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Signalizer Lattices in Finite Groups

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Dedicated to the memory of Donald G. Higman

Let G be a finite group and let H be a subgroup of G. We investigate constraints imposed upon the structure of G by restrictions on the lattice $\mathcal{O}_G(H)$ of overgroups of H in G. Call such a lattice a *finite group interval lattice*. In particular we would like to show that the following question has a positive answer.

Question I. Does there exist a nonempty finite lattice that is not isomorphic to a finite group interval lattice?

See [PPu] for the motivation behind Question I and for one consequence of proving that it has a positive answer. See [Sh] for some conjectures that would imply the Question has a positive answer.

Let Λ be finite lattice and $\mathcal{G}(\Lambda)$ the set of pairs (H, G) such that G is a finite group, $H \le G$, and $\mathcal{O}_G(H)$ is isomorphic to Λ or its dual Λ^*. Write $\mathcal{G}^*(\Lambda)$ for the set of pairs (H, G) such that $|G|$ is minimal subject to $(H, G) \in \mathcal{G}(\Lambda)$.

In [A2] we defined the notion of a "signalizer lattice" determined by a suitable tower $I_H \le N_H \le H$ of finite groups. We also defined a class of lattices we called "*CD*-lattices" and proved that, if Λ is a *CD*-lattice and $(H, G) \in \mathcal{G}^*(\Lambda)$, then either G is *almost simple* (i.e., G has a unique minimal normal subgroup D and D is a nonabelian simple group) or Λ (or Λ^*) is isomorphic to a signalizer lattice in H. Thus, to show Question I has a positive answer, it suffices to show there is a CD-lattice Λ such that:

($I\Lambda A$) there exists no almost simple finite group G with a subgroup H such that $\mathcal{O}_G(H)$ is isomorphic to Λ or its dual; and

($S\Lambda$) there exists no signalizer lattice isomorphic to Λ or its dual.

In this paper we initiate the study of signalizer lattices, with the hope of establishing ($I\Lambda A$) and ($S\Lambda$) for lattices Λ in a suitable family of *CD*-lattices and thereby proving that Question I has a positive answer. See [BL] for another possible approach.

Let L be a nonabelian finite simple group. Define $\mathcal{T}(L)$ to be the set of triples $\tau = (H, N_H, I_H)$ such that:

(T1) H is a finite group and $N_H \le H$;
(T2) $I_H \trianglelefteq N_H$ and $F^*(N_H/I_H) \cong L$.

The tuple $\tau \in \mathcal{T}(L)$ is said to be *faithful* if $\ker_{N_H}(H) = 1$.

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Assume $\tau \in \mathcal{T}(L)$ and write N_0 for the preimage in N_H of $\mathrm{Inn}(L)$ under the map of N_H into $\mathrm{Aut}(L)$ supplied by (T2). Define

$$\mathcal{W} = \mathcal{W}(\tau) = \{W \in \mathcal{I}_H(N_H) : W \cap N_H = I_H\}$$

and

$$\mathcal{P} = \mathcal{P}(\tau) = \{(V, K) : V \in \mathcal{W},\ K \in \mathcal{O}_{N_H(V)}(VN_H), \text{ and } N_0V/V = F^*(K/V)\}.$$

Here $\mathcal{I}_H(N_H)$ is the set of N_H-invariant subgroups of H.

Partially order $\mathcal{P}$ by $(V_1, K_1) \le (V_2, K_2)$ if $V_2 \le V_1$ and $K_2 \le K_1$. Let $\Lambda(\tau)$ be the poset obtained by adjoining an element 0 to $\mathcal{P}$ such that $0 < p$ for all $p \in \mathcal{P}$. The construction in [A2, 7.1] shows that, given a simple group L and $\tau = (H, N_H, I_H) \in \mathcal{T}(L)$, there exists an overgroup G of H such that the poset $\mathcal{O}_G(H)$ is isomorphic to $\Lambda(\tau)$. In particular, $\Lambda(\tau)$ is a lattice isomorphic as a lattice to $\mathcal{O}_G(H)$. We call lattices of the form $\Lambda(\tau)$ *signalizer lattices.*

Next we remark that Λ has a greatest element ∞ and least element 0. Set $\Lambda' = \Lambda - \{0, \infty\}$. Regard Λ as an undirected graph whose adjacency relation is the comparability relation on Λ. Define Λ to be *connected* if the subgraph Λ' is connected as a graph.

The notions of *D*-lattice, *C*-lattice, and *CD*-lattice are defined in Section 1. For example, a *D*-lattice is a disconnected lattice satisfying a certain nondegeneracy condition. We prove that if H admits a *CD*-signalizer lattice then the structure of H is highly restricted, as indicated in our first theorem.

THEOREM 1. *Assume L is a nonabelian finite simple group and Λ is a CD-lattice. Assume $\tau = (H, N_H, I_H) \in \mathcal{T}(L)$, Λ is isomorphic to $\Lambda(\tau)$ or its dual, and $|H|$ is minimal subject to this constraint. Then $F^*(H)$ is the direct product of nonabelian simple subgroups permuted transitively by H.*

Given $\tau = (H, N_H, I_H) \in \mathcal{T}(L)$, define

$$\mathcal{W}_1 = \mathcal{W}_1(\tau) = \{W \in \mathcal{W} : W \le F^*(H)I_H\}$$

and order $\mathcal{W}_1$ by inclusion. Let $\Xi(\tau)$ be the poset obtained by adjoining a greatest member ∞ to $\mathcal{W}_1$. By 2.11 (to follow), $\Xi(\tau)$ is a lattice isomorphic to the dual of a sublattice of $\Lambda(\tau)$. Call $\Xi(\tau)$ a *lower signalizer lattice*. Set $\mathcal{K}(\tau) = \langle \mathcal{W}_1, N_H\rangle$.

Given a positive integer n, an *n-set* is a set of order n. Let $\Delta(n)$ be the lattice of all subsets of an n-set, partially ordered by inclusion. Given integers t and $m_1, \dots, m_t$ with $t > 1$ and $m_i > 2$ for each i, a *$D\Delta(m_1, \dots, m_t)$-lattice* is a finite lattice Λ such that Λ' has t connected components $\mathcal{C}_1, \dots, \mathcal{C}_t$ such that $\mathcal{C}_i \cong \Delta(m_i)'$. As shown in Section 1, $D\Delta(m_1, \dots, m_t)$-lattices are *CD*-lattices. Shareshian's conjectures B and C in [Sh] suggest that the class of $D\Delta(m_1, \dots, m_t)$-lattices supplies a good collection of candidates for lattices Λ satisfying $(I\Lambda A)$ and $(S\Lambda)$. The following theorem reinforces that suggestion.

THEOREM 2. *Assume t and m_i, $1 \le i \le t$, are integers with $t > 1$ and $m_i > 2$, and assume Λ is a $D\Delta(m_1, \dots, m_t)$-lattice that is a finite group interval lattice. Then there exists an almost simple finite group G such that either*

(1) $\Lambda \cong \mathcal{O}_G(H)$ *for some subgroup* H *of* G *or*
(2) *there exists a nonabelian finite simple group* L *and a* $\gamma = (G, N_G, I_G) \in \mathcal{T}(L)$ *such that* $G = F^*(G)N_G$, $G = \mathcal{K}(\gamma)$, *and* $\Lambda \cong \Xi(\gamma)$.

In particular: by Theorem 2, to show that a $D\Delta(m_1, \ldots, m_t)$-lattice Λ supplies a positive answer to Question I, it suffices to verify $(I\Lambda A)$ and $(S\Lambda A)$.

$(S\Lambda A)$ There exists no nonabelian simple group L and no $\gamma = (G, N_G, I_G) \in \mathcal{T}(L)$ such that G is almost simple, $G = F^*(G)N_G = \mathcal{K}(\gamma)$, and $\Lambda \cong \Xi(\gamma)$.

John Shareshian and the author are currently in the midst of a program to verify $(I\Lambda A)$ and $(S\Lambda A)$ for most $D\Delta(m_1, \ldots, m_t)$-lattices Λ.

For notation and terminology involving finite groups, see [A1]. Theorem 1 is proved in Section 4, and Theorem 2 is proved in Section 6.

1. Lattices

In this section, Λ is a finite lattice.

For $x, y \in \Lambda$, we write $x \vee y$ for the least upper bound of x and y in Λ and write $x \wedge y$ for the greatest lower bound of x and y in Λ. Set $\Lambda^\# = \Lambda - \{0\}$. The *atoms* of Λ are the minimal members of Λ', and the *co-atoms* are the atoms of the dual of Λ. Define the *depth* of $x \in \Lambda$ in Λ to be the length d of the longest chain $x = x_0 < \cdots < x_n = \infty$ in Λ.

We say Λ is a *D-lattice* if there exists a partition $\Lambda' = \Lambda'_1 \cup \Lambda'_2$ of Λ' such that, for $i = 1$ and 2:

(D1) Λ'_i is a union of connected components of Λ', and
(D2) there exists a nontrivial chain $k_i < m_i$ in Λ'_i.

Define Λ to be a *C^*-lattice* if,

(C^*) for all $x \in \Lambda'$, there exist maximal elements $m_1, \ldots, m_n$ of Λ' such that $x = m_1 \wedge \cdots \wedge m_n$.

A C_*-lattice is a lattice dual to a C^*-lattice, and a *C-lattice* is a lattice that is both a C^*-lattice and a C_*-lattice. In the literature, C_*-lattices are often called *atomic lattices.*

Finally if X and Y are classes of lattices, then Λ is a *XY-lattice* if Λ is both an X-lattice and a Y-lattice.

1.1. *Assume* Λ *is a* C_**-lattice such that* Λ' *has no greatest element. Assume* $\varphi \colon \Lambda^\# \to \Lambda^\#$ *is a map of posets such that, for each* $p \in \Lambda^\#$, $\varphi(p) \leq p$. *Then* φ *is the identity.*

Proof. Let $p \in \Lambda^\#$. If $p \neq \infty$ then, since Λ is a C_*-lattice, there exist atoms $x_1, \ldots, x_n$ with $p = x_1 \vee \cdots \vee x_n$. If $p = \infty$ then, since Λ' has no greatest element, such atoms also exist.

Now $\varphi(x_i) \leq x_i$ and so, since x_i is an atom, φ fixes x_i. Then, since φ is a map of posets, $x_i = \varphi(x_i) \leq \varphi(p) \leq p$ and so $p = x_1 \vee \cdots \vee x_n \leq \varphi(p) \leq p$. That is, φ fixes p. □

1.2. *Assume Λ is a C^*D-lattice and $\mathcal{C}$ is a connected component of Λ'. Then there exist a connected component $\mathcal{B}$ of Λ', distinct from $\mathcal{C}$, and distinct co-atoms x_1 and x_2 of $\mathcal{B}$ such that $x_1 \wedge x_2 \neq 0$.*

Proof. Because Λ is a D-lattice, there exists a connected component $\mathcal{B}$, distinct from $\mathcal{C}$, containing an edge $x < x_1$ with x of depth 2 in Λ. Because Λ is a C^*-lattice, there exist co-atoms $x_2, \dots, x_n$ in $\mathcal{B}$ with $x = x_1 \wedge \cdots \wedge x_n$. Then $x \leq x_1 \wedge x_2$ and so, since x is of depth 2, $x = x_1 \wedge x_2$. □

2. Basic Properties of Signalizer Lattices

In this section we assume the following hypothesis.

HYPOTHESIS 2.1. L is a nonabelian finite simple group, and $\tau = (H, N_H, I_H) \in \mathcal{T}(L)$.

In addition, we adopt some notational conventions as follows.

NOTATION 2.2. Write N_0 for the preimage in N_H of $\mathrm{Inn}(L)$ under the map of N_H into $\mathrm{Aut}(L)$ supplied by (T2). Set $\mathcal{W} = \mathcal{W}(\tau)$ and $\mathcal{P} = \mathcal{P}(\tau)$. Write $\mathcal{W}_*$ for the set of minimal members of $\mathcal{W} - \{I_H\}$ under inclusion. Write ∞ for (I_H, N_H) and set $\mathcal{P}' = \mathcal{P} - \{\infty\}$. Write $\mathcal{P}^*$ for the set of maximal members of $\mathcal{P}'$. Thus, in the language of Section 1, $\mathcal{P}^*$ is the set of co-atoms of the poset $\mathcal{P}$.

For $p = (V, K) \in \mathcal{P}$, set $\mathcal{P}(\geq p) = \{q \in \mathcal{P} : q \geq p\}$, $\mathbf{M}(p) = N_H(V) \cap N_H(VN_0)$, $\mathbf{Q}(p) = C_{\mathbf{M}(p)}(N_0V/V)$, and $l(p) = (\mathbf{Q}(p), \mathbf{M}(p))$. Set

$$\mathcal{H}(\tau) = \langle K : (V, K) \in \mathcal{P} \rangle \quad \text{and} \quad \mathcal{H}_*(\tau) = \langle K : (V, K) \in \mathcal{P}^* \rangle.$$

For $N_H \leq M \leq H$, define $\tau_M = (M, N_H, I_H)$. Given $D \trianglelefteq H$, define

$$\Delta^\#(\tau) = \{(V, K) \in \mathcal{P} : K = VN_H\} \quad \text{and}$$

$$\Gamma^\#(\tau, D) = \{(V, K) \in \mathcal{P} : V \leq DI_H \text{ and } K \leq DN_H\}.$$

Let $\Delta(\tau)$ and $\Gamma(\tau, D)$ be the subposets of $\Lambda(\tau)$ obtained by adjoining 0 to $\Delta^\#(\tau)$ and $\Gamma^\#(\tau, D)$, respectively, and set $\Delta(\tau, D) = \Delta(\tau) \cap \Gamma(\tau, D)$. Set

$$\mathcal{K}(\tau, D) = \langle K : (V, K) \in \Delta(\tau, D)^\# \rangle,$$

$$\mathcal{K}_*(\tau, D) = \langle K : (V, K) \in \mathcal{P}^* \cap \Delta(\tau, D) \rangle.$$

The proof of the following observation is straightforward.

