Michigan Math. J. 58 (2009)

Higmanian Rank-5 Association Schemes on 40 Points

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Dedicated to the memory of Donald G. Higman

1. Introduction

We pay our tribute to the mathematical heritage of D. G. Higman in an investigation of imprimitive association schemes on 40 points with four classes that are proper class II in the sense of [H4].

We prove that there are exactly four possible sets of intersection numbers, all of which correspond to a parabolic of type $10 \circ K_4$. There exist exactly 15 association schemes for the first parameter set. The scheme in the case of the second parameter set is unique, up to isomorphism. We provide an example for the third parameter set, but the existence of a scheme for the fourth feasible parameter set remains an open question.

Our results were originally obtained with the aid of a computer, but in many cases we have been able to give computer-free constructions, which are presented here. Additional supporting material is available from ⟨http://www.math.bgu.ac.il/~zivav/math/⟩.

Coherent configurations and coherent algebras are two of the significant concepts introduced by Higman. These concepts are considered in Section 2 in an effort to make our presentation more clear to the reader.

Many of the proofs in this paper are geometric in nature. In a number of cases we even manage to use pictorial arguments, presenting a certain element of a considered group as a visible symmetry of a depicted diagram. In this fashion, following in the spirit of H. S. M. Coxeter, a number of nice auxiliary objects (including the configuration 8_3, the 4-dimensional cube, the Clebsch graph, and the cages on 50 and 40 vertices) are inspected in Section 3.

A short digest of part of Higman's classification of rank-5 imprimitive association schemes is given in Section 4 (a few regrettable typos have been corrected). With the aid of Higman's classification we prove Proposition 4.2, wherein four feasible parameter sets for the schemes on 40 points are enumerated.

In fact, only one such scheme (denoted by $\mathfrak{m}$) was known before. The scheme $\mathfrak{m}$ is generated by the classical Deza graph on 40 vertices. This serves to justify a new axiomatic system for a Deza family in a Higmanian house, as suggested in

Received November 2, 2007. Revision received April 16, 2008.

Section 5. Among our first wave of main results, we provide a new interpretation of $\mathfrak{m}$ using a master coherent configuration $\mathfrak{n}$, which is defined on the edges, quadrangles, and skew systems of quadrangles in the Clebsch graph $\square_5$. We thereby discover a new partial linear space as a geometrization of the classical Deza graph.

In Section 6, the entire family of 15 association schemes algebraically isomorphic to $\mathfrak{m}$ (aka Higmanian houses for Deza families on 40 points) is considered as a follow-up to the extended abstract [KZ], in which an outline of the algorithm for the enumeration of all such schemes is presented.

Elements of the recently developed theory of WFDF (Wallis and Fon-Der-Flaas) coherent configurations are considered in Section 7, which may be of independent interest. In Section 8, as an example, the scheme $\mathfrak{m}$ and two of its algebraic twins are revealed as merging schemes of a certain WFDF configuration.

In Section 9, the second wave of our results is presented. Such results are based on newly discovered properties of an amazing graph $\mathcal{R}$ on 40 vertices: the Anstee–Robertson cage of valency 6 and girth 5, which was also constructed independently by C. W. Evans. The coherent closure of this graph $\mathcal{R}$ is a non-Schurian association scheme of rank 5 (Theorem 9.4). This unexpected association scheme is uniquely determined, up to isomorphism, by its intersection numbers, a nice simple consequence of the uniqueness of the cage on 40 vertices (Theorem 9.5).

We provide a computer-free geometric description of the group $\mathrm{Aut}(\mathcal{R})$, using group amalgams and amalgamating brilliant images coined by our predecessors: Anstee, Evans, and Robertson (Theorem 9.9). An interesting link to the cage on 40 vertices with the unique locally icosahedral graph on 40 vertices (see [BlBrBuC]) is also described. A number of association schemes considered in this section are interpreted with the aid of Schur rings.

Finally, in Section 10 we consider a few other rank-5 association schemes. One of these illustrates the third feasible set of parameters (as it is presented in Proposition 4.2).

Acknowledgments. We thank E. Bannai, A. Blokhuis, E. van Dam, A. Deza, W. Haemers, P. Hafner, T. Huang, G. A. Jones, L. Jörgensen, D. Mesner, N. Robertson, A. Schwenk, and T. Spence for providing helpful information. We are very grateful to Ch. Pech, S. Reichard, and A. Woldar for a long-standing fruitful cooperation. This project was conceived at the time when the author M.K. was visiting the University of Texas at Brownsville. The kind attention and support of J. Mogilski, J. Navarro, and P.-H. Zieschang are gratefully acknowledged. Remarks of the editors and an anonymous referee helped to clarify portions of the text. Finally, we are pleased to thank John Dixon and (once more) Andrew Woldar for very helpful assistance in improving linguistical aspects of the paper.

2. Background Concepts

We provide a brief discussion of the most significant notions and notation. The survey [FKM] may serve as a source for more details. We denote the dihedral group of order $2n$ by D_n. A cyclic group of order n is denoted by $\mathbb{Z}_n$, thus reserving C_n for its traditional graph-theoretic notation as a regular connected graph of valency 2 with n vertices. Our notation for strongly regular graphs follows [HeH]. The concept of a partial linear space is considered in [BrCN].

2.1. Main Notions

We refer to [BanI; BrCN] for more details.

A *color graph* is a pair $(\Omega, \mathcal{R})$, where Ω is a set of vertices and $\mathcal{R}$ a partition of Ω^2 into a set of binary relations on Ω.

According to Higman [H1], a *coherent configuration* is a color graph $\mathfrak{m} = (\Omega, \mathcal{R})$, $\mathcal{R} = \{R_i \mid i \in I\}$, such that the following conditions are satisfied:

1. The identity relation $\mathrm{Id}_\Omega = \{(x,x) \mid x \in \Omega\}$ is a union of suitable relations R_i with $i \in I'$ and $I' \subseteq I$.
2. For each $i \in I$ there exists an $i' \in I$ such that $R_i^t = R_{i'}$, where $R_i^t = \{(y,x) \mid (x,y) \in R_i\}$.
3. For any $i, j, k \in I$, the number p_{ij}^k of elements $z \in \Omega$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is constant provided that $(x,y) \in R_k$.

The numbers p_{ij}^k are called *intersection numbers* of $\mathfrak{m}$. We refer to R_i as *basic relations,* graphs $\Gamma_i = (\Omega, R_i)$ as *basic graphs,* and adjacency matrices $A_i = A(\Gamma_i)$ as *basic matrices* of $\mathfrak{m}$. The notion of a fiber of a coherent configuration, and that of its type, appears in [H2]. The number $|I|$ is called the *rank* of $\mathfrak{m}$, and the number $|\Omega|$ is the *order* of $\mathfrak{m}$.

If (G, Ω) is a permutation group, 2-orb(G, Ω) denotes the set of all *2-orbits* of (G, Ω)—that is, orbits of the induced action of G on Ω^2. It is easy to see that $(\Omega, \text{2-orb}(G, \Omega))$ is a coherent configuration. Such coherent configurations will be called *Schurian* (cf. [FKM]).

The particular case of a coherent configuration $\mathfrak{m}$ in which the identity relation Id_Ω is one of the basic relations of Ω is called a *homogeneous coherent configuration* or an *association scheme*. Typically, the basic relation Id_Ω is denoted by R_0. All remaining basic relations are called *classes.* As in [BanI], an association scheme is not presumed to be commutative.

2.2. Coherent Closure

The notion of a coherent configuration may be reformulated in terms of matrices. A *coherent algebra* W (see [H3]) is a set of square matrices of order n over the field $\mathbb{C}$ that forms a matrix algebra, is closed with respect to the operations of Schur–Hadamard multiplication and transposition, and contains both the identity matrix I and the all-1 matrix J. The set of basic matrices $\{A_i \mid i \in I\}$ of a coherent configuration $\mathfrak{m}$ serves as a *standard basis* of the corresponding coherent algebra W, and in this case we write $W = \langle A_0, A_1, \ldots, A_{r-1}\rangle$. We may abuse notation, referring to W and $\mathfrak{m}$ as one and the same object.

The intersection of coherent algebras is again a coherent algebra. This implies the existence of the *coherent closure* of B; denoted by $\langle\langle B\rangle\rangle$, this is the smallest coherent algebra containing a prescribed set B of matrices of order n. An efficient polynomial-time algorithm for the computation of coherent closure, often referred to as WL-stabilization, is described in [Wei].

We call a graph $\Gamma = (V, E)$ a *coherent graph* if E is a basic relation of the coherent closure $\langle\langle\Gamma\rangle\rangle$. For example, each distance-regular graph is coherent.

2.3. Isomorphisms, Automorphisms, and Mergings

An *isomorphism* of color graphs $(\Omega, \mathcal{R})$ and $(\Omega', \mathcal{R}')$ is a bijection ϕ from Ω to Ω' that induces a bijection of colors (relations) in $\mathcal{R}$ onto colors in $\mathcal{R}'$. A *weak* (or *color*) *automorphism* of $\Gamma = (\Omega, \mathcal{R})$ is an isomorphism of Γ with itself. If the induced permutation of colors is the identity permutation, then we speak of a (*strong*) *automorphism*.

We denote by $\mathrm{CAut}(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ the groups of all weak and strong automorphisms of Γ, respectively. Clearly, $\mathrm{Aut}(\Gamma) \trianglelefteq \mathrm{CAut}(\Gamma)$. In the case where Γ is a Schurian coherent configuration, the group $\mathrm{CAut}(\Gamma)$ coincides with the normalizer of $\mathrm{Aut}(\Gamma)$ in symmetric group $S(\Omega)$.

An *algebraic isomorphism* between coherent configurations $(\Omega, \mathcal{R})$ and $(\Omega', \mathcal{R}')$ is a bijection $\phi\colon \mathcal{R} \to \mathcal{R}'$ such that $p_{ij}^k = p_{i^\phi j^\phi}^{k^\phi}$ for all $i, j, k \in I$. An algebraic isomorphism of a coherent configuration $\mathfrak{m} = (\Omega, \mathcal{R})$ with itself is called an *algebraic automorphism* of $\mathfrak{m}$. The group of algebraic automorphisms of $\mathfrak{m}$ is denoted by $\mathrm{AAut}(\mathfrak{m})$.

Clearly, $\mathrm{CAut}(\mathfrak{m})/\mathrm{Aut}(\mathfrak{m}) \hookrightarrow \mathrm{AAut}(\mathfrak{m})$. If the quotient group $\mathrm{CAut}(\mathfrak{m})/\mathrm{Aut}(\mathfrak{m})$ is a proper subgroup of $\mathrm{AAut}(\mathfrak{m})$ then the algebraic automorphisms of $\mathfrak{m}$ that are not induced by $\phi \in \mathrm{CAut}(\mathfrak{m})$ are called *proper* algebraic automorphisms. See [K+] for more details.

If W' is a coherent subalgebra of a coherent algebra W, then the corresponding coherent configuration $\mathfrak{m}'$ is called a *fusion* (or *merging configuration*) of $\mathfrak{m}$. If $\mathfrak{m} = (\Omega, \text{2-orb}(G, \Omega))$ for a suitable permutation group G, then overgroups of G in $S(\Omega)$ lead to fusions of $\mathfrak{m}$. Thus the most interesting fusions are the *non-Schurian* fusions—in other words, those that do not emerge from a suitable overgroup of (G, Ω).

For each subgroup $K \leq \mathrm{AAut}(\mathfrak{m})$, its orbits on the set of relations define a merging coherent configuration, called the *algebraic merging* defined by K. Again, those algebraic mergings that are non-Schurian are of special interest because they are, in a sense, "less predictable" combinatorial objects.

Let W' and W'' be coherent subalgebras of a coherent algebra W. Suppose W' and W'' are not isomorphic but there exists a $\phi \in \mathrm{AAut}(W)$ that maps W' to W''. Then we say W' and W'' form a pair of *twins* inside of W.

2.4. Decomposable Schemes

If W_1, W_2 are homogeneous coherent algebras of orders n_1, n_2 and ranks r_1, r_2, respectively, then their tensor (direct) product $W_1 \otimes W_2$ is a homogeneous coherent algebra of order $n_1 n_2$ and rank $r_1 r_2$.

