ is polynomially convex since $C:=M_{N+1}(0,\delta_{N+1}')$ is a simply connected smooth curve and since $K\cap C=\emptyset$ (see e.g. the main theorem in [13] or [14, Thm. 3.1.1]). Let U,V be open sets such that $U\cap V=\emptyset$, and let $K\subset U$ and $C\subset V$. There exists a neighborhood Ω of $K\cup C$ such that $\Omega\subset\subset U\cup V$ and such that $\overline{\Omega}$ is polynomially convex. If ε_{N+1}' is small enough then $M_{N+1}(\varepsilon_{N+1}',\delta_{N+1}')\subset\Omega$, and since $M_{N+1}(\varepsilon_{N+1}',\delta_{N+1}')$ is polynomially convex (it projects onto a smooth arc) it follows that the union with K is polynomially convex (this follows from the Oka–Weil theorem).

Choose $\varepsilon_{N+1}<\varepsilon_{N+1}'$ and $\delta_{N+1}'<\delta_{N+1}<s_2$. Finally, choose μ_{N+1} small enough. □

At this point we fix sequences of numbers as in the previous lemma, and we refer to our sets simply as M_j, M_j', E_j, A_j, and $B_j:=B_{\mu_j}(q_j)$. (Here M_j' denotes the set $M_j(\delta_j',\varepsilon_j')$.) We let $\tilde{M}$ denote the union $\tilde{M}=\bigcup_{i=0}^\infty M_i\cup q_i$.

LEMMA 3.3. *Uniform approximation is not possible on $\tilde{M}$.*

Proof. Define $g(q_j) = j$ and $g|_{M_j} \equiv 0$ for all $j \geq 0$. Assume to get a contradiction that there is an $h \in \mathcal{O}(\mathbb{C}^2)$ such that $\sup_{x \in \tilde{M}}\{|h(x) - g(x)|\} < \frac{1}{2}$.

Let $N \geq 1$ be a natural number such that $\|h\|_{E_0} \leq N$. By assumption we have that $|h(q_{N+1})| > N$. By the maximum principle, $h|_{\overline{A^0_{N+1}}}$ must take its maximum on $\partial A^0_{N+1} \subset M_{N+1} \cup E_{N+1}$. By assumption $\|h\|_{M_{N+1}} < \frac{1}{2}$, so there exists a point $p_N \in E_{N+1} \subset A_N$ (by Lemma 3.2(1)) such that $|h(p_N)| > N$. Then p_N lies in $A_N^{t_N}$ for some $t_N \in [-\varepsilon_N, \varepsilon_N]$.

Again $h|_{\overline{A_N^{t_N}}}$ must take its maximum on $\partial A_N^{t_N} \subset M_N \cup E_N$, and this must happen at a point $p_{N-1} \in E_N \subset A_{N-1}$. Continuing this argument, we end up with a point $p_0 \in E_0$ such that $|h(p_0)| > N$, which is a contradiction. □

Now we regard the set $\tilde{M}$ as being contained in $\mathbb{C}^2 \times \{0\} \subset \mathbb{C}^2 \times \mathbb{C} = \mathbb{C}^3$.

LEMMA 3.4. *There exists a $\mathcal{C}^\infty$-smooth and proper embedding $\phi\colon \mathbb{R}^2 \to \mathbb{C}^3$ such that $\tilde{M} \subset M := \phi(\mathbb{R}^2)$ and such that M is totally real and polynomially convex.*

Proof. We start by choosing a $\mathcal{C}^\infty$-smooth proper immersion $g\colon \mathbb{R} \to \mathbb{C}$ such that $g(\mathbb{R})$ contains all the curves Σ'_j. The image $g(\mathbb{R})$ should look like the curve in Figure 3.2.