2.3. *Let $N_H \leq M \leq H$. Then*:

(1) $\tau_M = (M, N_H, I_H) \in \mathcal{T}(L)$;
(2) *$\mathcal{P}(\tau_M)$ is a subposet of $\mathcal{P}$*;
(3) *if $p \in \mathcal{P}(\tau_M)$, then $\mathcal{P}(\geq p) \subseteq \mathcal{P}(\tau_M)$*;
(4) *the inclusion map is an isomorphism of $\Lambda(\tau_{\mathcal{H}(\tau)})$ with $\Lambda(\tau)$*;
(5) *if $D \trianglelefteq H$ then the inclusion map is an isomorphism of $\Delta(\tau_{\mathcal{K}(\tau,D)}, D \cap \mathcal{K}(\tau, D))$ with $\Delta(\tau, D)$.*

2.4. *Let $\hat{N}_H \le \hat{H} \le H$ such that $\hat{N}_H \le N_H$; $N_0 = \hat{N}_0 I_H$, where $\hat{N}_0 = N_0 \cap \hat{N}_H$; and $\hat{I}_H = I_H \cap \hat{H} \le \hat{N}_H$. Set $\hat{\tau} = (\hat{H}, \hat{N}_H, \hat{I}_H)$. Then*:

(1) $\hat{\tau} \in \mathcal{T}(L)$.

(2) *For $p = (V, K) \in \mathcal{P}$ define $\varphi(p) = (\hat{V}, \hat{K})$, where $\hat{V} = V \cap \hat{H}$ and $\hat{K} = K \cap \hat{H}$. Then $\varphi \colon \mathcal{P} \to \hat{\mathcal{P}} = \mathcal{P}(\hat{\tau})$ is a map of posets.*

Proof. Since $I_H \trianglelefteq N_H$ and $\hat{I}_H = I_H \cap \hat{H} \le \hat{N}_H \le N_H$, it follows that also $\hat{I}_H \trianglelefteq \hat{N}_H$. Furthermore,

$$\frac{\hat{N}_H}{\hat{I}_H} = \frac{\hat{N}_H}{\hat{N}_H \cap I_H} \cong \frac{\hat{N}_H I_H}{I_H}.$$

Similarly, since $\hat{N}_0 = N_0 \cap \hat{N}_H$ and $N_0 = \hat{N}_0 I_H$, we have $\hat{N}_0 \trianglelefteq \hat{N}_H$ and $\hat{N}_0/\hat{I}_H \cong N_0/I_H \cong L$. Then, since $N_0 = \hat{N}_0 I_H$ and $N_0/I_H = F^*(N_H/I_H)$, we have $\hat{N}_0/\hat{I}_H = F^*(\hat{N}_H/\hat{I}_H)$. Thus (1) holds.

Let $p = (V, K) \in \mathcal{P}$. Then $V \in \mathcal{I}_H(N_H)$ and $\hat{N}_H \le N_H$, so $\hat{V} = V \cap \hat{H} \in \mathcal{I}_{\hat{H}}(\hat{N}_H)$. Also, $V \cap N_H = I_H$ and $\hat{N}_H \le N_H$, so $V \cap \hat{N}_H = I_H \cap \hat{N}_H = \hat{I}_H$. Therefore, $\hat{V} \in \hat{\mathcal{W}} = \mathcal{W}_{\hat{H}}(\hat{N}_H, \hat{I}_H)$. Then $K \in \mathcal{O}_{N_H(V)}(VN_H)$, so

$$\hat{K} = K \cap \hat{H} \in \mathcal{O}_{N_{\hat{H}}(\hat{V})}(VN_H \cap \hat{H}) \subseteq \mathcal{O}_{N_{\hat{H}}(\hat{V})}(\hat{V}\hat{N}_H).$$

Furthermore, $N_0 V/V = F^*(K/V)$. Since $N_0 = \hat{N}_0 I_H$, we also have $N_0 V = \hat{N}_0 V$ and so $\hat{N}_0 V/V = F^*(\hat{K}V/V)$. Now $\hat{K}V/V \cong \hat{K}/(\hat{K} \cap V) = \hat{K}/\hat{V}$ with $N_0 V/V$ mapping to $\hat{N}_0\hat{V}/\hat{V}$, so $\hat{N}_0\hat{V}/\hat{V} = F^*(\hat{K}/\hat{V})$. Thus $\varphi(p) \in \hat{\mathcal{P}}$.

If $p \le q = (U, J)$ then $U \le V$ and $J \le K$, so $\hat{U} = U \cap \hat{H} \le V \cap \hat{H} = \hat{V}$ and similarly $\hat{J} \le \hat{K}$. Therefore, φ is a map of posets, completing the proof of (2). □

2.5. *Assume $V \in \mathcal{W}$ and $I_H \le U \in \mathcal{I}_V(N_H)$. Then*:

(1) $U \in \mathcal{W}$;

(2) *if $(U, K) \in \mathcal{P}$ then $V \cap K = U$.*

Proof. The proof of (1) is trivial. Assume the hypothesis of (2) and let $K^* = K/U$ and $X = V \cap K$. Because $(U, K) \in \mathcal{P}$, $N_0^* = F^*(K^*) \cong L$ is simple and so, since X is N_0-invariant, either $X^* = 1$ or $N_0^* \le X^*$. In the former case (2) holds; in the latter case $N_0 \le V$, contradicting $V \in \mathcal{W}$. □

2.6. *For $W \in \mathcal{W}$, $(W, WN_H) \in \mathcal{P}$.*

Proof. If $W \in \mathcal{W}$ then $W \trianglelefteq WN_H$ and $WN_H/W \cong N_H/(W \cap N_H) = N_H/I_H$. Thus, since $L \cong N_0/I_H = F^*(N_H/I_H)$, the lemma holds. □

2.7. *Let $D \trianglelefteq H$ and $\Phi \in \{\Delta(\tau), \Gamma(\tau, D), \Delta(\tau, D)\}$. Then, for each $p \in \Phi$, $\mathcal{P}(\ge p) \subseteq \Phi$.*

Proof. Let $p = (V, K) \in \Phi$ and $q = (U, J) \ge p$. Then $U \le V$ and $J \le K$. If $\Phi = \Delta(\tau)$ then $K = VN_H$, so $J = J \cap VN_H = (J \cap V)N_H = UN_H$ by 2.5(2) and hence $q \in \Phi$. If $\Phi = \Gamma(\tau, D)$ then $V \leq DI_H$ and $K \leq DN_H$. Thus $U = U \cap DI_H = (U \cap D)I_H \leq DI_H$ and similarly $J \leq DN_H$, so $q \in \Phi$. The lemma follows. □

2.8. *Let $p = (V, K) \in \mathcal{P}$ and set $\mathcal{Q} = \mathcal{P}(\geq p)$. Then*:

(1) $\mathcal{Q} = \{(V \cap J, J) : J \in \mathcal{O}_K(N_H)\}$;
(2) *the map $\psi : J \mapsto (V \cap J, J)$ is an isomorphism of the dual of $\mathcal{O}_K(N_H)$ with $\mathcal{Q}$*;
(3) *if $q_i = (V_i, K_i) \in \mathcal{Q}$ for $i = 1, 2$, then $q_1 \vee q_2 = (V_1 \cap V_2, K_1 \cap K_2)$ and $q_1 \wedge q_2 = (V_{1,2}, K_{1,2})$, where $K_{1,2} = \langle K_1, K_2\rangle$ and $V_{1,2} = K_{1,2} \cap V$*;
(4) *if $K = VN_H$, then $K_i = V_iN_H$, $\langle K_1, K_2\rangle = \langle V_1, V_2\rangle N_H$, and $V \cap \langle K_1, K_2\rangle = \langle V_1, V_2\rangle$.*

Proof. Let $J \in \mathcal{O}_K(N_H)$. Then $V \cap J \cap N_H \leq V \cap N_H = I_H$ and, since $J \in \mathcal{O}_K(N_H)$, $I_H = N_H \cap V \leq J \cap V$ so $V \cap J \cap N_H = I_H$. Thus $V \cap J \in \mathcal{W}$. Also, $N_0 \leq J$ and $N_0V/V = F^*(K/V)$, so $C_{K/V}(N_0V/V) = 1$. Thus $C_{JV/V}(N_0V/V) = 1$ and so $N_0V/V = F^*(JV/V)$. Furthermore, the map $\pi : jV \mapsto j(J \cap V)$, $j \in J$, is an isomorphism of JV/V with $J/(J \cap V)$ such that $(N_0V/V)\pi = N_0(J \cap V)/(J \cap V)$, so $F^*(J/(J \cap V)) = N_0(J \cap V)/(J \cap V)$. That is, $(V \cap J, J) \in \mathcal{Q}$.

Conversely, let $(U, X) \in \mathcal{Q}$. Then $U \leq V$ and $X \leq K$; moreover, $X \in \mathcal{O}_H(N_H)$ and so $X \in \mathcal{O}_K(N_H)$. By 2.5(2) we have $U = X \cap V$, completing the proof of (1) and showing the map ψ of (2) is surjective.

Clearly ψ is injective, so $\psi : \mathcal{O}_K(N_H) \to \mathcal{Q}$ is a bijection. Furthermore, for $q_i = (V_i, K_i) \in \mathcal{Q}$ we have $q_1 \leq q_2$ if and only if $K_2 \leq K_1$ and $V_2 \leq V_1$ iff $K_2 \leq K_1$ because $V_i = V \cap K_i$. This completes the proof of (2).

Next, in the lattice $\mathcal{O}_K(N_H)$ we have $K_1 \wedge K_2 = K_1 \cap K_2$ and $K_1 \vee K_2 = \langle K_1, K_2\rangle$. Then, applying the isomorphism ψ and recalling that ψ is applied to the *dual* of $\mathcal{O}_K(N_H)$ yields

$$\begin{aligned} q_1 \vee q_2 &= K_1\psi \vee K_2\psi = (K_1 \wedge K_2)\psi = (K_1 \cap K_2)\psi \\ &= (V \cap K_1 \cap K_2, K_1 \cap K_2) = (V_1 \cap V_2, K_1 \cap K_2), \end{aligned}$$

and

$$q_1 \wedge q_2 = K_1\psi \wedge K_2\psi = (K_1 \vee K_2)\psi = \langle K_1, K_2\rangle\psi = (V_{1,2}, K_{1,2});$$

this establishes (3).

Finally, suppose $K = VN_H$. By 2.7, $K_i = V_iN_H$. Also $\langle K_1, K_2\rangle = \langle N_H, V_1, V_2\rangle = UN_H$, where $U = \langle V_1, V_2\rangle \in \mathcal{I}_K(N_H)$. Then

$$V_{1,2} = K_{1,2} \cap V = UN_H \cap V = U(N_H \cap V) = UI_H = U,$$

establishing (4). □

2.9. *Let $q_i = (V_i, K_i) \in \mathcal{P}$ for $i = 1, 2$. Then*:

(1) $q_1 \vee q_2 = (U, K)$, *where $U = V_1 \cap V_2$ and $K = N_{K_1 \cap K_2}(N_0(V_1 \cap V_2))$*;
(2) *if $N_{V_1}(V_2)V_2 \in \mathcal{W}$, then $K = K_1 \cap K_2$*;
(3) *if $K_i = V_iN_H$, then $K = (V_1 \cap V_2)N_H$.*

Proof. Let $q = q_1 \vee q_2 = (U, K)$. Then $q_i \leq q$, so $U \leq V_i$ and $K \leq K_i$ and hence $U \leq W = V_1 \cap V_2$ and $K \leq J = N_{K_1 \cap K_2}(W)$. By 2.5(1), $W \in \mathcal{W}$ and so, by 2.6, $p = (W, WN_H) \in \mathcal{P}$. Then, since $q_i \leq p$ for $i = 1, 2$, it follows that $q \leq p$ and so $W \leq U$; hence $U = W$.

Let $J^* = J/U$ and $Y = C_J(N_0^*)$. Then $[Y, N_0] \leq U \leq V_i$ for $i = 1, 2$, so $Y \leq V_i$ since $F^*(K_i/V_i) = N_0V_i/V_i$. Thus $Y = U$ and $r = (U, J) \in \mathcal{P}$ with $q_i \leq r \leq q$, so $r = q$. This completes the proof of (1).

Assume the hypothesis of (2), and let $X = K_1 \cap K_2$. Then

$$[N_0, X] \leq N_0V_1 \cap N_0V_2 = N_0Z,$$

where $Z = V_1 \cap N_0V_2$. But $Z \leq A = N_{V_1}(V_2)V_2$ and $A \in \mathcal{W}$ by hypothesis; hence, by 2.5(2), $Z \leq A \cap N_HV_2 = V_2$. Thus $Z = V_1 \cap V_2 = U$ and so $[N_0, X] \leq N_0Z = N_0U$. That is, $X \leq J$, so $X = J = K$ and (2) holds.

Finally, (3) follows from (1) and 2.7. □

2.10. *Assume that* $p_i = (V_i, K_i) \in \mathcal{P}$ *for* $1 \leq i \leq n$ *and that* $p_1 \wedge \cdots \wedge p_n = p = (V, K) \neq 0$. *Then*:

(1) $K = \langle K_1, \ldots, K_n \rangle$ *and* $V_i = K_i \cap V$ *for* $1 \leq i \leq n$;

(2) $p_1 \vee \cdots \vee p_n = (U, J)$, *where*

$$J = \bigcap_{i=1}^{n} K_i \quad \text{and} \quad U = J \cap V = \bigcap_{i=1}^{n} V_i;$$

(3) *if* $K_i = V_iN_H$ *for each* i, $1 \leq i \leq n$, *then* $V = \langle V_1, \ldots, V_n \rangle$, $K = VN_H$, *and* $J = UN_H$.

Proof. Since $p_i \in \mathcal{P}(\geq p)$ for each i, (1) and (2) follow from 2.8(3) by induction on n.