Let $\mathfrak{m}_1 = (\Omega_1, R)$ and $\mathfrak{m}_2 = (\Omega_2, S)$ be two association schemes of orders n_1, n_2 and ranks r_1, r_2, respectively. Let $\Omega = \Omega_1 \times \Omega_2$. We define on Ω the basic relations Id_Ω, $\Delta_i = \{((u_1, v_1), (u_2, v_2)) \mid (u_1, u_2) \in R_i, 1 \le i \le r_1 - 1\}$, and $\Theta_j = \{((u, v_1), (u, v_2)) \mid (v_1, v_2) \in S_j, u \in \Omega_1, 1 \le j \le r_2 - 1\}$. The system $\mathfrak{m} = (\Omega, \{\mathrm{Id}_\Omega\} \cup \{\Delta_i \mid 1 \le i \le r_1 - 1\} \cup \{\Theta_j \mid 1 \le j \le r_2 - 1\})$ turns out to be an association scheme of order $n_1 n_2$ and rank $r_1 + r_2 - 1$. We use the notation $\mathfrak{m} = \mathfrak{m}_1 \wr \mathfrak{m}_2$ (or $\mathfrak{m} = \mathfrak{m}_1 \,\mathrm{wr}\, \mathfrak{m}_2$) and call $\mathfrak{m}$ the *wreath product* of $\mathfrak{m}_1$ and $\mathfrak{m}_2$.

Note that a more general definition of wreath product (see e.g. [EvPT; Wei]) provides an association scheme that is algebraically (but not necessarily combinatorially) isomorphic to $\mathfrak{m}_1 \wr \mathfrak{m}_2$.

An association scheme is called *primitive* if all its nonreflexive (directed) basic graphs are (strongly) connected; otherwise it is *imprimitive*. An association scheme is imprimitive if and only if it admits nontrivial equivalence relations as a union of suitable basic relations. Such equivalence relations are called *imprimitivity systems* [BanI]; alternative names are *parabolics* [H4] or *closed sets* [Zie]. For each imprimitive association scheme $\mathfrak{m}$ with corresponding imprimitivity system σ, we may define a *quotient scheme* $\mathfrak{m}/\sigma$ on the set of equivalence classes of σ. Following Higman, for a pair $(\mathfrak{m}, \sigma)$ we speak of its *rank* as the rank of the association scheme induced by $\mathfrak{m}$ on an arbitrary class of σ. In this case, we refer to its *corank* as the rank of $\mathfrak{m}/\sigma$. If $\mathfrak{m}$ has rank r, then the sum of the rank and corank of $(\mathfrak{m}, \sigma)$ is at most $r + 1$, with equality if and only if $\mathfrak{m}$ is algebraically isomorphic to $\mathfrak{m}_1 \wr \mathfrak{m}_2$, where $\mathfrak{m}_1$ is the quotient scheme and $\mathfrak{m}_2$ is isomorphic to an induced scheme on the classes of σ.

An imprimitive association scheme is called *decomposable* if it can be represented as a tensor or as a (generalized) wreath product of smaller schemes. Decomposable schemes are commonly regarded as trivial objects. Thus, our real interest shall be in imprimitive schemes that are indecomposable.

2.5. Computer Tools

Several computer programs were used in this work. First, COCO is a collection of programs designed to investigate coherent configurations. It was developed in Moscow in the early 1990s, mainly by Faradžev and Klin [FK; FKM]. Using COCO allows one to find all association schemes that are invariant with respect to a given permutation group, together with their automorphism groups.

Second, GAP (see [Gap; Sch]) is an acronym for Groups, Algorithms, and Programming. It is a system for computation in discrete abstract algebra. The system supports many diverse and mobile extensions (“packages”, in GAP nomenclature). One such package is GRAPE [Soi], which is designed for the construction and analysis of finite graphs. The GRAPE package is itself dependent on an external program, nauty [Mc], which is designed to calculate the automorphism group of a graph.

The COCO version 2 initiative (due to S. Reichard et al.) aims to re-implement the algorithms in COCO, WL-stabilization, and so forth as a GAP package. In addition, it should reflect new theoretical results obtained since [FK].

3. Preliminaries

3.1. Möbius–Kantor Configuration 8_3

The configuration 8_3 is a regular uniform incidence structure, with parameters $(v, b, k, r) = (8, 8, 3, 3)$, that forms a partial linear space. A classical model of 8_3 is formed by taking as points all nonzero vectors of $\mathbb{GF}(3)^2$ (the 2-dimensional vector space over $\mathbb{GF}(3)$) and as lines all sets of the form $U + v$, where U is any 1-dimensional subspace of $\mathbb{GF}(3)^2$ and v is any nonzero vector. This model easily implies the following properties of 8_3.

PROPOSITION 3.1. (a) *The configuration* 8_3 *is unique up to isomorphism.*

(b) *The group* $\mathrm{Aut}(8_3)$ *has order 48 and is isomorphic to* $\mathrm{GL}(2, 3)$.

(c) *The configuration* 8_3 *is self-dual. The automorphism group of its Levi graph is a group* H *of order 96 and isomorphic to* $\mathrm{GL}(2, 3) : 2$.

(d) *The group* H *contains a unique* (*up to conjugacy*) *transitive subgroup* K *of degree 16 that is isomorphic to* $\mathrm{GL}(2, 3)$.

Proof. For the copy of the Levi graph that is depicted in Figure 3.1(a) we have $K = \langle g_1, g_2 \rangle$, where $g_1 = (0, 1, 9, 13, 15, 14, 6, 2)(8, 5, 11, 12, 7, 10, 4, 3, 8)$ and $g_2 = (1, 8, 2)(3, 9, 10)(5, 12, 6)(7, 13, 14)$. $\square$

We call the Levi graph of 8_3 the *Möbius–Kantor* (MK) graph. The diagram shown in Figure 3.1(a) is borrowed from [Cox2].

REMARK. The MK graph may be represented as a regular map of type $\{8, 3\}$ on an orientable surface of genus 2. The group K preserves orientation of the map on the surface (cf. [Cox2, Fig. 20]).

3.2. 4-dimensional Cube

We consider the 4-dimensional cube Q_4. The vertices are binary sequences of length 4, with two sequences adjacent if they differ in exactly one position. Also,

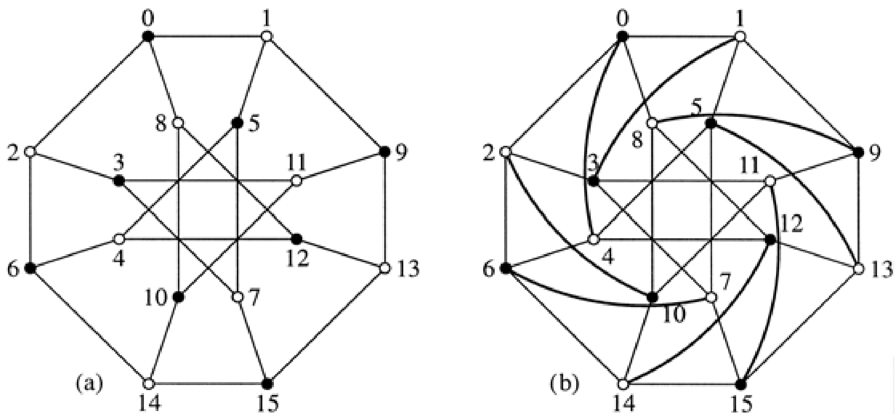

Figure 3.1 The MK graph without (a) and with (b) the skew 1-factor of Q_4

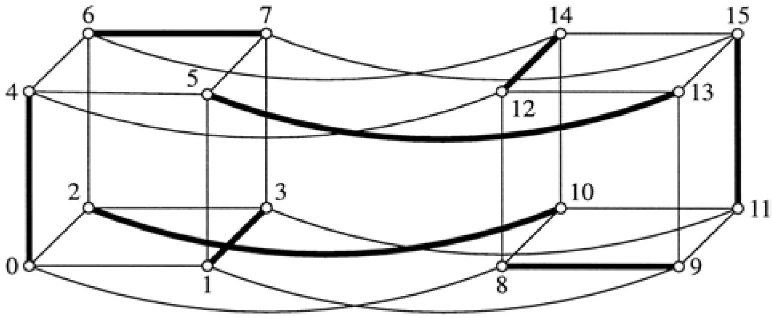

Figure 3.2 4-dimensional cube Q_4 with a skew 1-factor

a canonical labeling of the vertices of Q_4 by the numbers $0, \ldots, 15$ is used via decimal representations of the corresponding binary sequences.

The graph Q_4 is bipartite of girth 4. All quadrangles of Q_4 are readily identified: two binary coordinates take a prescribed value while the remaining two coordinates vary. Clearly, Q_4 has $\binom{4}{2} \cdot 2^2 = 24$ quadrangles.

We may represent Q_4 as a Cayley graph over the group E_{2^4} (the elementary abelian group of order 16) with the connection set $X_4 = \{0001, 0010, 0100, 1000\}$. The full automorphism group P of Q_4 is the exponentiation $S_2 \uparrow S_4$ of order $2^4 \cdot 4! = 384$ (cf. [FKM]).

We look at the smallest orbits of P on the 1-factors of Q_4. The orbit of length 4 is formed by the Cayley graphs over E_{2^4} with connection set $\{x\}$, $x \in X_4$, which we refer to as *direct 1-factors.* Removal of a direct 1-factor evidently splits Q_4 into two disjoint copies of the 3-dimensional cube Q_3. The orbit of length 8 will be called the orbit of *skew 1-factors.* A representative of this orbit is visible in Figure 3.1(b) (bold edges), and in Figure 3.2 one can immediately recognize the same graph with the same 1-factor as Q_4.

PROPOSITION 3.2. *The group $P = \mathrm{Aut}(Q_4)$ contains a transitive subgroup K, which is isomorphic to* $\mathrm{GL}(2,3)$ *and is the stabilizer in P of a skew 1-factor.*

Proof. To begin, consider the following explicit description of the structure of a skew 1-factor. Remove a direct 1-factor, and start from the resulting pair of disjoint copies of Q_3. Take a pair of antipodal vertices in one copy of Q_3 (say, $\{2, 5\}$). The remaining six vertices of this copy form an induced hexagon. Split this hexagon into two copies of $3 \circ K_2$, and choose one of them (say, with edge set $\{\{0,4\}, \{1,3\}, \{6,7\}\}$). The neighbors of the remaining copy of $3 \circ K_2$ in the corresponding direct 1-factor give three more edges ($\{8, 9\}$, $\{11, 15\}$, and $\{12, 14\}$). Now there is unique way to add two more edges in order to obtain the depicted skew 1-factor.

Easy combinatorial counting shows that there are a total of $\frac{4 \cdot 2 \cdot 2}{2} = 8$ ways to obtain a skew 1-factor. All skew 1-factors are isomorphic with respect to the group $P = \mathrm{Aut}(Q_4)$; hence the stabilizer K' of a skew 1-factor is a subgroup of order 48 in P.

To identify the group K', consider Q_4 as it is depicted in Figure 3.2. Remove from it the skew 1-factor and observe (using Figure 3.1(b)) that the remaining subgraph coincides with a copy of the MK graph in Figure 3.1(a). Thus we conclude that K' is a subgroup of index 2 in H, and with the aid of Proposition 3.1 we obtain $K' = K$. $\square$

REMARK. This embedding of the MK graph into Q_4 goes back to [Cox1].

3.3. Clebsch Graph

The Clebsch graph $\square_5$ is the unique strongly regular graph with parameters $(v, k, l, \lambda, \mu) = (16, 5, 10, 0, 2)$. The Clebsch graph is a Cayley graph over E_{2^4} with the connection set $X_5 = \{0001, 0010, 0100, 1000, 1111\}$. The group $G = \mathrm{Aut}(\square_5) \cong E_{2^4} \rtimes S_5$ is a rank-3 group. We may use also the auxiliary graph $5 \circ K_2$. In these terms, G is the subgroup $(S_5 \wr S_2)^{\mathrm{pos}}$ of even permutations in $\mathrm{Aut}(5 \circ K_2)$. Note that G is isomorphic to the irreducible Coxeter group $\mathcal{D}_5$.

The two smallest orbits of G on 1-factors are inherited from Q_4. The orbit of length 5 consists of *direct 1-factors*—that is, Cayley graphs over E_{2^4} with connection set $\{x\}$, $x \in X_5$. Using the labeling of the vertices of $\square_5$ borrowed from Q_4, we define the orbit of length 40 consisting of *skew 1-factors.* A skew 1-factor in Q_4 may be considered as a representative of this orbit. Each skew 1-factor of $\square_5$ has a natural *mate* in the form of the corresponding direct 1-factor. Two such 1-factors together provide a subgraph of $\square_5$ of form $4 \circ C_4$, where C_4 is a quadrangle. A list of such *skew systems of quadrangles* is provided in the Appendix.

PROPOSITION 3.3. *The group G acts transitively on each of the following systems: 40 quadrangles, 40 edges, and 40 skew systems of quadrangles. The stabilizers of representatives of these orbits are (respectively) the groups* $D_4 \times S_3$, $S_4 \times S_2 \cong S_3 \wr S_2$, *and* $K = \mathrm{GL}(2,3)$.