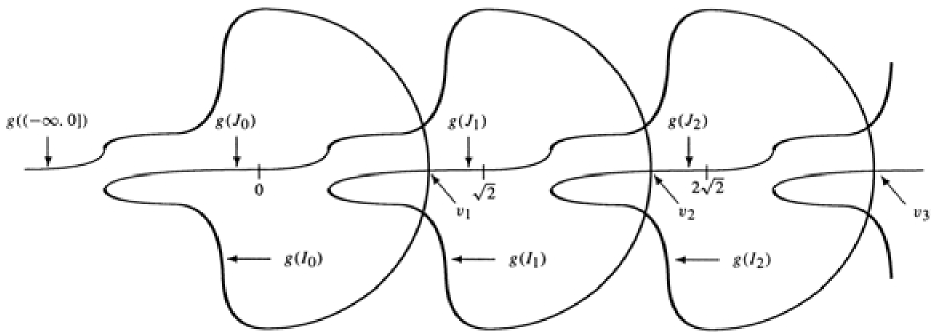

Figure 3.2

For each j we first connect Σ'_j to Σ'_{j+1} by a smooth curve l_j between the endpoint of Σ'_j with negative imaginary part and the endpoint of Σ'_{j+1} with positive imaginary part. The curve l_j will intersect Σ'_{j-1} at a single point v_j for $j \geq 1$. Let l_{-1} be a smooth curve between the point $\{-1\}$ and the endpoint of Σ'_0 with positive imaginary part. Let Γ denote the union of all these curves with the interval $(-\infty, -1)$. The image $g(\mathbb{R})$ should be a smoothing of Γ that creates no new self-intersection points; $g(\mathbb{R})$ should contain the curves Σ'_j, and we make sure that it passes through all points $\{j \cdot \sqrt{2}\}$ for $j \geq 0$. Moreover we make sure that, for all real numbers t, there are only a finite number of points on $g(\mathbb{R})$ whose real part equals t. If we let $I_k := [2k, 2k+1]$ and $J_k := [2k+1, 2k+2]$, then we may assume the following:

(a) $g|_{I_k}$, $g|_{J_k}$, and $g|_{\{t\leq 0\}}$ are all embeddings;
(b) $g(I_k) = \Sigma'_k$;
(c) $g(2k+\frac{4}{3}) = v_k$ for $k \geq 1$;
(d) $g(2k+\frac{5}{3}) = k\cdot\sqrt{2}$ for $k \geq 0$;
(e) $g(I_j)\cap g(I_k) = \{y_1^j, y_2^j\}$ if $k = j+1$, and if $k \neq j\pm 1$ then the intersection is empty;
(f) $g(I_j)\cap g(J_k) = \{v_k\}$ if $j = k-1 \geq 0$, and otherwise the intersection is empty;
(g) $g(J_i)\cap g(J_k) = \emptyset$ for all $i \neq k$; and
(h) $g(\{t<0\})\cap g(\{t>0\}) = \emptyset$.

Choose a $\mathcal{C}^\infty$-smooth function $h\colon \mathbb{R}\to\mathbb{C}$ such that the following hold:

(i) $h(t) = f_j(g(t))$ for all $t\in I_j$;
(j) $\operatorname{Re}(h(2j+\frac{4}{3})) \neq \operatorname{Re}(f_{j-1}(v_j))$; and
(k) $h(2j+\frac{5}{3}) = f(0)+j$.

We define first a smooth, proper embedding $\psi\colon \mathbb{R}^2\to\mathbb{C}^2$ by

$$\psi(x,y) := (g(x), h(x)+y\cdot i).$$

This map is injective because our choices have been made such that if $g(x_1) = g(x_2)$ then $\operatorname{Re}(h(x_1)) \neq \operatorname{Re}(h(x_2))$ if $x_1\neq x_2$.

It is clear that $\psi(\mathbb{R}^2)$ is totally real. Since $\tilde{M}\subset\psi(\mathbb{R}^2)$, it follows from Lemma 3.3 that uniform approximation is not possible on $\psi(\mathbb{R}^2)$. To get a polynomially convex surface we will now bend "most" of $\psi(\mathbb{R}^2)$ into $\mathbb{C}^3$, keeping $\tilde{M}$ in $\mathbb{C}^2$.

Let $P'_j\subset\mathbb{R}^2$ denote the set $P'_j := I_j\times[-\varepsilon'_j, \varepsilon'_j]$. We have:

(l) $\psi(P'_j) = M'_j$ for all $j\geq 0$; and
(m) $\psi((2j+\frac{5}{3}, 0)) = q_j$ for all $j\geq 0$.