Assume the hypothesis of (3). Then $K = \langle N_H, V_1, \ldots, V_n \rangle$ and N_H acts on $W = \langle V_1, \ldots, V_n \rangle$, so $K = WN_H$. Also $V_i = V \cap K_i$, so $W \leq V$. Thus $K = VN_H$. Now $V = V \cap K = \langle V_1, \ldots, V_n \rangle$ by 2.8(4). Finally, $J = UN_H$ by 2.9(3) and induction on n. □

2.11. (1) *Let* $D \trianglelefteq H$. *Then* $\Delta(\tau)$ *and* $\Delta(\tau, D)$ *are sublattices of* $\Lambda(\tau)$.

(2) *The poset* $\Xi(\tau)$ *is isomorphic as a poset to the dual of* $\Delta(\tau, F^*(G))$.

(3) $\Xi(\tau)$ *is a lattice. Indeed, if* $\infty \neq W_i \in \Xi(\tau)$ *then* $W_1 \wedge W_2 = W_1 \cap W_2$, *and if* $W_1 \vee W_2 \neq \infty$ *then* $W_1 \vee W_2 = \langle W_1, W_2 \rangle$.

Proof. Let $\Phi = \Delta(\tau)$ or $\Delta(\tau, D)$. We first prove (1). By 2.7, Φ is closed under $\vee$, so it suffices to take $q_i = (V_i, K_i) \in \Phi$ with $q = q_1 \wedge q_2 \neq 0$ and to show $q \in \Phi$. By 2.10(3), $q = (V, VN_H)$, where $V = \langle V_1, V_2 \rangle$. In particular, $q \in \Delta(\tau)$, so we may take $\Phi = \Delta(\tau, D)$. Then $V_i \leq DI_H$, so $V \leq DI_H$ and $VN_H \leq DI_HN_H = DN_H$; hence $q \in \Delta(\tau, D)$. Thus (1) is established.

Take $D = F^*(H)$. Then the map $V \mapsto (V, VN_H)$ is an isomorphism of posets from the dual of $\Xi(\tau) - \{\infty\}$ to $\Delta(\tau, D)^{\#}$. Thus (2) holds, and from (1) and (2) it follows that $\Xi(\tau)$ is a lattice. Then 2.9 and 2.10(3) complete the proof of (3). □

2.12. *Let* $p = (V, K) \in \mathcal{P}$. *Then*:

(1) $p \in \mathcal{P}^*$ *iff* N_H *is maximal in* K;
(2) *if* $p \in \mathcal{P}^*$ *then either* $K = VN_H$ *and* $V \in \mathcal{W}_*$ *or* $V = I_H$.

Proof. Part (1) follows from 2.8(2).

Suppose $p \in \mathcal{P}^*$. By 2.6, $q = (V, VN_H) \in \mathcal{P}$ and then $q \geq p$, so $q \in \{\infty, p\}$ since $p \in \mathcal{P}^*$. If $q = \infty$ then $V = I_H$, whereas if $q = p$ then $K = VN_H$. In the latter case, since N_H is maximal in K we have $V \in \mathcal{W}_*$. □

2.13. *Assume* $H = \mathcal{H}(\tau)$ *and* X *is a normal subgroup of* H *contained in* N_H. *Set* $H^* = H/X$. *Then one of the following two statements holds.*

(1) $X \leq I_H$, $\tau^* = (H^*, N_H^*, I_H^*) \in \mathcal{T}(L)$, *and the map* $(V, K) \mapsto (V^*, K^*)$ *is an isomorphism of* $\mathcal{P}$ *with* $\mathcal{P}(\tau^*)$.
(2) $X \not\leq I_H$; *then, setting* $I_1 = X \cap I_H$, $N_1 = X \cap N_0$, *and* $Q_1 = C_H(N_1/I_1)$, *we have that* I_1 *and* Q_1 *are normal in* H, (Q_1, H) *is the least element of* $\mathcal{P}$, *and* $\mathcal{P}$ *is isomorphic to the dual of* $\mathcal{O}_H(N_H)$.

Proof. If $X \leq I_H$ then it is an easy exercise to check that (1) holds. Thus we may assume that $X \not\leq I_H$ and adopt the notation in (2). Since $N_0/I_H = F^*(N_H/I_H) \cong L$, we have $N_0 = N_1 I_H$.

Let $(V, K) \in \mathcal{P}$. Since $X \leq N_H$, it follows that $V \cap X = V \cap N_H \cap X = I_H \cap X = I_1$. Then, since K acts on V and X, we have $K \leq N_H(I_1)$. Hence, since $H = \mathcal{H}(\tau)$, $I_1 \trianglelefteq H$. Set $H^+ = H/I_1$.

Similarly, K acts on $N_0 V$ and X and hence on $N_0 V \cap X = N_1 V \cap X = N_1(V \cap X) = N_1 I_1 = N_1$. Thus $[V, N_1] \leq N_1 \cap V = I_1$, so $V \leq Q_1$. Now $Q_1 \cap N_H$ centralizes N_1^+ and so, since $N_0 = N_1 I_H$, $Q_1 \cap N_H$ also centralizes N_0/I_H. Hence $Q_1 \cap N_H = I_H$; that is, $Q_1 \in \mathcal{W}$. Because $N_1^+ \cong L$ is normal in H^+ and $Q_1^+ = C_{H^+}(N_1^+)$, it follows that $N_0 Q_1/Q_1 = N_1 Q_1/Q_1 = F^*(H/Q_1)$, so $q = (Q_1, H) \in \mathcal{P}$. Then, since each member of $\mathcal{W}$ is contained in Q_1, we have q as the least element of $\mathcal{P}$. Finally, $\mathcal{P}$ is isomorphic to the dual of $\mathcal{O}_H(N_H)$ by 2.8(2). □

2.14. *Assume that* Λ *is a finite lattice and that* $\tau \in \mathcal{T}(L)$ *with* $|H|$ *minimal, subject to* $\Lambda(\tau)$ *being isomorphic to* Λ *or* Λ^*. *Then*:

(1) $H = \mathcal{H}(\tau)$.
(2) *Assume* Λ' *has neither a least element nor a greatest element. Then* τ *is faithful.*

Proof. Part (1) follows from 2.3(4) and the minimality of $|H|$.

Suppose $X = \ker_{N_H}(H) \neq 1$. Then either conclusion (1) or (2) of 2.13 holds and, by minimality of $|H|$, it must be conclusion (2). In particular, $\mathcal{P}$ has a least element; hence the hypothesis of (2) does not hold, since $\Lambda(\tau)$ is isomorphic to Λ or its dual. Thus (2) is established. □

2.15. (1) *If* $\Lambda(\tau)$ *is a* C^**-lattice, then* $\mathcal{H}(\tau) = \mathcal{H}_*(\tau)$.
(2) *If* $H = \mathcal{H}_*(\tau)$, *then* $H = \langle \mathcal{W}_*, \mathbf{M}(\infty) \rangle$.

Proof. Assume $\Lambda(\tau)$ is a C^*-lattice. Then, for each $p = (V, K) \in \mathcal{P}$, we have $p = p_1 \wedge \cdots \wedge p_n$ for some $p_i = (V_i, K_i) \in \mathcal{P}^*$. Hence, by 2.10(1), $K = \langle K_1, \ldots, K_n \rangle$ and so (1) holds.

Next assume $H = \mathcal{H}_*(\tau)$ and $p \in \mathcal{P}^*$. Then 2.12(2) says that either $K = VN_H$ and $V \in \mathcal{W}_*$ or $V = I_H$. Moreover, if $V = I_H$ then $N_0/I_H = F^*(K/I_H)$, so $K \leq \mathbf{M}(\infty)$. Thus (2) holds. □

2.16. *For each $p \in \mathcal{P}$, $l(p) \in \mathcal{P}$.*

Proof. Let $p = (V, K)$ and $l(p) = (Q, M)$. By definition of M, N_0V and V are normal in M. Let $M^* = M/V$. Again by definition, $Q^* = C_{M^*}(N_0^*)$; then, since $N_0^* = F^*(N_H^*)$ is nonabelian, $Q \cap N_H \leq V \cap N_H = I_H$ (i.e., $Q \in \mathcal{W}$). Also, N_0^* is a nonabelian simple normal subgroup of M^* and $Q^* = C_{M^*}(N_0^*)$, so $N_0Q/Q = F^*(M/Q)$, completing the proof. □

2.17. *Let $X \leq H$ and $p = (V, K) \in \mathcal{P}$ such that $K \leq N_H(X)$, $W = XI_H \in \mathcal{W}$, and $WV \in \mathcal{W}$. Set $r = (W, WN_H)$ and $q = (WV, WK)$. Then $q, r \in \mathcal{P}$ and $q = p \wedge r$.*

Proof. Since VW and W are in $\mathcal{W}$, it follows from 2.6 that $s = (VW, VWN_H)$ and r are in $\mathcal{P}$. Because $K \leq N_H(X)$, K acts on $VX = VW$. Then, since K acts on VN_0, K also acts on VWN_0. Hence $K \leq M = \mathbf{M}(s)$ and $V \leq VW \leq Q = \mathbf{Q}(s)$. By 2.16, $l = (Q, M) \in \mathcal{P}$, and we just showed that $p \geq l$.

Next, $WK \in \mathcal{O}_M(N_H)$, so $q' = (WK \cap Q, WK) \in \mathcal{P}$ by an application of 2.8(1) to l in the role of "p". Also, $WK \cap Q = W(K \cap Q)$ and, by 2.5(2), $K \cap Q = V$, so that $WK \cap Q = VW$ and hence $q' = q$. By 2.8(3),

$$p \wedge r = (Q \cap WK, WK) = (VW, WK) = q,$$

completing the proof. □

3. Normal Subgroups of *H*

In this section we continue to assume Hypothesis 2.1 and adopt Notation 2.2.

DEFINITION 3.1. Define $\mathcal{W}_-$ to be the set of $W \in \mathcal{W}$ such that $W \leq N_H(I_H)$, $W \not\leq N_H(N_0)$, and $W/I_H \cong L$.

3.2. *Assume $W \in \mathcal{W}_-$. Then*:

(1) *$WN_0/I_H \cong L \times L$ has two components, W/I_H and W'/I_H (write $\theta(W)$ for W')*;
(2) *N_0/I_H is a full diagonal subgroup of $W/I_H \times \theta(W)/I_H$*;
(3) *$W \in \mathcal{W}_*$*;
(4) *$\theta(W) \in \mathcal{W}_-$.*

Proof. Let $X = WN_0$ and $Y = WN_H$. Since $W \leq N_H(I_H)$, also $Y \leq N_H(I_H)$. Set $Y^* = Y/I_H$.

Since $W \in \mathcal{W}_-$, we have $L \cong W^* \trianglelefteq Y^*$. Since $X = WN_0$ and $W \cap N_0 = I_H$, N_0^* is a complement to W^* in X^*. Then, since $W^* \cong L$, (1) follows from the Schreier conjecture. Let $T = \theta(W)$. Since $W \not\leq N_H(N_0)$, $N_0^* \neq T^*$ and so (2) follows. By (2), N_H^* is maximal in Y^*, so (3) follows from 2.12. By (2), $N_H \cap T = I_H$ and $T \not\leq N_H(N_0)$, so (4) holds because $T^* \cong L$. □

NOTATION 3.3. Given $W \in \mathcal{W}_-$, define $\theta(W)$ as in 3.2(1).

3.4. *Assume $W \in \mathcal{W}_-$ and let $T = \theta(W)$, $p = (W, WN_H)$, and $q = (T, TN_H)$. Then*:

(1) $\theta(\theta(W)) = W$;
(2) $p \vee q = \infty$ *and* $p \wedge q = 0$.

Proof. Let $Y = WN_H$ and $Y^* = Y/I_H$. From 3.2, $X = WN_0 = TN_0$ and W^*, T^* are the components of X^*, so (1) holds.

Let $p \vee q = (U, J)$. By parts (1) and (3) of 2.9, $U = T \cap W = I_H$ and $J = UN_H = N_H$, so $p \vee q = \infty$. Suppose $p \wedge q = (V, K) \neq 0$. Then, by 2.10(3), $V = WT$, contradicting $N_0 \leq WT$. Thus (2) holds. □

3.5. *Let $W \in \mathcal{I}_H(N_H)$. Then either*

(1) $WI_H \in \mathcal{W}$ *or*
(2) $N_0 = (W \cap N_0)I_H$.

Proof. Let $X = W \cap N_H$. Since $W \in \mathcal{I}_H(N_H)$, we have $X \trianglelefteq N_H$. Therefore, because $F^*(N_H/I_H) = N_0/I_H$ is a nonabelian simple group, either $X \leq I_H$ or $N_0 \leq XI_H$. In the former case

$$WI_H \cap N_H = (W \cap N_H)I_H = XI_H = I_H,$$

so that (1) holds. In the latter case,

$$N_0 = N_0 \cap XI_H = (N_0 \cap X)I_H = (N_0 \cap W)I_H,$$

so (2) holds. □

3.6. *Assume $V \in \mathcal{W}$ and $W \in \mathcal{I}_H(N_H)$ with $\langle V, W\rangle = VW$ and $VW \notin \mathcal{W}$. Then*:

(1) $N_0 \leq VN_W(V) \cap WN_V(W) = N_W(V)N_V(W)$;
(2) $N_W(V)$ *is N_H-invariant and* $L \cong N_W(V)/(V \cap W) = [N_0, N_W(V)/(V \cap W)]$;
(3) *if $Y \in \mathcal{I}_{C_H(W)}(N_H)$ with $\langle V, Y\rangle = VY$, then $N_0 \neq (VY \cap N_0)I_H$ and so $VY \in \mathcal{W}$.*

Proof. Let $N_1 = VW \cap N_0$. Since $V, W \in \mathcal{I}_H(N_H)$, also $\langle V, W\rangle \in \mathcal{I}_H(N_H)$. Then, since $VW = \langle V, W\rangle$ is not in $\mathcal{W}$, it follows from 3.5 that $N_0 = N_1 I_H$. Next, $N_1 \leq N_H(V) \cap VW = VN_W(V)$, so $N_0 = N_1 I_H \leq VN_W(V)$. Similarly, $N_0 \leq WN_V(W)$, establishing (1). Then

$$L \cong \frac{N_0 V}{V} \leq \frac{VN_W(V)}{V} \cong \frac{N_W(V)}{V \cap W},$$

so (2) holds.