Proof. The group $P = \mathrm{Aut}(Q_4)$ is embedded in the group $G = \mathrm{Aut}(\square_5)$. By the definition of a skew system of quadrangles (inherited from skew 1-factors), the stabilizer of such a system in G coincides with the stabilizer of the corresponding skew 1-factor in Q_4, that is, with K (see Proposition 3.2). The proofs for quadrangles and edges are straightforward. $\square$

3.4. Anstee–Robertson Graph

The Hoffman–Singleton graph, HoSi [HoS], is the unique strongly regular graph with parameters $(50, 7, 42, 0, 1)$. The group $\mathrm{Aut}(\mathrm{HoSi}) \cong P\Gamma U(3, 5^2)$ has order 252,000. The Robertson model of HoSi ([R]; see also [BoMu]) consists of five pentagons P_j and five pentagrams Q_k, where vertex i of P_j is joined to vertex $i + jk \pmod 5$ of Q_k.

Following [R], let us consider the Robertson decomposition of HoSi. After removing from HoSi one pentagon and one pentagram (together, forming a Petersen graph P), we obtain a graph $\mathcal{R}$ induced on the remaining 40 vertices. This

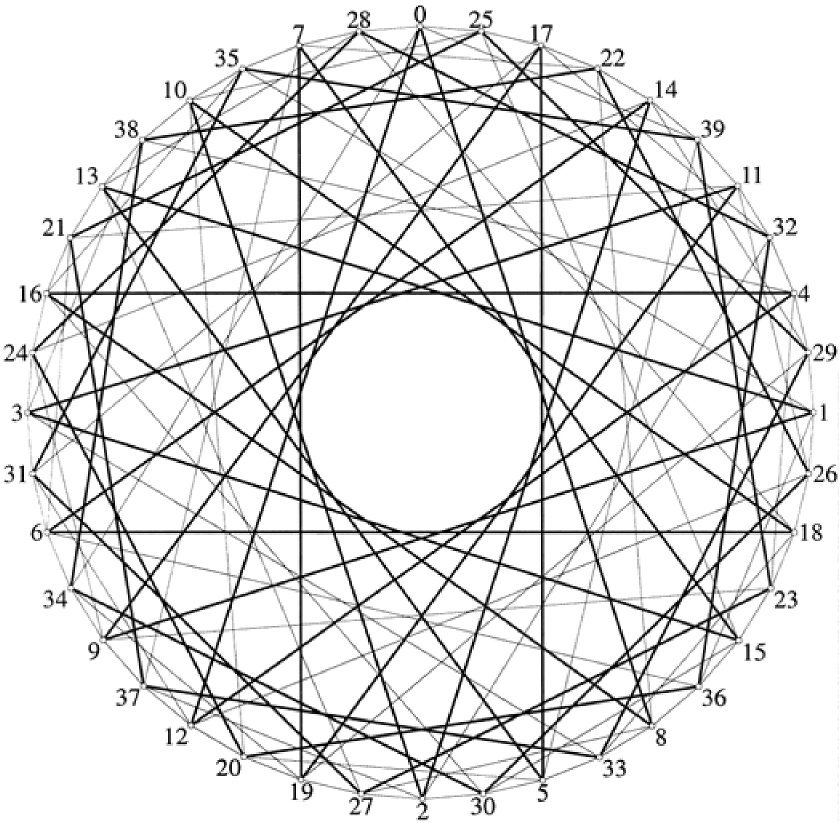

Figure 3.3 Hamiltonian model of the Anstee–Robertson graph

graph $\mathcal{R}$ is regular of valency 6 with girth 5 and can be characterized as the unique $(5, 6)$-cage.

R. P. Anstee found his own way to graph $\mathcal{R}$ (see [A]). He showed that its adjacency matrix S satisfies

$$S^2 + S = J_{40} - A + 6I_{40},$$

where $A = A(10 \circ K_4)$. Anstee claimed that $\mathrm{Aut}(\mathcal{R}) \cong \mathbb{Z}_4 \times S_5$, but we used GAP to find that $\mathrm{Aut}(\mathcal{R})$ is a nonsplit extension of $\mathbb{Z}_4$ by S_5 (see Section 9.4). As correctly observed by Anstee, the quotient graph of $\mathcal{R}$ (with respect to the imprimitivity system consisting of ten disjoint independent sets of size 4) is isomorphic to the complement $\bar{P}$ of the Petersen graph. However, $\mathrm{Aut}(P)$ does not embed in $\mathrm{Aut}(\mathcal{R})$.

For the Hamiltonian model of Evans (see e.g. [Eva]), we work with the Anstee model, wherein labels of the vertices have been shifted by -1 to the segment $[0, 39]$. Figure 3.3 represents a factorization of $\mathcal{R}$ into three regular spanning subgraphs of valency 2. The first and second subgraphs are disjoint Hamiltonian cycles, and the remaining subgraph (shown with bold lines) is the graph $8 \circ C_5$ consisting of eight disjoint cycles of length 5. Four of these cycles are Robertson pentagons and the other four are pentagrams.

PROPOSITION 3.4. *The group* $\mathrm{Aut}(\mathcal{R})$ *contains the following subgroups*: $\mathbb{Z}_{20}$ *with two orbits of length 20*; D_5, *the dihedral group of order 10*; *and a group* $H \cong \mathbb{Z}_4 \times D_5$ *of order 40 with two orbits of length 20.*

Proof. Let

$$\begin{aligned} g_1 = {}& (0,17,14,11,4,1,18,15,8,5,2,19,12,9,6,3,16,13,10,7) \\ & (20,37,34,31,24,21,38,35,28,25,22,39,32,29,26,23,36,33,30,27) \end{aligned}$$

and

$$\begin{aligned} g_2 = {}& (4,16)(5,17)(6,18)(7,19)(8,12)(9,13)(10,14)(11,15)(20,22)(21,23) \\ & (24,38)(25,39)(26,36)(27,37)(28,34)(29,35)(30,32)(31,33). \end{aligned}$$

The diagram shows that $\mathbb{Z}_{20} = \langle g_1 \rangle$ preserves both Hamiltonian cycles and that the permutation g_2 exchanges these cycles. In the established notation, $D_5 \cong \langle g_1^4, g_2 \rangle$ and $H \cong \mathbb{Z}_4 \times D_5 = \langle g_1, g_2 \rangle$. □

REMARK. Group H is the largest subgroup of $\mathrm{Aut}(\mathcal{R})$ of the form $\mathbb{Z}_4 \times Y$, where $Y \leq \mathrm{Aut}(P)$ (see Section 9).

4. Rank-5 Imprimitive Association Schemes: Higman's Classification

In comparison with ranks 3 and 4, the classification of rank-5 association schemes is not as fully developed. Following [H4], we consider symmetric imprimitive schemes of rank 5. Because 5 is prime, each decomposable scheme is wreath decomposable. Higman suggested considering three classes of symmetric indecomposable imprimitive schemes containing a parabolic E:

Class of E	I	II	III
rank of E	3	2	2
corank of E	2	3	2

If a quotient scheme from class II is imprimitive then the global scheme $\mathcal{H}$ has one additional parabolic E' of rank 3. In this case, $\mathcal{H}$ may also be attributed to class I. Clearly class III has empty intersection with classes I and II.

For each of the three classes, a description of the corresponding intersection matrices $M_i = (p_{si}^t)$, $0 \leq i, s, t \leq 4$, and character multiplicity tables is provided. There are interesting examples of schemes belonging to class I and to the intersection of classes I and II. We also mention a family of class III schemes that goes back to the Ph.D. thesis of Chang [Ch], written under the supervision of Higman.

We will regard schemes belonging to class II but not to class I as *proper class II schemes.* No example of such a scheme is mentioned in [H4].

Assume that $\mathcal{H}$ is a proper class II association scheme on a set Ω of nv points, and let E be a parabolic in $\mathcal{H}$ with n classes of size v. Let one of the classes of the quotient scheme Ω/E be a strongly regular graph with n vertices and the traditional parameters $k, l, \lambda, \mu, r, s, f, g$. Here $n = 1 + k + l$.

PROPOSITION 4.1. (a) *The valencies of* $\mathcal{H}$ *have the form* $v_0 = 1, v_1 = v - 1, v_2 = kS$, $v_3 = k(v - S)$, *and* $v_4 = lv$.

(b) *The character multiplicity table of* $\mathcal{H}$ *is*:

$$\begin{array}{c} \begin{pmatrix} 1 & v-1 & kS & k(v-S) & lv \\ 1 & v-1 & rS & r(v-S) & -(r+1)v \\ 1 & v-1 & sS & s(v-S) & -(s+1)v \\ 1 & -1 & x_1 & -x_1 & 0 \\ 1 & -1 & x_2 & -x_2 & 0 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ f \\ g \\ z_1 \\ z_2 \end{array}$$

Here S $(1 \leq S < v)$ *is an extra parameter,* x_1, x_2 *are the roots of the equation* $x^2 - \big(\frac{rv}{S} - \lambda(v - S)\big)x - \frac{kS(v-S)}{v-1} = 0$, $\tau = p^2_{33}$ *is one more extra parameter, and* $z_1 + z_2 = n(v - 1)$, *where*

$$z_1 = \frac{n(v-1)x_2}{x_2 - x_1} \quad \textit{and} \quad z_2 = \frac{n(v-1)x_1}{x_1 - x_2}.$$

(c) $0 < \mu < k$.

Proof. The formulas in parts (a) and (b) are given in [H4] (note that we have corrected a typo in the value for v_3 that occurred on page 213). Part (c) follows immediately from the definition of a proper class II scheme. □

In fact, [H4] provides a lot more information that is helpful in the enumeration of feasible sets of parameters. Unfortunately, there were a couple of misprints in the text, as follows.

- Page 213: In matrix M_2, the entry $(2, 2)$ should be $(2\lambda S - \tau) - \frac{(\lambda S - \tau)v}{S}$.
- Page 214, line –11: The end of the sentence should read "and $\frac{\tau v}{S} = \lambda(v - S)$."

In [KZ] we discovered an example of a rank-5 proper class II scheme on 40 points. It turns out that this example was known before (see [ChHu]), and it seems to be the only such example that appears in the literature. This motivated us to consider more systematically the Higmanian schemes on 40 points.

PROPOSITION 4.2. *There are just four feasible parameter sets for a putative rank-5 proper class II scheme on 40 points*:

Case	k	l	S	v_0	v_1	v_2	v_3	v_4	τ
a1.1	6	3	2	1	3	12	12	12	2
a1.2	6	3	2	1	3	12	12	12	3
a2	6	3	1	1	3	6	18	12	1
a3	3	6	2	1	3	6	6	24	0

Proof. In our case $nv = 40, 1 < v < 40$. We immediately observe that $n \in \{5, 10\}$.

Assume that $n = 5$, so $k = l = 2$, $\lambda = 0$, $\mu = 5$, and $v = 8$. Note that $p_{23}^2 = \lambda S - \tau$. Hence in this case $p_{23}^2 = -\tau \geq 0$, implying that $\tau = 0$. For each feasible value of S, the solution to the equation $x^2 = \frac{2S(8-5)}{7}$ is irrational. Therefore $z_1 = z_2 = \frac{35}{2} \notin \mathbb{Z}$, a contradiction.

Now assume that $n = 10$ and $v = 4$. Let us first consider the Petersen graph P in the role of the quotient graph. Thus $k = 3, l = 6, \lambda = 0$, and $\mu = 1$. We use the intersection number $p_{24}^2 = \frac{\mu S^2}{v}$ to conclude that $S = 2$, and the formula for p_{23}^2 again implies that $\tau = 0$. This yields the parameter set corresponding to case a3.

Now we consider the complement $\bar{P}$ of the Petersen graph in the role of the quotient graph; that is, $k = 6, l = 3, \lambda = 3$, and $\mu = 4$.

Let $S = 1$. Since both τ and $\lambda S - \tau$ are intersection numbers, we have $0 \leq \tau \leq \lambda S = 3$. For $\tau = 0, 3$ the values x_1, x_2 are irrational, yet $\frac{\tau v}{S} \neq \lambda(v - S)$. For $\tau = 1$ we obtain $x_{1,2} = 6, -1$; however, $z_1 = \frac{10 \cdot 3 \cdot (-1)}{-7} \notin \mathbb{Z}$. The value $\tau = 2$ leads to the case a2.

Now let $S = 2$, so that $0 \leq \tau \leq 6$. For $\tau \in \{0, 1, 5, 6\}$ we have irrational x_1, x_2, yet $\frac{\tau v}{S} \neq \lambda(v - S)$. For $\tau = 3$ we have irrational $x_1 = -x_2$, which yields the parameter set corresponding to case a1.2.

Finally, for $\tau = 2$ and $\tau = 4$ we obtain rational values of x_1 and x_2. We calculated corresponding tensors of the structure constants and recognized that they are algebraically isomorphic, with the algebraic isomorphism exchanging the relations R_2 and R_3. Therefore, we were able to restrict ourselves to the case $\tau = 2$ alone, thus obtaining the parameter set corresponding to case a1.1. □

In what follows we will completely characterize the cases a1.1 and a2; we also discuss the remaining two cases.