Let $P_j := \psi^{-1}(M_j)\subset\subset P'_j$. For each j let $S_j\subset\mathbb{R}^2$ be a closed disk centered at $(2j+\frac{5}{3}, 0)$ such that $\psi(S_j)\subset\subset B_j$. Let $\chi:\mathbb{R}^2\to[0,1]$ be a $\mathcal{C}^\infty$-smooth function such that

(n) $\chi|_{P_j\cup\{(2j+5/3,0)\}}\equiv 0$ and
(o) $\chi(x,y) = 1$ for all $(x,y)\in\mathbb{R}^2\setminus\left(\bigcup_{i=0}^\infty(P'_i\cup S_i)\right)$.

We define our final embedding by

$$\phi := \psi + \chi\cdot\operatorname{Re}(g)\cdot \mathrm{e}_3.$$

Then $\phi(\mathbb{R}^2)$ is polynomially convex. Note first that $Z' := \phi\left(\bigcup_{i=0}^m(P'_i\cup S_i)\right)$ is polynomially convex for all $m\in\mathbb{N}$. It follows from Lemma 3.2 that $Z := \psi\left(\bigcup_{i=0}^m(P'_i\cup S_i)\right)\subset\mathbb{C}^2$ is polynomially convex, and it follows from [14, Thm. 6.3.1] together with the Oka–Weil theorem that $\mathcal{P}(Z) = \mathcal{C}(Z)$. Since Z' is a graph over $Z \subset \mathbb{C}^2$ in $\mathbb{C}^3$ it follows that $\mathcal{P}(Z') = \mathcal{C}(Z')$, and Z' is polynomially convex by [14, Thm. 1.2.10].

Let π_3 denote the projection onto the third coordinate in $\mathbb{C}^3$. Restricted to M we have that π_3 is a real-valued function. According to [14, Thm. 1.2.16], a compact set $K \subset M$ is polynomially convex if and only if $\pi_3^{-1}(t_0) \cap K$ is polynomially convex for each $t_0 \in \mathbb{R}$. Let X_j be a compact exhaustion of $\mathbb{R}^2$ by squares that pick up exactly one pair (P_j', S_j) for each increase of j. The following shows that $\phi(X_j)$ is polynomially convex for each j. Consider first a set $Y := \left(\bigcup_{i=1}^k \{x_i\} \times [-R, R]\right) \cup \left(\bigcup_{i=0}^m P_i' \cup S_i\right)$, where $x_1, \dots, x_k$ is a finite set of points and where $R \in \mathbb{R}^+$ and $m \in \mathbb{N}^+$. We have seen that $\phi\left(\bigcup_{i=0}^m P_i' \cup S_i\right)$ is polynomially convex, so by [13] the set $\phi(Y)$ is polynomially convex. It now follows from [14, Thm. 6.3.1] and [13] that $\mathcal{P}(\phi(Y)) = \mathcal{C}(\phi(Y))$; hence any relatively closed subset of $\phi(Y)$ is polynomially convex.

Fix a t_0 and assume that $\pi_3^{-1}(t_0) \cap \phi(X_j) \neq \emptyset$. Define

$$L := \phi^{-1}(\pi_3^{-1}(t_0) \cap \phi(X_j)) = \{(x, y) \in X_j \mid \chi(x, y) \cdot \mathrm{Re}(g(x)) = t_0\}.$$

By assumption there are only a finite number of x_j such that $\mathrm{Re}(g(x_j)) = t_0$. Since $\chi \equiv 1$ on $L \setminus \left(\bigcup_{i=0}^j P_i' \cup S_i\right)$, it follows that $L \setminus \left(\bigcup_{i=0}^j P_i' \cup S_i\right) \subset \bigcup_{j=1}^k \{x_j\} \times [-R, R]$ for some $R \in \mathbb{R}$. So L is a closed subset of a set on the same form as Y, and by the preceding arguments $\phi(L)$ is polynomially convex.

Adding a third component to the embedding ψ does not change the fact that the image is totally real. $\square$

References

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