Assume the hypothesis of (3). Let $N_2 = VY \cap N_0$. If $N_0 \neq N_2 I_H$, then $VY \in \mathcal{W}$ by 3.5, so that (3) holds. Thus we may assume $N_0 = N_2 I_H$. By the previous paragraph applied to Y in the role of "W", we have $N_2 \leq VN_Y(V)$. Let $M = N_H(V)$, $K = N_0 V$, and $M^* = M/V$. Then $K^* \cong L$ and $K = N_i I_H V = N_i V$ for $i = 1, 2$, so $K^* = N_1^* = N_2^*$. But $N_1 \leq VN_W(V)$ and $N_2 \leq VN_Y(V)$, so $K^* \leq N_W(V)^* \cap N_Y(V)^*$. Then, since $[W, Y] = 1$, K^* is abelian—contradicting $K^* \cong L$. This completes the proof of (3). □

Notation 3.7. Set $\mathcal{W}' = \mathcal{W} - \{I_H\}$. For $p \in \mathcal{P}$, write $\mathcal{C}(p)$ for the connected component of $\mathcal{P}'$ containing p. For $W \in \mathcal{W}'$, set

$$\mathcal{C}(W) = \{V \in \mathcal{W}' : \mathcal{C}(W, WN_H) = \mathcal{C}(V, VN_H)\}.$$

3.8. *Assume $p_i = (V_i, K_i) \in \mathcal{P}'$ for $i = 1, 2$ such that $\mathcal{C}(p_1) \neq \mathcal{C}(p_2)$. Then $\langle V_1, V_2 \rangle \notin \mathcal{W}$ and $V_1 \cap V_2 = I_H$.*

Proof. Let $q_i = (V_i, V_i N_H)$. Then $p_i \leq q_i$ and so, replacing p_i by q_i, we may assume that $K_i = V_i N_H$. Since $\mathcal{C}(p_1) \neq \mathcal{C}(p_2)$, we have $p_1 \vee p_2 = \infty$ and $p_1 \wedge p_2 = 0$. Hence $V_1 \cap V_2 = I_H$ by 2.9(1). Furthermore, $U = \langle V_1, V_2 \rangle \notin \mathcal{W}$ or else $(U, UN_H) \leq p_i$ for $i = 1, 2$. □

3.9. *Assume $V_i \in \mathcal{W}$ for $i = 1, 2$ such that $\langle V_1, V_2 \rangle = V_1 V_2$. Then the following statements hold.*

(1) *$W_i = N_{V_i}(V_{3-i})$ and $V_{1,2} = V_1 \cap V_2$ are in $\mathcal{W}$.*
(2) *$\langle W_1, W_2 \rangle = W_1 W_2$.*
(3) *Assume $V_1 V_2 \notin \mathcal{W}$ and let $X = W_1 W_2$ and $X^* = X/V_{1,2}$. Then*:
  (a) *$N_0 \leq W_1 W_2$, so $W_1 W_2 \notin \mathcal{W}$;*
  (b) *$X^* = W_1^* \times W_2^*$. Let U_i be the preimage in W_i of the projection of N_0^* on W_i^*. Then $N_0^* \cong L$ is a full diagonal subgroup of $U_1^* \times U_2^* \cong L \times L$.*
  (c) *$U_i \in \mathcal{W}$ for $i = 1, 2$, but $\langle U_1, U_2 \rangle = U_1 U_2 \notin \mathcal{W}$.*
  (d) *If $\mathcal{C}(V_1) \neq \mathcal{C}(V_2)$, then $U_i \in \mathcal{W}_-$ and $U_{3-i} = \theta(U_i)$.*

Proof. First, W_i and $V_{1,2}$ are in $\mathcal{W}$ by 2.5(1), so (1) holds. Since W_1 acts on W_2, we have $\langle W_1, W_2 \rangle = W_1 W_2$ and so (2) holds.

Assume $V_1 V_2 \notin \mathcal{W}$. Then, by 3.6(1), $N_0 \leq X$ and so (3a) holds. Since $W_i \in \mathcal{W}$, $N_0 \cap W_i = I_H \leq V_{1,2}$, so (3b) follows. Since N_H acts on W_i and N_0, also N_H acts on U_i, so that $U_i \in \mathcal{W}$ by 2.5(1). Since $X^* = W_1^* \times W_2^*$, $\langle U_1, U_2 \rangle = U_1 U_2$, and since $N_0 \leq U_1 U_2$, $U_1 U_2 \notin \mathcal{W}$, establishing (3c).

Assume the hypothesis of (d). Then it follows from 3.8 that $U_1 \cap U_2 = I_H$. In particular, since U_i acts on U_{3-i}, we have $I_H \trianglelefteq U_i$. Next, $L \cong U_i^*$ and, since N_0^* is a full diagonal subgroup of $U_1^* \times U_2^*$, we have $U_i \not\leq N_H(N_0)$. Therefore $U_i \in \mathcal{W}_-$ and, since U_j^* $(j = 1, 2)$ are the components of $X^* = U_j N_0 / I_H$, it follows that $U_{3-i} = \theta(U_i)$, completing the proof of (3). □

3.10. *Assume that $X \trianglelefteq H$ and $X \not\leq I_H$, and assume that X satisfies one of the following*:

(a) *X is solvable; or*
(b) *X has no L-section; or*
(c) *for each $V \in \mathcal{W}$, $XV \in \mathcal{W}$.*

Then:

(1) *$Y = XI_H \in \mathcal{W}'$ and $x = (Y, XN_H) \in \mathcal{P}$;*
(2) *for each $p = (V, K) \in \mathcal{P}$, $x \wedge p \in \mathcal{P}'$;*
(3) *$\Lambda(\tau)$ is connected.*

Proof. Observe that (a) implies (b), so it suffices to assume that (b) or (c) holds. Let $V \in \mathcal{W}$. Since $X \trianglelefteq H$, we have $\langle X, V\rangle = XV$. Thus, if (b) holds then it follows from (b) and 3.6(2) that $XV \in \mathcal{W}$. That is, (b) implies (c), so we may assume that (c) holds. In particular, by applying (c) when $V = I_H$, we conclude that $Y \in \mathcal{W}$. Then, since $X \not\leq I_H$, also $Y \in \mathcal{W}'$. Now (1) follows from 2.6.

Let $p = (V, K) \in \mathcal{P}$. Then $K \leq N_H(X)$ and, by (c), Y and YV are in $\mathcal{W}$. Therefore, by 2.17, $q = (XV, XK) \in \mathcal{P}$ and $q = x \wedge p$. Thus (2) holds, and (2) implies (3). □

3.11. *Assume $B \trianglelefteq H$ such that $BI_H \notin \mathcal{W}$ and $X = C_H(B) \not\leq I_H$. Then X satisfies condition* (c) *of 3.10, and hence the conclusions of 3.10 are also satisfied.*

Proof. Because B is normal in H, so is X. Since $BI_H \notin \mathcal{W}$, it follows from 3.5 that $N_0 \leq BI_H$. Thus, for each $V \in \mathcal{W}$, we have $N_0 \leq BV$ and so $BV \notin \mathcal{W}$. Hence $VX \in \mathcal{W}$ by 3.6(3); that is, X satisfies 3.10(c), so the lemma follows from 3.10. □

3.12. *Assume* $\ker_{I_H}(H) = 1$ *and $\Lambda(\tau)$ is disconnected. Let $\mathcal{B} = \{B_1, \dots, B_n\}$ be the set of minimal normal subgroups of H. Then:*

(1) *$F(H) = 1$;*
(2) *each component of H has an L-section;*
(3) *if $n > 1$, then $B_i I_H \in \mathcal{W}$ for all i;*
(4) *$n \leq 2$.*

Proof. Since $\ker_{I_H}(H) = 1$, no member of $\mathcal{B}$ is contained in I_H. In particular, if (1) fails then $F(H) \not\leq I_H$, so $F(H)$ satisfies 3.10(a). Then 3.10(3) contradicts the hypothesis that $\Lambda(\tau)$ is disconnected. This establishes (1).

Similarly, if A is a component of H that contains no L-section, then $B = \langle A^H\rangle \in \mathcal{B}$ satisfies 3.10(b), and we obtain a contradiction as in the previous paragraph. Thus (2) holds.

Assume $n > 1$. Then, for each i, $1 \neq D_i = \langle \mathcal{B} - \{B_i\}\rangle \leq C_H(B_i)$. Hence (3) follows from 3.11.

Finally, assume $n > 2$ and let $B = B_1 B_2$. Then $B_3 \trianglelefteq H$ with $B_3 \leq C_H(B)$, so arguing as in the previous paragraph yields $BI_H \in \mathcal{W}$. Thus $\mathcal{C} = \mathcal{C}(B_1 I_H) = \mathcal{C}(BI_H) = \mathcal{C}(B_2 I_H)$. Now let $V \in \mathcal{W}'$. If $VB_1 \notin \mathcal{W}$ then, by 3.6(3), $VB_2 \in \mathcal{W}$. Hence, for $i = 1$ or 2, $\mathcal{C}(V) = \mathcal{C}(VB_i) = \mathcal{C}$. Let $r = (B_1 I_H, B_1 N_H)$. Then $\mathcal{R} = \mathcal{C}(r) = \mathcal{C}(p)$ for each $p = (V, K) \in \mathcal{P}'$ with $V \neq I_H$. Therefore, since $\Lambda(\tau)$ is disconnected, we conclude that there exists a $q = (I_H, J) \in \mathcal{P}^*$ with $\mathcal{C}(q) \neq \mathcal{R}$. But $B_1 I_H \in \mathcal{W}$ and so, by 2.17, $q \wedge r \neq 0$, contradicting $\mathcal{C}(q) \neq \mathcal{R}$. □

3.13. *Let $c\colon H \to G$ be a surjective homomorphism with kernel A. Set $N_G = N_H c$, $I_G = I_H c$, and $\gamma = (G, N_G, I_G)$. Assume $A \cap N_H \leq I_H$ and set $B = AI_H$ and $r = (B, AN_H)$. Then*:

(1) $\gamma \in \mathcal{T}(L)$.
(2) $r \in \mathcal{P}$.
(3) *Let $\mathcal{R} = \mathcal{P}(\leq r)$ and $\mathcal{Q} = \{(V, K) \in \mathcal{P} : AV \in \mathcal{W}\}$; then $\mathcal{Q} = \{p \in \mathcal{P} : p \wedge r \neq 0\}$ and, for $p = (V, K) \in \mathcal{Q}$, we have $p \wedge r = (AV, AK) \in \mathcal{R}$.*
(4) *For $p = (V, K) \in \mathcal{Q}$, define $\psi(p) = (Vc, Kc)$; then $\psi\colon \mathcal{Q} \to \mathcal{P}(\gamma)$ is a map of posets that restricts to an isomorphism $\psi\colon \mathcal{R} \to \mathcal{P}(\gamma)$ with inverse $\nu\colon (V_1, K_1) \mapsto (V_1 c^{-1}, K_1 c^{-1})$.*

Proof. Since $A \cap N_H \leq I_H$, where $B \cap N_H = (A \cap N_H)I_H = I_H$ and so $B \in \mathcal{W}$. Then (2) follows from 2.6.

By 2.17, $\mathcal{Q} \subseteq \mathcal{Q}_1 = \{p \in \mathcal{P} : p \wedge r \neq 0\}$. Conversely, if $p = (V, K) \in \mathcal{Q}_1$, then $p \wedge r = (AV, AK)$ by 2.10(1), so $AV \in \mathcal{W}$ and hence $\mathcal{Q} = \mathcal{Q}_1$. That is, (3) holds.

Since $\tau \in \mathcal{T}(L)$, also $\alpha = (N_H, N_H, I_H) \in \mathcal{T}(L)$ by 2.3(1). Let $D = A \cap I_H$. Then $\beta = (N_H/D, N_H/D, I_H/D) \in \mathcal{T}(L)$ by 2.13. Since $D = A \cap N_H$, also $\beta = (N_H A/A, N_H A/A, I_H A/A) = (N_G, N_G, I_G)$, so (1) holds.

Let $p = (V, K) \in \mathcal{Q}$. Then $(AV, AK) \in \mathcal{P}$ by (3), so $AV \cap AN_H = A(AV \cap N_H) = AI_H$ and hence $Vc \cap N_G = (AI_H)c = I_H c = I_G$; therefore, $Vc \in \mathcal{W}_G(N_G, I_G)$. Also, $Kc \in \mathcal{O}_{N_G(Vc)}(VcN_G)$ and, since $(AV, AK) \in \mathcal{P}$,

$$\frac{N_0 cVc}{Vc} = \frac{N_0 AV}{AV} = F^*\left(\frac{AK}{AV}\right) = F^*\left(\frac{Kc}{Vc}\right),$$

so $\psi(p) \in \mathcal{S} = \mathcal{P}(\gamma)$. Furthermore, if $p \leq q = (U, J)$ then $U \leq V$ and $J \leq K$, so $Uc \leq Vc$ and $Jc \leq Kc$; hence $\psi(p) \leq \psi(q)$. That is, $\psi\colon \mathcal{Q} \to \mathcal{S}$ is a map of posets.

Let $p_1 = (V_1, K_1) \in \mathcal{S}$ and set $V = V_1 c^{-1}$ and $K = K_1 c^{-1}$. Then $V_1 \cap N_G = I_G$, so $V \cap AN_H = AI_H = B$; hence $V \in \mathcal{W}$. Also, $B \leq V \trianglelefteq K$ and, since $N_G \leq K_1$, we have $AN_H = N_G c^{-1} \leq K$. In addition, $F^*(K_1/V_1) = N_0 cV_1/V_1$, so $F^*(K/V) = N_0 V/V$ and hence $\nu(p_1) = (V, K) \in \mathcal{R}$. Moreover, $\psi(V, K) = (Vc, Kc) = p_1$, so $\psi \circ \nu = 1$. Also, for $p \in \mathcal{R}$, $\nu(\psi(p)) = \nu(Vc, Kc) = p$, so $\nu = \psi^{-1}$ and hence $\psi\colon \mathcal{R} \to \mathcal{S}$ is a bijection. Clearly ν is a map of posets, completing the proof of (4). □

4. Disconnected Lattices

In this section we often assume the following hypothesis.