5. Classical Deza Graph on 40 Vertices

The notion of a Deza graph goes back to [DeD]. In an explicit form, the concept was introduced in [E+] as a generalization of strongly regular graphs. Namely, a regular graph is a *Deza graph* if the number of common neighbors of two distinct vertices takes one of two possible values (not necessarily dependent on the adjacency of the two vertices). This notion also has a natural formulation in terms of matrices. Suppose that Γ is an n-vertex graph with adjacency matrix M while A and B are symmetric $(0,1)$-matrices such that $A + B + I = J$. Then Γ is an (n, k, b, a) Deza graph if

$$M^2 = aA + bB + kI. \qquad (*)$$

Note that Γ is strongly regular if and only if A or B is M in $(*)$.

Both matrices A and B may be regarded as adjacency matrices of suitable graphs Γ_A and Γ_B, in which case Γ_A and Γ_B are called *Deza children* (recall that [DeD] was written by Antoine and Michel Deza). The cases when Deza children are strongly regular are of special interest. This is exactly the graph considered in [DeD] and

Table 5.1 Valencies of relations in $\mathfrak{n}$

Fiber	Quadrangles	Edges	Skew 1-factors
Quadrangles	$1, 3, 12, 12, 12$	$4, 12, 12, 12$	$4, 12, 24$
Edges	$4, 12, 12, 12$	$1, 3, 4, 8, 24$	$8, 8, 24$
Skew 1-factors	$4, 12, 24$	$8, 8, 24$	$1, 3, 4, 8, 24$

that is the subject of our consideration. We will call this original Deza graph the *classical Deza graph*.

5.1. Deza Family in a Higmanian House

We consider Higmanian association schemes of type a1.1. The classical Deza graph and its two Deza children may be embedded into such an association scheme (see [ChHu; DHu]). This motivates our axiomatization.

Definition 5.1. Assume we have a Deza family on nv points consisting of Deza graph Γ and Deza children Γ_A and Γ_B with adjacency matrices M, A, and B, respectively. Assume that S is the adjacency matrix of the graph $n \circ K_v$ such that $S + M + A' + B + I = J$ and $S + A' = A$.

Assume in addition that one of the matrices A and B (say B) is the adjacency matrix of a suitable strongly regular graph $\Delta = \Gamma_B$. If $\langle S, M, A', B, I \rangle$ is a symmetric Higmanian rank-5 association scheme of class II, then the scheme will be called a *Higmanian house* for the Deza family $(\Gamma, \Gamma_A, \Gamma_B)$.

5.2. Master Coherent Configuration on 120 Points

Let $G = \mathrm{Aut}(\square_5)$ as in Section 3.3. Let $K = \mathrm{GL}(2, 3)$ be the stabilizer of a skew 1-factor (i.e., a skew system of quadrangles) in $\square_5$, let $Q = D_4 \times S_3$ be the stabilizer of a quadrangle, and let $T = S_4 \times S_2$ be the stabilizer of an edge in $\square_5$.

We consider the Schurian coherent configuration $\mathfrak{n}$ with three fibers of size 40 that appears from the action of G on the cosets of Q, T, and K in G, interpreted as all possible quadrangles, edges, and skew systems in $\square_5$ (see the Appendix). Initial information about $\mathfrak{n}$ was obtained with the aid of COCO: $\mathfrak{n}$ has rank 35 with the valencies of basic relations presented in Table 5.1, and $\mathrm{Aut}(\mathfrak{n}) = G$. Restriction of $\mathfrak{n}$ to the first fiber defines a Higmanian association scheme $\mathfrak{m}$ of type a1.1. Restrictions of $\mathfrak{n}$ to the second or third fibers define schemes of rank 5. The rank-16 configuration on the first and third fibers has a merging scheme of rank 5 that comes from $GQ(3)$. The rank-16 configuration on first and second fibers has two non-Schurian merging schemes of rank 6 and 5, respectively.

Proposition 5.1. *Let G act on the set Ω of 40 quadrangles of $\square_5$. Then the following statements hold.*

Table 5.2 2-orbits of the association scheme $\mathfrak{m}$

		Invariants of representatives				
No.	Representative	Common vertices	Common edges	Skew or direct system	Valency	Name
0	$(0,1,5,4)$	4	4	N/A	1	Loops
1	$(2,3,7,6)$	0	0	Direct	3	Spread
2	$(0,1,3,2)$	2	1	N/A	12	Deza
3	$(8,9,11,10)$	0	0	Skew	12	$GQ(3)$
4	$(3,11,4,12)$	1	0	N/A	12	Petersen$[E_4]$

(a) 2-orb$(G,\Omega) = (R_0, R_1, R_2, R_3, R_4, R_5)$; *see Table 5.2.*
(b) *The relation* $R_0 \cup R_1$ *is an equivalence relation. The relations* R_2, R_3, R_4 *can be described* (*respectively*) *as the edge set of a Deza graph, the point graph of a generalized quadrangle, and the wreath product of the Petersen graph with the empty graph* $4 \circ K_1$ *on four vertices.*

Proof. The original proof was obtained with the aid of COCO. It is a straightforward exercise to prove (a) with the aid of the Appendix and the invariants introduced in Table 5.2. Consider $Q = (0,1,5,4)$ as a reference quadrangle, and count the number of common vertices and common edges of other quadrangles with Q. When the number of common vertices is zero, we take into account additional information. Namely, consider $\square_5$ as a Cayley graph over E_{2^4} with connection set X_5 (see Section 3.3). Consider subgroups of order 4 generated by two elements of X_5, and identify quadrangles as cosets of such subgroups. This implies that $R_0 \cup R_1$ is an equivalence relation. Its equivalence classes may be called *direct systems* of disjoint quadrangles. Similarly, we explain relation R_4 as the wreath product of the Petersen graph with the empty graph $4 \circ K_1$.

To prove the property of the basic graph $\Gamma_2 = (\Omega, R_2)$, inspect the 14 paths of length 2 of quadrangles having a common edge (i.e., adjacent in Γ_2). The interpretation of relation R_3 will then follow from Proposition 5.2. $\square$

5.3. Two Incidence Structures

Let us define an incidence structure $\mathfrak{S}_1$ on two fibers of the master coherent configuration $\mathfrak{n}$. The points are quadrangles in $\square_5$, the lines are skew systems of quadrangles in $\square_5$, and incidence is defined as set inclusion.

PROPOSITION 5.2. $\mathfrak{S}_1$ *is a generalized quadrangle of order 3 that is isomorphic to* $Q(4,3)$. Aut$(\mathfrak{S}_1)$ *is a rank-3 group of order 51,840.*

Proof. Group G acts transitively on the lines of $\mathfrak{S}_1$. Let $L = \mathrm{GL}(2,3)$ be the stabilizer of a reference line l. According to Table 5.1, L has orbits of length 4, 12, and 24 on the point set Ω. Choose representatives of these orbits, and identify lines that contain said representatives and also intersect l. Next count the number of flags in $\mathfrak{S}_1$ to confirm that all axioms of a generalized quadrangle of order 3 are fulfilled. The dual incidence structure to $\mathfrak{S}_1$ contains spreads (aka direct systems of quadrangles). Thus $\mathfrak{S}_1$ is isomorphic to $Q(4,3)$, and its automorphism group is well known. $\square$

We define the incidence structure $\mathfrak{S}_2 = (\Omega, \mathcal{S})$ on the remaining fibers of $\mathfrak{n}$. Again, points will be the elements of the set Ω of quadrangles in $\square_5$, while lines will be the edges of $\square_5$. Incidence is the dual of inclusion.

PROPOSITION 5.3. (a) $\mathfrak{S}_2$ *is a symmetric incidence structure with 40 points and 40 lines of size 4, forming a partial linear space.*

(b) *For each line* $l \in \mathcal{S}$, *there are precisely*:

- *12 points* $p \notin l$ *through which there are no lines intersecting* l;
- *12 points* $p \notin l$ *through which there is exactly one line intersecting* l; *and*
- *12 points* $p \notin l$ *through which there are exactly two lines intersecting* l.

Proof. Since (G, S) is transitive, we may select an arbitrary reference line—say, $\{12, 14\}$. The stabilizer T of this line has orbits in Ω of length 4, 12, 12, and 12 with representatives $(1,14,12,3)$, $(0,2,3,1)$, $(0,4,5,1)$, and $(0,15,14,1)$, respectively. Now check that the lines intersecting the reference line and containing the given representative points have the claimed properties. $\square$

PROPOSITION 5.4. *The classical Deza graph is the point graph* Γ_2 *of* $\mathfrak{S}_2$. *Moreover,* $\mathfrak{S}_2$ *is uniquely reconstructed from its point graph* Γ_2, *and* $\mathrm{Aut}(\mathfrak{S}_2) = \mathrm{Aut}(\Gamma_2) = G$.

Proof. It was proved in Proposition 5.1 that R_2 is the edge set of a Deza graph. According to Proposition 5.2, the graph Γ_3 is the point graph of $Q(4,3)$. These two facts uniquely characterize the classical Deza graph.

Each edge of $\square_5$ corresponds to a clique of size 4 in Γ_2. Thus we have at least 40 such cliques. Since Γ_2 has exactly 40 cliques of size 4, it follows that $\mathfrak{S}_2$ may be recovered from Γ_2 and thus $\mathrm{Aut}(\mathfrak{S}_2) = \mathrm{Aut}(\Gamma_2)$ contains G.

Let us consider the edge $\{0,1\} \in S$ as a reference point. Then the edges $\{1,5\}$, $\{4,5\}$, $\{13,15\}$, and $\{10,14\}$ correspond to the 2-orbits of (G, S) with valencies 8, 4, 24, and 3, respectively. Check that the edge $l_0 = \{0,1\}$ (regarded as a line of $\mathfrak{S}_2$) has intersection size 1 with lines $l_1 = \{1,5\}$ and $l_2 = \{4,5\}$ and that its intersection with the remaining two lines is empty. Each of the three points on line l_1 not belonging to l_0 is incident to exactly one point on l_0. Similarly, each of the three points on the line l_2 is incident with no point on l_0.

We define an auxiliary graph Δ with vertex set S and edge set $\{\{0,1\},\{1,5\}\}^G$. Thus Δ is isomorphic to the line graph of $\square_5$. Note that $\mathrm{Aut}(\mathfrak{S}_2)$ acts faithfully on Δ as a subgroup of $\mathrm{Aut}(\Delta)$. Using the classical Whitney–Jung theorem (see e.g. [Ha]), we obtain $\mathrm{Aut}(\Delta) \cong \mathrm{Aut}(\square_5) \cong G$. $\square$

Table 6.1 Higmanian schemes

Γ_i	$\mathfrak{m}_i$	$\lvert\mathrm{Aut}(\Gamma)\rvert$	$\lvert\mathrm{orb}(\mathrm{Aut}(\Gamma))\rvert$	$\lvert\mathrm{Aut}(\mathfrak{m})\rvert$	$\lvert\mathrm{orb}(\mathrm{Aut}(\mathfrak{m}))\rvert$	Geom.	4-cliques
Γ_1	1.1	48	$4, 12^3$	48	same	no	32
Γ_2	2.1	384	$16, 24$	384	same	no	8
Γ_2	2.2	384	$16, 24$	192	same	yes	40
Γ_3	3.1	8	$2^8, 4^2, 8^2$	8	same	no	20
Γ_4	4.1	12	$1, 3^3, 6^3, 12$	12	same	no	24
Γ_5	5.1	64	$8, 16^2$	64	same	yes	40
Γ_5	5.2	64	$8, 16^2$	32	$4^2, 8^2, 16$	no	24
Γ_6	6.1	51,840	40	1920	same	yes	40
Γ_7	7.1	192	$4, 12, 24$	192	same	no	24
Γ_7	7.2	192	$4, 12, 24$	32	$4^2, 8^2, 16$	yes	40
Γ_8	8.1	8	$2^8, 4^2, 8^2$	8	same	no	28
Γ_9	9.1	48	$2, 4, 6, 12, 16$	16	$2^4, 4^4, 16$	no	32
Γ_{10}	10.1	16	$4^2, 8^4$	16	same	no	32
Γ_{10}	10.2	16	$4^2, 8^4$	8	$4^8, 8$	no	32
Γ_{11}	11.1	144	$4, 12, 24$	48	same	no	32

COROLLARY. *The association scheme* $(\Omega, 2\text{-orb}(G, \Omega))$ *represents the Schurian Higmanian association scheme* $\mathfrak{m}$, *which is a Higmanian house for the classical Deza family defined by* $\mathfrak{S}_2$ *and its point graph* Γ_2.

6. A Family of Algebraically Isomorphic Association Schemes on 40 Points

We wish to classify all association schemes that are algebraically isomorphic to $\mathfrak{m}$. It is easy to see that $\mathfrak{m} = \langle\langle R_2 \rangle\rangle$. We may replace Γ_3 (the point graph of $Q(4,3)$) with an arbitrary strongly regular graph of valency 12 on 40 vertices. There are precisely 28 such graphs that are nonisomorphic (see [Sp]). In [KZ] we described the algorithm (implemented in GAP) that we use here to search for all association schemes that are algebraically isomorphic to $\mathfrak{m}$.