Hypothesis 4.1. Hypothesis 2.1 holds, $\ker_{I_H}(H) = 1$, and $\Lambda = \Lambda(\tau)$ is disconnected.

We adopt Notations 2.2, 3.3, and 3.7 in addition to the following.

NOTATION 4.2. *$\mathcal{B} = \{B_1, \dots, B_n\}$ is the set of minimal normal subgroups of H.*

4.3. *Assume Hypothesis 4.1. Then*:

(1) $F(H) = 1$;
(2) $n \leq 2$;
(3) *if $n = 2$, then $B_i I_H \in \mathcal{W}$ for $i = 1, 2$.*

Proof. This is immediate from 3.12. □

4.4. *Assume Hypothesis 4.1, and let $V, W \in \mathcal{W}'$ be such that $\langle V, W \rangle = VW$ and $\mathcal{C}(V) \neq \mathcal{C}(W)$. Then*:

(1) $V \cap W = I_H$;
(2) *there exists a $U \in \mathcal{W}_-$ with $U \leq W$ and $\theta(U) \leq V$.*

Proof. Part (1) follows from 3.8, while (2) follows from 3.9(3d). □

HYPOTHESIS 4.5. Hypothesis 4.1 holds, and $\mathcal{B} = \{B_1, B_2\}$ is of order 2. Let $r_i = (B_i, B_i N_H)$ and $\mathcal{C}_i = \mathcal{C}(r_i)$ for $i = 1, 2$.

4.6. *Assume Hypothesis 4.5. Then*:

(1) *for each $p = (V, K) \in \Lambda'$ there exists a unique $i = i(p) \in \{1, 2\}$ such that $(VB_i, KB_i) \in \mathcal{P}$;*
(2) $VB_{3-i} \notin \mathcal{W}$;
(3) $\mathcal{C}(p) = \mathcal{C}_i$;
(4) *$\mathcal{C}_1$ and $\mathcal{C}_2$ are the connected components of Λ'.*

Proof. Let $j \in \{1, 2\}$. If $VB_j \notin \mathcal{W}$ then, by 3.6(3), $VB_{3-j} \in \mathcal{W}$; hence, by 2.17, $(VB_{3-j}, KB_{3-j}) \in \mathcal{P}$. So in this case (1) and (2) hold with $i = 3 - j$, and then (3) follows from (1). We conclude that $\Lambda' \subseteq \mathcal{C}_1 \cup \mathcal{C}_2$; therefore, since Λ is disconnected, the lemma holds. □

4.7. *Assume Hypothesis 4.5. Then*:

(1) $\mathcal{W}_* = \mathcal{W}_-$.
(2) *Let $V \in \mathcal{W}_*$ and set $i = i(V) = i(V, VN_H)$; then $V \leq B_i I_H$ and $\theta(V) \leq B_{3-i} I_H$.*
(3) $N_0 \leq B_1 B_2 I_H$.

Proof. By 3.2(3), $\mathcal{W}_- \subseteq \mathcal{W}_*$. Let $V \in \mathcal{W}_*$ and $i = i(V, VN_H)$. Then $VB_{3-i} \notin \mathcal{W}$ by 4.6(2); so by 4.4(2) and the minimality of V, $V \in \mathcal{W}_-$ and $\theta(V) \leq B_{3-i}$. By 3.4(1) and the symmetry between V and $\theta(V)$, $V = \theta(\theta(V)) \leq B_i I_H$, completing the proof of (1) and (2). Then, since $N_0 \leq U\theta(U)$ for $U \in \mathcal{W}_-$ by 3.2, (3) follows. □

4.8. *Assume Hypothesis 4.5. Then, for $(V, K) \in \mathcal{P}^*$, $V \neq I_H$.*

Proof. Suppose $(V, K) \in \mathcal{P}^*$ with $V = I_H$. Then $VB_i = I_H B_i \in \mathcal{W}$ for $i = 1, 2$ by 4.3(3), contrary to 4.6(2). □

4.9. *Assume Hypothesis 4.5 and $H = \mathcal{H}_*(\tau)$. Then*:

(1) $I_H = 1$ *so* $N_0 \cong L$.
(2) *Let U_i be the projection of N_0 on B_i. Then $\mathcal{W}_* = \{U_1, U_2\}$ and $\mathcal{P}^* = \{p_1, p_2\}$, where $p_i = (U_i, U_i N_H)$.*

Proof. For each $V \in \mathcal{W}_-$, $I_H \trianglelefteq V$. Since $H = \mathcal{H}_*(\tau)$, we have $H = \langle \mathcal{W}_*, N_H(I_H)\rangle$ by 2.15(2). Thus $I_H \trianglelefteq H$ by 4.7(1), so (1) follows because $\ker_{I_H}(H) = 1$ by Hypothesis 4.1. Let $V \in \mathcal{W}_-$ and $i = i(V, VN_H)$. Then $V\theta(V) = VN_0$, with $V \le B_i$ and $\theta(V) \le B_{3-i}$ by 4.7. It follows that $V = U_i$ and $\theta(V) = U_{3-i}$. Hence $\mathcal{W}_* = \{U_1, U_2\}$ by 4.7(1), and then 4.8 completes the proof. □

THEOREM 4.10. *Assume Hypothesis 2.1 and that $\Lambda = \Lambda(\tau)$ is a disconnected C^*-lattice. Let $H_* = \mathcal{H}_*(\tau)$ and set $K_* = \ker_{N_H}(H_*)$ and $H^* = H_*/K_*$. Then the following statements hold.*

(1) $\tau^* = (H^*, N_H^*, I_H^*) \in \mathcal{T}(L)$.
(2) $\Lambda \cong \Lambda(\tau^*)$ *and* $K_* \le I_H$.
(3) $F(H^*) = 1$.
(4) *Either*
  (a) *there exists a unique minimal normal subgroup of H^* or*
  (b) *there are exactly two minimal normal subgroups B_1^* and B_2^* of H^*. Furthermore, $K_* = I_H$, $B_i^* \cong L$, and N_0^* is a full diagonal subgroup of $B_1^* \times B_2^*$. Moreover, $\Lambda' = \{r_1, r_2\}$, where $r_i = (B_i, B_i N_H)$ and $H_* = B_1 B_2 N_H$.*

Proof. Let $\mu = \tau_{H_*}$. Since Λ is a C^*-lattice, we conclude from 2.3(4) and 2.15(1) that $\Lambda \cong \Lambda(\mu)$. Observe that $H_* = \mathcal{H}_*(\mu)$ because $\mathcal{P}^*(\mu) \subseteq \mathcal{P}^*(\tau)$. Therefore, since Λ is disconnected, $K_* \le I_H$ by 2.13, which also means that (1) holds and $\Lambda(\mu) \cong \Lambda(\tau^*)$. Thus (1) and (2) are established.

By construction, $\ker_{I_H^*}(H^*) = 1$, so τ^* satisfies Hypothesis 4.1. Then (3) follows from 4.3(1). Assume that (4a) does not hold. Then τ^* satisfies Hypothesis 4.5 by 4.3(2), with minimal normal subgroups B_1^* and B_2^*. By construction, $H^* = \mathcal{H}_*(\tau^*)$, so $I_H^* = 1$ by 4.9(1). Hence $K_* = I_H$. Let U_i^* be the projection of N_0^* on B_i^* and let $p_i^* = (U_i^*, U_i^* N_H^*)$. By 4.9(2), $\mathcal{P}^*(\tau^*) = \{p_1^*, p_2^*\}$. Since Λ is a C^*-lattice and since $B_i^* \in \mathcal{W}_{H^*}(N_H^*, I_H^*)$ by 4.3(3), it follows that $B_i^* = U_i^*$, so $\Lambda' = \{r_1, r_2\}$. Thus (4b) holds, completing the proof. □

COROLLARY 4.11. *Assume Hypothesis 2.1 and that $H = \mathcal{H}(\tau)$. Assume in addition that $\Lambda = \Lambda(\tau)$ is a C^*D-lattice and* $\ker_{I_H}(H) = 1$. *Then $F^*(H)$ is the direct product of the set $\mathcal{L}$ of components of H, each component is simple, and H is transitive on $\mathcal{L}$.*

Proof. As $H = \mathcal{H}(\tau)$ and $\Lambda = \Lambda(\tau)$ is a C^*-lattice, 2.15(1) says that $H = \mathcal{H}_*(\tau)$. Thus the hypotheses of Theorem 4.10 are satisfied and $H = H_*$. Since $\ker_{I_H}(H) = 1$, $K_* = \ker_{N_H}(H) = 1$ by 4.10(2), so H is the group H^* of 4.10. Since Λ is a D-lattice, $|\Lambda'| > 2$ and so, by 4.10(4), there is a unique minimal normal subgroup E of H. By 4.10(3), $F(H) = 1$, so $E = F^*(H)$ and the corollary holds. □

We are now in a position to prove Theorem 1. Assume the hypotheses of that theorem. By 2.14(1), $H = H(\tau)$. Because Λ is disconnected, Λ' has neither a least nor a greatest element, so $\ker_{N_H}(H) = 1$ by 2.14(2). Thus the hypotheses of Corollary 4.11 are satisfied, and that result implies Theorem 1.

4.12. *Assume Hypothesis 2.1 and that* $H = \mathcal{H}(\tau)$. *Assume in addition that* $\Lambda = \Lambda(\tau)$ *is a* C^*D*-lattice and* $\ker_{I_H}(H) = 1$. *Then* $F^*(H)I_H \notin \mathcal{W}$.

Proof. Let $E = F^*(H)$ and assume $A = EI_H \in \mathcal{W}$. Set $a = (A, AN_H)$, so that $a \in \Lambda'$ by 2.6. Let $\mathcal{C} = \mathcal{C}(a)$ and let $\mathcal{E} = \{p \in \mathcal{P}^* : \mathcal{C}(p) \neq \mathcal{C}\}$.

Since Λ is disconnected, $\mathcal{E} \neq \emptyset$. Pick $p = (V, K) \in \mathcal{E}$. If $V = I_H$ then $A = EV \in \mathcal{W}$, so $p \in \mathcal{C}$ by 2.17—a contradiction. Thus $V \in \mathcal{W}_*$ and $K = VN_H$ by 2.12(2).

Let

$$\mathcal{V} = \{V \in \mathcal{W}_* : (V, VN_H) \in \mathcal{E}\}.$$

Applying 4.4 to V and A, we conclude that $V \in \mathcal{W}_-$ and $\theta(V) \leq A$.

By 1.2, there exists a connected component $\mathcal{C}_1$ of Λ' distinct from $\mathcal{C}$ such that $\mathcal{C}_1$ contains distinct co-atoms x_1 and x_1 with $x = x_1 \wedge x_2 \neq 0$. Set

$$\mathcal{V}_1 = \{V \in \mathcal{V} : p(v) = (V, VN_H) \in \mathcal{C}_1\}.$$

By paragraph two, the set $\mathcal{C}_1^*$ of maximal members of $\mathcal{C}_1$ is $\{p(V) : V \in \mathcal{V}_1\}$. Thus $p(x_i) = (V_i, V_iN_H)$ for some $V_i \in \mathcal{V}_1$. Then, by 2.10, $p(x) = (V, VN_H)$, where $V = \langle V_1, V_2\rangle$. Set $X = N_G(I_H)$ and $X^* = X/I_H$.

Next, $V \cap A = I_H(V \cap E) \in \mathcal{W}$, so $p = (V \cap A, (V \cap A)N_H) \in \Lambda$. Thus, if $p \neq \infty$ then $a \leq p \geq x$, contradicting $x \in \mathcal{C}_1$. Hence $p = \infty$, so $V \cap A = I_H$.

By 3.8 and 3.9, for $i = 1, 2$ we have $V_i \leq X \geq \theta(V_i)$ and $V_i(\theta(V_i) \cap E) = V_i\theta(V_i) = V_iN_0 \leq N_X(V)$. Thus $F_0 = \langle \theta(V_i) \cap E : i = 1, 2\rangle \leq F = N_{X \cap E}(V)$ and $N_0 \leq VF_0$. Then F and V are normal in $\langle F, V\rangle$ with $F \cap V \leq A \cap V = I_H$, so $\langle F^*, V^*\rangle = F^* \times V^* \geq N_0^*$. Now $N_0 \cap V = I_H$ and $N_0 \cap FI_H \leq N_0 \cap A = I_H$, so $N_0^* \cap F^* = N_0^* \cap V^* = 1$. Therefore, N_0^* is a diagonal subgroup of $F^* \times V^*$. Let N_V^* be the projection of N_0^* on V^*. Since N_0^* is a full diagonal subgroup of $V_i^* \times \theta(V_i)^* = V_i^* \times E_i^*$, where $E_i = E \cap \theta(V_i)$, it follows that $V_i^* = N_V^*$. But then $V_1 = V_2$, so $x_1 = x_2$, a contradiction. This contradiction completes the proof of 4.12. □

5. *CD*-lattices

In this section we assume the following hypothesis.

HYPOTHESIS 5.1. Hypothesis 2.1 holds, $\ker_{I_H}(H) = 1$, $\Lambda(\tau)$ is disconnected, $H = \mathcal{H}_*(\tau)$, and D is a minimal normal subgroup of H such that $DI_H \notin \mathcal{W}$.

We adopt Notations 2.2, 3.3, and 3.7. Observe that Hypothesis 4.1 is satisfied.

5.2. $D = F^*(H)$ *is the direct product of the set* $\mathcal{L}$ *of components of* H, *the components of* H *are simple, and* H *is transitive on* $\mathcal{L}$.

Proof. By 4.3(1), $F(H) = 1$; thus, since D is a minimal normal subgroup of H, it follows that D is the direct product of its set $\mathcal{L}$ of components, which are simple and transitively permutated by H. If $D \neq F^*(G)$ then $1 \neq X = C_H(D) \trianglelefteq H$. Since $\ker_{I_H}(H) = 1$, $X \nleq I_H$. Therefore, since $DI_H \notin \mathcal{W}$, $\Lambda(\tau)$ is connected by 3.11 and 3.10(3), contrary to Hypothesis 4.1. □

NOTATION 5.3. Let $\mathcal{L}$ be the set of components of D. For $E \in \mathcal{L}$ and $X \leq H$, set $X_E = X \cap E$ and $X_D = X \cap D$, and write $\bar{X}_E$ for the projection of X_D on E with respect to the direct product decomposition of 5.2. Write N_D for $N_0 \cap D$ and I_D for $I_H \cap D$, and write $\bar{N}_E$ and $\bar{I}_E$ for the corresponding projections on E. Set

$$\bar{X} = \prod_{E \in \mathcal{L}} \bar{X}_E, \quad \bar{N}_D = \prod_{E \in \mathcal{L}} \bar{N}_E, \quad \bar{I} = \prod_{E \in \mathcal{L}} \bar{I}_E.$$

For $\gamma \subseteq \mathcal{L}$, set $D_\gamma = \langle \gamma \rangle$. Set $M = N_H(I_D) \cap N_H(N_D)$, $Q = C_M(N_D/I_D)$, and $d = (Q, M)$. Let

$$\mathcal{P}_D = \Gamma(\tau, D)' \quad \text{and} \quad \mathcal{P}^*_\infty = \mathcal{P}^* - \Delta(\tau, D).$$

5.4. (1) *$N_0 = N_D I_H$ and $N_D/I_D \cong L$.*
(2) *$d \in \mathcal{P}$.*
(3) *$\mathbf{M}(\infty) \leq M$ and $\mathbf{Q}(\infty) \leq Q$, so $l(\infty) \geq d$.*
(4) *$\mathcal{P}(\tau_M) = \mathcal{P}(\geq d)$, and the map $X \mapsto (X \cap Q, X)$ is an isomorphism of the dual of $\mathcal{O}_M(N_H)$ with $\mathcal{P}(\tau_M)$.*

Proof. Since $DI_H \notin \mathcal{W}$, (1) follows from 3.5.

Let $M^* = M/I_D$. Now $[I_H, N_D] \leq I_H \cap N_D = I_D$, so $I_H \leq Q$ and $N_D Q = N_0 Q$. Next, by construction $[Q, N_D] \leq I_D$, so $[Q \cap N_H, N_0] \leq I_H$ by (1). Therefore, since $F^*(N_H/I_H) = N_0/I_H \cong L$, we have $Q \cap N_H = I_H$ and so $Q \in \mathcal{W}$. By definition we have $Q^* = C_{M^*}(N_D^*)$ and N_D^* is a nonabelian simple normal subgroup of M^*, so $N_D Q/Q = F^*(M/Q)$, establishing (2). The proof of (3) is straightforward.

For part (4), let $p = (V, K) \in \mathcal{P}(\tau_M)$. Then $K \leq M$ and

$$[V, N_D] \leq V \cap N_D = V \cap N_H \cap N_D = I_H \cap N_D = I_D,$$

so $V \leq Q$. Thus $p \geq d$, so $\mathcal{P}(\tau_M) \subseteq \mathcal{P}(\geq d)$. The opposite inclusion is trivial, so now (4) follows from 2.8(2). □

5.5. *Let $p = (V, K) \in \mathcal{P}^*_\infty$. Then*:
(1) *either $V = I_H$ or $K = VN_H$ and $V \in \mathcal{W}_*$;*
(2) *$V \cap DI_H = I_H$;*
(3) *$p \geq d$;*
(4) *$H = DQ\mathbf{M}(\infty) = DM$.*

Proof. Part (1) follows from 2.12. Next $I_H \leq V \cap DI_H \in \mathcal{I}_V(N_H)$, so $V \cap DI_H \in \mathcal{W}$ by 2.5. Hence, because $V = I_H$ or $V \in \mathcal{W}_*$ (and in the latter case $V \nleq DI_H$ as $p \in \mathcal{P}^*_\infty$), it follows that (2) holds. By (2),

$$[N_D, V] \leq V \cap D = I_H \cap D = I_D,$$

so $V \leq Q$. Then $K \leq M$ if $K = VN_H$; if $V = I_H$ then $N_0 \trianglelefteq K$, so also $N_D = N_0 \cap D \trianglelefteq K$ and again $K \leq M$. Hence (3) holds.

By 5.1, $H = \mathcal{H}_*(\tau)$ and so, by 2.15(2), $H = \langle \mathcal{W}_*, \mathbf{M}(\infty)\rangle$. Let $U \in \mathcal{W}_*$. If $U \leq DI_H$ then $U \leq DQ$. On the other hand, if $U \not\leq DI_H$ then $U \leq Q$ by (3). Thus $H \leq DQ\mathbf{M}(\infty)$, so (4) holds because $\mathbf{M}_\infty \leq M$ by 5.4(3). □

5.6. (1) *For each proper subset γ of $\mathcal{L}$, $N_D \neq (N_D \cap D_\gamma)I_D$.*

(2) *Assume that N_H is transitive on $\mathcal{L}$ and let $V \in \mathcal{W}$. Then, for each proper subset γ of $\mathcal{L}$, $N_DV_D \neq (N_DV_D \cap D_\gamma)V_D$.*

Proof. Write N_γ for $N_DV_D \cap D_\gamma$, and let $*\colon N_DV_D \to N_DV_D/V_D$ be the natural surjection. Let

$$S_V = \{\gamma \subseteq \mathcal{L} : N_DV_D = N_\gamma V_D\},$$

and write S_V^* for the set of minimal members of S_V under inclusion. Now $V_\gamma = V \cap D_\gamma$, the N_γ are N_DV_D-invariant, and for $\gamma \in S_V$ we have

$$L \cong N_D^* = \frac{N_DV_D}{V_D} = \frac{N_\gamma V_D}{V_D} = N_\gamma^*.$$

Let $\alpha, \beta \in S_V^*$. Then

$$[N_\alpha, N_\beta] \leq N_\alpha \cap N_\beta = N_DV_D \cap D_\alpha \cap D_\beta = N_DV_D \cap D_{\alpha\cap\beta} = N_{\alpha\cap\beta}.$$

Since also $N_\alpha^* = N_D^* = N_\beta^* \cong L$,

$$N_{\alpha\cap\beta}^* \geq [N_\alpha, N_\beta]^* = [N_\alpha^*, N_\beta^*] = N_D^*$$

and hence $\alpha \cap \beta \in S_V$, so $\alpha = \alpha \cap \beta = \beta$ because $\alpha, \beta \in S_V^*$. However, $Q\mathbf{M}(\infty)$ acts on N_D and I_D by 5.4(3), and it is transitive on $\mathcal{L}$ by 5.5(4). Furthermore, N_H acts on N_D and V_D and, under the hypothesis of (2), N_H is transitive on $\mathcal{L}$. Thus, if either $V = I_H$ or the hypothesis of (2) holds, then if $\alpha \neq \mathcal{L}$ is in S_V^* we can pick $h \in N_H(V_D) \cap N_H(N_D)$ with $\alpha \neq \alpha^h = \beta$ and, since $\alpha \in S_V^*$, also $\beta \in S_V^*$, a contradiction.

We conclude that if either $V = I_H$ or the hypothesis of (2) is satisfied, then $S_V^* = \{\mathcal{L}\}$ and hence $S_V = \{\mathcal{L}\}$. This completes the proof of the lemma. □

5.7. *Assume $|\mathcal{L}| > 1$. Then the following statements hold.*

(1) *For each $E \in \mathcal{L}$, $\bar{N}_E/\bar{I}_E \cong L$.*
(2) *$\bar{I}I_H \in \mathcal{W}$.*
(3) *$N_D\bar{I}/\bar{I}$ is a full diagonal subgroup of $\bar{N}_D/\bar{I} = \prod_{E\in\mathcal{L}} \bar{N}_E\bar{I}/\bar{I}$.*
(4) *Assume that $V \in \mathcal{W}$ and that N_H is transitive on $\mathcal{L}$. Then:*
   (a) *for each $E \in \mathcal{L}$, $\bar{N}_E\bar{V}_E/\bar{V}_E \cong L$;*
   (b) *$\bar{V}I_H \in \mathcal{W}$;*
   (c) *$N_D\bar{V}/\bar{V}$ is a full diagonal subgroup of $\bar{N}_D/\bar{V}$.*

Proof. Let $E \in \mathcal{L}$ and let $\pi_E\colon D \to E$ be the projection map. Let $V \in \mathcal{W}$ and assume that either $V = I_H$ or N_H is transitive on $\mathcal{L}$. If $\bar{N}_E\bar{V}_E = \bar{V}_E$ then $N_DV_D = V_D(N_DV_D \cap \ker(\pi_E))$, so $\mathcal{L} - \{E\}$ is in the set S_V defined in the proof of 5.6, contrary to 5.6. Therefore, $\bar{V}_E$ is a proper normal subgroup of $\bar{N}_E\bar{V}_E$ and so, since $N_DV_D/V_D \cong L$, (1) and (4a) follow by applying π_E.

Let $P = N_DV_D \cap \bar{V}$. Then $P\pi_E \le \bar{V}\pi_E = V_D\pi_E = \bar{V}_E$ and, if $P \nleq V_D$, then (since $N_DV_D/V_D \cong L$) we have $N_DV_D = PV_D$. But now

$$\bar{N}_E\bar{V}_E = (N_DV_D)\pi_E = P\pi_E V_D\pi_E = P\pi_E\bar{V}_E = \bar{V}_E,$$

contrary to (1) and (4a). Therefore $P \le V_D$, so

$$\begin{aligned} N_0 \cap \bar{V}I_H = N_DI_H \cap \bar{V}I_H &= (N_DI_H \cap \bar{V})I_H = (N_DI_H \cap D \cap \bar{V})I_H \\ &= (N_D \cap \bar{V})I_H = (N_D \cap N_DV_D \cap \bar{V})I_H = (N_D \cap P)I_H \\ &\le (N_D \cap V_D)I_H = I_H, \end{aligned}$$

establishing (2) and (4b).

Now, by (2) and (4b), $N_D\bar{V}/\bar{V} \cong N_D/(N_D \cap \bar{V}) = N_D/I_D \cong L$ and

$$\frac{N_D\bar{V}}{\bar{V}} \le \frac{\bar{N}_D\bar{V}}{\bar{V}} = \frac{\left(\prod_{E\in\mathcal{L}} \bar{N}_E\right)\bar{V}}{\bar{V}} \cong \prod_{E\in\mathcal{L}} \frac{\bar{N}_E\bar{V}}{\bar{V}},$$

with $(N_D\bar{V})\pi_E = \bar{N}_E\bar{V}_E$ for each $E \in \mathcal{L}$. Therefore (3) and (4c) follow from (1) and (4a) together with [AS, 1.4]. □

5.8. *Assume $\Lambda(\tau)$ is a C^*-lattice in which no connected component has a least element. Then N_H is transitive on $\mathcal{L}$.*

Proof. We may assume $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ is an N_H-invariant partition of $\mathcal{L}$ with $\mathcal{L}_i \neq \mathcal{L}$ for $i = 1, 2$. Let $D_i = \langle \mathcal{L}_i \rangle$, $G = N_H(D_1)$, and $\mu = \tau_G$. By 2.3, $\mu \in \mathcal{T}(L)$. Because $\Lambda = \Lambda(\tau)$ is a C^*-lattice, so is $\Sigma = \Lambda(\mu)$. Furthermore, D_1 and D_2 contain distinct minimal normal subgroups of G. Hence, applying Theorem 4.10 to μ, we conclude that either:

(i) Σ is connected; or
(ii) $I_H \trianglelefteq G_* = \mathcal{H}_*(\mu)$, $D_i \cong L$, and $\mathcal{P}'(\mu) = \{r_1, r_2\}$, where $r_i = (D_iI_H, D_iN_H)$.

By 5.5(4), M is transitive on $\mathcal{L}$; thus, since N_H is not transitive, we have $N_H < M$. Therefore, using 5.4(2), $d \in \mathcal{Q} = \mathcal{P}'(\tau_M)$ and so $\mathcal{Q}$ is nonempty. Next, by 5.4(4), $\mathcal{Q}$ is connected with least element d. Hence $\mathcal{Q}$ is contained in a connected component $\mathcal{C}_1$ of $\mathcal{P}'$. Furthermore, $\mathcal{P}^* = \mathcal{P}^*_D \cup \mathcal{P}^*_\infty$, where $\mathcal{P}^*_D \subseteq \mathcal{P}(\mu)$ and, by 5.5(3), $\mathcal{P}^*_\infty \subseteq \mathcal{Q}$. On the other hand, $\mathcal{P}'$ is disconnected, so it contains a second component $\mathcal{C}_2$. It follows in case (i) that $\mathcal{C}_1 = \mathcal{Q}$ and $\mathcal{C}_2 = \mathcal{P}'(\mu)$; and in case (ii), $\mathcal{C}_2 = \{r_i\}$ for $i = 1$ or 2. But by hypothesis, neither $\mathcal{C}_1$ or $\mathcal{C}_2$ has a least element, whereas $\mathcal{C}_1$ has least element d in case (i) and $\mathcal{C}_2$ has least element r_i in case (ii). □

HYPOTHESIS 5.9. Hypothesis 5.1 holds, $\Lambda(\tau)$ is a C_*-lattice, and N_H is transitive on $\mathcal{L}$.

5.10. *Assume Hypothesis 5.9. Then, for each $V \in \mathcal{W}$, V_D is the direct product of the subgroups V_E as E varies over $\mathcal{L}$.*

Proof. We may assume $|\mathcal{L}| > 1$. Let $p = (V, K) \in \mathcal{P}$ and define $\varphi(p) = (\bar{V}V, \bar{V}K)$. By Hypothesis 5.9, N_H is transitive on $\mathcal{L}$ and so, by 5.7(4b), $\bar{V}I_H \in \mathcal{W}$.