In terms of the labeling in [Sp] of strongly regular graphs, precisely the first 11 graphs admit at least one scheme algebraically isomorphic to $\mathfrak{m}$. Altogether we obtain 15 schemes. Of special interest are those Deza graphs that are also *geometric* (i.e., point graphs of a structure analogous to $\mathfrak{S}_2$, as described in Section 5.3).

Our main computational results are recorded in Table 6.1. For each graph and each association scheme, we give the order of the automorphism group and the lengths of its orbits on vertices. The classical scheme $\mathfrak{m}$ coincides with the scheme $\mathfrak{m}_{6.1}$. The number of 4-cliques in each of the 15 Higmanian graphs is also provided. Of these, 14 schemes are non-Schurian.

The geometric Deza graphs are precisely those that have exactly 40 4-cliques, and they generate the Higmanian schemes $\mathfrak{m}_{2.2}$, $\mathfrak{m}_{5.1}$, $\mathfrak{m}_{6.1}$, and $\mathfrak{m}_{7.2}$. All four geometric Deza graphs share the same intersection property as the one formulated in

Proposition 5.3(b). Two of the schemes ($\mathfrak{m}_{2.1}$ and $\mathfrak{m}_{2.2}$) have reasonably large automorphism groups; they may be described as certain mergings of the rank-16 coherent configuration $\mathcal{X}_{2,1}$ and the rank-20 coherent configuration $\mathcal{X}_{2,2}$, respectively. Note that $\mathrm{AAut}(\mathcal{X}_{2,1})$ has order 2 whereas $\mathrm{AAut}(\mathcal{X}_{2,2})$ has order 24. Both configurations have proper algebraic automorphisms. In particular, $\mathfrak{m}_{2.1}$ and $\mathfrak{m}$ are algebraic twins in $\mathcal{X}_{2,1}$.

In the next two sections we will split $\mathfrak{m}$ into a coherent configuration of rank 190. This will allow us to interpret a few of the detected Higmanian schemes as remarkable mergings of the resulting high-rank configuration.

7. WFDF Coherent Configurations

We describe a concept recently introduced in [M], following ideas of Wallis and Fon-Der-Flaas. A specific example will be investigated in Section 8. Another particular significant example was considered in [K+].

As the first (internal) ingredient we consider a complete affine amorphic association scheme of order n (see [GIvK]). It has n^2 vertices and $n+1$ classes, and it bijectively corresponds to an affine plane of order n.

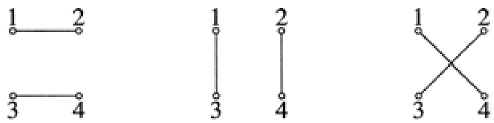

Figure 7.1 Parallel classes

EXAMPLE 7.1. Figure 7.1 depicts three parallel classes of an affine plane of order 2, also known as classes of a complete affine scheme on four points.

EXAMPLE 7.2. The complement of the Petersen graph (the triangular graph $T(5)$) is geometrical. Indeed, its maximal cliques with respect to inclusion are $\{\{i,j\},\{i,k\},\{j,k\}\}$ for each subset $\{i,j,k\} \subseteq [1,5]$ of size 3. The ten cliques represent blocks of a uniform partial linear space.

We are now in a position to define WFDF configurations as a kind of a "blow up" of an external structure, namely of a partial linear space. Each point of the space will be replaced by a copy of an affine plane. In this way we obtain the fibers of the resulting configuration.

DEFINITION 7.1. Let $\mathfrak{O}_i = (V_i, \{C_{i,1}, \ldots, C_{i,n+1}\})$, $1 \le i \le m$, be a copy of an affine plane of order n, where $|V_i| = n^2$. Here $C_{i,j}$, $1 \le j \le n+1$, is a parallel class of lines in $\mathfrak{O}_i$, regarded as a disconnected graph $n \circ K_n$ consisting of n copies of the complete graph K_n. The m planes $\mathfrak{O}_i$ are labeled by elements from $[1,m]$. Let $\mathfrak{S}$ be a partial linear space with m points and assume that each point of $\mathfrak{S}$ is incident to at most $n+1$ lines of $\mathfrak{S}$.

Consider the set $V = V_1 \cup V_2 \cup \cdots \cup V_m$, where the V_i are pairwise disjoint. Identify V with the Cartesian product $V = [1, n^2] \times [1, m]$, in which case we have $V_i = [1, n^2] \times \{i\}$ for each i. We now define an arc partition of the complete graph $K_{n^2 m}$ with vertex set V—in other words, a complete color graph with $n^2 \cdot m$ vertices.

Each set V_i plays the role of a fiber in our forthcoming coherent configuration. Inside of this fiber we naturally have $n+2$ colors (i.e., relations $C_{i,j}$, $1 \leq j \leq n+1$ and also the identity relation Δ_i on V_i). It remains to define relations between different fibers.

Consider the Levi graph $L(\mathfrak{S})$ of the partial linear space $\mathfrak{S} = ([1,m], \mathcal{H})$. For each point $i \in [1,m]$, we label all blocks from $\mathcal{H}$ that are incident to i by distinct elements of $[1, n+1]$ (this is possible by our assumption on $\mathfrak{S}$). Let us denote by f_i the bijection used in this labeling. Take an arbitrary block $h \in \mathcal{H}$, and let $i, j \in [1,m]$ be incident to h. Assume that f_i assigns s_i to (i,h) while f_j assigns s_j to (j,h). Take the class C_{s_i} from fiber V_i and the class C_{s_j} from V_j. Each such class can be regarded as a partition of $[1, n^2]$ into n subsets of cardinality n. Consider a bijection σ_{ij} between the partitions associated with the classes C_{s_i} and C_{s_j} with $\sigma_{ij} = \sigma_{ji}^{-1}$.

With the aid of σ_{ij} we define a directed regular bipartite graph R_{ij} of valency n: each vertex from class x of partition C_{s_i} is joined by an arc with each vertex from class $x^{\sigma_{ij}}$ in partition C_{s_j}. Define $\overline{R_{ij}} = (V_i \times V_j) \setminus R_{ij}$. Note that if i and j are not collinear in $\mathfrak{S}$ then R_{ij} is empty. In this case, we obtain just one relation $\overline{R_{ij}} = V_i \times V_j$ between the fibers V_i and V_j.

When $|h| > 2$ we require some additional conditions on the bijection σ_{ij}. For all $i, j, k \in h$ we shall assume that

$$\sigma_{ij} \cdot \sigma_{jk} = \sigma_{ik}. \tag{$**$}$$

Thus, if we fix i and choose σ_{ij} arbitrarily for each $j \neq i$, then $(**)$ determines σ_{jk}.

PROPOSITION 7.1. *The color graph $\mathfrak{C} = (V, R)$ defines a coherent configuration.*

Proof. Let W be spanned by the adjacency matrices of relations from R. We must check that the product of any two basic matrices belongs to W.

We consider separate cases. If both relations are from the same fiber then apply [GIvK, Lemma 3.2]. Let both relations be between the same pair of distinct fibers V_i and V_j. If i and j are not collinear then $A(\overline{R_{ij}}) \cdot A(\overline{R_{ij}}) = 0$ and $A(\overline{R_{ij}}) \cdot A(\overline{R_{ji}}) = n^2 A((V_i)^2)$. Otherwise we get $A(R_{ij}) \cdot A(R_{ji}) = nA(n \circ K_n)$, where $n \circ K_n$ is a partition of V_i defined by the bijection f_i. In this case, all other nonzero products of the relations (i.e., $A(R_{ij}) \cdot A(\overline{R_{ji}})$ and $A(R_{ij}) \cdot A(\overline{R_{ij}})$) are immediate consequences of the preceding equalities. If the first relation is between fibers V_i and V_j and the second relation is between fibers V_j and V_k, where i, j, k are pairwise distinct, then: (1) if i, j, k are not collinear then $A(R_{ij}) \cdot A(R_{jk}) = A(V_i \times V_k)$; and (2) if i, j, k are collinear then $A(R_{ij}) \cdot A(R_{jk}) = nA(R_{ik})$. Finally, if the first relation is between fibers V_i and V_j and the second is between fibers V_k and V_l, where i, j, k, l are all distinct, then the product of the two matrices is zero. $\square$

8. WFDF Configuration on 40 Points and Some of Its Mergings

We started from the classical Higmanian association scheme $\mathfrak{m}$ on 40 points, colored the ten cells of the parabolic relation E with ten distinct colors, and then constructed the coherent closure of the resulting colored graph. As a result, we obtained a rank-190 coherent algebra W with ten fibers that turns out to be Schurian. Using some experimental programs from COCO-II, we described the groups $\mathrm{Aut}(W)$, $\mathrm{CAut}(W)$, $\mathrm{CAut}(W)/\mathrm{Aut}(W)$, and $\mathrm{AAut}(W)$. We checked that the configuration corresponding to W satisfies all the axioms presented in Section 7 for the particular case in which the internal and external structures coincide with the objects presented in Examples 7.1 and 7.2, respectively. Finally, we constructed a model for our WFDF configuration on 40 points. With the aid of this model, we were able to achieve a computer-free interpretation of a major part of our discoveries.

Model of W. Start with group $H = E_{2^4}$ regarded as a 4-dimensional vector space over $\mathbb{F}_2$. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of H, and consider the set $X_5 = \{e_1, e_2, e_3, e_4, e_1 + e_2 + e_3 + e_4\}$. Clearly these five vectors are linearly dependent; however, any four are independent. Every pair of distinct vectors from X_5 generates a subgroup of order 4 in H, and this subgroup has four cosets in H. Since there are ten possible pairs, we have altogether a set Ω of 40 cosets. The group H acts faithfully on Ω via translation.

Proposition 8.1. *Let* $W' = (\Omega, 2\text{-orb}(H, \Omega))$.

(a) *W' is a rank-190 Schurian coherent configuration with ten fibers of size 4. Each fiber corresponds to four cosets of the same subgroup. The association scheme induced by each fiber is isomorphic to a complete affine scheme on four points as described in Example 7.1.*
(b) *Let Ω_i and Ω_j be two fibers generated by disjoint 2-element subsets of X_5. Then all arcs starting in Ω_i and ending in Ω_j form one relation from* $2\text{-orb}(H, \Omega)$. *Let Ω_i and Ω_j be two fibers generated by 2-element subsets having a common point. Then* $2\text{-orb}(H, \Omega)$ *contains exactly two relations formed by arcs starting in Ω_i and ending in Ω_j.*
(c) *One has $W' \cong W$. Thus W' forms a WFDF coherent configuration with internal structure isomorphic to an affine plane of order 2 and external structure isomorphic to one described in Example 7.2.*

Proof. As a control sum the reader may count $10 \cdot 4 + 30 \cdot 1 + 60 \cdot 2 = 190$ relations of three different kinds. □

In what follows we will identify the coherent configurations W and W'.

Proposition 8.2. *The group* (H, Ω) *is 2-closed*; *that is,* $\mathrm{Aut}(W) = H$.

Proof. The group $\mathrm{Aut}(W)$ is the stabilizer of all cells of the parabolic $10 \circ K_4$ in $\mathrm{Aut}(\mathfrak{m})$. Further, $\mathrm{Aut}(\mathfrak{m})$ coincides with the group G of order 1920. Since H is the maximal normal 2-subgroup of G, we obtain $H = \mathrm{Aut}(W)$. □

PROPOSITION 8.3. *The centralizer* $C_{\mathrm{Sym}(\Omega)}(H)$ *of* (H,Ω) *is an elementary abelian group of order* 2^{20}.

Proof. The action of H on each of its ten orbits is unfaithful and coincides with the regular group E_4, which is self-centralizing. Now apply [Wie, Prop. 4.3]. □

PROPOSITION 8.4. $\mathrm{CAut}(W) = N_{\mathrm{Sym}(\Omega)}(H)$, *and*

$$\mathrm{CAut}(W) \cong \left(S_5, \left\{ {[1,5] \atop 2} \right\}\right) \wr (E_4, E_4) \cong E_{2^{20}} \rtimes S_5,$$

a group of order $2^{23} \cdot 3 \cdot 5$.