Observe that $N_H(V)$ acts on V_D and hence permutes the groups $\bar{V}_E$, $E \in \mathcal{L}$, so $N_H(V)$ acts on $\bar{V}$. If $N_0 \leq \bar{V}V$ then $N_D \leq \bar{V}V \cap D = \bar{V}(V \cap D) = \bar{V}$, contrary to $\bar{V}I_H \in \mathcal{W}$. Thus $N_0 \not\leq \bar{V}V$, so $\bar{V}V \in \mathcal{W}$ by 3.6(1). Therefore $\varphi(p) \in \mathcal{P}$ by 2.17. By construction, $\varphi(p) \leq p$.

Let $q = (U, J)$ and suppose $p \leq q$. Then $U \leq V$ and $J \leq K$. Hence $U_D \leq V_D$, so for each $E \in \mathcal{L}$ we have $\bar{U}_E \leq \bar{V}_E$ and hence $\bar{U} \leq \bar{V}$. Therefore $\varphi(p) \leq \varphi(q)$, so $\varphi\colon \mathcal{P} \to \mathcal{P}$ is a map of posets. Thus, since $\Lambda(\tau)$ is a C_*-lattice by Hypothesis 5.9, it follows from 1.1 that φ is the identity map on $\mathcal{P}$. Hence $V = \bar{V}V$, so $\bar{V} \leq V$, establishing the lemma. □

NOTATION 5.11. Pick $E \in \mathcal{L}$ and let $\pi : D \to E$ be the projection of D on E. For $X \leq H$, set $\hat{X} = N_X(E)$. Set $\hat{\tau} = (\hat{H}, \hat{N}_H, \hat{I}_H)$. Set $G = \mathrm{Aut}_H(E)$ and let $c\colon \hat{H} \to G$ be the conjugation map. Set $N_G = \hat{N}_H c$, $I_G = \hat{I}_H c$, and $\gamma = (G, N_G, I_G)$. Let $A = C_H(E)$, $B = A\hat{I}_H$, and $r = (B, A\hat{N}_H)$. Identify E with $\mathrm{Inn}(E) \leq G$ via c. Define $\mathcal{P}(\gamma)_E = \Gamma(\gamma, E)'$.

5.12. (1) $\hat{\tau} \in \mathcal{T}(L)$.

(2) $\varphi\colon \mathcal{P} \to \hat{\mathcal{P}} = \mathcal{P}(\hat{\tau})$ *is a map of posets, where* $\varphi(p) = (\hat{V}, \hat{K})$ *for* $p = (V, K) \in \mathcal{P}$.

(3) $\gamma \in \mathcal{T}(L)$ *and* $r \in \hat{\mathcal{P}}$.

(4) *Let* $\hat{\mathcal{R}} = \hat{\mathcal{P}}(\leq r)$,

$$\hat{\mathcal{V}} = \{V_1 \in \mathcal{W}(\hat{\tau}) : AV_1 \cap \hat{N}_H = \hat{I}_H\},$$

and $\hat{\mathcal{Q}} = \{(V_1, K_1) \in \hat{\mathcal{P}} : V_1 \in \hat{\mathcal{V}}\}$. *For* $p_1 = (V_1, K_1) \in \hat{\mathcal{Q}}$, *define* $\psi(p_1) = (V_1 c, K_1 c)$. *Then* $\psi\colon \hat{\mathcal{Q}} \to \mathcal{P}(\gamma)$ *is a map of posets that restricts to an isomorphism* $\psi\colon \hat{\mathcal{R}} \to \mathcal{P}(\gamma)$.

(5) *Assume Hypothesis 5.9.*

(a) *For each* $p \in \mathcal{P}$, $\varphi(p) \in \hat{\mathcal{Q}}$.

(b) *For* $p \in \mathcal{P}$, *define* $\phi(p) = \psi(\varphi(p))$; *then* $\phi\colon \mathcal{P} \to \mathcal{P}(\gamma)$ *is a map of posets.*

Proof. Parts (1) and (2) follow from the corresponding parts of 2.4.

Let $\sigma : N_D \to \bar{N}_E$ be the restriction of π to N_D. Because π is $\hat{H}$-equivariant, σ is $\hat{N}_H$ equivariant. Since $N_D\sigma = \bar{N}_E$ with $\bar{N}_E/\bar{I}_E \cong L \cong N_D/I_D$ by 5.7(1), σ induces an isomorphism of N_D/I_D with $\bar{N}_E/\bar{I}_E$. In particular, $C_{N_H}(E)$ centralizes N_D/I_D, so $C_{N_H}(E) \leq Q$. Thus, by 5.4(2), $C_{N_H}(E) \leq \hat{I}_H$ and so, since $C_{N_H}(E) = A \cap \hat{N}_H$, we have $A \cap \hat{N}_H \leq \hat{I}_H$. Therefore, (3) follows from parts (1) and (2) of 3.13 while (4) follows from 3.13(4).

Finally, assume Hypothesis 5.9 and let $p = (V, K) \in \mathcal{P}$. By 5.10, $V_D\pi = V_E$. Since π is $\hat{H}$-equivariant and $[V, N_D] \leq V_D$, it follows that $[A\hat{V}, \bar{N}_E] = [\hat{V}, \bar{N}_E] = [\hat{V}, N_D\pi] \leq V_D\pi = V_E$. Suppose $\varphi(p) \notin \hat{\mathcal{Q}}$. Then, by 3.6(1), $N_D \leq A\hat{V}$. But $X = \langle \bar{N}_F : F \in \mathcal{L} - \{E\}\rangle \leq A$, so $\bar{N}_E X = N_D X \leq A\hat{V}$. Thus $[\bar{N}_E, \bar{N}_E] \leq [A\hat{V}, \bar{N}_E] \leq V_E$ whereas, by 5.7(4) and 5.10, $\bar{N}_E/V_E = \bar{N}_E/\bar{V}_E \cong L$—a contradiction.

Thus (5a) holds. Then (5b) follows from (5a), (2), and (4). □

5.13. *Assume Hypothesis 5.9 and let* I_+ *be the kernel of the action of* I_H *on* $\mathcal{L}$. *Then* $I_+ = C_{N_H}(\bar{N}_D/I_D)$.

Proof. By 5.10, $I_D \trianglelefteq \bar{N}_D$. Let $\bar{N}_D^* = \bar{N}_D/I_D$. Since $\bar{N}_D^*$ is the direct product of the groups $\bar{N}_F^*$, $F \in \mathcal{L}$, it follows that $I = C_{N_H}(\bar{N}_D^*)$ is contained in the kernel of the action of N_H on $\mathcal{L}$. Since $F^*(N_H/I_H) = N_D I_H/I_H$, we have $I_H = C_{N_H}(N_D^*)$, so $I \leq I_H$ and hence $I \leq I_+$. Finally, 5.7(4c) says that, for each $F \in \mathcal{L}$, π_F induces an I_+-equivariant isomorphism of N_D^* with $\bar{N}_F^*$. Therefore, $I_+ \leq I$. □

5.14. *Assume Hypothesis 5.9. For $q = (U, J) \in \mathcal{P}(\gamma)_E$, define*

$$U\eta = \langle (U \cap E)^{N_H} \rangle, \quad J\eta = \langle (J \cap E)^{N_H} \rangle, \quad J\mu = N_{J\eta}(N_D U\eta),$$

and $\eta(q) = (U\eta I_H, J\mu N_H)$. Then:

(1) *the image of $\mathcal{P}_D$ under the map ϕ of 5.12(5) is contained in $\mathcal{P}(\gamma)_E$*;
(2) *$\eta\colon \mathcal{P}(\gamma)_E \to \mathcal{P}_D$ is a map of posets*;
(3) *$\eta \circ \phi = 1$ on $\mathcal{P}_D$, so ϕ is injective on $\mathcal{P}_D$ and induces an isomorphism of $\mathcal{P}_D$ with $\phi(\mathcal{P}_D) \leq \mathcal{P}(\gamma)$*;
(4) *$\phi(\eta(q)) = (U, J_+)$, $J_+ \leq J$, so $\phi(\eta(q)) \geq q$*;
(5) *ϕ induces an isomorphism of $\Delta(\tau, D)$ with $\Delta(\gamma, E)$ that has inverse η.*

Proof. Let $p = (V, K) \in \mathcal{P}_D$. Then $V \leq DI_H$, so $V = V \cap DI_H = (V \cap D)I_H = V_D I_H$. Then, since $V_D \leq \hat{H}$, we have $\hat{V} = V_D I_H \cap \hat{H} = V_D(I_H \cap \hat{H}) = V_D \hat{I}_H$. Similarly, $K = K_D N_H$ and $\hat{K} = K_D \hat{N}_H$. Thus $\phi(p) = \psi(V_D \hat{I}_H, K_D \hat{N}_H) = (V_E I_G, K_D \pi N_G) \in \mathcal{P}(\gamma)_E$ by 5.12(5). This establishes (1).

On the other hand, let $q = (U, J) \in \mathcal{P}(\gamma)_E$. Then U_E is N_G-invariant, so $U\eta$ is the direct product of the group U_E^n, $n \in N_H$, with $U_E^n = (U\eta)\pi_{E^n}$. Since U_E is N_G-invariant, $U\eta$ is N_H-invariant, and since $U \cap \bar{N}_E = I_E$, we have $U\eta \cap \bar{N}_D = I_D$. Thus $N_D \not\leq U\eta = U\eta I_H \cap D$, so $N_D \not\leq U\eta I_H$. Hence $W = U\eta I_H \in \mathcal{W}$ by 3.5. Furthermore, $W_D = W \cap D = U\eta I_D = U\eta$ and $W \leq DI_H$.

Similarly, $J\eta$ is the direct product of the groups $J_E^n, n \in N_H$, with $J_E^n = (J\eta)\pi_{E^n}$. Write J_{E^n} for J_E^n. Define I_+ as in 5.13. Since $I_+ \leq \hat{I}_H \leq C_H(J_E/U_E)$ and $I_+ \trianglelefteq N_H$, it follows that $[I_+, J\eta] \leq W_D$. Therefore, $W_D I_+ \trianglelefteq X = J\eta N_H$. Set $X^* = X/W_D I_+$. Then $(J\eta)^* \trianglelefteq X^*$ is the direct product of the groups $J_F^* \cong J_E/U_E$ and, by 5.13, $I_+ = C_{N_H}(\bar{N}_D^*)$. Thus $F^*(X^*) = \bar{N}_D^*$ is the direct product of the groups $F^*(J_F^*) = \bar{N}_F^* \cong L$. By 5.7(4c), N_D^* is a full diagonal subgroup of $\bar{N}_D^*$.

Let $Y = J\mu\hat{N}_H$. Now π induces a Y-equivariant isomorphism of N_D^* with $\bar{N}_E/U_E$, so $W_D\hat{I}_H = C_Y(\bar{N}_E/U_E) = C_Y(N_D^*)$ and hence $U\eta I_H = W_D I_H = C_{J\mu N_H}(N_D^*)$; thus $N_D U\eta I_H/U\eta I_H = F^*(J\mu N_H/U\eta I_H)$ and therefore $\eta(q) \in \mathcal{P}_D$. Clearly η is map of posets, so (2) holds.

Let $U = I_G \cap E$. Then $[U, \bar{N}_E] \leq \bar{N}_E \cap U = \bar{I}_E$, so $W = \langle U^{N_H} \rangle I_H$ centralizes $N_D/\bar{I}$. By 5.10, $\bar{I} = I_D$, so $r = (W, WN_H) \in \Delta(\tau, D)$ and $r \geq d$. Claim $U \leq I_E$. Suppose not. Then $r \neq \infty$. But for each $v = (V, N_H) \in \Delta(\tau, D)$, U acts on V_E, so W acts on $\bar{V}$. By 5.10, $V_D = \bar{V}$; then, since W centralizes N_D/I_D, we have $r \leq (WV, WVN_H) \geq v$ by 3.5 and so $\mathcal{C}(d) = \mathcal{C}(r) = \mathcal{C}(s)$. Then, by 5.5(3), $\Lambda(\tau)$ is connected, contrary to 5.1. Therefore, $I_G \cap E \leq I_E$.

Recall that $\phi(p) = (V_E I_G, K_D \pi N_G)$. Since $I_G \cap E \leq I_E$, we have $V_E I_G \cap E = V_E$ and so, by 5.10, $V_E \eta = V_D$. Next, $K_D \pi N_G \cap E = K_D \pi(N_G \cap E)$. Let P be the preimage in $\hat{N}_H$ under c of $N_G \cap E$. Then $K_D \leq (K_D \pi N_G)\mu$ and, since π induces a $K_D \hat{N}_H$-equivariant isomorphism of $N_D V_D / V_D$ with N_E / V_E, it follows that $(K_D \pi N_G)\mu \leq K_D P$. Therefore $(K_D \pi N_G)\mu N_H = K_D N_H$. That is, $\eta(\phi(p)) = p$, establishing (3).

Now $J\mu N_H \cap D = J\mu(N_H \cap D) = J\mu N_D = J\mu$ because $N_D \leq J\mu$. By construction, $J\mu c \leq J$. Similarly, $U\eta I_H \cap J\eta = U\eta$ and $U = (U\eta)c$. Thus $\phi(\eta(q)) = (U, J_+)$, where $J_+ = J\mu c N_G \leq J$. Hence (4) holds. In particular, if $J = UN_G$ then $J = J_+$, so (5) follows from (2) and (3). □

5.15. *Assume* Γ *is a sublattice of* $\Lambda(\tau)$ *that is isomorphic to* $\Delta(m)$ *for some* $m > 2$ *and contains* $0, \infty$. *Assume that* $\mathcal{P}(\geq x) \subseteq \Gamma$ *for each* $x \in \Gamma^\#$ *and that* $d \in \Gamma$. *Then*:

(1) $\mathcal{P}(\geq d) \subseteq \Delta(\tau, D)$;
(2) $Q = Q_D I_H$ *and* $M = QN_H$, *so* $M = Q_D N_H$;
(3) $H = DN_H$ *and* $G = EN_G$;
(4) *if Hypothesis 5.9 is satisfied and* $\Lambda(\tau)$ *is a C-lattice then* $\Lambda(\tau) = \Delta(\tau, D)$, $H = \mathcal{K}_*(\tau, D), G = \mathcal{K}_*(\gamma, E)$, *and* $\phi \colon \Lambda(\tau) \to \Delta(\gamma, E)$ *is an isomorphism.*

Proof. Let $J = \{1, \dots, m\}$ and let $(x_j = (V_j, K_j) : j \in J)$ be the set of co-atoms of Γ. For $\alpha \subseteq J$, set $x_\alpha = \bigwedge_{a \in \alpha} x_a = (V_\alpha, K_\alpha)$. Set $J_1 = \{j \in J : V_j = I_H\}$ and $J_2 = J - J_1$. We first observe that, by 2.10(3),

(5) if $J \neq \alpha \subseteq J_2$ then $V_\alpha = \langle V_a : a \in \alpha \rangle$ and $K_\alpha = V_\alpha N_H$.

Suppose $\alpha \subseteq J_1$. Then, by 5.5(3), $d \leq x_a$ for $a \in \alpha$ and so $d \leq x_\alpha$. Then we can apply 2.10(1) to conclude that:

(6) if $\alpha \subseteq J_1$ then $d \leq x_\alpha$, $K_\alpha = \langle K_a : a \in \alpha \rangle$, and $V_\alpha = K \cap Q$;
(7) if $j \in J_1$ and $i \in J_2$, then $V_{i,j} = V_i$ is K_j-invariant and $K_i = V_i N_H$.

For as $j \in J_1$, we have $V_j = I_H$ and so $K_j \neq N_H$ since $x_j \neq \infty$. Then, since $K_j \leq K_{i,j}$ and since $V_{i,j} \cap K_j = V_j$ by 2.10(1), we also have $K_{i,j} \neq V_{i,j} N_H$. But by hypothesis $\mathcal{P}(\geq x_{i,j}) \subseteq \Gamma$, so $\mathcal{P}(\geq x_{i,j}) \cong \Delta(2)$. Therefore, since $x_{i,j} < (V_{i,j}, V_{i,j} N_H) = x$, it follows that $x = x_i$; hence $V_i = V_{i,j}$ is $K_{i,j}$-invariant and $K_i = V_i N_H$. Then, since $K_j \leq K_{i,j}$, (7) follows.

(8) For $\alpha \subseteq J_1$ and $\beta \subseteq J_2$, V_β is K_α-invariant.

By (5), $V_\beta = \langle V_b : b \in \beta \rangle$, and by (6), $K_\alpha = \langle K_a : a \in \alpha \rangle$. Then (8) follows from (7).

(9) For $\beta \subseteq J_1$, $V_\beta = I_H$.

Choose a counterexample with $|\beta|$ minimal. Then $|\beta| > 1$, so $\beta = \alpha \cup \{j\}$ for some $j \in J_1$ and $\alpha \subseteq J_1$ with $V_\alpha = I_H$. Set $x = (V_\beta, V_\beta N_H)$. By (6), $d \leq x_\beta$. Thus $V_\beta \leq Q$, so $d \leq x$ and hence $0 \neq x \in \Gamma$. Also $x_\beta \leq x$, so $x = x_\gamma$ or $x_\gamma \wedge x_j$ for some $\gamma \subseteq \alpha$. But for $\gamma \subseteq \alpha$ we have $V_\gamma = I_H$, so $x = x_\gamma \wedge x_j$. Therefore $x \leq x_j$, so $K_j \leq V_\beta N_H$, a contradiction.

(10) For all $j \in J$ we have $V_j \neq I_H$, so $x_j = (V_j, V_j N_H)$ and $J = J_2$.

Assume otherwise, so that $J_1 \neq \emptyset$. Now $0 = x_{J_1} \wedge x_{J_2}$, where $x_{J_1} = (I_H, K_{J_1})$ by (9) and $x_{J_2} = (V_{J_2}, V_{J_2}N_H)$ by (5). By (8), K_{J_1} acts on V_{J_2}. Thus, by 2.17, $x_{J_1} \wedge x_{J_2} = (V_{J_2}, K_{J_2}) \neq 0$, a contradiction.

Set $J_3 = \{j \in J : V_j \nleq DI_H\}$ and $J_4 = J - J_3$. Set $\Delta = \Delta(\tau, D)$.

(11) $V_{J_4} = XI_H$, where $X = \langle V_j \cap D : j \in J_4\rangle \leq D$; in particular, $x_{J_4} \in \Delta$.

By (10), $J = J_2$, so by (5), $V_{J_4} = \langle V_j \cap D : j \in j_4\rangle I_H = XI_H$.

(12) Let $j \in J_4$ and $\alpha = J_3 \cup \{j\}$; then $V_\alpha \cap D = V_j \cap D \trianglelefteq K_\alpha$.

Set $U = V_\alpha \cap D$. Since $j \in J_4$ we have $V_j = (V_j \cap D)I_H$ and so, since $V_j \leq V_\alpha$, it follows that $V_j \leq UI_H$. Let $x = (UI_H, UN_H)$. Then $x_\alpha \leq x \leq x_j$ and $x \in \Delta$ as $U \leq D$. Because $x_\alpha \leq x$, $x = x_\beta$ for some $\beta \subseteq \alpha$. If $\beta \neq \{j\}$ then $x \leq x_i$ for some $i \in J_3$; thus, since $x \in \Delta$, also $x_i \in \Delta$ by 2.7, a contradiction. Then $\beta = \{j\}$ and so $x = x_j$. Therefore $(V_\alpha \cap D)I_H = V_j$, so

$$V_j \cap D = (V_\alpha \cap D)I_H \cap D = (V_\alpha \cap D)(I_H \cap D) = V_\alpha \cap D \trianglelefteq K_\alpha,$$

completing the proof of (12).

(13) Let $\alpha \subseteq J_3$ and $\emptyset \neq \beta \subseteq J_4$. Then:
  (a) $V_\beta \cap D$ is K_α-invariant; and
  (b) $N_D \nleq (V_\beta \cap D)V_\alpha$.

By (5) and (10), $V_\beta = \langle V_j : j \in \beta\rangle$. For $j \in J_4$, we have $V_j = (V_j \cap D)I_H$ and so $V_\beta \cap D = \langle V_j \cap D : j \in \beta\rangle$. Now (13a) follows from (12).

Next, $(V_\beta \cap D)V_\alpha \cap D = (V_\beta \cap D)(V_\alpha \cap D)$ and, by (12), for $j \in \beta$ we have $V_\alpha \cap D \leq V_{\alpha\cup\{j\}} \cap D = V_j \cap D \leq V_\beta$, so $(V_\beta \cap D)(V_\alpha \cap D) \leq V_\beta \cap D$. Thus, if $N_D \leq (V_\beta \cap D)V_\alpha$ then $N_D \leq V_\beta$, a contradiction. This establishes (13b).

We now establish (1). Assume (1) fails. Then $d \neq \infty$ and so, since $d \in \Gamma$, we have $d = x_\gamma$ for some $\gamma \subseteq J$. By 2.7, $d \notin \Delta$; then, since Δ is a sublattice, $\gamma \nsubseteq J_4$. Thus $J_3 \neq \emptyset$. If $J_4 = \emptyset$ then $d \leq x_i$ for each $i \in J$ by 5.5(3), contradicting $\Gamma \cong \Delta(m)$. Thus $J_4 \neq \emptyset$.

Let $\alpha = J_3$ and $\beta = J_4$. Then $0 = x_\alpha \wedge x_\beta$. But K_α acts on $X = V_\beta \cap D$ by (13), and $N_D \nleq XV_\alpha$. Thus, by 2.17, $x_\alpha \wedge x_\beta = (X, XN_H) \neq 0$—a contradiction. This completes the proof of (1).

By (1), $(Q, M) = d \in \Delta$ and hence $Q = Q_D I_H$ and $M = QN_H$. Thus (2) holds. Then (2) and 5.5(4) imply $H = DN_H$. Now, since $D \leq \hat{H}$, it follows that $\hat{H} = \hat{H} \cap DN_H = D\hat{N}_H$ and so $G = \hat{H}c = Dc\hat{N}_H c = EN_G$, establishing (3).

Assume the hypothesis of (4). Then $\phi\colon \Delta(\tau, D) \to \Delta(\gamma, E)$ is an isomorphism with inverse η by 5.14(5). We claim that $\Delta = \Delta(\tau, D) = \Lambda(\tau)$. Suppose the contrary; then, since $\Lambda(\tau)$ is a C-lattice, there exists a $p = (V, K) \in \mathcal{P}^*$ with $p \notin \Delta$. By 5.5(3), $p \geq d$. But now (1) supplies a contradiction, establishing the claim.

Since $\Delta = \Lambda(\tau)$, we have $\mathcal{H}_*(\tau) = \mathcal{K}_*(\tau, D)$ and so $H = \mathcal{K}_*(\tau, D)$ by 5.1. Let $G_1 = \mathcal{K}_*(\gamma, E)$ and suppose $G \neq G_1$. Then, since $G = EN_G$, we have $X_E = G_1 \cap E \neq E$ and so

$$X = \langle X_E^{N_H}\rangle = \prod_{F \in \mathcal{L}} X_F \neq D,$$

where $X_{E^n} = X_E^n$ for $n \in N_H$. Let $p = (V, K) \in \mathcal{P}^*$ and $p_1 = (V_1, K_1) = \phi(p)$. Then $V_1 \cap E \leq X_E$, so $V_D = V_1\eta \leq X$ and hence, since $\eta = \phi^{-1}$, $K = V_D N_H \leq XN_H$. Therefore $H = \mathcal{K}_*(\tau, D) = XN_H$. Then, since $X \trianglelefteq XN_H$, the unique minimal normal subgroup D of H is contained in X, contradicting X proper in D. This completes the proof of (4). □

6. Proof of Theorem 2

In this section we assume the following hypothesis.

HYPOTHESIS 6.1. For some integers $t > 1$ and $m_i > 2$, Λ is a $D\Delta(m_1, \ldots, m_t)$-lattice, L is a nonabelian finite simple group, and $\tau = (H, N_H, I_H) \in \mathcal{T}(L)$ with $\Lambda \cong \Lambda(\tau)$ and $|H|$ minimal subject to this constraint.

6.2. *Hypothesis 5.1 is satisfied.*

Proof. We begin by remarking that Hypothesis 6.1 implies Hypothesis 2.1. Because Λ is a $D\Delta(m_1, \ldots, m_t)$-lattice, Λ is a CD-lattice. By 2.14, $H = \mathcal{H}(\tau)$ and $\ker_{N_H}(H) = 1$. Then, by 4.11, $D = F^*(H)$ is a minimal normal subgroup of H; and, by 4.12, $DI_H \notin \mathcal{W}$. By 2.15, $H = \mathcal{H}_*(\tau)$, completing the proof. □

Set $D = F^*(H)$ and let $\mathcal{L}$ be the set of components of H.

6.3. *Hypothesis 5.9 is satisfied.*

Proof. By 5.8, N_H is transitive on $\mathcal{L}$. Therefore, since Λ is a C-lattice, the lemma follows from 6.2. □

6.4. *Adopt Notation 5.11. Then*:

(1) $\Lambda(\tau) = \Delta(\tau, D)$;
(2) $H = DN_H$;
(3) $G = EN_G$;
(4) $\phi\colon \Lambda(\tau) \to \Delta(\gamma, E)$ *is an isomorphism*;
(5) $G = \mathcal{K}_*(\gamma, E)$.

Proof. Let $\mathcal{C}$ be a connected component of $\Lambda(\tau)'$ with $d \in \mathcal{C}$ if $d \neq \infty$, and set $\Gamma = \mathcal{C} \cup \{0, \infty\}$. Since Λ is a $D\Delta(m_1, \ldots, m_t)$-lattice, $\Gamma \cong \Delta(m_i)$ for some i. Furthermore, $\mathcal{P}(\geq x) \subseteq \Gamma$ for all $x \in \Gamma^\#$, so the lemma follows from 6.3 and 5.15. □

Observe that Theorem 2 follows from [A2, Thm. 3] and 6.4. To see this, assume the hypotheses of Theorem 2 and let $(H, \tilde{G}) \in \mathcal{G}^*(\Lambda)$. We may assume that $\tilde{G}$ is not almost simple. Then [A2, Thm. 3] shows that $\tau = (H, N_H, I_H) \in \mathcal{T}(L)$, where L is a component of $\tilde{G}$, $N_H = N_H(L)$, and $I_H = C_H(L)$. Replace τ by a tuple in $\mathcal{T}(L)$ with $|H|$ minimal. Then Hypothesis 6.1 is satisfied. We adopt Notation 5.11 and appeal to 6.4: by construction, $E \leq G \leq \mathrm{Aut}(E)$, so G is almost simple with $F^*(G) = E$. By 5.12(3), $\gamma = (G, N_G, I_G) \in \mathcal{T}(L)$; by 6.4(3),

$G = EN_G$. Moreover, $\Lambda \cong \Lambda(\tau) \cong \Delta(\gamma, E)$ by 6.4(4), and $G = \mathcal{K}_*(\gamma, E)$ by 6.4(5). Finally, by 2.11(2), $\Xi(\gamma)$ is isomorphic to the dual of $\Delta(\gamma, E)$ and so, since $\Delta(\gamma, E) \cong \Lambda$ and since the $D\Delta(m_1, \ldots, m_t)$-lattice Λ is self-dual, it follows that $\Lambda \cong \Xi(\gamma)$. Similarly, $\mathcal{K}(\gamma) = \mathcal{K}_*(\gamma, E)$, so $G = \mathcal{K}(\gamma)$. Hence γ satisfies conclusion (2) of Theorem 2, so the proof of Theorem 2 is complete.

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