Proof. The color group of an arbitrary Schurian coherent configuration coincides with the normalizer of its automorphism group in the corresponding symmetric group. Apply Proposition 8.2. The group $\left(S_5, \left\{ {[1,5] \atop 2} \right\}\right) \wr (E_4, E_4)$ is a semidirect product of $E_{2^{20}}$ with S_5; hence S_5 normalizes $E_{2^{20}}$, which in turn coincides with the group $C_{\mathrm{Sym}(\Omega)}(H)$. By Proposition 8.1(c), the group $\mathrm{CAut}(W)$ acts on the fibers as a subgroup of the automorphism group of the point graph of the external structure, which is known to be isomorphic to $T(5)$. Because $\left(S_5, \left\{ {[1,5] \atop 2} \right\}\right) = \mathrm{Aut}(T(5))$, it follows that $\mathrm{CAut}(W) \cong E_{2^{20}} \rtimes L$, where $L \leq S_5$. Finally we check that $L = S_5$. □

COROLLARY 8.5. $\mathrm{CAut}(W)/\mathrm{Aut}(W)$ *is a group of order* $2^{19} \cdot 3 \cdot 5$.

PROPOSITION 8.6. (a) $\mathrm{AAut}(W)$ *is isomorphic to the group* $E_{2^{20}} \rtimes S_5$.

(b) $\mathrm{CAut}(W)/\mathrm{Aut}(W)$ *is a nonnormal subgroup of index 16 in* $\mathrm{AAut}(W)$.

Proof. Both parts were confirmed using COCO-II and GAP. For a computer-free proof, count the structure constants of W and then consider particular algebraic automorphisms of W that generate the group A. □

REMARK. In its action on 40 squares, $G = \mathrm{Aut}(\square_5)$ is a subgroup of $\mathrm{CAut}(W)$. If we now consider the unfaithful action of G on the 190 colors of W, we see that its kernel $H \cong E_{2^4}$ coincides with $\mathrm{Aut}(W)$. Thus G/H embeds in $\mathrm{AAut}(W)$ as a subgroup S isomorphic to S_5.

PROPOSITION 8.7. (a) *There are* 2^{20} *subgroups of* $A = \mathrm{AAut}(W)$ *that are conjugate in* A *to the subgroup* S. *These subgroups split into three* Q*-conjugacy classes under the action of the subgroup* $Q = \mathrm{CAut}(W)/\mathrm{Aut}(W)$ *of* A. *The cardinalities of these classes are 65,536, 655,360, and 327,680.*

(b) *Representatives of the* Q*-conjugacy classes lead to three algebraic mergings that are algebraically isomorphic to* $\mathfrak{m}$. *The mergings are 2.1, 6.1, and 7.1 as designated in Table 6.1. In fact, these three schemes are algebraic "triplets". More explicitly, we obtain two non-Schurian schemes from the classical Schurian one with the aid of suitable algebraic automorphisms from* A.

Proof. Confirmed via COCO-II. □

9. Coherent Closure of Cage on 40 Vertices

9.1. Computer-aided Results

In what follows we consider the same copy of the graph $\mathcal{R}$ as in Section 3. Set $G = \mathrm{Aut}(\mathcal{R})$. GAP shows that G has order 480. Furthermore, using COCO we find that G acts transitively of rank 7 on the vertex set Ω of $\mathcal{R}$ with subdegrees 1, 2, 1, 12, 6, 12, and 6. Representatives of the corresponding 2-orbits are $(0,0)$, $(0,1)$, $(0,2)$, $(0,4)$, $(0,8)$, $(0,9)$, and $(0,10)$, respectively. COCO returns seven nontrivial merging schemes of the symmetric association scheme $\mathcal{X} = (\Omega, 2\text{-orb}(G,\Omega))$. Among these is the scheme $\mathfrak{m}_{a2} = (\Omega, \{R_0, R_1 \cup R_2, R_3, R_4, R_5 \cup R_6\})$. COCO shows that $\mathrm{Aut}(\mathfrak{m}_{a2}) = G$, so $\mathfrak{m}_{a2}$ is a non-Schurian association scheme. Because $\{0,8\}$ is an edge of $\mathcal{R}$, we know that $\mathcal{R} = \Gamma_4$. Finally, we conclude that $\mathfrak{m}_{a2}$ is a proper class II scheme that represents case a2 of Proposition 4.2. Using GAP once more, we find that $G = (\mathrm{SL}(2,5) : \mathbb{Z}_2) : \mathbb{Z}_2$. We now give a computer-free explanation of these results using the fact that $\mathcal{R}$ is the unique cage on 40 vertices.

9.2. A Few Subgroups of G

PROPOSITION 9.1. (a) *The group G has an imprimitivity system S consisting of ten blocks of size 4. Each block of the system induces an empty subgraph of size 4. There is only one such system in G. The stabilizer in G of a block is a group $\mathbb{Z}_4$ acting semiregularly on Ω.*

(b) *The quotient graph $\mathcal{R}/S$ of the graph $\mathcal{R}$ with respect to S is the complement $\bar{P}$ of the Petersen graph. One has* $|\mathrm{Aut}(\mathcal{R})| \leq 480$.

Proof. This proposition is a summary of results from [A] after certain corrections (see Section 3.4). The system S consists of blocks of order 4 that form the matrix given by Anstee. The group $\mathbb{Z}_4 = \langle g_1^5 \rangle$ is the stabilizer of each block in the system S. Uniqueness of S was proved by Anstee. Thus $\mathrm{Aut}(\mathcal{R}) \cong \mathbb{Z}_4.Y$, where $Y \leq S_5$. □

Recall the subgroup $H \leq \mathrm{Aut}(\mathcal{R})$ with $H \cong \mathbb{Z}_4 \times D_5 = \langle g_1, g_2 \rangle$ from Proposition 3.4.

PROPOSITION 9.2. *There exists a subgroup $Q \leq \mathrm{Aut}(\mathcal{R})$ of order 80 acting transitively on Ω. One has $Q \cong \mathbb{Z}_4.\mathrm{AGL}(1,5)$. Moreover, Q contains a regular subgroup R.*

Proof. Set

$$\begin{aligned} g_3 = {} & (0,20,3,21,2,22,1,23)(4,28,19,33,6,30,17,35) \\ & (5,31,16,32,7,29,18,34)(8,36,15,25,10,38,13,27) \\ & (9,39,12,24,11,37,14,26) \end{aligned}$$

and check that $g_3 \in G$. Let $Q = \langle g_1, g_2, g_3 \rangle$. Note that g_3 normalizes $\langle g_1, g_2 \rangle$ and interchanges two orbits of length 20 of $\langle g_1, g_2 \rangle$ on Ω. Therefore Q is isomorphic to $\mathbb{Z}_4.Y$, where $Y \cong \mathrm{AGL}(1,5)$. Define $R = \langle g_1^4, g_3 \rangle$. The group R is transitive and a proper subgroup of Q, so R is the required regular subgroup of order 40. □

9.3. Coherent Closure of the Graph $\mathcal{R}$

The following lemma is regarded as folklore.

LEMMA 9.3. *Let $\mathfrak{m} = (\Omega, \{R_i \mid i \in I\})$ be an association scheme and let $\tau = \{\tau_0 = \{0\}, \tau_1, \dots, \tau_s\}$ be a partition of I. Define $S_j = \bigcup_{i \in \tau_j} R_i$ for all $0 \le j \le s$. Let $\Gamma_j = (\Omega, S_j)$, $0 \le j \le s$. Let $x \in \Omega$ be a reference vertex and let $\sigma = \{\{x\}, \Gamma_1(x), \dots, \Gamma_s(x)\}$ be a partition of Ω into the neighbor sets of x in the graphs $\Gamma_0, \dots, \Gamma_s$, respectively.*

Assume that the relations $S_1, \dots, S_s$ are symmetric and that the adjacency matrices $A(\Gamma_i)$ for $2 \le i \le s$ are expressible as suitable polynomials in $A(\Gamma_1)$. If σ is an equitable partition with respect to Γ_1, then $\mathfrak{m}' = (\Omega, \{S_j \mid j \in [0,s]\})$ is a merging association scheme of $\mathfrak{m}$. Moreover, in this case $\langle\langle \Gamma_1 \rangle\rangle = \mathfrak{m}'$.

The intersection diagram of $\mathcal{R}$ in Figure 9.1 was constructed as follows. Fix vertex 0 and consider the set $N(0) = N_1(0)$ of neighbors of 0. Consider the sets $N_2(0)$ and $N_3(0)$ of the vertices at distance 2 and 3 from 0. Split the set $N_2(0)$ into two sets $N_{2,1}(0)$ and $N_{2,2}(0)$, where $N_{2,2}(0)$ contains precisely those vertices of $N_2(0)$ that have a neighbor in $N_3(0)$. Check that the obtained partition is equitable. The sets in the partition are as follows: $N_0(0) = \{0\}$, $N_1(0) = \{8, 12, 25, 28, 34, 39\}$,

$$N_{2,1}(0) = \{4, 5, 6, 7, 16, 17, 18, 19, 20, 21, 22, 23\},$$

$$N_{2,2}(0) = \{9, 10, 11, 13, 14, 15, 24, 26, 27, 29, 30, 31, 32, 33, 35, 36, 37, 38\},$$

and $N_3(0) = \{1, 2, 3\}$.

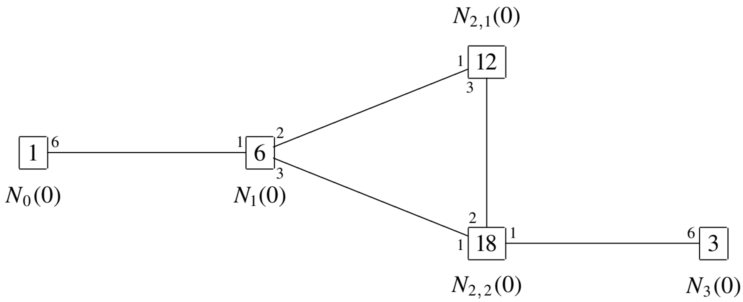

Figure 9.1 Intersection diagram of graph $\mathcal{R}$

THEOREM 9.4. *The coherent closure $\langle\langle \mathcal{R} \rangle\rangle$ of graph $\mathcal{R}$ is an association scheme with four classes and valencies 1, 6, 12, 18, and 3. This association scheme is a proper class II Higmanian scheme of rank 5 that belongs to type a2.*

Proof. We apply Lemma 9.3 in conjunction with Proposition 9.2, together with the above observation about the intersection diagram of $\mathcal{R}$ depicted in Figure 9.1. □

We use the notation $\mathfrak{m}_{\text{a2}}$ for the resulting Higmanian association scheme $\langle\langle\mathcal{R}\rangle\rangle$. This is our first example of a scheme of type a2.

THEOREM 9.5. *Any association scheme* $\mathfrak{m}'$ *that is algebraically isomorphic to* $\mathfrak{m}_{\text{a2}}$ *is also combinatorially isomorphic to* $\mathfrak{m}_{\text{a2}}$.

Proof. Let $A_0 = I$, A_1', $A_2' = S$, A_3', and A_4' be the basic matrices of the scheme $\mathfrak{m}'$. Consider the basic graph $\mathcal{R}'$ defined by the matrix A_2'; this is a connected regular graph of valency 6. We construct for $\mathcal{R}'$ the intersection diagram as was done for $\mathcal{R}$. Any algebraic isomorphism from $\mathfrak{m}_{\text{a2}}$ to $\mathfrak{m}'$ sends $\mathcal{R}$ to $\mathcal{R}'$, so $\mathcal{R}'$ has the same intersection diagram as $\mathcal{R}$. From the diagram we conclude that $\mathcal{R}'$ does not contain triangles and quadrangles but does contain cycles of length 5. Therefore $\mathcal{R}'$ is a regular graph of valency 6 and girth 5; hence it is a cage on 40 vertices and so is unique up to isomorphism. Thus, both $\mathfrak{m}_{\text{a2}} = \langle\langle\mathcal{R}\rangle\rangle$ and $\mathfrak{m}' = \langle\langle\mathcal{R}'\rangle\rangle$ are isomorphic. $\square$

9.4. Full Automorphism Group

Observe that $N = \mathrm{SL}(2,5)$ is the unique subgroup of index 4 in $G = \mathrm{Aut}(\mathfrak{m}_{\text{a2}})$. (Note that $\mathrm{Aut}(\mathcal{R}) = \mathrm{Aut}(\mathfrak{m}_{\text{a2}})$ because $\langle\langle\mathcal{R}\rangle\rangle = \mathfrak{m}_{\text{a2}}$.) Our goal is to substantiate this computer-aided knowledge. We will construct a new model of the graph $\mathcal{R}$. Consider the subgroup $K = \mathrm{HL}(2,5)$ of $\mathrm{GL}(2,5)$ consisting of the matrices whose determinants belong to $\mathbb{F}_5^{*2}$ (here "H" stands for "half"). Clearly, $|K| = \frac{1}{2}|\mathrm{GL}(2,5)| = 240$. Let $V = (\mathbb{GF}(5))^2 \setminus \{0\}$ be the set of nonzero row vectors of $(\mathbb{GF}(5))^2$. Then K acts transitively on V via right multiplication of a row by a matrix.

Let $O = \left\{\{x,y,z\} \in \left\{\binom{V}{3}\right\} \mid x+y+z=0)\right\}$. Three elements of a typical subset $\{x,y,z\} \in O$ are pairwise independent, and the element z is uniquely determined by x and y. Therefore $|O| = \frac{24\cdot 20}{3!} = 80$.

Now consider the natural action of K on O by $\{x,y,z\}^A = \{xA, yA, zA\}$. We regard $o_0 = \{(1,0),(0,1),(4,4)\}$ as a reference point in O.

PROPOSITION 9.6. *Group* (K,O) *has two orbits, each of length 40.*

Let $\Omega = o_0^K$ be the orbit containing o_0 under the action of K for $|\Omega| = 40$.

PROPOSITION 9.7. *The transitive permutation group* (K,Ω) *has rank 10, with four 2-orbits of valency 1 and six 2-orbits of valency 6.*

Proof. The stabilizer K_{o_0} of o_0 is of order $\frac{240}{40} = 6$. Note that any matrix whose two rows are elements of o_0 is in K_{o_0}; thus we get all of K_{o_0} explicitly, as follows:

$$K_{o_0} = \left\{ I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 1 & 0 \end{pmatrix} \right\}.$$

There are three other elements of Ω stabilized by K_{o_0}, namely

$$o_1 = \{(2,0),(0,2),(3,3)\}, \quad o_2 = \{(3,0),(0,3),(2,2)\},$$
$$\text{and} \quad o_3 = \{(4,0),(0,4),(1,1)\}.$$

Similarly, we have six remaining representatives of orbits of length 6. A list of representatives of the ten 2-orbits of (K, Ω) follows: $R_i = (o_0, o_i)^K$ for $i = 0, 1, 2, 3$; $R_4 = (o_0, \{(0,1), (1,1), (4,3)\})^K$; $R_5 = (o_0, \{(0,1), (1,2), (4,2)\})^K$; $R_6 = (o_0, \{(0,2), (2,1), (3,2)\})^K$; $R_7 = (o_0, \{(0,2), (2,2), (3,1)\})^K$; $R_8 = (o_0, \{(0,3), (2,1), (3,1)\})^K$; $R_9 = (o_0, \{(0,4), (1,4), (4,2)\})^K$. □

PROPOSITION 9.8. $\Gamma = (\Omega, R_5)$ *is a simple graph of valency 6 and girth 5.*

Proof. Observe that R_5 is a symmetric relation and that Γ has no triangles or quadrangles. Use the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 4 & 2 \end{smallmatrix}\right) \in K$ to obtain a cycle of length 5 in Γ. □

COROLLARY. Γ *is isomorphic to the unique cage* $\mathcal{R}$ *on 40 vertices.*

THEOREM 9.9. $G = \mathrm{Aut}(\mathcal{R})$ *is a group of order 480 that is isomorphic to* $\mathbb{Z}_4.S_5$.

Proof. According to Proposition 9.1, $G \cong \mathbb{Z}_4.Y$ for $Y \leq S_5$. We know from Proposition 9.2 that $Q \leq G$, where $Q \cong \mathbb{Z}_4.\mathrm{AGL}(1,5)$. It follows from Propositions 9.6 and 9.7 that $K \leq G$. Note that $K \cong \mathbb{Z}_4.Y$, where Y is a subgroup of index 2 in $\mathrm{PGL}(2,5) \cong S_5$. Thus $Y \cong A_5$ and so $G/\mathbb{Z}_4$ is a subgroup of S_5 that is an amalgam of A_5 and $\mathrm{AGL}(1,5)$. □

COROLLARY. *The group G is not isomorphic to* $\mathbb{Z}_4 \times S_5$.

9.5. Locally Icosahedral Graph on 40 Vertices

The group $G = \mathrm{Aut}(\mathcal{R})$ has a normal subgroup $N \cong \mathrm{SL}(2,5)$ with $G/N \cong E_{2^2}$. For each of the three involutions in E_{2^2} we get a subgroup of index 2 in G. One of these groups is K (considered in Section 9.4) and the other two, say L and M, have respective ranks 9 and 11. Now $\mathcal{X}_K = (\Omega, 2\text{-orb}(K, \Omega))$ admits 15 merging schemes, including two non-Schurian schemes of rank 9 and 8. The first of these latter schemes is obtained by merging relation R_4 with R_7, the second by merging relation R_1 with R_2 and R_6 with R_8. The automorphism group in the latter case is K. During our attempts to explain these observations, we became aware of [BlBrBuC].

PROPOSITION 9.10. (a) *There exists a unique locally icosahedral graph* Δ *on 40 vertices,* $\mathrm{Aut}(\Delta) = K = \mathrm{HL}(2,5)$.

(b) *The intersection diagram of* Δ *with respect to the orbits of the stabilizer of a point in* $\mathrm{Aut}(\Delta)$ *is presented in Figure 9.2. The coherent closure of* Δ *is the non-Schurian association scheme with seven classes described previously.*

Proof. It is proved in [BlBrBuC] that there are precisely three locally icosahedral graphs: the 600-cell on 120 vertices and its respective quotients on 60 and 40 vertices. Using GAP and the construction of the unique locally icosahedral graph Δ on 40 vertices provided in [BlBrBuC], we obtained that its automorphism group is isomorphic (as a permutation group) to (K, Ω). Thus, graph Δ may be described as a merging of 2-orbits of (K, Ω), namely R_4 with R_9. Using the provided description of the 2-orbits, the reader may easily verify that we indeed obtain a locally icosahedral graph.

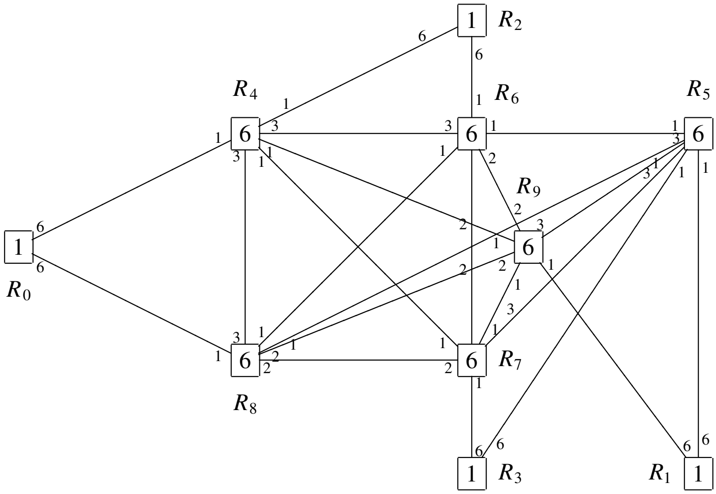

Figure 9.2 Intersection diagram of graph Δ with respect to $\mathrm{Aut}(\Delta)$

Analysis of the mergings of $(\Omega, 2\text{-orb}(G, \Omega))$ shows that $\langle\langle\Delta\rangle\rangle$ is a non-Schurian rank-8 scheme and $\mathrm{Aut}(\Delta) = K$. The intersection diagram depicted in Figure 9.2 was constructed with the aid of GAP. It is easy to see that two cells of size 1 at distance 3 from the reference vertex, as well as two cells of size 6 at distance 2 from the reference vertex (with external valencies $1^3, 2^2, 3$), may be compressed to a diagram with eight cells exactly as depicted in [BlBrBuC, p. 22]. This, together with Lemma 9.3, provides another justification for the existence of a rank-8 non-Schurian merging. □

REMARKS. 1. Using COCO-II, we found that the color automorphism group of the scheme $(\Omega, 2\text{-orb}(K, \Omega))$ has order 480 while the algebraic automorphism group of this scheme is isomorphic to E_{2^2}. The latter group (in action on 2-orbits) consists of the permutations e, $\tau_1 = (4, 7)$, $\tau_2 = (1, 2)(6, 8)$, and $\tau_3 = (4, 7)(1, 2)(6, 8)$. Note that τ_3 is induced by $\mathrm{CAut}(X_K) = G$. The centralizer algebra $V(K, \Omega)$ is commutative, so existence of τ_2 follows from the well-known fact that the symmetrization of a commutative association scheme is an association scheme (see e.g. [BanI]). Thus $\langle\langle\Delta\rangle\rangle$ provides a nice illustration of the fact that the symmetrization of a commutative Schurian scheme is not necessarily Schurian. The automorphisms τ_2 and τ_3 generate the entire algebraic automorphism group. Finally, existence of τ_1 and the corresponding non-Schurian rank-9 merging are simple by-products of all our presented observations.

2. The group K is mentioned in [BlBrBuC] as $\mathrm{SL}(2, 5) \circ \mathbb{Z}_4$. We believe that here we shed some new light on its origin and structure while also revealing some interesting links to the Anstee–Robertson graph.

9.6. Some S-rings on 40 Points

The group G of order 480 has two conjugacy classes of regular subgroups of order 40, each of size 6. The groups in each class are isomorphic to the group R (see

Proposition 9.2). GAP identifies this group as $\mathbb{Z}_5 : \mathbb{Z}_8$, or group number 3 in the catalog of groups of order 40. Of the three index-2 subgroups in G, only L and M admit the regular group R as a subgroup. The group R can be defined by generators and relations as follows:

$$\langle x, y \mid x^5 = y^8 = 1,\ xy = yx^3\rangle.$$

We may interpret the considered association schemes as S-rings over R. The following proposition is a presentation of computer results.

PROPOSITION 9.11. (a) *Group L contains a regular subgroup $R \cong \mathbb{Z}_5 : \mathbb{Z}_8$. The transitivity module $\mathcal{T}_1$ of L has the following basic sets*:

$$T_0 = \{e\},\quad T_1 = \{y, xy^2, xy^4, x^4y^6, y^3, x^4\},\quad T_2 = \{y^2, y^6\},\quad T_3 = \{y^4\},$$
$$T_4 = \{y^5, xy^6, x, x^4y^2, y^7, x^4y^4\},\quad T_5 = \{xy, x^2y^7, x^3y^3, x^2y^4, x^4y^5, x^3y^4\},$$
$$T_6 = \{xy^3, x^2y^6, x^3y^2, x^2y, x^4y^3, x^3y\},\quad T_7 = \{xy^5, x^2y^3, x^3y^7, x^2, x^4y, x^3\},$$
$$T_8 = \{xy^7, x^2y^2, x^3y^6, x^2y^5, x^4y^7, x^3y^5\}.$$

(b) *The following S-rings appear as mergings of basic sets of $\mathcal{T}_1$*:

$$\mathfrak{I}_1 = \{T_0, T_1 \cup T_4, T_2, T_3, T_5, T_6 \cup T_8, T_7\},$$
$$\mathfrak{I}_2 = \{T_0, T_2 \cup T_3, T_1 \cup T_4, T_5 \cup T_6 \cup T_8, T_7\}.$$

(c) *Group M contains a regular subgroup $R \cong \mathbb{Z}_5 : \mathbb{Z}_8$. The transitivity module $\mathcal{T}_2$ of M has the following basic sets*:

$$S_0 = \{e\},\quad S_1 = \{y, y^3, x^4y^4, xy^2, x^4y^6, x\},\quad S_2 = \{y^2, y^6\},\quad S_3 = \{y^4\},$$
$$S_4 = \{y^5, y^7, x^4, xy^6, x^4y^2, xy^4\},\quad S_5 = \{xy, xy^5, x^2y^7, x^2y^3, x^3y^2, x^3y^6\},$$
$$S_6 = \{xy^3, x^3, x^2y\},\quad S_7 = \{xy^7, x^3y^4, x^2y^5\},\quad S_8 = \{x^2, x^3y^5, x^4y^7\},$$
$$S_9 = \{x^2y^2, x^3y^3, x^3y^7, x^4y^5, x^2y^6, x^4y\},\quad S_{10} = \{x^2y^4, x^3y, x^4y^3\}.$$

(d) *The following S-rings appear as mergings of basic sets of $\mathcal{T}_2$*:

$$\mathfrak{I}'_1 = \{S_0, S_1 \cup S_4, S_2, S_3, S_5 \cup S_9, S_6 \cup S_8, S_7 \cup S_{10}\},$$
$$\mathfrak{I}'_2 = \{S_0, S_2 \cup S_3, S_1 \cup S_4, S_5 \cup S_9 \cup S_7 \cup S_{10}, S_6 \cup S_8\}.$$

(e) *Choosing either T_7 or $S_6 \cup S_8$ as a connection set, one obtains graph $\mathcal{R}$ as a Cayley graph over R.*

Proof. Observe that $\mathfrak{I}_1$ is the transitivity module arising from G, while $\mathfrak{I}_2$ is a non-Schurian merging isomorphic to $\mathfrak{m}_{a2}$. A similar explanation holds for $\mathfrak{I}'_1$ and $\mathfrak{I}'_2$. $\square$

10. Other Association Schemes on 40 Points

We once more consider the action of group $G = \mathrm{Aut}(\square_5)$ on $\mathfrak{n}$.

PROPOSITION 10.1. *Restrictions of $\mathfrak{n}$ on the second and third fibers define association schemes with four classes and valencies 1, 3, 4, 8, and 24; however, these schemes are not algebraically isomorphic. The association scheme on the third* *fiber is unique up to isomorphism and has a merging rank-3 scheme corresponding to the generalized quadrangle $W(3)$.*

Proof. We use COCO and GAP. The uniqueness of a certain rank-5 scheme on 40 points was proved in [BanBaB]. In [Z] it was proved that this scheme is isomorphic to the one that appears on the third fiber of $\mathfrak{n}$. □

REMARK. The description of the scheme in [Z] was given in terms of $\overline{5 \circ K_2}$ instead of $\square_5$. We intend to consider this scheme once more in a forthcoming joint paper of K. Abdukhalikov, E. Bannai, M.K., and M.Z-A.

Finally, we describe the origin of one more Higmanian association scheme.

The Schurian association scheme $\mathfrak{m}_{a3}$ with valencies $1, 3, 6, 6, 24$ appears twice as a merging of classes in the total graph coherent configuration $\mathcal{T}(5)$ (as described in [KZ]). These two mergings provide isomorphic schemes with automorphism group of order 7680. GAP shows that this group is a split extension $E_{64} \rtimes S_5$. The stabilizer of a point is a subgroup of order 192, identified by GAP as $D_4 \times S_4$. We hope to consider scheme $\mathfrak{m}_{a3}$ in more detail in a forthcoming paper.

Unfortunately, we were not able to find an example of a scheme of type a1.2 as in Proposition 4.2.

Appendix

40 quadrangles of $\square_5$

0	0, 1, 2, 3	1	0, 1, 4, 5	2	0, 1, 8, 9	3	0, 1, 14, 15
4	0, 2, 4, 6	5	0, 2, 8, 10	6	0, 2, 13, 15	7	0, 4, 8, 12
8	0, 4, 11, 15	9	0, 7, 8, 15	10	1, 3, 5, 7	11	1, 3, 9, 11
12	1, 3, 12, 14	13	1, 5, 9, 13	14	1, 5, 10, 14	15	1, 6, 9, 14
16	2, 3, 6, 7	17	2, 3, 10, 11	18	2, 3, 12, 13	19	2, 5, 10, 13
20	2, 6, 9, 13	21	2, 6, 10, 14	22	3, 4, 11, 12	23	3, 7, 8, 12
24	3, 7, 11, 15	25	4, 5, 6, 7	26	4, 5, 10, 11	27	4, 5, 12, 13
28	4, 6, 9, 11	29	4, 6, 12, 14	30	5, 7, 8, 10	31	5, 7, 13, 15
32	6, 7, 8, 9	33	6, 7, 14, 15	34	8, 9, 10, 11	35	8, 9, 12, 13
36	8, 10, 12, 14	37	9, 11, 13, 15	38	10, 11, 14, 15	39	12, 13, 14, 15

40 skew systems of quadrangles of $\square_5$ as lines of $\mathfrak{S}_1$

0	0, 26, 33, 35	1	0, 27, 32, 38	2	1, 18, 33, 34	3	0, 28, 31, 36
4	1, 17, 32, 39	5	0, 29, 30, 37	6	2, 18, 25, 38	7	4, 12, 31, 34
8	1, 20, 24, 36	9	2, 16, 26, 39	10	4, 11, 30, 39	11	1, 21, 23, 37
12	3, 17, 25, 35	13	5, 12, 25, 37	14	4, 14, 24, 35	15	6, 10, 29, 34
16	2, 19, 24, 29	17	3, 16, 27, 34	18	5, 10, 28, 39	19	4, 13, 23, 38
20	2, 21, 22, 31	21	6, 11, 25, 36	22	7, 14, 16, 37	23	5, 15, 24, 27
24	8, 10, 21, 35	25	3, 19, 23, 28	26	7, 10, 20, 38	27	5, 13, 22, 33
28	3, 20, 22, 30	29	8, 13, 16, 36	30	7, 15, 17, 31	31	6, 15, 23, 26
32	9, 11, 21, 27	33	7, 11, 19, 33	34	6, 14, 22, 32	35	9, 13, 17, 29
36	8, 15, 18, 30	37	9, 12, 20, 26	38	8, 12, 19, 32	39	9, 14, 18, 28

40 edges of $\square_5$ as lines of $\mathfrak{S}_2$

0	0, 1	0, 1, 2, 3	1	0, 2	0, 4, 5, 6	2	0, 4	1, 4, 7, 8	3	0, 8	2, 5, 7, 9
4	0, 15	3, 6, 8, 9	5	1, 3	0, 10, 11, 12	6	1, 5	1, 10, 13, 14	7	1, 9	2, 11, 13, 15
8	1, 14	3, 12, 14, 15	9	2, 3	0, 16, 17, 18	10	2, 6	4, 16, 20, 21	11	2, 10	5, 17, 19, 21
12	2, 13	6, 18, 19, 20	13	3, 7	10, 16, 23, 24	14	3, 11	11, 17, 22, 24	15	3, 12	12, 18, 22, 23
16	4, 5	1, 25, 26, 27	17	4, 6	4, 25, 28, 29	18	4, 11	8, 22, 26, 28	19	4, 12	7, 22, 27, 29
20	5, 7	10, 25, 30, 31	21	5, 10	14, 19, 26, 30	22	5, 13	13, 19, 27, 31	23	6, 7	16, 25, 32, 33
24	6, 9	15, 20, 28, 32	25	6, 14	15, 21, 29, 33	26	7, 8	9, 23, 30, 32	27	7, 15	9, 24, 31, 33
28	8, 9	2, 32, 34, 35	29	8, 10	5, 30, 34, 36	30	8, 12	7, 23, 35, 36	31	9, 11	11, 28, 34, 37
32	9, 13	13, 20, 35, 37	33	10, 11	17, 26, 34, 38	34	10, 14	14, 21, 36, 38	35	11, 15	8, 24, 37, 38
36	12, 13	18, 27, 35, 39	37	12, 14	12, 29, 36, 39	38	13, 15	6, 31, 37, 39	39	14, 15	3, 33, 38, 39

References

[A] R. P. Anstee, *An analogue of group divisible designs for Moore graphs,* J. Combin. Theory Ser. B 30 (1981), 11–20.

[BanBaB] E. Bannai, E. Bannai, and H. Bannai, *Uniqueness of certain association schemes,* European J. Combin. 29 (2008), 1379–1395.

[BanI] E. Bannai and T. Ito, *Algebraic combinatorics. I. Association schemes,* Benjamin-Cummings, Menlo Park, CA, 1984.

[BlBrBuC] A. Blokhuis, A. E. Brouwer, D. Buset, and A. M. Cohen, *The locally icosahedral graphs,* Finite geometries (Winnipeg, 1984), Lecture Notes in Pure and Appl. Math., 103, pp. 19–22, Dekker, New York, 1985.

[BoMu] J. A. Bondy and U. S. R. Murty, *Graph theory with applications,* Elsevier, New York, 1976.

[BrCN] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-regular graphs,* Ergeb. Math. Grenzgeb. (3), 18, Springer-Verlag, Berlin, 1989.

[Ch] Y. Chang, *Imprimitive symmetric association schemes of rank 4,* Ph.D. thesis, Univ. of Michigan, 1994.

[ChHu] Y. Chang and T. Huang, *Imprimitive association schemes of low ranks and Higmanian graphs,* Conference on combinatorics and physics (Los Alamos, 1998), Ann. Comb. 4 (2000), 317–326.

[Cox1] H. S. M. Coxeter, *Self-dual configurations and regular graphs,* Bull. Amer. Math. Soc. 56 (1950), 413–455.

[Cox2] ———, *The Pappus configuration and the self-inscribed octagon. I, II, III,* Indag. Math. 39 (1977), 256–300.

[DeD] A. Deza and M. Deza, *The ridge graph of the metric polytope and some relatives,* Polytopes: Abstract, convex and computational (Scarborough, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 440, pp. 359–372, Kluwer, Dordrecht, 1994.

[DHu] M. Deza and T. Huang, *A generalization of strongly regular graphs,* Southeast Asian Bull. Math. 26 (2002), 193–201.

[E+] M. Erickson, S. Fernando, W. H. Haemers, D. Hardy, and J. Hemmeter, *Deza graphs: A generalization of strongly regular graphs,* J. Combin. Des. 7 (1999), 395–405.

[Eva] C. W. Evans, *Net structure and cages,* Discrete Math. 27 (1979), 193–204.

[EvPT] S. Evdokimov, I. Ponomarenko, and G. Tinhofer, *Forestal algebras and algebraic forests (on a new class of weakly compact graphs),* Discrete Math. 225 (2000), 149–172.

[FK] I. A. Faradžev and M. H. Klin, *Computer package for computations with coherent configurations,* Proc. ISSAC-91, pp. 219–223, ACM Press, Bonn, 1991.

[FKM] I. A. Faradžev, M. H. Klin, and M. E. Muzichuk, *Cellular rings and groups of automorphisms of graphs,* Investigations in algebraic theory of combinatorial objects (I. A. Faradžev et al., eds.), pp. 1–152, Kluwer, Dordrecht, 1994.

[Gap] ⟨http://www.gap-system.org⟩.

[GIvK] Ja. Ju. Gol'fand, A. V. Ivanov, and M. H. Klin, *Amorphic cellular rings,* Investigations in algebraic theory of combinatorial objects (I. A. Faradžev et al., eds.), pp. 167–186, Kluwer, Dordrecht, 1994.

[Ha] F. Harary, *Graph theory,* Addison-Wesley, Reading, MA, 1969.

[HeH] M. D. Hestenes and D. G. Higman, *Rank* 3 *groups and strongly regular graphs,* Computers in algebra and number theory (Proc. SIAM-AMS Sympos. Appl. Math., New York, 1970), SIAM-AMS Proc., vol. 4, pp. 141–159, Amer. Math. Soc., Providence, RI, 1971.

[H1] D. G. Higman, *Coherent configurations, I,* Rend. Sem. Mat. Univ. Padova 44 (1970), 1–25.

[H2] ———, *Coherent configurations. I. Ordinary representation theory,* Geom. Dedicata 4 (1975), 1–32.

[H3] ———, *Coherent algebras,* Linear Algebra Appl. 93 (1987), 209–239.

[H4] ———, *Rank 5 association schemes and triality,* Linear Algebra Appl. 226–228 (1995), 197–222.

[HoS] A. J. Hoffman and R. R. Singleton, *On Moore graphs with diameters* 2 *and* 3, IBM J. Res. Develop. 4 (1960), 497–504.

[K+] M. Klin, M. Muzychuk, C. Pech, A. Woldar, and P.-H. Zieschang, *Association schemes on 28 points as mergings of a half-homogeneous coherent configuration,* European J. Combin. 28 (2007), 1994–2025.

[KZ] M. Klin and M. Ziv-Av, *A family of Higmanian association schemes on 40 points: A computer algebra approach,* Algebraic combinatorics. Proceedings of an international conference in Honor of Eiichi Bannai's 60th birthday (Sendai, 2006), pp. 190–203, Sendai International Center, Sendai, Japan.

[Mc] B. D. McKay, *nauty user's guide,* ver. 1.5, Technical Report TR-CS-90-02, Computer Science Department, Australian National Univ., 1990.

[M] M. E. Muzychuk, *On half-homogeneous coherent configurations,* Unpublished manuscript.

[R] N. Robertson, *Graphs minimal under girth, valency and connectivity constraints,* Ph.D. thesis, Univ. of Waterloo, 1969.

[Sch] M. Schönert et al., *GAP—Groups, algorithms, and programming,* 5th ed., Lehrstuhl D für Mathematik, Rheinisch-Westfälische Technische Hochschule, Aachen, Germany, 1995.

[Soi] L. H. Soicher, *GRAPE: A system for computing with graphs and groups,* Groups and computation (New Brunswick, 1991), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 11, pp. 287–291, Amer. Math. Soc., Providence, RI, 1993.

[Sp] E. Spence, *The strongly regular* $(40, 12, 2, 4)$ *graphs,* Electron. J. Combin. 7 (2000), R22.

[Wei] B. Weisfeiler (ed.), *On construction and identification of graphs,* Lecture Notes in Math., 558, Springer-Verlag, Berlin, 1976.

[Wie] H. Wielandt, *Finite permutation groups,* Academic Press, New York, 1964.

[Zie] P.-H. Zieschang, *An algebraic approach to association schemes,* Lecture Notes in Math., 1628, Springer-Verlag, Berlin, 1996.

[Z] M. Ziv-Av, *Two association schemes on 40 and 64 points: A supplement to the paper by Bannai–Bannai–Bannai,* Poster presentation (jointly with M. Klin), Linz, 2006, ⟨http://www.ricam.oeaw.ac.at/specsem/srs/groeb/download/ZivAv_poster.pdf⟩.

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